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# KORTEWEG-DE VRIES AND BENJAMIN-ONO EQUATIONS ON ZHIDKOV SPACES

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**Abstract.** Motivated by the study of the Cauchy problem with bore-like initial data, we show the "well-posedness" for Korteweg-de Vries and Benjamin-Ono equations with initial data in Zhidkov spaces $X^s$, with respectively $s > 1$ and $s > 5/4$. Here, "well-posedness" includes local (global in some cases) existence, uniqueness under a supplementary assumption, and continuity with respect to the initial data.

## 1. Introduction

In [9], the authors considered the Cauchy problems for Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations with bore-like initial data, namely

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(x,0) = g(x) \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t u + H\partial_x^2 u + u\partial_x u = 0 \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(x,0) = g(x) \end{cases} \tag{1.2}$$

where $H$ denotes the Hilbert transform, and $g$ satisfies

$$\begin{cases} i) & g(x) \to C_\pm \text{ as } x \to \pm\infty \ . \\ ii) & g' \in H^{s-1} \text{ where } s \geqslant 1 \ . \\ iii) & g - C_+ \in L^2([0,\infty)), \ g - C_- \in L^2((-\infty,0]) \ . \end{cases} \tag{1.3}$$

They showed the local well-posedness of (1.1) and (1.2) with initial data $g$ satisfying (1.3) , under the assumption $s > 3/2$. Global well-posedness was obtained for $s \geqslant 2$.

Our aim here is to improve this result by weakening the assumptions on the initial data $g$: we replace (1.3) by

$$g \in X^s, \tag{1.4}$$

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where $X^s$ denotes the Zhidkov space we introduced in [7] (see also [22]) for integer values of $s$:

$$X^s := \{f \in \mathcal{D}'(\mathbb{R}),\ f \in L^\infty,\ f' \in H^{s-1}(\mathbb{R})\}. \tag{1.5}$$

Moreover, (and that is probably the most interesting improvement), we assume only that $s > 1$ in the KdV case, and $s > 5/4$ in the BO case, instead of $s > 3/2$.

We first consider the KdV equation (1.1). Our strategy is as follows. Thanks to Lemma 2.1 below, for $g \in X^s$, there exists a function $\psi \in C^\infty(\mathbb{R})$ with $\psi' \in H^\infty$ such that $\phi = g - \psi \in H^s$ (we remark that this implies that $\psi$ is bounded, since $s \geqslant 1$). Similarly to [9], we write a solution $u$ of (1.1) as $u = v + \psi$, and we study the Cauchy problem associated with $v$, namely

$$\begin{cases} \partial_t v + \partial_x^3 v + v\partial_x v + \partial_x(v\psi) = -(\partial_x^3\psi + \psi\psi')\ ,\ x \in \mathbb{R},\ t \in \mathbb{R} \\ v(x,0) = \phi(x) = g(x) - \psi(x) \end{cases} \tag{1.6}$$

Our main result is as follows:

**Theorem 1.1.** *Let $\psi \in C_b^\infty(\mathbb{R})$ such that $\psi' \in H^\infty$, $s > 1$ and $\phi \in H^s(\mathbb{R})$. Then there exists $T = T(\psi, s, ||\phi||_{H^s}) > 0$ and a unique $v \in C([-T,T], H^s) \cap C^1([-T,T], H^{s-3})$ solving (1.6), and such that $v_x \in L^1([-T,T], L^\infty)$.*

*Moreover, for any $R > 0$, the map $\phi \to v$ is continuous from the ball of radius $R$ in $H^s(\mathbb{R})$ to $C([-T(R), T(R)], H^s)$.*

From this local well-posedness result for (1.6) we deduce a local well-posedness result for (1.1):

**Theorem 1.2.** *Let $s > 1$, $g \in X^s$. Then there exists $\tilde{T} = \tilde{T}(s, ||g||_{X^s}) > 0$ and a unique solution $u$ of (1.1) such that $u \in C([-\tilde{T}, \tilde{T}], X^s)$, $u - g \in C([-\tilde{T}, \tilde{T}], H^s)$ and $u_x \in L^1([-\tilde{T}, \tilde{T}], L^\infty)$.*

*Moreover, for any $R > 0$, the map $g \to u$ is continuous from the ball of radius $R$ in $X^s$ to $C([-\tilde{T}(R), \tilde{T}(R)], X^s)$.*

Equation (1.6) is just the KdV equation perturbed by some terms. In the case of KdV, Kenig, Ponce, and Vega ([16]) showed the local well-posedness in $H^s$, with $s > -3/4$, by the contraction principle. As was mentioned in [9], this method fails here. Indeed, it does not seem to be possible to get an appropriate estimate on the $H^s$ norm of the term $\psi\partial_x v$, because $\psi$ does not vanish at infinity. Bourgain's method (see [8]) fails for the same reason, as well as the method used by Kenig and Koenig in [12]. Iorio, Linares, and Scialom ([9]) used a parabolic regularization and the Bona-Smith approximation. To improve their result, we use here the method that was employed by Koch and Tzetvkov in [17] to show the local well-posedness of BO in $H^s$, $s > 5/4$. Namely, we show Strichartz estimates for a linearized version of (1.6). Next we derive the crucial nonlinear estimate, using the Littlewood-Paley decomposition. To get an a priori estimate on the $H^s$ norm of a solution of (1.6), a commutator lemma due to Kato ([10]) was used in [9]. This lemma fails for $s \leqslant 3/2$. We use and prove here a new commutator lemma (Lemma 2.4 below), which is a variant of a lemma due to Kato and Ponce (see Lemma 2.3 below or [11]).

Our method gives similar results for the BO equation:

**Theorem 1.3.** *Let $\psi \in C_b^\infty(\mathbb{R})$ such that $\psi' \in H^\infty$, $s > 5/4$ and $\phi \in H^s(\mathbb{R})$. Then there exists $T = T(\psi, s, ||\phi||_{H^s})$ and a unique solution $v$ of*

$$\begin{cases} \partial_t v + H\partial_x^2 v + v\partial_x v + \partial_x(v\psi) = -(H\partial_x^2\psi + \psi\psi') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(x,0) = \phi(x) = g(x) - \psi(x), \end{cases} \tag{1.7}$$

*such that $v \in C([-T,T], H^s) \cap C^1([-T,T], H^{s-2})$ and $v_x \in L^1([-T,T], L^\infty)$.*

*Moreover, for any $R > 0$, the map $\phi \to v$ is continuous from the ball of radius $R$ in $H^s(\mathbb{R})$ to $C([-T(R), T(R)], H^s)$.*

**Theorem 1.4.** *Let $s > 5/4$, $g \in X^s$. Then there exists $\tilde{T} = \tilde{T}(s, ||g||_{X^s}) > 0$ and a unique solution $u$ of (1.2) such that $u \in C([-\tilde{T}, \tilde{T}], X^s)$, $u - g \in C([-\tilde{T}, \tilde{T}], H^s)$ and $u_x \in L^1([-\tilde{T}, \tilde{T}], L^\infty)$.*

*Moreover, for any $R > 0$, the map $g \to u$ is continuous from the ball of radius $R$ in $X^s$ to $C([-\tilde{T}(R), \tilde{T}(R)], X^s)$.*

In fact, it also works for all the dispersions between BO and KdV. Namely, for $\alpha \in [1,2]$, the problem

$$\begin{cases} \partial_t v - D^\alpha \partial_x v + v\partial_x v + \partial_x(v\psi) = D^\alpha\partial_x\psi - \psi\psi' \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ v(0) = \phi \in H^s, \end{cases} \tag{1.8}$$

where $D = (-\partial_x^2)^{1/2}$, is locally well posed in $H^s$ for $s > 3/2 - \alpha/4$.

In [9], a global well-posedness result was obtained for (1.7) in $H^s$ for $s \geqslant 2$. Since our local well-posedness result goes below $3/2$, using the invariant of the Benjamin-Ono equation associated with the $H^{3/2}$ norm, we improve this result:

**Theorem 1.5.** *Let $\phi \in H^s$, $s \geqslant 3/2$. Then the solution of* (1.7) *obtained in Theorem* 1.3 *can be extended to $\mathbb{R}$.*

**Corollary 1.1.** *Let $g \in X^s$, $s \geqslant 3/2$. Then the solution of* (1.2) *obtained in Theorem* 1.3 *can be extended to $\mathbb{R}$.*

Note that if Theorem 1.1 were true for $s = 1$, we would have a global well-posedness result. We failed to show this for general $\psi \in C_b^\infty$ with $\psi' \in H^\infty$. However, if $\psi \equiv a$ is a constant, a change of variables shows that $v$ solves (1.6) if and only if $w(x,t) := v(x + at, t)$ solves the classical KdV equation, which is known to be globally well-posed in $H^1$. For other dispersions, if we assume that $\psi$ is constant, the results of Kenig and Koenig similarly ensure that (1.8) is locally well-posed in $H^s$, $s > 3/2 - 3\alpha/8$.

The proof of Theorem 1.3 (resp. 1.4) is similar and simpler to that of Theorem 1.1 (resp. 1.2), so we will omit it.

This paper is organized as follows. In Section 2, we state some preliminary results, including the crucial commutator lemmas. In Section 3, we prove Strichartz estimates for a linearized version of (1.6). In Section 4, we prove a nonlinear estimate. The proof is based on the Littlewood-Paley theory. In the last two sections, the main ideas are these of Koch and Tzvetkov explained in [17]. In Section 5 and 6, we prove Theorem 1.1. In Section 7, we derive Theorem 1.2. In Section 8, we prove Theorem 1.5. Finally, we prove Lemma 2.4 in the appendix. The proof of this commutator lemma is inspired by that of Lemma 2.3 given in [11].

**Notations.** Throughout this paper, the notation $A \lesssim B$ means that there exists a harmless constant $c > 0$ such that $A \leqslant cB$.

We denote by $H^\infty$ the space $H^\infty = \bigcap_{s \geqslant 0} H^s$, $C_b^\infty$ the space of $C^\infty$ bounded functions and $\mathcal{S}$ the space of Schwartz functions.

If $X$ is a Banach space, $T$ a positive number and $I \subset \mathbb{R}$ an interval, we define $L_T^p X := L^p([-T,T], X)$ and $L_I^p X := L^p(I, X)$ equipped with their natural norms. We denote $L_T^\infty H^\infty := \bigcap_{s \geqslant 0} L_T^\infty H^s$.

For $\sigma \geqslant 0$, we denote $J^\sigma := (1 - \partial_x^2)^{\sigma/2}$, $D^\sigma := (-\partial_x^2)^{\sigma/2}$.
The letters $\lambda$ and $\mu$ will denote dyadic integers. The notation $\sum_\lambda f(\lambda)$ should be understood as $\sum_{k=0}^\infty f(2^k)$.

We call $(q,p) \in \mathbb{R}^2$ an admissible pair if

$$(q,p) = \left(\frac{6}{\theta(\beta+1)}, \frac{2}{1-\theta}\right),\ (\theta,\beta) \in [0,1] \times [0,1/2]\ .$$

Note that to prove Theorems 1.3 and 1.4 dealing with the BO equation, we should replace this definition by:

$$\frac{2}{q} + \frac{1}{p} = \frac{1}{2},\ q \in (4,\infty),\ p \in (2,\infty)\ .$$

We recall that the solution of the initial value problem

$$\begin{cases} \partial_t v + \partial_x^3 v = 0 \ , \ t, x \in \mathbb{R} \\ v(x,0) = v_0(x) \end{cases} \tag{1.9}$$

is given by the unitary Airy group which will be denoted by $\{W(t)\}_{t\in\mathbb{R}}$, i.e., $v(.,t) = W(t)v_0 = S_t \star v_0$ where for $t > 0$

$$S_t = t^{-1/3}K(t^{-1/3}.) \ \text{ and } \ K(x) = c\int_{-\infty}^{\infty} e^{i(\xi^3 + x\xi)} d\xi, \ x \in \mathbb{R} \ .$$

## 2. Preliminary results

We now state a result that gives a decomposition of the initial data $g$, which is used to reduce the study of (1.1) (respectively (1.2)) to that of (1.6) (respectively (1.7)).

**Lemma 2.1.** *Let $g \in X^s, \ s \geqslant 1$. Then there exists $\psi \in C^\infty(\mathbb{R}), \ \phi \in H^s(\mathbb{R})$ such that*

*i)* $\psi' \in H^\infty(\mathbb{R})$.
*ii)* $g = \psi + \phi$.

*Moreover, the maps $g \to \psi$ and $g \to \phi$ can be defined as linear maps such that for every $s_1 \geqslant 1$, $g \to \psi$ is continuous from $X^s$ into $X^{s_1}$ and $g \to \phi$ is continuous from $X^s$ into $H^s$.*

**Proof.** Let $k(x) = (4\pi)^{-1/2}e^{-x^2/4}$, $\psi := k \star g$. Then $\psi \in C_b^\infty(\mathbb{R})$, $\psi' = k \star g' \in H^\infty$, therefore, $g - \psi \in L^\infty \subset \mathcal{S}'$ and

$$\widehat{g-\psi}(\xi) = (\hat{g} - \hat{\psi})(\xi) = (1 - \hat{k})\hat{g}(\xi) = \underbrace{\frac{1 - e^{-\xi^2}}{\xi}}_{\in L^\infty} \underbrace{\xi\hat{g}(\xi)}_{\in L^2} \ .$$

Hence, $g - \psi \in L^2$, and $g - \psi \in H^s$. Moreover, $||\psi||_{L^\infty} \leqslant ||g||_{L^\infty}$ and

$$||\psi'||^2_{H^{s_1-1}} = \int (1+\xi^2)^{s_1-1} e^{-2\xi^2} |\xi\hat{g}(\xi)|^2 d\xi \leqslant \sup_{\xi\in\mathbb{R}} \Big( (1+\xi^2)^{s_1-1} e^{-2\xi^2} \Big) ||g'||^2_{L^2},$$

which shows the continuity of $g \to \psi$ from $X^s$ into $X^{s_1}$. Similarly,

$$||\phi||^2_{H^s} \leqslant \sup_{\xi\in\mathbb{R}} \Big( (1+\xi^2) \Big( \frac{1 - e^{-\xi^2}}{\xi} \Big)^2 \Big) ||g'||^2_{H^{s-1}}$$

gives the continuity of $g \to \phi$ from $X^s$ into $H^s$. □

**Lemma 2.2.** *Let $(a_\lambda)_\lambda$, $(d_\lambda)_\lambda$ be two sequences indexed on dyadic integers $\lambda = 2^j,\ j \in \mathbb{N}$. Let $s > 0$. Then*

$$\sum_\lambda \lambda^s \sum_{\mu \geqslant \lambda/8} a_\mu d_\lambda \lesssim \Big(\sum_\lambda \lambda^{2s} a_\lambda^2\Big)^{1/2} \Big(\sum_\lambda d_\lambda^2\Big)^{1/2},$$

*and hence by duality*

$$\sum_\lambda \lambda^{2s} \Big(\sum_{\mu \geqslant \lambda/8} a_\mu\Big)^2 \lesssim \sum_\lambda \lambda^{2s} a_\lambda^2 \ .$$

**Proof.** Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_\lambda \lambda^s \sum_{\mu \geqslant \lambda/8} a_\mu d_\lambda &= \sum_\lambda \lambda^s \sum_{\substack{k=-3 \\ 2^k\lambda \geqslant 1}}^{\infty} a_{2^k\lambda} d_\lambda = \sum_{k=-3}^{\infty} 2^{-ks} \sum_{\lambda \geqslant 2^{-k}} (2^k\lambda)^s a_{2^k\lambda} d_\lambda \\
&\leqslant \sum_{k=-3}^{\infty} 2^{-ks} \Big(\sum_{\lambda \geqslant 2^{-k}} \Big((2^k\lambda)^s a_{2^k\lambda}\Big)^2\Big)^{1/2} \Big(\sum_{\lambda \geqslant 2^{-k}} d_\lambda^2\Big)^{1/2} \\
&\leqslant \frac{2^{3s}}{1-2^{-s}} \Big(\sum_\lambda \lambda^{2s} a_\lambda^2\Big)^{1/2} \Big(\sum_\lambda d_\lambda^2\Big)^{1/2}.
\end{aligned}$$

**Commutator and Bilinear Estimates.** We will use in the sequel two commutator lemmas. The first one is due to Kato and Ponce and is proved in [11].

**Lemma 2.3.** *If $s > 0$,*

$$||\,[J^s, f]\, g||_{L^2} \lesssim ||\partial_x f||_{L^\infty} ||J^{s-1} g||_{L^2} + ||J^s f||_{L^2} ||g||_{L^\infty} \ . \tag{2.1}$$

The second commutator lemma is proved in the appendix.

**Lemma 2.4.** *Let $s > 0$ and let $s_0 \geqslant \max(0, 3-s)$. Then*

$$||\,[J^s, f]\, g||_{L^2} \lesssim ||\partial_x f||_{L^\infty} ||J^{s-1} g||_{L^2} + ||J^{s_0+s-1} \partial_x f||_{L^2} ||g||_{L^\infty} \ . \tag{2.2}$$

The next lemmas (see [21]) will be used in the proof of Theorem 1.5.

**Lemma 2.5.** *Let $a, b, c \in \mathbb{R}$ such that $a \geqslant c$, $b \geqslant c$, $a + b \geqslant 0$ and $a + b - c > n/2$. Then the map $(f, g) \longrightarrow fg$ is a continuous bilinear form from $H^a(\mathbb{R}^n) \times H^b(\mathbb{R}^n)$ into $H^c(\mathbb{R}^n)$.*

**Lemma 2.6.** *If $s \geqslant 1$, then there is a constant $C$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$,*

$$||\,[J^s, f]\, g||_{L^2} \leqslant C ||f'||_{H^s} ||g||_{H^{s-1}} \ .$$

We will need a generalized version of Lemma 2.6.

**Lemma 2.7.** *Let $n = 1$. In Lemma 2.6 above, the assumption $f \in \mathcal{S}$ can be replaced by $f \in X^\infty := \bigcap_{s \geqslant 1} X^s$.*

**Proof.** Let $f \in X^\infty$. For $\varepsilon > 0$, $f_\varepsilon(x) := e^{-\varepsilon x^2} f(x)$ belongs to $\mathcal{S}$, therefore we can apply Lemma 2.6 to $f_\varepsilon$. Passing to the limit, we obtain Lemma 2.7. □

In Sections 3 to 6 below, we are interested in solving (1.6) where the functions $\psi$ and $\phi$ are fixed.

## 3. The linear estimate

We first recall the Strichartz estimate with smoothing for the Airy group (see [15]).

**Lemma 3.1.** *For any admissible pair $(q,p) = (6/(\theta(\beta+1)), 2/(1-\theta))$ with parameters $(\theta,\beta) \in [0,1] \times [0,1/2]$, and $u_0 \in L^2(\mathbb{R})$, we have*

$$||D^{\frac{\theta\beta}{2}} W(t) u_0||_{L^q_t(\mathbb{R},L^p)} \lesssim ||u_0||_{L^2(\mathbb{R})} \ .$$

As in [17], we deduce next from Lemma 3.1 a Strichartz inequality for a linearized version of (1.6).

**Lemma 3.2.** *Let $\lambda \geqslant 1$, $T > 0$, $\sigma > 1/2$. Let $u : [-T,T] \times \mathbb{R} \to \mathbb{R}$ be a regular solution of*

$$\partial_t u + \partial_x^3 u + V_1 \partial_x u + V_2 \partial_x u + V_3 u = f \tag{3.1}$$

*where $V_1 \in L^\infty_T H^\sigma$, $V_2 \in L^\infty_T L^\infty$, $V_3 \in L^\infty_T H^\infty$ and $f \in L^1_T L^2$.*

*We assume moreover that there exists a constant $C > 0$ such that*

$$Supp(\hat{u}(.,t)) \subset C[-\lambda,\lambda], \ t \in [-T,T].$$

*Then for every admissible pair $(q,p)$ with parameter $\beta = 1/2$ fixed, for any interval $I \subset [-T,T]$ such that $|I| \lesssim \lambda^{-1}$,*

$$\begin{aligned} ||D^{\theta/4} u||_{L^q_I L^p} \lesssim \Big(1 + ||J^\sigma V_1||_{L^\infty_T L^2} + ||V_2||_{L^\infty_T L^\infty} + ||V_3||_{L^1_T L^\infty}\Big) \\ \times \Big(||u||_{L^\infty_I L^2} + ||f||_{L^1_I L^2}\Big). \end{aligned} \tag{3.2}$$

*Moreover,*

$$\begin{aligned} ||D^{\theta/4} u||_{L^q_T L^p} \lesssim (1+T)^{1/q} \lambda^{1/q} \big(1 + ||J^\sigma V_1||_{L^\infty_T L^2} + ||V_2||_{L^\infty_T L^\infty} + ||V_3||_{L^1_T L^\infty}\big) \\ \times \Big(||u||_{L^\infty_T L^2} + ||f||_{L^1_T L^2}\Big). \end{aligned} \tag{3.3}$$

**Proof.** The solution $v$ of the Cauchy problem

$$\partial_t v + \partial_x^3 v = h \in L^1L^2, \quad v(0) = v_0 \in L^2$$

is given by

$$v(t) = W(t)v_0 + \int_0^t W(t-s)h(s)ds.$$

Applying $D^{\theta/4}$ to this equation, we obtain that for any $t \in [-T,T]$,

$$||D^{\theta/4}v(t)||_{L_x^p} \leqslant ||D^{\theta/4}W(t)v_0||_{L_x^p} + \int_{-T}^{T} ||D^{\theta/4}W(t)W(-s)h(s)||_{L_x^p} ds,$$

hence Lemma 3.1 yields

$$||D^{\theta/4}v||_{L_T^qL^p} \lesssim ||v_0||_{L^2} + ||h||_{L_T^1L^2}.$$

We apply this to $u$:

$$\begin{aligned} ||D^{\theta/4}u||_{L_I^qL^p} \quad \leqslant \quad & ||u||_{L_I^\infty L^2} + (||V_1||_{L_I^\infty L^\infty} + ||V_2||_{L_I^\infty L^\infty})||\partial_x u||_{L_I^1L^2} \\ & + ||V_3||_{L_I^1L^\infty}||u||_{L_I^\infty L^2} + ||f||_{L_I^1L^2}. \end{aligned}$$

Now, as in [KT], using the Sobolev embedding, we have

$$||V_1||_{L_I^\infty L^\infty} \lesssim ||J^\sigma V_1||_{L_I^\infty L^2},$$

Next, using the assumptions on the support of $\hat{u}(.,t)$ and on the length of $I$ and Plancherel's theorem, we get

$$||\partial_x u||_{L_I^1L^2} = ||\xi\hat{u}||_{L_I^1L^2} \lesssim \lambda|I|||\hat{u}||_{L_I^\infty L^2} \lesssim ||u||_{L_I^\infty L^2},$$

which completes the proof of (3.2). The proof of (3.3) is the same as that in [17]: we write $[-T,T] = \bigcup_{k=1}^{n} I_k$ where $|I_k| \leqslant \lambda^{-1}$. We may assume that $n < 1 + 2\lambda T \leqslant 2\lambda(1+T)$, and hence using (3.2) applied to $I_k$ and summing over $k$, we get (3.3). □

## 4. The nonlinear estimate

In the sequel, we use the same notations as that in [17] about the Littlewood -Paley decomposition. Namely,

$$u = \sum_\lambda u_\lambda,$$

where $u_\lambda := \Delta_\lambda u$, and the Fourier multiplier $\Delta_\lambda$ is defined by

$$\widehat{\Delta_\lambda u}(\xi) := \begin{cases} \phi(\xi/\lambda)\hat{u}(\xi) & \lambda = 2^k, k \geqslant 1 \\ \chi(\xi)\hat{u}(\xi) & \lambda = 1 \end{cases},$$

where $\chi$ and $\phi$ are nonnegative, $C_c^\infty$ functions on $\mathbb{R}$ satisfying

$$\chi(\xi) + \sum_{\lambda > 1} \phi(\xi/\lambda) = 1$$

and

$$\phi(\xi) = \begin{cases} 0 & \text{if } |\xi| < 5/8 \text{ or } |\xi| > 2 \\ 1 & \text{if } 1 < |\xi| < 5/4 \end{cases} .$$

For a dyadic integer $\lambda$, we also define

$$\tilde{\Delta}_\lambda := \begin{cases} \Delta_{\lambda/2} + \Delta_\lambda + \Delta_{2\lambda} & \text{if } \lambda > 1 \\ \Delta_1 + \Delta_2 & \text{if } \lambda = 1 \end{cases} .$$

Let $u$ be a regular solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u + \psi\partial_x u + \psi' u = -(\partial_x^3 \psi + \psi\psi') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ u(0) = u_0 \in H^\infty \end{cases} . \tag{4.1}$$

By "regular solution" we mean that $u \in \underset{s \geqslant 0}{\cap} C(\mathbb{R}, H^s)$. Theorem 1.2 in [9] ensures that such a solution does exist (throughout [9], the assumption (1.3) can be replaced by "$g \in X^s$" for free).

Our aim in this section is to prove the following estimate on $u$.

**Theorem 4.1.** *Let $\sigma > 1/2$, $0 < T \leqslant 1$, $(q,p)$ an admissible pair with parameters $\theta \in [0,1)$, $\beta = 1/2$, $s = \sigma + 1/q$ and $u$ a regular solution of* (4.1). *Then*

$$||D^{\theta/4} J^\sigma u||_{L^q_T L^p} \lesssim (1 + ||J^\sigma u||_{L^\infty_T L^2})(1 + ||u_x||_{L^1_T L^\infty})^{3/2}(1 + ||J^s u||^2_{L^\infty_T L^2})^{1/2}. \tag{4.2}$$

We split the proof into several lemmas.

**Lemma 4.1.** *Let $(q,p)$ be an admissible pair with parameter $\theta \in [0,1)$ (i.e., $p < \infty$) and let $\sigma > 1/2$. Then*

$$||D^{\theta/4} J^\sigma u||_{L^q_T L^p} \lesssim \Big( \sum_\lambda \lambda^{2\sigma} ||D^{\theta/4} u_\lambda||^2_{L^q_T L^p} \Big)^{1/2}. \tag{4.3}$$

**Proof.** A similar lemma was stated in [17]. We recall the proof. We define $v := D^{\theta/4} u$. We have

$$\begin{aligned} ||J^\sigma v||_{L^q_T L^p} &= \Big( \int_{-T}^T || \sum_\lambda J^\sigma v_\lambda(t) ||^q_{L^p_x} dt \Big)^{1/q} \\ &\leqslant \Big( \int_{-T}^T || \Big( \sum_\lambda |J^\sigma v_\lambda(t)|^2 \Big)^{1/2} ||^q_{L^p_x} dt \Big)^{1/q} \end{aligned} \tag{4.4}$$

$$= \Big( \int_{-T}^{T} \Big( \int_x \Big( \sum_\lambda |J^\sigma v_\lambda(t)|^2 \Big)^{p/2} dx \Big)^{q/p} dt \Big)^{1/q}$$

$$\leqslant \Big( \int_{-T}^{T} \Big( \sum_\lambda \Big( \int_x |J^\sigma v_\lambda(t)|^{2.p/2} dx \Big)^{2/p} \Big)^{q/2} dt \Big)^{1/q} \tag{4.5}$$

$$||J^\sigma v||_{L^q_T L^p} \leqslant \Big( \int_{-T}^{T} \Big( \sum_\lambda ||J^\sigma v_\lambda(t)||^2_{L^p_x} \Big)^{q/2} dt \Big)^{1/q}$$

$$\leqslant \Big( \sum_\lambda \Big( \int_{-T}^{T} ||J^\sigma v_\lambda(t)||^{2.q/2}_{L^p_x} dt \Big)^{2/q} \Big)^{1/2} \tag{4.6}$$

$$= \Big( \sum_\lambda ||J^\sigma v_\lambda||^2_{L^q_T L^p_x} \Big)^{1/2} . \tag{4.7}$$

Here we have used the square functions theorem for the Littlewood-Paley decomposition (see [20]) to obtain (4.4) ($1 < p < \infty$), and the Minkowski inequality (see [18]) to get (4.5) and (4.6) (it works because $p/2 \geqslant 1$ and $q/2 \geqslant 1$, respectively). Next, using the Mikhlin-Hörmander theorem (or more precisely Lemma 6.2.1 in [4]), we get, for all $t$,

$$||J^\sigma v_\lambda(t)||_{L^p} \lesssim \lambda^\sigma ||v_\lambda(t)||_{L^p} . \tag{4.8}$$

(4.7) and (4.8) complete the proof of Lemma 4.1. □

**Lemma 4.2.** *There exists a constant $C > 0$ such that for all $w \in L^2$ and $v \in C_b^\infty$ such that $v_x \in H^\infty$,*

$$|| \, [\Delta_\lambda, v\partial_x] \, w||_{L^2} \leqslant C||v_x||_{L^\infty}||w||_{L^2}.$$

**Proof.** By the density of $\mathcal{S}$ in $L^2$, it suffices to show this for $w \in \mathcal{S}$. As in the proof of Lemma 2 in [17], we write for $\lambda \geqslant 2$

$$[\Delta_\lambda, v\partial_x] \, w(x) = \int_{-\infty}^{\infty} K(x,y)w(y)dy \ ,$$

where

$$K(x,y) = c \int_{-\infty}^{\infty} e^{i\lambda(x-y)\eta} \phi(\eta) \left[ i\lambda^2 \eta (v(y) - v(x)) - \lambda v_x(y) \right] d\eta \ .$$

Therefore, using the mean-value theorem,

$$|K(x,y)| \leqslant c\lambda ||v_x||_{L^\infty} g(\lambda(y-x)),$$

where $g$ is in $L^1$. Hence

$$\sup_y \int_{-\infty}^{\infty} |K(x,y)|dx + \sup_x \int_{-\infty}^{\infty} |K(x,y)|dy \lesssim ||v_x||_{L^\infty},$$

and the Schur lemma completes the proof of the lemma in the case $\lambda \geqslant 2$. The proof is similar in the case $\lambda = 1$. □

**Lemma 4.3.** *There exists a constant $C > 0$ such that for all $v \in C_b^\infty$ and $w \in L^2$,*

$$||\,[\Delta_\lambda, v]\, w||_{L^2} \leqslant C||v||_{L^\infty}||w||_{L^2}.$$

**Proof.** The proof is similar to, and easier than, that of Lemma 4.2:

$$\begin{aligned}[\Delta_\lambda, v]\, w(x) &= c\lambda \int_{-\infty}^{\infty} \hat{\phi}(\lambda(y-x))(v(y)-v(x))w(y)dy \\ &\leqslant 2c||v||_{L^\infty} \int_{-\infty}^{\infty} \lambda|\hat{\phi}(\lambda(y-x))||w(y)|dy,\end{aligned}$$

and the Schur lemma completes the proof similarly to Lemma 4.2. □

**Lemma 4.4.** *Let $\sigma > 1/2$, $(q,p)$ an admissible pair, $T > 0$ and $u$ a regular solution of* (4.1)*. Then*

$$\begin{aligned}\sum_\lambda \lambda^{2\sigma}||D^{\theta/4}u_\lambda||^2_{L^q_T L^p} &\lesssim (1+T)^{\frac{2}{q}}\big(1 + ||J^\sigma u||_{L^\infty_T L^2} + ||\psi||_{L^\infty} + T||\psi'||_{L^\infty}\big)^2 \\ \times \Big[\big(1+||u_x||^2_{L^1_T L^\infty} &+T^2||\psi'||^2_{L^\infty}\big)\sum_\lambda \lambda^{2\sigma+\frac{2}{q}}||u_\lambda||^2_{L^\infty_T L^2} + ||\psi'||^2_{H^{\sigma+1/q}}||u_x||^2_{L^1_T L^\infty} \\ &+ T^2||\psi\psi' + \partial_x^3\psi||^2_{H^{\sigma+1/q}} + T^2||J^\sigma u||^2_{L^\infty_T L^2}||\psi'||^2_{H^{\sigma+1/q}}\Big].\end{aligned}$$

**Proof.** We apply $\Delta_\lambda$ to (4.1):

$$\begin{aligned}&\partial_t u_\lambda + \partial_x^3 u_\lambda + (u+\psi)\partial_x u_\lambda + \psi' u_\lambda \\ &\quad = -\Delta_\lambda(\partial_x^3\psi + \psi\psi') - [\Delta_\lambda, (u+\psi)\partial_x]\, u - \big[\Delta_\lambda, \psi'\big]\, u.\end{aligned} \tag{4.9}$$

Therefore, we can apply Lemma 3.2 to $u_\lambda$, with $V_1 = u$, $V_2 = \psi$, $V_3 = \psi'$, $f = -\Delta_\lambda(\partial_x^3\psi + \psi\psi') - [\Delta_\lambda, (u+\psi)\partial_x]\, u - [\Delta_\lambda, \psi']\, u$. Hence,

$$\begin{aligned}\sum_\lambda \lambda^{2\sigma}||D^{\frac{\theta}{4}}u_\lambda||^2_{L^q_T L^p} &\lesssim (1+T)^{\frac{2}{q}}\Big(1 + ||J^\sigma u||_{L^\infty_T L^2} + ||\psi||_{L^\infty} + T||\psi'||_{L^\infty}\Big)^2 \\ \times \sum_\lambda \lambda^{2\sigma+2/q}\Big(||u_\lambda||_{L^\infty_T L^2} &+ T||\Delta_\lambda(\psi\psi' + \partial_x^3\psi)||_{L^2} + ||\,[\Delta_\lambda, u\partial_x]\, u||_{L^1_T L^2} \\ &+||\,[\Delta_\lambda, \psi\partial_x]\, u||_{L^1_T L^2} + ||\,\big[\Delta_\lambda, \psi'\big]\, u||_{L^1_T L^2}\Big)^2.\end{aligned} \tag{4.10}$$

We give now a bound for each term in the right-hand side parenthesis of (4.10). As in the Lemma 3 in [17], we write

$$[\Delta_\lambda, v\partial_x] = [\Delta_\lambda, v\partial_x]\,\widetilde{\Delta_\lambda} + \Delta_\lambda v\partial_x(1-\widetilde{\Delta_\lambda}),$$

where $v = u$ or $\psi$. Therefore, thanks to Lemma 4.2, we get

$$||\,[\Delta_\lambda, u\partial_x]\,u||_{L^1_T L^2} \lesssim ||u_x||_{L^1_T L^\infty}||\widetilde{\Delta_\lambda} u||_{L^\infty_T L^2} + ||\Delta_\lambda u\partial_x(1-\widetilde{\Delta_\lambda})u||_{L^1_T L^2} \tag{4.11}$$

and

$$||\,[\Delta_\lambda, \psi\partial_x]\,u||_{L^1_T L^2} \lesssim T||\psi'||_{L^\infty}||\widetilde{\Delta_\lambda} u||_{L^\infty_T L^2} + ||\Delta_\lambda \psi\partial_x(1-\widetilde{\Delta_\lambda})u||_{L^1_T L^2}. \tag{4.12}$$

For $v = u$ or $\psi$ and $\lambda \geqslant 4$,

$$\begin{aligned}&\mathcal{F}\left\{\Delta_\lambda\left(v\partial_x(1-\widetilde{\Delta_\lambda})u\right)\right\}\\ &\quad= \underbrace{\phi(\xi/\lambda)}_{\text{Supp}\,\subset\{\xi,\frac{5\lambda}{8}\leqslant|\xi|\leqslant 2\lambda\}}\quad \hat{v}\star\underbrace{\left(i\xi(1-(\phi(\frac{2\xi}{\lambda})+\phi(\frac{\xi}{\lambda})+\phi(\frac{\xi}{2\lambda})))\hat{u}\right)}_{\text{Supp}\,\subset\mathbb{R}\{\xi,\frac{\lambda}{2}\leqslant|\xi|\leqslant\frac{5\lambda}{2}\}},\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. If $\mu \leqslant \lambda/16$, Supp $\hat{v}_\mu \subset [-\lambda/8, \lambda/8]$, and

$$\mathbb{R}\{\xi, \frac{\lambda}{2} \leqslant |\xi| \leqslant \frac{5\lambda}{2}\} + [-\lambda/8, \lambda/8] \subset \mathbb{R}\{\xi, \frac{5\lambda}{8} \leqslant |\xi| \leqslant 2\lambda\}.$$

Therefore,

$$\mathcal{F}\Big\{\Delta_\lambda\Big(v\partial_x(1-\widetilde{\Delta_\lambda})u\Big)\Big\} = \phi(\xi/\lambda)\sum_{\mu\geqslant\lambda/8}\widehat{v_\mu}\star\Big(i\xi(1-(\phi(\frac{2\xi}{\lambda})+\phi(\frac{\xi}{\lambda})+\phi(\frac{\xi}{2\lambda})))\hat{u}\Big)$$

and

$$\Delta_\lambda\left(v\partial_x(1-\widetilde{\Delta_\lambda})u\right) = \sum_{\mu\geqslant\lambda/8}\Delta_\lambda(v_\mu\partial_x(1-\widetilde{\Delta_\lambda})u)\ . \tag{4.13}$$

This equality trivially remains true in the cases $\lambda = 1$ or 2.

In the case $v = u$, using the fact that $\Delta_\lambda$ defines an operator on $L^\infty$, with a norm uniformly bounded in $\lambda$ (see Lemma II.1.1.2 in [2]), we get

$$\begin{aligned}||\Delta_\lambda\left(u\partial_x(1-\widetilde{\Delta_\lambda})u\right)||_{L^1_T L^2} &\leqslant \sum_{\mu\geqslant\lambda/8}||u_\mu||_{L^\infty_T L^2}||(1-\widetilde{\Delta_\lambda})u_x||_{L^1_T L^\infty}\\ &\lesssim ||u_x||_{L^1_T L^\infty}\sum_{\mu\geqslant\lambda/8}||u_\mu||_{L^\infty_T L^2}. \end{aligned}\tag{4.14}$$

The case $v = \psi$ is a bit more difficult, because $\psi_1 \notin L^2$. Nevertheless, if $\mu \geqslant 2$, it follows that

$$\widehat{\psi_\mu}(\xi) = \phi(\xi/\mu)\hat{\psi}(\xi) = \frac{\phi(\xi/\mu)}{\xi}\xi\hat{\psi}(\xi) \in L^2$$

because $\phi \equiv 0$ near 0 and $\psi' \in L^2$. Therefore, $\psi_\mu \in L^2$ for $\mu \geqslant 2$, and if $\lambda \geqslant 16$, we get

$$||\Delta_\lambda \left(\psi \partial_x (1 - \widetilde{\Delta_\lambda}) u\right) ||_{L^1_T L^2} \lesssim ||u_x||_{L^1_T L^\infty} \sum_{\mu \geqslant \lambda/8} ||\psi_\mu||_{L^2}. \quad (4.15)$$

If $\lambda \in \{1, 2, 4, 8\}$, we write $\psi_1 = T\psi_1 + (1 - T)\psi_1$, where $\widehat{Tf}(\xi) = \tilde{\chi}(\xi)\hat{f}(\xi)$, $0 \leqslant \tilde{\chi} \leqslant 1$, $\tilde{\chi} \equiv 1$ on $[-1/4, 1/4]$ and $\mathrm{Supp}\tilde{\chi} \subset [-1/2, 1/2]$. Proceeding the same way as to obtain (4.13), we have

$$\Delta_\lambda \left(T\psi_1 \partial_x (1 - \widetilde{\Delta_\lambda}) u\right) = 0,$$

and $(1-T)\psi_1 \in L^2$, with $||(1-T)\psi_1||_{L^2} \leqslant 4||\psi'||_{L^2}$. Therefore, as for (4.15), we have

$$||\Delta_\lambda \left(\psi \partial_x (1 - \widetilde{\Delta_\lambda}) u\right) ||_{L^1_T L^2} \lesssim ||u_x||_{L^1_T L^\infty} \Big( ||\psi'||_{L^2} + \sum_{\mu \geqslant 2} ||\psi_\mu||_{L^2} \Big). \quad (4.16)$$

Similarly, we write

$$\left[\Delta_\lambda, \psi'\right] = \left[\Delta_\lambda, \psi'\right] \widetilde{\Delta_\lambda} + \Delta_\lambda \psi' (1 - \widetilde{\Delta_\lambda})$$

and Lemma 4.3 yields

$$|| \left[\Delta_\lambda, \psi'\right] u||_{L^1_T L^2} \lesssim ||\psi'||_{L^\infty} ||\widetilde{\Delta_\lambda} u||_{L^1_T L^2} + ||\Delta_\lambda \psi' (1 - \widetilde{\Delta_\lambda}) u||_{L^1_T L^2}.$$

Using the same arguments as for stating (4.14) and Sobolev embedding ($\sigma > 1/2$), we get

$$\begin{aligned} ||\Delta_\lambda \psi' (1 - \widetilde{\Delta_\lambda}) u||_{L^1_T L^2} &\lesssim T ||u||_{L^\infty_T L^\infty} \sum_{\mu \geqslant \lambda/8} ||(\psi')_\mu||_{L^2} \\ &\lesssim T ||J^\sigma u||_{L^\infty_T L^2} \sum_{\mu \geqslant \lambda/8} ||(\psi')_\mu||_{L^2}. \quad (4.17) \end{aligned}$$

Now, using (4.10), (4.11), (4.12), (4.14), (4.15), (4.16), and (4.17), we obtain

$$\sum_\lambda \lambda^{2\sigma} ||D^{\theta/4} u_\lambda||^2_{L^q_T L^p} \lesssim (1+T)^{\frac{2}{q}} \Big(1 + ||J^\sigma u||_{L^\infty_T L^2} + ||\psi||_{L^\infty} + T||\psi'||_{L^\infty}\Big)^2$$

$$\times \sum_\lambda \lambda^{2\sigma+2/q}\Big[||u_\lambda||^2_{L^\infty_T L^2} + T^2||(\psi\psi' + \partial_x^3\psi)_\lambda||^2_{L^2}$$

$$+ ||u_x||^2_{L^1_T L^\infty}\Big(||\widetilde{\Delta_\lambda} u||^2_{L^\infty_T L^2} + \Big(\sum_{\mu\geqslant\lambda/8} ||u_\mu||_{L^\infty_T L^2}\Big)^2\Big) + T^2||\psi'||^2_{L^\infty}||\widetilde{\Delta_\lambda} u||^2_{L^\infty_T L^2}$$

$$+ \|u_x\|^2_{L^1_T L^\infty}\Big(\Big(\sum_{\mu\geqslant\lambda/8}\|\psi_\mu\|_{L^\infty_T L^2}\Big)^2\mathbf{1}_{\{\lambda\geqslant 16\}} + \Big(\|\psi'\|_{L^2} + \sum_{\mu\geqslant 2}\|\psi_\mu\|_{L^2}\Big)^2\mathbf{1}_{\{1\leqslant\lambda<16\}}\Big)$$

$$+ T^2||J^\sigma u||^2_{L^\infty_T L^2}\Big(\sum_{\mu\geqslant\lambda/8}||(\psi')_\mu||_{L^\infty_T L^2}\Big)^2\Big]. \tag{4.18}$$

We will now majorize each term of the sum in the right hand side of (4.18). We first clearly have

$$\sum_\lambda \lambda^{2\sigma+2/q}||\widetilde{\Delta_\lambda} u||^2_{L^\infty_T L^2} \lesssim \sum_\lambda \lambda^{2\sigma+2/q}||u_\lambda||^2_{L^\infty_T L^2}. \tag{4.19}$$

Using the fact that $\phi$ vanishes near 0, if $\lambda \geqslant 2$,

$$||\psi_\lambda||^2_{L^2} = \int \frac{1}{\xi^2}\phi(\xi/\lambda)^2\xi^2|\hat{\psi}(\xi)|^2 d\xi \lesssim \lambda^{-2}||(\psi')_\lambda||^2_{L^2} \lesssim ||(\psi')_\lambda||^2_{L^2}\ . \tag{4.20}$$

We also use the properties of the support of $\phi$ to obtain that if $v \in H^s$, $s \in \mathbb{R}$,

$$\sum_\lambda \lambda^{2s}||v_\lambda||^2_{L^2} \lesssim ||v||^2_{H^s}\ . \tag{4.21}$$

The lemma easily follows from (4.18), (4.19), (4.20), (4.21), and Lemma 2.2. □

To complete the proof of Theorem 4.1, it remains to control the quantity

$$\sum_\lambda \lambda^{2\sigma+2/q}||u_\lambda||^2_{L^\infty_T L^2}.$$

That is what we do in the following lemma.

**Lemma 4.5.** *Let $u$ be a regular solution of* (4.1), *let $s > 1/2$. Then, noting $(u_\lambda)_\lambda$ the Littlewood-Paley decomposition of $u$,*

$$\sum_\lambda \lambda^{2s}||u_\lambda||^2_{L^\infty_T L^2} \lesssim ||J^s u||^2_{L^\infty_T L^2}\Big(1 + ||u_x||_{L^1_T L^\infty} + T||\psi'||_{H^s}\Big)$$
$$+||J^s u||_{L^\infty_T L^2}(T(1 + ||\psi\psi' + \partial_x^3\psi||^2_{H^s}) + ||\psi'||_{H^s}||u_x||_{L^1_T L^\infty})\ . \tag{4.22}$$

**Proof.** During the proof of Lemma 4.4, we saw that $u_\lambda$ solves (4.9). We multiply now (4.9) by $\overline{u_\lambda}$, we take the real part and we sum in $x$ and $t$ variables, taking into account the fact that $u$ and $\psi$ are real-valued:

$$||u_\lambda(t)||_{L^2}^2 = ||u_\lambda(0)||_{L^2}^2 - \int_0^t \int_{-\infty}^{\infty} \big\{(u+\psi)(\partial_x u_\lambda \overline{u_\lambda} + \partial_x \overline{u_\lambda} u_\lambda) + 2\psi' |u_\lambda(s)|^2$$
$$+2\mathfrak{Re}\left(\Delta_\lambda(\psi\psi' + \partial_x^3\psi)\overline{u_\lambda}(s) + [\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda} + [\Delta_\lambda, \partial_x\psi]u\overline{u_\lambda}\right)\big\} dxds. \tag{4.23}$$

Therefore, integrating by parts, multiplying by $\lambda^{2s}$, taking the supremum in the $t$ variable and summing over $\lambda$, we get

$$\begin{aligned}\sum_\lambda \lambda^{2s}||u_\lambda||_{L_T^\infty L^2}^2 \lesssim\ & \sum_\lambda \lambda^{2s}||u_\lambda(0)||_{L^2}^2 \\ & + \int_{-T}^{T}\left(||u_x(t)||_{L^\infty} + ||\psi'||_{L^\infty}\right)\sum_\lambda \lambda^{2s}||u_\lambda(t)||_{L^2}^2 dt \\ & + \int_{-T}^{T}\sum_\lambda \lambda^{2s}||\Delta_\lambda(\psi\psi' + \partial_x^3\psi)||_{L^2}||u_\lambda(t)||_{L^2}dt \\ & + \int_{-T}^{T}\sum_\lambda \lambda^{2s}\left|\int_{-\infty}^{\infty}[\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda}dx\right|dt \\ & + \int_{-T}^{T}\sum_\lambda \lambda^{2s}\left|\int_{-\infty}^{\infty}[\Delta_\lambda, \psi']u(t)\overline{u_\lambda}(t)dx\right|dt. \end{aligned} \tag{4.24}$$

We will now control each term in the right-hand side of (4.24). We first use (4.21) to get the following inequalities:

$$\begin{aligned}&\int_{-T}^{T}\sum_\lambda \lambda^{2s}||\Delta_\lambda(\psi\psi' + \partial_x^3\psi)||_{L^2}||u_\lambda(t)||_{L^2}dt \\ \leqslant\ & \frac{1}{2}\int_{-T}^{T}\sum_\lambda \lambda^{2s}\left(||J^s u||_{L_T^\infty L^2}||\Delta_\lambda(\psi\psi' + \partial_x^3\psi)||_{L^2}^2 + \frac{||u_\lambda(t)||_{L^2}^2}{||J^s u||_{L_T^\infty L^2}}\right)dt \\ \lesssim\ & \int_{-T}^{T}\left(||J^s u||_{L_T^\infty L^2}||\psi\psi' + \partial_x^3\psi||_{H^s}^2 + \frac{||u(t)||_{H^s}^2}{||J^s u||_{L_T^\infty L^2}}\right)dt \\ \lesssim\ & T||J^s u||_{L_T^\infty L^2}(1 + ||\psi\psi' + \partial_x^3\psi||_{H^s}^2), \end{aligned} \tag{4.25}$$

$$\sum_\lambda \lambda^{2s}||u_\lambda(0)||_{L^2}^2 \lesssim ||J^s u(0)||_{L^2}^2 \leqslant ||J^s u||_{L_T^\infty L^2}^2 \tag{4.26}$$

and

$$\int_{-T}^{T} \left(||u_x(t)||_{L^\infty} + ||\psi'||_{L^\infty}\right) \sum_\lambda \lambda^{2s} ||u_\lambda(t)||_{L^2}^2 dt$$
$$\lesssim \left(||u_x||_{L^1_T L^\infty} + T||\psi'||_{L^\infty}\right) ||J^s u||^2_{L^\infty_T L^2} . \tag{4.27}$$

As in the proof of Lemma 4.4, we write

$$[\Delta_\lambda, (u+\psi)\partial_x]u(t) = [\Delta_\lambda, (u+\psi)\partial_x]\widetilde{\Delta_\lambda} u(t) + \Delta_\lambda \left((u+\psi)\partial_x(1-\widetilde{\Delta_\lambda})u(t)\right).$$

Then, Lemma 4.2 and a simplified version of (4.14) yield

$$||[\Delta_\lambda, (u+\psi)\partial_x]u(t)||_{L^2} \lesssim (||u_x(t)||_{L^\infty} + ||\psi'||_{L^\infty})||\widetilde{\Delta_\lambda} u(t)||_{L^2}$$
$$+||u_x(t)||_{L^\infty} \sum_{\mu \geqslant \lambda/8} ||u_\mu(t)||_{L^2} + ||\Delta_\lambda(\psi\partial_x(1-\widetilde{\Delta_\lambda})u(t))||_{L^2}.$$

Next, we bound up the last term in the right-hand side of the above inequality as in (4.15) and (4.16)

$$||\Delta_\lambda(\psi\partial_x(1-\widetilde{\Delta_\lambda})u(t))||_{L^2}$$
$$\lesssim ||u_x||_{L^\infty}\Big(\sum_{\mu \geqslant \lambda/8} ||\psi_\mu||_{L^2}\mathbf{1}_{\lambda \geqslant 16} + \Big(||\psi'||_{L^2} + \sum_{\mu \geqslant 2} ||\psi_\mu||_{L^2}\Big)\mathbf{1}_{\lambda \leqslant 8}\Big).$$

Therefore,

$$\int_{-T}^{T} \sum_\lambda \lambda^{2s} \Big| \int_{-\infty}^{\infty} [\Delta_\lambda, (u+\psi)\partial_x]u\overline{u_\lambda} dx \Big| dt$$
$$\lesssim \int_{-T}^{T} \sum_\lambda \lambda^{2s} \Big[ (||u_x(t)||_{L^\infty} + ||\psi'||_{L^\infty})||\widetilde{\Delta_\lambda} u(t)||_{L^2}$$
$$+||u_x(t)||_{L^\infty}\Big( \sum_{\mu \geqslant \lambda/8} ||u_\mu(t)||_{L^2} + \sum_{\mu \geqslant \lambda/8} ||\psi_\mu||_{L^2}\mathbf{1}_{\lambda \geqslant 16}$$
$$+\Big(||\psi'||_{L^2} + \sum_{\mu \geqslant 2} ||\psi_\mu||_{L^2}\Big)\mathbf{1}_{\lambda \leqslant 8}\Big)\Big] ||u_\lambda(t)||_{L^2} dt. \tag{4.28}$$

Using again (4.21), it is easy to see that

$$\sum_\lambda \lambda^{2s} ||\widetilde{\Delta_\lambda} u(t)||_{L^2} ||u_\lambda(t)||_{L^2} \lesssim ||u(t)||^2_{H^s} . \tag{4.29}$$

Lemma 2.2 with $a_\mu = ||u_\mu(t)||_{L^2}$, $d_\lambda = \lambda^s ||u_\lambda(t)||_{L^2}$, and (4.21) yield

$$\sum_\lambda \lambda^{2s} \sum_{\mu \geqslant \lambda/8} ||u_\mu(t)||_{L^2} ||u_\lambda(t)||_{L^2} \quad \lesssim \quad ||u(t)||^2_{H^s} \ . \tag{4.30}$$

Using once again Lemma 2.2, (4.21), and (4.20),

$$\begin{aligned}
&\sum_\lambda \lambda^{2s} \Big[ \sum_{\mu \geqslant \lambda/8} ||\psi_\mu||_{L^2} \mathbf{1}_{\lambda \geqslant 16} + (||\psi'||_{L^2} + \sum_{\mu \geqslant 2} ||\psi_\mu||_{L^2}) \mathbf{1}_{\lambda \leqslant 8} \Big] ||u_\lambda(t)||_{L^2} \\
&\quad \lesssim ||u(t)||_{H^s} \Big( \sum_{\lambda \geqslant 2} \lambda^{2s} ||\psi_\lambda||^2_{L^2} \Big)^{1/2} + \Big( ||\psi'||_{L^2} + \sum_{\mu \geqslant 2} ||\psi_\mu||_{L^2} \Big) ||u(t)||_{L^2} \\
&\quad \lesssim ||u(t)||_{H^s} ||\psi'||_{H^s}.
\end{aligned} \tag{4.31}$$

Thanks to (4.28), (4.29), (4.30), and (4.31), we have

$$\begin{aligned}
&\int_{-T}^{T} \sum_\lambda \lambda^{2s} \Big| \int_{-\infty}^{\infty} [\Delta_\lambda, (u+\psi)\partial_x] u \overline{u_\lambda} dx \Big| dt \qquad (4.32) \\
&\lesssim \Big( ||u_x||_{L^1_T L^\infty} + T ||\psi'||_{L^\infty} \Big) ||J^s u||^2_{L^\infty_T L^2} + ||\psi'||_{H^s} ||u_x||_{L^1_T L^\infty} ||J^s u||_{L^\infty_T L^2}.
\end{aligned}$$

As in the proof of Lemma 4.4,

$$|| \left[ \Delta_\lambda, \psi' \right] u(t) ||_{L^2} \quad \lesssim \quad ||\psi'||_{L^\infty} ||\widetilde{\Delta_\lambda} u(t)||_{L^2} + ||u(t)||_{L^\infty} \sum_{\mu \geqslant \lambda/8} ||(\psi')_\mu||_{L^2}$$

and therefore, using also the Sobolev embedding ($s > 1/2$),

$$\begin{aligned}
&\int_{-T}^{T} \sum_\lambda \lambda^{2s} \left| \int_{-\infty}^{\infty} [\Delta_\lambda, \psi'] u(t) \overline{u_\lambda}(t) dx \right| dt \\
&\lesssim \quad T ||\psi'||_{L^\infty} ||J^s u||^2_{L^\infty_T L^2} + ||\psi'||_{H^s} ||u||_{L^1_T L^\infty} ||J^s u||_{L^\infty_T L^2} \\
&\lesssim \quad T ||\psi'||_{H^s} ||J^s u||^2_{L^\infty_T L^2} \ .
\end{aligned} \tag{4.33}$$

Concatenating (4.24), (4.25), (4.26), (4.27), (4.32), and (4.33), we finally obtain the announced inequality. □

Theorem 4.1 directly follows from Lemmas 4.1, 4.4, 4.5.

## 5. The proof of Theorem 1.1 (existence and uniqueness)

We begin the proof of Theorem 1.1 with a lemma that gives an a priori estimate on the $H^s$ norm of a regular solution of (1.6).

**Lemma 5.1.** *Let $u_0 \in H^\infty$, $s > 1/2$ and $u \in C([-T,T],H^\infty)$ a regular solution of* (4.1). *Then there exists $C = C(\psi) > 0$ such that*

$$||u(t)||_{H^s}^2 \leqslant \left(||u_0||_{H^s}^2 + 1\right) \exp\left(C(T + ||u_x||_{L^1_T L^\infty})\right)$$

**Proof.** We apply $J^s$ to (4.1), we multiply the obtained equation by $J^s u$, and we sum over $\mathbb{R}$:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}||J^s u||_{L^2}^2 + \int \frac{(u+\psi)}{2}\partial_x (J^s u)^2 + \int \psi'(J^s u)^2 \\ = \quad -\int J^s(\psi\psi' + \partial_x^3\psi)J^s u - \int [J^s,(u+\psi)]\partial_x u J^s u - \int [J^s,\psi']uJ^s u.\end{aligned}$$

After an integration by parts, we get

$$\begin{aligned}\frac{d}{dt}||J^s u||_{L^2}^2 \quad &= \quad \int \partial_x (u-\psi)(J^s u)^2 - 2\int J^s(\psi\psi' + \partial_x^3\psi)J^s u \\ &\quad -2\int [J^s,(u+\psi)]\partial_x u J^s u - 2\int [J^s,\psi']uJ^s u.\end{aligned}$$

We use Lemma 2.3 to estimate the $L^2$ norms of $[J^s,u]\partial_x u$, and $[J^s,\psi']u$. We get

$$||[J^s,u]\partial_x u||_{L^2} \lesssim ||J^s u||_{L^2}||u_x||_{L^\infty}$$

and

$$||[J^s,\psi']u||_{L^2} \lesssim ||\psi''||_{L^\infty}||J^{s-1}u||_{L^2} + ||J^s\psi'||_{L^2}||u||_{L^\infty}\ .$$

Since $\psi$ does not belong to $L^2$, it does not work to estimate the $L^2$ norms of $[J^s,\psi]\partial_x u$. Lemma 2.4 (note that Lemma 2.7 could also have been used) yields

$$||[J^s,\psi]\partial_x u||_{L^2} \lesssim ||\psi'||_{L^\infty}||u||_{H^s} + ||\psi'||_{H^{s_0+s-1}}||u_x||_{L^\infty}\ .$$

Using the Cauchy-Schwarz inequality and Sobolev embedding ($s > 1/2$), we obtain

$$\begin{aligned}\frac{d}{dt}||u||_{H^s}^2 \quad &\lesssim \quad ||u||_{H^s}^2\left(||u_x||_{L^\infty} + ||\psi'||_{L^\infty} + 1 + ||\psi'||_{H^s} + ||\psi''||_{L^\infty}\right) \\ &\quad +||\psi\psi' + \partial_x^3\psi||_{H^s}^2 + ||u_x||_{L^\infty}||\psi'||_{H^{s_0+s-1}}^2,\end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}(1+||u(t)||_{H^s}^2) \leqslant C(\psi)(1+||u_x(t)||_{L^\infty})(1+||u(t)||_{H^s}^2),\ t \in (-T,T). \quad (5.1)$$

Next, Gronwall's lemma concludes the proof of the lemma. □

We are now ready to begin the proof of Theorem 1.1 itself. Let $s > 1$, $\theta \in (0,1)$ such that $(1-\theta)/2 < s-1$, $p := 2/(1-\theta)$ and $\sigma := s - \theta/4 > 3/4$.

We also define $q := 4/\theta > 4$ and $\alpha := (1-1/q)/2 < 2$. For $T \geqslant 0$, and $u$ a regular solution of (4.1), we define

$$F(T) := ||u_x||_{L^1_T L^\infty} + T^\alpha ||J^\sigma u||_{L^\infty_T L^2}.$$

Our choice of $p$ and Sobolev embedding ensure that $W^{\sigma-(1-\theta/4),p} \subset L^\infty$ with continuous embedding, because $\sigma - (1-\theta/4) = s - 1 > 1/p$. Hence, using also the continuity of the Hilbert transform on $L^p$ and Hölder inequality in the $t$ variable,

$$\begin{aligned} ||u_x||_{L^1_T L^\infty} &\lesssim ||D^{1-\theta/4} H D^{\theta/4} u||_{L^1_T W^{\sigma-(1-\theta/4),p}} \lesssim ||H D^{\theta/4} J^\sigma u||_{L^1_T L^p} \\ &\lesssim ||D^{\theta/4} J^\sigma u||_{L^1_T L^p} \lesssim T^{1-1/q} ||D^{\theta/4} J^\sigma u||_{L^q_T L^p} \ . \end{aligned}$$

Thanks to Theorem 4.1, for $T \leqslant 1$,

$$||u_x||_{L^1_T L^\infty} \quad \lesssim \quad (T^{1-1/q} + T^{1-1/q-\alpha} F(T))(1+F(T))^{3/2}(1 + ||u||_{L^\infty_T H^s}) \ .$$

Therefore, since $\sigma < s$,

$$F(T) \quad \lesssim \quad T^\alpha (1+F(T))^{5/2}(1+||u||_{L^\infty_T H^s}) \ .$$

We control the quantity $||u||_{L^\infty_T H^s}$ by Lemma 5.1:

$$||u||_{L^\infty_T H^s} \quad \lesssim \quad (1 + ||u_0||_{H^s}) \exp(CF(T)) \ .$$

Finally, we obtain that there exists a constant $C' > 0$ (which only depends on $\psi$) such that

$$F(T) \leqslant C' T^\alpha (1+F(T))^{5/2}(1+||u_0||_{H^s}) \exp(CF(T)), \ T \in [0,1]. \tag{5.2}$$

We define $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(x) := \frac{x}{(1+x)^{5/2} \exp(Cx)}, \ x \in \mathbb{R} \ .$$

Let $T(||u_0||_{H^s}) \in [0,1]$ be small enough such that

$$C' T(||u_0||_{H^s})^\alpha (1 + ||u_0||_{H^s}) < ||g||_\infty \ .$$

Then by continuity of $g(F(T))$, since $g(F(0)) = g(0) = 0$, there exists some $A > 0$, which does not depend on $u_0$, such that

$$F(t) \leqslant A, \ t \in [0, T(||u_0||_{H^s})] \ .$$

In particular, $||u_x||_{L^1_{T(||u_0||_{H^s})} L^\infty} \leqslant A$. Therefore, thanks to Lemma 5.1,

$$||u||_{L^\infty_{T(||u_0||_{H^s})} H^s} \leqslant \tilde{A}(||u_0||_{H^s}) < \infty. \tag{5.3}$$

Let now $u_0 \in H^s$, and $u_{0,n} \to u_0$ in $H^s$, where $u_{0,n}$ is regular ($u_{0,n} \in H^\infty$). We denote by $u_n$ the solution of (4.1) with initial data $u_n(0) = u_{0,n}$. The results in [9] ensure that $u_n$ is global. Thanks to (5.3), we can extract a subsequence (that we also denote by $u_n$) such that $u_n$ converges for the weak $\star$ topology of $L^\infty_{T(||u_0||_{H^s}+1)}H^s$.

Now, if $v$ and $w$ are two solutions of (4.1), then

$$\frac{1}{2}\frac{d}{dt}||(v-w)(t)||^2_{L^2}+\int(v-w)^2\partial_x v-\frac{1}{2}\int(v-w)^2\partial_x w+\frac{1}{2}\int(v-w)^2\psi'=0.$$

Therefore,

$$\frac{d}{dt}||(v-w)(t)||^2_{L^2}\lesssim||(v-w)(t)||^2_{L^2}\left(||v_x(t)||_{L^\infty}+||w_x(t)||_{L^\infty}+||\psi'||_{L^\infty}\right),$$

and the Gronwall lemma yields

$$\begin{aligned}&||(v-w)(t)||^2_{L^2} \\ &\lesssim||(v-w)(0)||^2_{L^2}\exp\left(C(||v_x(t)||_{L^1_TL^\infty}+||w_x(t)||_{L^1_TL^\infty}+||\psi'||_{L^\infty})\right).\end{aligned} \tag{5.4}$$

Applying this to $v=u_n$, $w=u_m$, we get that for $n$ and $m$ large enough, $|t|\leqslant T(||u_0||_{H^s}+1)$,

$$||u_n(t)-u_m(t)||^2_{L^2}\lesssim||u_{0,n}-u_{0,m}||^2_{L^2}\exp(C(2F(T(||u_0||_{H^s}+1))+||\psi'||_{L^\infty})),$$

which shows that $(u_n)_n$ is a Cauchy sequence, and hence converges to $u$ strongly in the Banach space $L^\infty_T L^2$, because $u_{0,n}\to u$ in $L^2$. We deduce that the map $t\to u(t)$ is weakly continuous from $[-T,T]$ into $H^s$. Indeed, if we choose some $\varepsilon>0$, if $\phi\in H^s$, let $\tilde\phi\in H^{2s}$ be such that $||\phi-\tilde\phi||_{H^s}<\varepsilon/(2||u||_{L^\infty_T H^s})$. For $t,\tau\in[-T,T]$,

$$\begin{aligned}|(u(t)-u(\tau),\phi)_{H^s}|&\leqslant\varepsilon+\left|\left(u(t)-u(\tau),J^{2s}\tilde\phi\right)_{L^2}\right| \\ &\leqslant\varepsilon+2||u_n-u||_{L^\infty_TL^2}||\tilde\phi||_{H^{2s}}+||u_n(t)-u_n(\tau)||_{L^2}||\tilde\phi||_{H^{2s}}.\end{aligned}$$

Choosing $n$ large enough, and $t-\tau$ small enough, we have

$$|(u(t)-u(\tau),\phi)_{H^s}|\leqslant 2\varepsilon.$$

In order to show that $u$ is continuous with values in $H^s$, we define the norm

$$|||v|||:=\Big(\sum_\lambda\lambda^{2s}||v_\lambda||^2_{L^2}\Big)^{1/2},$$

which is equivalent to the $H^s$ norm (indeed, (4.21) gives one of the inequalities and the proof of the other one is similar). It suffices to show that $|||v(t)|||\underset{t\to\tau}{\longrightarrow}|||v(\tau)|||$. That is what we do in the following lemma.

**Lemma 5.2.** *Let $u$ be as above the weak $\star$ limit in $L^\infty_T H^s$ of $u_n$ and $(u_\lambda)_\lambda$ its Littlewood-Paley decomposition, $t, \tau \in [-T, T]$. Then*

$$\sum_\lambda \lambda^{2s} ||u_\lambda(t)||^2_{L^2} \leqslant \exp\Big(C||u_x||_{L^1_I L^\infty}\Big) \sum_\lambda \lambda^{2s} ||u_\lambda(\tau)||^2_{L^2} + g(t,\tau) \ ,$$

*where $I = [\tau, t]$ or $[t, \tau]$, depending on the sign of $t - \tau$, and $g(t,\tau) \underset{t-\tau\to 0}{\to} 0$.*

**Proof.** We first remark that $u$ solves (1.6), since $u_n$ solves (4.1). We assume for instance that $\tau \leqslant t$. As in the proof of Lemma 4.5, we show

$$\begin{aligned}\sum_\lambda \lambda^{2s} ||u_\lambda(t)||^2_{L^2} \ &\leqslant \ \sum_\lambda \lambda^{2s} ||u_\lambda(\tau)||^2_{L^2} + C(t-\tau) + C||u_x||_{L^1_I L^\infty} \\ &\quad + C \int_\tau^t (1 + ||u_x(\sigma)||_{L^\infty}) \sum_\lambda \lambda^{2s} ||u_\lambda(\sigma)||^2_{L^2} d\sigma \ .\end{aligned}$$

We conclude by Gronwall's lemma that

$$\begin{aligned}&\sum_\lambda \lambda^{2s} ||u_\lambda(t)||^2_{L^2} \leqslant \Big(\sum_\lambda \lambda^{2s} ||u_\lambda(\tau)||^2_{L^2} + C(t-\tau)\Big) \\ &\times\Big[1 + C\Big(||u_x||_{L^1_I L^\infty} + t - \tau\Big) \exp\Big(C(||u_x||_{L^1_I L^\infty} + t - \tau)\Big)\Big] \\ &= \exp\Big(C||u_x||_{L^1_I L^\infty}\Big) \sum_\lambda \lambda^{2s} ||u_\lambda(\tau)||^2_{L^2} + \underset{t-\tau\to 0}{o}(1).\end{aligned}$$

□

Coming back to the proof of Theorem 1.1, since $||u_x||_{L^1_I L^\infty} \underset{t-\tau\to 0}{\longrightarrow} 0$, using Lemma 5.2 and exchanging $t$ and $\tau$, we get that

$$\sum_\lambda \lambda^{2s} ||u_\lambda(t)||^2_{L^2} = \sum_\lambda \lambda^{2s} ||u_\lambda(\tau)||^2_{L^2} + \underset{t-\tau\to 0}{o}(1) \ .$$

To complete the first part of the proof of Theorem 1.1, it remains to show the uniqueness result. It is a straightforward consequence of (5.4). □

## 6. The proof of Theorem 1.1 (continuous dependence on the initial data).

We want now to prove the part of Theorem 1.1 about the continuous dependence on the initial data of a solution of (1.6). We use the method presented in [17]. We begin with two lemmas.

**Lemma 6.1.** *Let $u$ be a solution of* (1.6) *in $\mathcal{D}'$. Let $1 < \delta \leqslant \kappa$, and $(\omega_\lambda)_\lambda$ be a sequence of positive numbers such that for every dyadic integer $\lambda$, $\delta\omega_\lambda \leqslant \omega_{2\lambda} \leqslant \kappa\omega_\lambda$. Then for all $t, \tau \in [-T, T]$,*

$$\sum_\lambda \omega_\lambda^2 ||u_\lambda(t)||_{L^2}^2 \leqslant \big(1 + C(||u_x||_{L_I^1 L^\infty} + |t-\tau|)\big)$$
$$\times \exp\big(C(||u_x||_{L_I^1 L^\infty} + |t-\tau|)\big)\Big(\sum_\lambda \omega_\lambda^2 ||u_\lambda(\tau)||_{L^2}^2 + C|t-\tau|\Big),$$

*where $I = [\tau, t]$ or $[t, \tau]$, depending on the sign of $t - \tau$.*

**Proof.** The proof is identical to that of Lemma 5.2. The slight difference is that we use a variant version of Lemma 2.2 which is obtained by replacing the sequence $(\lambda^s)$ by $(\omega_\lambda)$. This works because $\sum_\lambda \omega_\lambda^{-1}$ is finite. □

The second lemma is proved in [17].

**Lemma 6.2.** *Assume that $v_n \to v$ in $H^s$. Then there exists a sequence $(\omega_\lambda)$ of positive numbers which satisfies $2^s\omega_\lambda \leqslant \omega_{2\lambda} \leqslant 2^{s+1}\omega_\lambda$ , $\forall\lambda$ and $\omega_\lambda/\lambda^s \to \infty$ such that*

$$\sup_n \sum_\lambda \omega_\lambda^2 ||(v_n)_\lambda||_{L^2}^2 < \infty.$$

Let $u_{0n}$ be a sequence such that $u_{0n} \to u_0$ in $H^s$. Let $u_n, u \in C([0,T], H^s)$ be the associated solutions of (1.6). Then

$$u_n \to u \text{ in } C([-T(R), T(R)], L^2)\ , \tag{6.1}$$

because of (5.4) and $||(u_n)_x||_{L^1_{T(R)} L^\infty} \leqslant A$, where $R$ is such that $R \geqslant \sup_n ||u_{0n}||_{H^s}$. We define a sequence $(\omega_\lambda)_\lambda$ as in Lemma 6.2, with $v_n = u_{0n}$. Lemma 6.1 with $\tau = 0$ yields

$$\sup_{n,t} \sum_\lambda \omega_\lambda^2 (||(u_n)_\lambda(t)||_{L^2}^2 + ||u_\lambda(t)||_{L^2}^2) < \infty. \tag{6.2}$$

Let

$$u_\Lambda := \sum_{\lambda \leqslant \Lambda} u_\lambda.$$

We first have

$$||u_n - u||_{L_T^\infty H^s} \leqslant ||u_n - (u_n)_\Lambda||_{L_T^\infty H^s} + ||(u_n)_\Lambda - u_\Lambda||_{L_T^\infty H^s} + ||u_\Lambda - u||_{L_T^\infty H^s}.$$

Let $\varepsilon > 0$.

$$\text{Max}\Big(\sup_{n,t} ||(u_n)_\Lambda(t) - u_n(t)||_{H^s}, \sup_t ||u_\Lambda(t) - u(t)||_{H^s}\Big)$$

$$\leqslant \sup_{n,t} \underbrace{\Big( \sup_{\lambda>\Lambda} \Big(\frac{\lambda^s}{\omega_\lambda}\Big)\Big)}_{\substack{\longrightarrow 0 \\ \Lambda\to\infty}} \Big( \sum_{\lambda>\Lambda} \omega_\lambda^2 (||(u_n)_\lambda(t)||_{L^2}^2 + ||u_\lambda(t)||_{L^2}^2) \Big)^{1/2},$$

hence, because of (6.2), we can choose $\Lambda$ large enough such that

$$\text{Max}\Big( \sup_{n,t} ||(u_n)_\Lambda(t) - u_n(t)||_{H^s}, \sup_t ||u_\Lambda(t) - u(t)||_{H^s} \Big) \leqslant \varepsilon/4 \ .$$

Next, thanks to (6.1), we can choose $n_0$ large enough such that for $n \geqslant n_0$, $t \in [-T(R), T(R)]$,

$$||(u_n)_\Lambda(t) - u_\Lambda(t)||_{H^s} \leqslant C\Lambda^s ||(u_n)_\Lambda(t) - u_\Lambda(t)||_{L^2} \leqslant \varepsilon/2 \ .$$

This completes the proof of Theorem 1.1. □

## 7. The proof of Theorem 1.2

**Existence.** We first remark that in Theorem 1.1 above,

$$T(\psi, s, ||\phi||_{H^s}) = T(||\psi||_{X^{s_1}}, s, ||\phi||_{H^s}) \ ,$$

where $s_1 > 1$ is large, and $T$ can be assumed to be nonincreasing with respect to $||\psi||_{X^{s_1}}$ and $||\phi||_{H^s}$. Hence, if we define

$$\tilde{T}(||g||_{X^s}) := T(C_{s_1}||g||_{X^s}, s, \tilde{C}||g||_{X^s}),$$

where $C_{s_1}$ (respectively $\tilde{C}$) is the norm of the bounded linear map $g \to \psi$ (respectively $g \to \phi$) from $X^s$ into $X^{s_1}$ (respectively $H^s$) defined in Lemma 2.1, we have

$$\tilde{T}(||g||_{X^s}) \leqslant T(||\psi||_{X^{s_1}}, s, ||\phi||_{H^s}) = T(\psi, s, ||\phi||_{H^s}) \ .$$

Let $v \in C([-T,T], H^s)$ be the solution of (1.6) given in Theorem 1.1. Then $u := v + \psi \in C([-\tilde{T}, \tilde{T}], X^s)$ solves (1.1). Moreover, since $\psi - g \in H^s$, $u - g \in C([-\tilde{T}, \tilde{T}], H^s)$. $u_x \in L^1_{\tilde{T}} L^\infty$ because $v_x$ is.

**Uniqueness.** We choose $\psi$ and $\tilde{T}$ as above. Let $u$ be a solution of (1.1) as required. Let $\tilde{v} = u - \psi$. It is easy to see that $\tilde{v}$ solves (1.6), and the uniqueness in Theorem 1.1 yields $\tilde{v} = v$, where $v$ is the solution of (1.6) we obtained in Theorem 1.1.

**Continuity with respect to the initial data.** Let $R > 0$ and $g \in X^s$ with $||g||_{X^s} < R$. Let $(g_n)_n$ be a sequence in $X^s$ such that $g_n \to g$ in $X^s$. We assume that for all $n$, $||g_n||_{X^s} \leqslant R$.

As in Lemma 2.1, we define $k(x) = (4\pi)^{-1/2}e^{-x^2/4}$, $\psi = k \star g$, $\psi_n = k \star g_n$, $\phi = g - \psi$, $\phi_n = g - \psi_n$.

Let $v \in C([-\tilde{T}, \tilde{T}], H^s)$ be the solution of (1.6), and $v_n \in C([-\tilde{T}, \tilde{T}], H^s)$ solving

$$\begin{cases} \partial_t w + \partial_x^3 w + w\partial_x w + \partial_x(w\psi_n) = -(\partial_x^3 \psi_n + \psi_n \psi_n') \ , \ x \in \mathbb{R}, \ t \in \mathbb{R} \\ w(x,0) = \phi_n(x). \end{cases}$$

Lemma 6.2 applied to the sequence $\phi_n \to \phi$ in $H^s$ and Lemma 6.1 yield

$$\sup_{n,t} \sum_{\lambda} \omega_\lambda^2 \left( ||(v_n)_\lambda(t)||_{L^2}^2 + ||v_\lambda(t)||_{L^2}^2 \right) < \infty \ ,$$

and we conclude as in Section 6 that $w_n := v_n - v \to 0$ in $C([-\tilde{T}, \tilde{T}], H^s)$. Since $s > 1$ and $\psi_n \to \psi$ in $X^s$, we deduce that

$$u_n - u = (v_n + \psi_n) - (v + \psi) \to 0 \ \text{ in } X^s.$$

## 8. Global well-posedness of (1.7) in $H^s$, $s \geqslant 3/2$.

The Benjamin-Ono equation possesses infinitely many invariants (see for instance [1]). There is an invariant associated with each $H^s$ norm, $s \in \mathbb{N}/2$. Here are the first ones.

$$I_0(u) := \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx, \qquad I_1(u) := -\int_{-\infty}^{\infty} \Big[\frac{u^3}{3} + uH(\partial_x u)\Big] dx,$$

$$I_2(u) := \int_{-\infty}^{\infty} \Big[\frac{u^4}{4} + \frac{3}{2}u^2 H(\partial_x u) + 2(\partial_x u)^2\Big] dx,$$

$$I_3(u) := \int_{-\infty}^{\infty} \Big[ -\frac{u^5}{5} - \frac{4}{3}u^3 H(\partial_x u) - u^2 H(u\partial_x u) - 2uH(\partial_x u)^2 \\ -6u(\partial_x u)^2 + 4uH\partial_x^3 u\Big] dx.$$

The first two of the following lemmas are proved in [9].

**Lemma 8.1.** *Let $\phi \in H^{s_0}$ where $s_0$ is large. Let $u \in C(\mathbb{R}, H^{s_0})$ be the solution of* (1.7) *given in* [9]*. Then for all $t \in \mathbb{R}$,*

$$||u(t)||_{L^2}^2 \leqslant h(\psi, t, ||\phi||_{L^2}). \tag{8.1}$$

**Lemma 8.2.** *Under the same assumptions as in Lemma* 8.1, *for all* $t \in \mathbb{R}$,

$$||u(t)||_{H^1}^2 \leqslant k(\psi, t, ||\phi||_{H^1}). \tag{8.2}$$

Here, $h$ and $k$ are valued in $\mathbb{R}_+$. They are even, nondecreasing functions of $t$ on $\mathbb{R}_+$, and continuous with respect to $t$ and $||\phi||_{L^2}$ or $||\phi||_{H^1}$.

Our aim is now to prove a similar lemma, where the $L^2$ and $H^1$ norms are replaced by the $H^{3/2}$ norm.

**Lemma 8.3.** *Under the same assumptions as in Lemmas* 8.1 *and* 8.2,

$$\begin{aligned}-I_3(u(t)) \leqslant -I_3(\phi) + C(\psi)\int_0^{|t|}(1 + h(\psi, s, ||\phi||_{L^2}) + k(\psi, s, ||\phi||_{H^1}))^5 ds \\ + C(\psi)\int_0^{|t|}(1 + h(\psi, s, ||\phi||_{L^2}) + k(\psi, s, ||\phi||_{H^1}))||u(s)||_{H^{3/2}}^2 ds.\end{aligned} \tag{8.3}$$

**Proof.** We will prove the lemma only for $t \geqslant 0$. We remark as in [9] that (1.7) can be rewritten as

$$\partial_t u - \frac{1}{2}\partial_x I_1'(u) + \partial_x(\psi u) + (H\psi'' + \psi\psi') = 0, \qquad u(0) = \phi. \tag{8.4}$$

For convenience, we will denote in the sequel $\rho := \psi\psi' + H\psi''$. Using the properties of the invariants of the Benjamin-Ono equation (see [1]), we compute

$$\begin{aligned}\frac{d}{dt}(-I_3(u)) &= \Big(I_3'(u), -\frac{1}{2}\partial_x I_1'(u) + \partial_x(\psi u) + \rho\Big) \\ &= \big(I_3'(u), \partial_x(\psi u)\big) + \big(I_3'(u), \rho\big).\end{aligned} \tag{8.5}$$

We note that

$$\begin{aligned}I_3'(u) = -u^4 - 4(u^2 H(\partial_x u) + uH(u\partial_x u) + H(u^2\partial_x u)) - 2H(\partial_x u)^2 \\ - 4H\partial_x(uH(\partial_x u)) + 6(\partial_x u)^2 + 12u\partial_x^2 u + 8H\partial_x^3 u.\end{aligned} \tag{8.6}$$

Using (8.6), the Cauchy-Schwarz inequality, Sobolev inequalities and integration by parts, we first show that

$$\begin{aligned}\big(I_3'(u), \rho\big) \lesssim ||u||_{H^1}^4||\rho||_{L^\infty} + ||u||_{H^1}^3||\rho||_{L^2} + ||u||_{H^{3/2}}^2||\rho||_{L^2} + ||u||_{H^1}^2||\rho'||_{L^2} \\ + ||u||_{H^1}^2||\rho||_{L^\infty} + ||u||_{H^1}(||\rho'||_{L^\infty}||u||_{L^2} + ||\rho||_{L^\infty}||u||_{H^1}) + ||u||_{L^2}||\rho'''||_{L^2}.\end{aligned} \tag{8.7}$$

Next, we control the quantity $(I_3'(u), \partial_x(\psi u))$. We will majorize separately each term that we obtain as we substitute (8.6) into $(I_3'(u), \partial_x(\psi u))$. Our arguments are again the Cauchy-Schwarz inequality, Sobolev inequalities, integration by parts and the continuity of the Hilbert transform on $L^p$, $1 < p < \infty$.

$$\left(-u^4, \partial_x(\psi u)\right) = -\frac{4}{5}\left(u^5, \psi'\right) \lesssim ||\psi'||_{L^\infty}||u||_{H^1}^5 , \tag{8.8}$$

$$-4\left(u^2 H(\partial_x u) + uH(u\partial_x u) + H(u^2\partial_x u), \partial_x(\psi u)\right) \lesssim ||u||_{H^1}^3(||\psi'||_{L^\infty}||u||_{L^2} + ||\psi||_{L^\infty}||u||_{H^1}) , \tag{8.9}$$

$$\left(-2H(\partial_x u)^2 + 6(\partial_x u)^2, \partial_x(\psi u)\right) \lesssim ||u||_{H^{5/4}}^2(||\psi'||_{L^\infty}||u||_{L^2} + ||\psi||_{L^\infty}||u||_{H^1}) \tag{8.10}$$

and

$$\left(u\partial_x^2 u, \partial_x(\psi u)\right) = -\left(\partial_x u, \partial_x(\psi' u^2)\right) - \frac{1}{2}\left((\partial_x u)^2, \partial_x(\psi u)\right) \tag{8.11}$$
$$\lesssim ||u||_{H^1}^3(||\psi''||_{L^2} + ||\psi'||_{L^\infty}) + ||u||_{H^{5/4}}(||\psi'||_{L^\infty}||u||_{L^2} + ||\psi||_{L^\infty}||u||_{H^1}).$$

It is a bit more delicate to estimate the two remaining terms, which are $(H\partial_x(uH(\partial_x u)), \partial_x(\psi u))$ and $\left(H\partial_x^3 u, \partial_x(\psi u)\right)$. We begin with the last one. Using the properties of the Hilbert transform, the fact that $D = H\partial_x$ and an integration by parts,

$$\begin{aligned}\left(H\partial_x^3 u, \partial_x(\psi u)\right) &= \left(\partial_x D^{3/2}u, D^{3/2}(\psi u)\right)\\ &= -\left(D^{3/2}u, D^{3/2}(\psi' u)\right) + \frac{1}{2}\left((D^{3/2}u)^2, \psi'\right) - \left(D^{3/2}u, \left[D^{3/2}, \psi\right]\partial_x u\right).\end{aligned}$$

Using Lemma 2.5, we have

$$-\left(D^{3/2}u, D^{3/2}(\psi' u)\right) \lesssim ||u||_{H^{3/2}}^2||\psi'||_{H^{3/2}} , \tag{8.12}$$

and it is easy to see that

$$\frac{1}{2}\left((D^{3/2}u)^2, \psi'\right) \lesssim ||\psi'||_{L^\infty}||u||_{H^{3/2}}^2 . \tag{8.13}$$

Next,

$$\left[D^{3/2}, \psi\right]\partial_x u = \left[J^{3/2}, \psi\right]\partial_x u - [R, \psi]\,\partial_x u$$

where $R$ is defined by

$$\widehat{Rf}(\xi) = \left[(1+\xi^2)^{3/4} - |\xi|^{3/2}\right]\widehat{f}(\xi) = |\xi|^{3/2}\left((1+\frac{1}{\xi^2})^{3/4} - 1\right)\widehat{f}(\xi) .$$

In particular, $R$ is bounded on $L^2$, and

$$||\,[R,\psi]\,\partial_x u||_{L^2} \lesssim ||\psi||_{L^\infty}||u||_{H^1} .$$

Using finally Lemma 2.7, we obtain

$$\left(H\partial_x^3 u, \partial_x(\psi u)\right) \lesssim (||\psi'||_{L^\infty} + ||\psi'||_{H^{3/2}})||u||_{H^{3/2}}^2 + ||\psi||_{L^\infty}||u||_{H^1}||u||_{H^{3/2}}. \tag{8.14}$$

We now control the last term.

$$(H\partial_x(uH(\partial_x u)), \partial_x(\psi u)) = \left(D^{1/2}(uDu), D^{1/2}\partial_x(\psi u)\right) \ .$$

Lemma 2.5 yields

$$||D^{1/2}(uDu)||_{L^2} \lesssim ||u||_{H^1}||u||_{H^{3/2}} \ , \tag{8.15}$$

and with Lemma 2.7, we obtain that

$$\begin{aligned} ||D^{1/2}\partial_x(\psi u)||_{L^2} \lesssim ||J^{3/2}(\psi u)||_{L^2} &\lesssim ||\psi J^{3/2}u||_{L^2} + ||[J^{3/2}, \psi]u||_{L^2} \\ &\lesssim ||\psi||_{L^\infty}||u||_{H^{3/2}} + ||\psi'||_{H^{3/2}}||u||_{H^{1/2}}. \end{aligned} \tag{8.16}$$

Bringing together (8.5) and the estimates (8.7-8.11), (8.14), (8.15) and (8.16), we get

$$\frac{d}{dt}(-I_3(u)) \quad \lesssim \quad C(\psi)\left[(1+h+k)^5 + (1+h+k)||u||_{H^{3/2}}^2\right]. \tag{8.17}$$

Integrating with respect to time, we obtain the announced result. □

**Lemma 8.4.** *Under the same assumptions as in Lemma* 8.3,

$$||u(t)||_{H^{3/2}} \leqslant m(\psi, t, ||\phi||_H^{3/2}), \tag{8.18}$$

*where $m$ is an even positive function which grows with $t$ on $\mathbb{R}_+$, and which is continuous with respect to $t$ and $||\phi||_H^{3/2}$. $m$ can be expressed as a function of $h$ and $k$, but we will omit this expression, which is of low interest.*

**Proof.** Using the conservation law $I_3$, Lemma 8.3 and the Cauchy-Schwarz inequality,

$$\begin{aligned} ||u(t)||_{H^{3/2}}^2 \quad &\lesssim \quad C(||\phi||_H^{3/2} + (1+k)^5) + C(\psi)\int_0^t (1+h+k)^5 ds \\ &\qquad + C(\psi)\int_0^t (1+h+k)||u(s)||_{H^{3/2}}^2 ds \ . \end{aligned}$$

The result follows by applying Gronwall's lemma. □

Passing to the proof of Theorem 1.5, take now $\phi \in H^{3/2}$. Since (1.7) has been shown to be locally well-posed in $H^{3/2}$, there exists $T^* > 0$ and a solution $u \in C([0, T^*[, H^{3/2})$ of (1.7). Moreover, either $T^* = \infty$ or $T^*$ is finite and $||u(t)||_{H^{3/2}} \underset{t\uparrow T^*}{\longrightarrow} +\infty$. We assume by contradiction that $T^*$ is finite. Let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of $H^{s_1}$ functions such that $\phi_j \to \phi$ in $H^{3/2}$, where $s_1$ is large. We denote by $u_j \in C(\mathbb{R}, H^{s_0})$ the solution of (1.7) with initial data $\phi_j$, which is given by the results of [9]. Then, using Theorem 1.3, it is easy to see that for all $T < T^*$, $u_j \to u$ in $C([0,T], H^{3/2})$. In particular,

$$||u||_{L^\infty_T H^{\frac{3}{2}}} \leqslant \liminf ||u_j||_{L^\infty_T H^{\frac{3}{2}}} \leqslant \lim m(\psi, t, ||\phi_j||_{H^{\frac{3}{2}}}) = m(\psi, t, ||\phi||_{H^{\frac{3}{2}}}).$$

Letting $T \to T^*$, we obtain a contradiction of the fact that $||u(t)||_{H^{3/2}}$ blows up as $t \uparrow T^*$.

We have shown that (1.7) is globally well-posed in $H^{3/2}$. We next prove that it is globally well-posed in $H^s$, where $s \in (3/2, 2)$.

Inequality (5.1) was shown for a regular solution of (1.6), but it clearly remains true for a regular solution of (1.7).

As in [19], we use the following inequality due to Brezis and Gallouët (see [3]): if $f \in H^a(\mathbb{R}^n)$ with $a > n/2$, then

$$||f||_{L^\infty} \leqslant C_{a,n}(1 + ||f||_{H^{n/2}} \sqrt{\log(2 + ||f||_{H^a})}) \ . \tag{8.19}$$

Since $s - 1 > 1/2$, it follows that

$$||\partial_x u||_{L^\infty} \lesssim (1 + ||u||_{H^{3/2}} \sqrt{\log(2 + ||u||_{H^s})}) \ . \tag{8.20}$$

Thanks to (8.20) and (5.1), we obtain that there is $C > 0$ such that for all $u_0 \in H^\infty$, $T > 0$, $t \in [-T, T]$,

$$\frac{d}{dt}||u(t)||^2_{H^s} \leqslant C(1 + ||u(t)||_{H^{3/2}}) \log(3 + ||u(t)||^2_{H^s})(3 + ||u(t)||^2_{H^s}) \ ;$$

using Lemma 8.4, we get

$$||u(t)||^2_{H^s} \leqslant (3 + ||u_0||^2_{H^s}) \exp\exp(C(1 + m(\psi, T, ||u_0||_{H^{3/2}}))t) \ .$$

By continuity with respect to the initial data, this inequality remains true for $u_0 \in H^s$. This shows that (1.7) is globally well-posed in $H^s$, and the proof of Theorem 1.5 is complete.

## 9. Appendix

The proof of Lemma 2.4 is similar to, and simpler than, that of Lemma 2.3 which was done in [11]. It is based on the following result due to Coifman and Meyer (see [5], [6]).

**Theorem 9.1.** *Let $\sigma(\eta,\xi) \in C^\infty((\mathbb{R}^n)^k \times \mathbb{R}^n(0,0))$ satisfy*

$$|\partial^\nu \sigma(\eta,\xi)| \leqslant C_\nu(|\xi|+|\eta|)^{-|\nu|} , \tag{9.1}$$

*where $\nu=(\nu_0,...,\nu_k)$, $\nu_j \in \mathbb{N}^n$, and*

$$|\nu| = |\nu_0| + .. + |\nu_k| \leqslant N := n(k+1)+1 .$$

*Then*

$$||\sigma(D)(a_1...a_k,f)||_{L^2} \lesssim ||a_1||_{L^\infty}...||a_k||_{L^\infty}||f||_{L^2} ,$$

*where $\sigma(D)(a,f)=\sigma(D)(a_1...a_k,f)$ is defined by*

$$\sigma(D)(a,f) := \int\int e^{ix(\xi+\tilde{\eta})}\sigma(\eta,\xi)\hat{a}(\eta)\hat{f}(\xi)d\eta d\xi ,$$

$$a(x) = a_1(x_1)...a_k(x_k) , \ x_j \in \mathbb{R}^n ,$$

$$\tilde{\eta} = \eta_1 + ... + \eta_k , \ \eta = (\eta_1,...,\eta_k) .$$

In our case, $n=k=1$, hence $N=3$.

**Proof of Lemma 2.4.** We begin as in the proof of Lemma 2.3 in [11]. We write

$$[J^s,f]g(x) = c\int\int e^{ix(\xi+\eta)}\left[(1+(\xi+\eta)^2)^{s/2}-(1+\eta^2)^{s/2}\right]\hat{f}(\xi)\hat{g}(\eta)d\eta d\xi$$
$$= c\sum_{j=1}^{2}\sigma_j(D)(f,g)(x),$$

where

$$\sigma_j(\xi,\eta) = \left[(1+(\xi+\eta)^2)^{s/2}-(1+\eta^2)^{s/2}\right]\phi_j(\frac{\xi}{\eta})$$

and the $\phi_j$ are positive even functions on $\mathbb{R}$ such that $\phi_1+\phi_2\equiv 1$ on $\mathbb{R}$ and $\mathrm{Supp}\phi_1 \subset [-1/3,1/3]$, $\mathrm{Supp}\phi_2 \subset \mathbb{R}[-1/4,1/4]$.

It was shown in [11] that

$$||\sigma_1(D)(f,g)||_{L^2} \lesssim ||\partial_x f||_{L^\infty}||J^{s-1}g||_{L^2}. \tag{9.2}$$

Next, we define

$$\sigma_2(D)(f,g)(x) = \int\int e^{ix(\xi+\eta)}\frac{(1+(\xi+\eta)^2)^{s/2}-(1+\eta^2)^{s/2}}{\xi(1+\xi^2)^{\frac{s_0+s-1}{2}}}$$
$$\times\phi_2(\frac{\xi}{\eta})(1+\xi^2)^{\frac{s_0+s-1}{2}}\xi\hat{f}(\xi)\hat{g}(\eta)d\eta d\xi =: \tilde{\sigma}_2(D)(J^{s_0+s-1}\partial_x f,g)(x).$$

It suffices to prove that $\tilde{\sigma}_2$ satisfies the assumptions of Theorem 9.1 to obtain that

$$||\sigma_2(D)(f,g)||_{L^2} \lesssim ||J^{s_0+s-1}\partial_x f||_{L^2}||g||_{L^\infty} , \tag{9.3}$$

which will complete the proof of Lemma 2.4, combined with (9.2).

It is clear that $\tilde{\sigma}_2 \in C^\infty(\mathbb{R}^2(0,0))$ (because it vanishes if $|\xi| \leqslant |\eta|/4$). In fact, $\tilde{\sigma}_2 \in C^\infty(\mathbb{R}^2)$. Indeed, if $|\xi| \leqslant 1/8$ and $\tilde{\sigma}_2(\xi,\eta) \neq 0$, then $|\eta| \leqslant 4|\xi| \leqslant 1/2$, and $|\xi+\eta| \leqslant 5/8$. Therefore, using integer series, for $|\xi| \leqslant 1/8$, we can write

$$\tilde{\sigma}_2(\xi,\eta) = \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}} \sum_{r=0}^{\infty} c_r \frac{(\xi+\eta)^{2r} - \eta^{2r}}{\xi} \phi_2(\frac{\xi}{\eta}) ,$$

which is clearly of class $C^\infty$ on $\{(\xi,\eta),\ |\xi| \leqslant 1/8\}$, thanks to the mean value theorem and because the ray of convergence of $\sum c_r X^r$ is 1, and $1/8+1/2 < 1$. Therefore, $\tilde{\sigma}_2$ is bounded on the compact set $K := \{(\xi,\eta),\ |\xi| \leqslant 1,\ |\eta| \leqslant 4\}$, hence on the set $\{(\xi,\eta),\ |\xi| \leqslant 1\}$, because it vanishes on $\{(\xi,\eta),\ |\xi| \leqslant 1\}$“$K$, and we have a similar result for the derivatives of $\tilde{\sigma}_2$.

It remains to see that the derivatives of $\tilde{\sigma}_2$ of order less than or equal to 3 are bounded on $\{(\xi,\eta),\ |\xi| \geqslant 1\}$. That is what we will do now.

Let $\nu$ be a multi index such that $|\nu| \leqslant 3$. Then by the Leibniz formula,

$$\partial^\nu \tilde{\sigma}_2(\xi,\eta) = \sum_{\alpha+\beta+\gamma+\delta=\nu} c_{\alpha\beta\gamma\delta} \partial^\alpha \left( (1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2} \right) \times \partial^\beta \frac{1}{\xi} \partial^\gamma \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}} \partial^\delta \phi_2(\frac{\xi}{\eta}). \tag{9.4}$$

We estimate now each derivate in the right-hand side of (9.4).

$$\left|\partial^\alpha \left( (1+(\xi+\eta)^2)^{s/2} - (1+\eta^2)^{s/2} \right)\right| \lesssim (1+(\xi+\eta)^2)^{\frac{s-|\alpha|}{2}} + (1+\eta^2)^{\frac{s-|\alpha|}{2}} ,$$

$$\left|\partial^\beta \frac{1}{\xi}\right| \lesssim \frac{1}{\xi^{1+|\beta|}}$$

and

$$\left|\partial^\gamma \frac{1}{(1+\xi^2)^{\frac{s_0+s-1}{2}}}\right| \lesssim \frac{1}{(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}}.$$

If $\delta = 0$, $\partial^\delta \phi_2(\xi/\eta) = \phi_2(\xi/\eta)$ is bounded and vanishes on the set $\{(\xi,\eta),$ $|\xi/\eta| \leqslant 1/4\}$. Otherwise, $\partial^\delta \phi_2(\xi/\eta)$ vanishes on $\{(\xi,\eta),\ |\xi/\eta| \leqslant 1/4$ or $|\xi/\eta| \geqslant 1/3\}$, and

$$\left|\partial^\delta \phi_2(\frac{\xi}{\eta})\right| \lesssim \frac{1}{\eta^{|\delta|}} \lesssim \frac{1}{\xi^{|\delta|}}.$$

Therefore, on the set $\{(\xi,\eta),\ |\xi| \geqslant 1\}$,

$$|\partial^\nu \tilde{\sigma}_2(\xi,\eta)|\,(|\xi|+|\eta|)^{|\nu|} \lesssim \operatorname*{Max}_{\alpha+\beta+\gamma+\delta=\nu} Q_{\alpha\beta\gamma\delta}\ , \tag{9.5}$$

where

$$Q_{\alpha\beta\gamma\delta} := \frac{(|\xi|+|\eta|)^{|\nu|}\left((1+(\xi+\eta)^2)^{\frac{s-|\alpha|}{2}} + (1+\eta^2)^{\frac{s-|\alpha|}{2}}\right)}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}}\ .$$

To control $Q_{\alpha\beta\gamma\delta}$, we distinguish two cases:

- if $s-|\alpha| \geqslant 0$,

$$Q_{\alpha\beta\gamma\delta} \lesssim \frac{|\xi|^{|\nu|}(1+|\xi|^2)^{\frac{s-|\alpha|}{2}}}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \lesssim (1+\xi^2)^{-s_0/2} \lesssim 1$$

  because $s_0 \geqslant 0$.
- if $s-|\alpha| < 0$,

$$Q_{\alpha\beta\gamma\delta} \lesssim \frac{|\xi|^{|\nu|}}{|\xi|^{1+|\beta|+|\delta|}(1+\xi^2)^{\frac{s_0+s-1+|\gamma|}{2}}} \lesssim (1+\xi^2)^{(|\alpha|-s_0-s)/2} \lesssim 1$$

  because $|\alpha| \leqslant 3$ and $3-s_0-s \leqslant 0$.

The proof of Lemma 2.4 is complete. □

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