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STABILITY OF STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL POWER NONLINEARITY AND POTENTIALS

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Abstract. We study the stability of standing waves $e^{i\omega t}\phi_\omega(x)$ for a nonlinear Schrödinger equation with critical power nonlinearity $|u|^{4/n}u$ and a potential $V(x)$ in $\mathbb{R}^n$. Here, $\omega \in \mathbb{R}$ and $\phi_\omega(x)$ is a ground state of the stationary problem. Under suitable assumptions on $V(x)$, we show that $e^{i\omega t}\phi_\omega(x)$ is stable for sufficiently large ω. This result gives a different phenomenon from the case $V(x) \equiv 0$ where the strong instability was proved by M. I. Weinstein [25].

1. Introduction and main results

The nonlinear Schrödinger equations with a real-valued potential $V(x)$,

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n}, \tag{1.1}$$

arises in various physical contexts. When $V(x) \equiv 0$, equation (1.1) appears in areas such as nonlinear optics and plasma physics (see, e.g., [6, 23, 26]). The nonlinearity enters due to the effect of changes in the field intensity on the wave-propagation characteristics of the medium. The potential $V(x)$ can be thought of as modeling inhomogeneities in the medium. In [20], equation (1.1) with a bounded potential $V(x)$ is studied as a model proposed to describe the local dynamics at a nucleation site. Equation (1.1) with a harmonic potential $V(x) = |x|^2$ is known as a model to describe the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap (see, e.g., [1, 12, 24]).

We always assume that $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$. In this paper, we particularly discuss the critical case $p = 1 + 4/n$ and prove the stability of a standing-wave solution under suitable assumptions on $V(x)$ and the frequency.

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By a standing wave, we mean a solution of (1.1) of the form

$$u_\omega(t,x) = e^{i\omega t}\phi_\omega(x),$$

where $\omega \in \mathbb{R}$ is a frequency, and $\phi_\omega(x)$ is a ground state of

$$-\Delta\phi + V(x)\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n \tag{1.2}$$

(see Definition 1 below).

Many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [2, 5, 7, 8, 9, 10, 11, 13, 14, 15, 18, 20, 21, 22, 25, 27, 28, 29]). We recall some known results. First, we consider the case $V(x) \equiv 0$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$-\Delta\psi + \omega\psi - |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^n \tag{1.3}$$

in $H^1(\mathbb{R}^n)$ (see [16] for the uniqueness), and the standing-wave solution $e^{i\omega t}\psi_\omega(x)$ of (1.1) with $V(x) \equiv 0$ is stable for any $\omega > 0$ if $p < 1 + 4/n$, and unstable for any $\omega > 0$ if $p \geq 1+4/n$ (see [2, 5, 25]). When $-\Delta+V(x)$ has the first eigenvalue λ_1, Ohta and the author [11] showed that the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ (see also Kunze et al. [15]). Moreover, we proved in [10, 11] that under suitable assumptions on $V(x)$, the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is unstable for sufficiently large $\omega > 0$ if $p > 1 + 4/n$ and that the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for sufficiently large $\omega > 0$ if $p < 1 + 4/n$.

The hypotheses for potential $V(x)$ that will be used are the following. There exist real-valued, radially symmetric functions $V_1(x) = V_1(|x|)$ and $V_2(x) = V_2(|x|)$ such that $V(x) = V_1(x) + V_2(x)$.

(V0) $V_j(x) \geq 0$ in $\mathbb{R}^n$ and $V_j(x) \in C^2(\mathbb{R}^n, \mathbb{R})$, for $j = 1, 2$.

(V1-1) For α with $|\alpha| \leq 2$, there exist $C_\alpha > 0$ and $m_\alpha > 0$ such that

$$|x^\alpha \partial_x^\alpha V_1(x)| \leq C_\alpha(1 + |x|^{m_\alpha}) \text{ for } |x| \geq 1.$$

(V1-2) $\Delta V_1(x) \in L^\infty(\{|x| \geq 1\})$.

(V2) $x^\alpha \partial_x^\alpha V_2(x) \in L^\infty(\{|x| \geq 1\})$ for $|\alpha| \leq 2$.

(V3-1) There exist $\delta_1 > 0$ and $\beta > 0$ such that

$$3x \cdot \nabla V(x) + \Sigma_{k,l} x_k x_l \partial_k \partial_l V(x) \geq \delta_1 |x|^\beta \text{ for } |x| \leq 1.$$

(V3-2) There exist $\delta_2 > 0$ and $\varepsilon > 0$ with $0 < \beta < 2(1 + \varepsilon)$ such that

$$|V(x) + (1/2)x \cdot \nabla V(x)| \leq \delta_2 |x|^\varepsilon \text{ for } |x| \leq 1.$$

Remark 1.1. The conditions (V3-1) and (V3-2) derive from differentiating $\omega^{-1}V(x/\sqrt{\omega})$ twice with respect to ω in order to verify the sufficient condition for stability (see the proof of Lemma 3.2 for details).

Here, we define a real Hilbert space X by

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C})\ ; V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$

with the inner product

$$(v, w)_X := \mathrm{Re} \int_{\mathbb{R}^n} (v(x)\overline{w(x)} + \nabla v(x) \cdot \overline{\nabla w(x)} + V_1(x)v(x)\overline{w(x)})dx.$$

The norm of X is denoted by $\|\cdot\|_X$. We define a closed subspace X_{rad} of X by

$$X_{\mathrm{rad}} = \{v \in X\ ;\ v(x) = v(|x|),\ x \in \mathbb{R}^n\}.$$

Moreover, we define the energy functional E and the charge Q on X by

$$E(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v(x)|^2dx - \frac{1}{p+1}\|v\|_{p+1}^{p+1}, \quad Q(v) := \frac{1}{2}\|v\|_2^2.$$

We remark that by the assumptions (V0), (V2), and $1 < p < 2^* - 1$, the functional E is well-defined on X.

Assumption (A1). For any $u_0 \in X$, there exist $T = T(\|u_0\|_X) > 0$ and a unique solution $u(t) \in C([0,T], X)$ of (1.1) with $u(0) = u_0$ satisfying

$$E(u(t)) = E(u_0), \quad Q(u(t)) = Q(u_0), \quad t \in [0,T].$$

Remark 1.2. The assumption (A1) is satisfied if $V_1(x)$ satisfies the following condition (A1.1) with (V0) (see Theorem 9.2.5 of [4]).

(A1.1) $V_1(x) \in C^\infty(\mathbb{R}^n)$, $\partial_x^\alpha V_1(x) \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \geq 2$.

Next, we consider the stationary problem (1.2).

Definition 1. For $\omega > 0$, we define two functionals on X:

$$S_\omega(v) := E(v) + \omega Q(v) \qquad \text{(action)},$$
$$I_\omega(v) := \|\nabla v\|_2^2 + \omega\|v\|_2^2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2dx - \|v\|_{p+1}^{p+1}.$$

Let $\mathcal{G}_\omega$ be the set of all nonnegative minimizers for

$$\inf\{S_\omega(v):\ v \in X_{\mathrm{rad}} \setminus \{0\},\ I_\omega(v) = 0\}. \tag{1.4}$$

Remark 1.3. (i) Note that $I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v\rangle$.
(ii) Let $\phi_\omega \in \mathcal{G}_\omega$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(\phi_\omega) = \Lambda I'_\omega(\phi_\omega)$. Thus, we have $\langle S'_\omega(\phi_\omega), \phi_\omega\rangle = \Lambda\langle I'_\omega(\phi_\omega), \phi_\omega\rangle$. Since $\langle S'_\omega(\phi_\omega), \phi_\omega\rangle = I_\omega(\phi_\omega) = 0$ and $\langle I'_\omega(\phi_\omega), \phi_\omega\rangle = -(p-1)\|\phi_\omega\|_{p+1}^{p+1} < 0$, we

have $\Lambda = 0$. Namely, ϕ_ω satisfies (1.2). Moreover, for any $v \in X_{\mathrm{rad}} \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $\mathcal{G}_\omega$, we have $S_\omega(\phi_\omega) \leq S_\omega(v)$. Namely, $\phi_\omega \in \mathcal{G}_\omega$ is a (radial) ground state (minimal action solution) of (1.2) in X_{rad}. It is easy to see that a ground state of (1.2) in X_{rad} is a minimizer of (1.4).

Remark 1.4. In (1.4), we consider the radial space X_{rad}. Therefore, under the assumptions (V0), (V1-1), and (V2), $\mathcal{G}_\omega$ is not empty for $n \geq 2$ and $\omega > 0$ since the embedding $X_{\mathrm{rad}} \subset L^r(\mathbb{R}^n)$ is compact for $2 < r < 2^*$.

Assumption (A2). $\phi_\omega(x) \in W^{2,r}(\mathbb{R}^n)$ for any $r \in [2, \infty)$ and $\phi_\omega(x) \in C^2(\mathbb{R}^n)$.

Remark 1.5. Concerning the regularity of $\phi_\omega(x)$, we can apply the method of Theorem 8.1.1 (i) of [4] to the case where $V(x)$ is bounded like $V_2(x)$. In the other case, an L^p estimate of the Schrödinger operator $-\Delta + V(x) + 1$ might be useful. As an example, such a property of the operator can be satisfied when $V(x)$ satisfies

$$|\nabla V(x)|^2 \leq a[V(x) + c_1]^2 + b[V(x) + c_2]^3 \quad \text{on} \quad \mathbb{R}^n,$$

where c_1, c_2, a, and b are nonnegative constants and b satisfies a certain condition (see Okazawa [19] for details). We remark that (A2) and (V0) ensure the positivity of $\phi_\omega \in \mathcal{G}_\omega$ by the maximum principle.

Remark 1.6. We prove in Section 2 that the solution $\phi_\omega \in \mathcal{G}_\omega$ is unique and nondegenerate uniformly in large ω. Namely, for large ω, we can verify that the inverse of the linearized operator is bounded independently of ω.

Examples. (i) (Harmonic potentials) For $c_1, \ldots, c_n \in \mathbb{R}$, $\sum_{j=1}^n c_j^2 x_j^2$ satisfies (A1), (A2), (V0), (V1-1), (V1-2), (V3-1), and (V3-2) with $V_2(x) \equiv 0$.

(ii) Let $n \geq 2$ and $U(x) \in C^2(\mathbb{R}^n)$ be a nonnegative, radially symmetric function which satisfies $|\partial_x^\alpha U(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}$ for $|\alpha| \leq 2$ and there exist $\theta \geq 2$ and $C > 0$ such that $U(x) = C|x|^\theta$ for $|x| \leq 1$. Then, $U(x)$ satisfies (A1), (V0), (V2), (V3-1), and (V3-2) with $V_1(x) \equiv 0$. Moreover, (A2) is satisfied for sufficiently large $\omega > 1$.

(iii) $V(x) \equiv 1$ satisfies (A1), (A2), (V0), (V1-1), (V1-2), and (V2), but does not satisfy (V3-1) and (V3-2), which bring out the difference from the pure power case.

Definition 2. For $\phi_\omega \in \mathcal{G}_\omega$ and $\delta > 0$, we put

$$U_\delta(\phi_\omega) := \Big\{ v \in X_{\mathrm{rad}} : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta}\phi_\omega\|_X < \delta \Big\}.$$

We say that a standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable in X_{rad} if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in U_\delta(\phi_\omega)$, the solution

$u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be unstable in X_{rad} (X).

Our main result in this paper is the following.

Theorem 1. *Assume $n \geq 2$, (A1), (A2), (V0), $(V1-1)$, $(V1-2)$, (V2), $(V3-1)$, $(V3-2)$, and $p = 1 + 4/n$. Let $\phi_\omega \in \mathcal{G}_\omega$. Then there exists $\omega_* \in (0, \infty)$ such that the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of* (1.1) *is stable in X_{rad} for any $\omega \in (\omega_*, \infty)$.*

Remark 1.7. In the case where $V_1(x) = |x|^2$, we may prove Theorem 1 with $V_2(x) \equiv 0$ under fewer assumptions. We can prove the result for $n \geq 1$, and we do not need to require radial symmetry of the space X. Namely, we do not need to assume $n \geq 2$ for the existence of ground states since the embedding $X \subset L^r(\mathbb{R}^n)$ is compact for $2 \leq r < 2^*$ and $n \geq 1$. Moreover, Li and Ni [17] proved the radial symmetry of positive solutions of (1.2). We remark that the proof of (ii) of Proposition 2 below does not work for $n = 1$. However, we might obtain (ii) using equation (1.2) and (i) of Proposition 2 because the regularity of ϕ_ω can be improved for large ω, uniformly in ω, by a bootstrap argument as in the text of Cazenave, Theorem 8.1.1 (i) and (ii) of [4]. Furthermore, a harmonic potential satisfies all the assumptions of Theorem 1 as we have seen in the examples. Concerning the verification of assumption (A2), see the author [9].

When $V(x) \equiv 0$ and $p = 1+4/n$, Weinstein [25] proved that the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ is strongly unstable for $\omega > 0$ (see also Berestycki and Cazenave [2]). However, the argument in [2] and [25] cannot be applied to the case $V(x) \not\equiv 0$. The standing-wave solution of (1.1) with $V(x) \equiv 0$ corresponds to 0 energy, but, for example, the standing-wave solution with $V(x) = |x|^2$ always corresponds to positive energy. In [29], Zhang discussed the instability of the standing-wave solution for (1.1) with $V(x) = |x|^2$ and $p \geq 1 + 4/n$. He constructed a kind of cross-constrained minimization problem following [2], but it is not easy to verify his sufficient condition for the strong instability. To our knowledge, the problem of whether the standing-wave solution of (1.1) with $V(x) \not\equiv 0$ and $p = 1+4/n$ is stable or unstable is still open for $\omega > 0$. Therefore, by Theorem 1, we may answer that if $V(x)$ satisfies the assumptions of Theorem 1, the standing-wave solution of (1.1) with $V(x) \not\equiv 0$ and $p = 1+4/n$ is stable for sufficiently large $\omega > 0$. Here, we remark that when $-\Delta + V(x)$ has the first eigenvalue λ_1, the standing-wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for ω such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$ even if $p = 1 + 4/n$ (see [11]).

For a bounded potential $V(x)$, Rose and Weinstein [20] studied by numerical simulations that if $p = 1 + 4/n$, $\|\phi_\omega\|_2^2$ would increase for large ω, so that $e^{i\omega t}\phi_\omega(x)$ would be stable. We can affirm that this numerical result is correct by Theorem 1 for large $\omega > 0$.

Put $d(\omega) = S_\omega(\phi_\omega)$ for $\phi_\omega \in \mathcal{G}_\omega$. To prove Theorem 1, we verify the following sufficient condition for stability which was obtained by Shatah [21].

Proposition 1. *Assume* $1 < p < 2^* - 1$, $n \geq 2$, *(A1), (A2), (V0), (V1−1), (V1−2), and (V2). Let* $\phi_\omega \in \mathcal{G}_\omega$. *If* $d''(\omega) > 0$ *at* $\omega = \omega_0$, *then the standing-wave solution* $e^{i\omega_0 t}\phi_{\omega_0}(x)$ *of* (1.1) *is stable in* X_{rad}.

In the case where $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)}\psi_1(\sqrt{\omega}x)$, we have $d_0(\omega) = \omega^{2/(p-1)-n/2+1}d_0(1)$, where we put $d_0(\omega) = S_\omega(\psi_\omega)$ with $V(x) \equiv 0$. Therefore, it is easy to check the increase and decrease of $d_0'(\omega)$. However, it seems difficult to check this property of $d(\omega)$ for general $V(x)$.

This paper is organized as follows. In Section 2, we investigate and summarize the properties of a ground state $\phi_\omega \in \mathcal{G}_\omega$. In Section 3, we check the stability condition by Shatah [21] and we prove Theorem 1. The method in this paper is inspired by the paper of Fibich and Wang [8], which studied the stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities. Following Fibich and Wang [8], we prove Proposition 1 in Section 4. Throughout this paper, different positive constants might be denoted by the same letter C.

2. Properties of a ground state

As we mentioned in Remark 1.4, $\mathcal{G}_\omega$ is not empty for $\omega > 0$ if $n \geq 2$. We always assume $n \geq 2$ from now on and let $\phi_\omega \in \mathcal{G}_\omega$. First we remark that $d(\omega)$ is simply rewritten by

$$d(\omega) = S_\omega(\phi_\omega) = \frac{p-1}{2(p+1)}\|\phi_\omega\|_{p+1}^{p+1} \tag{2.1}$$

since $\phi_\omega \in \mathcal{G}_\omega$ and satisfies $I_\omega(\phi_\omega) = 0$. In this section, we study the properties of a rescaled function of $\phi_\omega(x)$ to check the stability condition $d''(\omega) > 0$ in Proposition 1. Namely, we define the rescaled function $\tilde{\phi_\omega}(x)$ by

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi_\omega}(\sqrt{\omega}x), \qquad \omega \in (0, \infty).$$

Then $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\tilde{\phi_\omega} + \omega^{-1}V\left(\frac{x}{\sqrt{\omega}}\right)\tilde{\phi_\omega} + \tilde{\phi_\omega} - |\tilde{\phi_\omega}|^{p-1}\tilde{\phi_\omega} = 0, \tag{2.2}$$

and we have

$$(2.1) = \frac{p-1}{2(p+1)}\omega^\alpha\|\tilde{\phi_\omega}\|_{p+1}^{p+1}, \tag{2.3}$$

where $\alpha = (p+1)/(p-1) - n/2$.

Remark 2.1. We note that $\alpha > 1$ if $p < 1 + 4/n$, $\alpha = 1$ if $p = 1 + 4/n$ and that $\alpha < 1$ if $p > 1 + 4/n$.

Define the linearized operator $\tilde{L_\omega}$ on $\{v \in H^2_{\mathrm{rad}}(\mathbb{R}^n) : \ V(x)v \in L^2(\mathbb{R}^n)\}$ by

$$\tilde{L_\omega} := -\Delta + 1 + \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) - p\tilde{\phi_\omega}^{p-1}(x), \qquad \omega \in (0, \infty).$$

Proposition 2. *Assume $n \geq 2$, $1 < p < 2^* - 1$, (A2), (V0), (V1 − 1), (V1 − 2), and (V2). Let $\phi_\omega \in \mathcal{G}_\omega$ and $\psi_1(x)$ be the unique positive radial solution of* (1.3) *with $\omega = 1$. Then the following hold.*

(i) $\lim_{\omega \to \infty} \|\tilde{\phi_\omega} - \psi_1\|_{H^1_{\mathrm{rad}}} = 0.$

(ii) $\tilde{\phi_\omega}(r) \to 0$ *as* $r \to \infty$ (*independent of* ω).

(iii) *There exist* $C_0(n) > 0$, $r_0(n,p) > 0$, *and* $\omega_1(n,p) > 0$ *such that*

$$|\tilde{\phi_\omega}(r)| \leq C_0 r^{-(n-1)/2} e^{-r/2}$$

for any $r \geq r_0$ *and* $\omega \geq \omega_1$.

(iv) $\tilde{L_\omega}$ *is invertible and* $\tilde{L_\omega}^{-1}$ *is bounded for sufficiently large* ω; *i.e., there exist* $\omega_2 > 0$ *and* $C_2 > 0$ *such that for any* $\omega \geq \omega_2$

$$\|\tilde{L_\omega} v\|_2 \geq C_2 \|v\|_2$$

for any $v \in H^2_{\mathrm{rad}}(\mathbb{R}^n)$ *and* $V(x)v \in L^2(\mathbb{R}^2)$.

(v) *The positive radial solution of* (1.2) *is unique when ω is large enough.*

(vi) $\omega \mapsto \tilde{\phi_\omega}$ *is a* C^1 *mapping from* $(0,\infty)$ *to* X_{rad} *for sufficiently large* ω.

Proof. We have shown (i) in Remark 2.1 of [10]. To prove (ii) and (iii), we apply a method similar to that in Berestycki and Lions [3]. For the sake of completeness, we repeat their proof here.

(ii) We note that $\tilde{\phi_\omega}(x) \in X_{\mathrm{rad}}$. We use the Sobolev embedding for $\tilde{\phi_\omega}(\cdot) : [0,\infty) \to \mathbb{R}$. For $L > 0$, we have

$$\|\tilde{\phi_\omega}\|^2_{L^\infty(L,\infty)} \leq C \|\tilde{\phi_\omega}\|^2_{H^1(L,\infty)}.$$

Here,

$$\begin{aligned}\|\tilde{\phi_\omega}\|^2_{H^1(L,\infty)} &= \int_L^\infty (|\tilde{\phi_\omega}'|^2 + |\tilde{\phi_\omega}|^2) dr \leq L^{-(n-1)} \int_L^\infty (|\tilde{\phi_\omega}'|^2 + |\tilde{\phi_\omega}|^2) r^{n-1} dr \\ &\leq L^{-(n-1)} \|\tilde{\phi_\omega}\|^2_{H^1_{\mathrm{rad}}}.\end{aligned}$$

Therefore, the right-hand side converges to 0 as $L \to \infty$; in particular, this convergence is independent of ω since we have (i).

(iii) By assumption (A2), $\tilde{\phi}_\omega(x)$ is of class $C^2(\mathbb{R}^n)$. Accordingly, it satisfies the equation

$$-u'' - \frac{n-1}{r}u' + \Big\{1 + \omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}u - u^p = 0, \qquad r > 0. \tag{2.4}$$

For simplicity, we denote the solution of (2.4) by $u_\omega(r)$. Set $v_\omega = r^{(n-1)/2}u_\omega$. Then v_ω satisfies

$$(v_\omega)_r r = \Big[q(r) + \frac{b}{r^2}\Big]v_\omega,$$

where $q(r) = \left[\left\{1 + \omega^{-1}V(x/\sqrt{\omega})\right\} u_\omega - u_\omega^p\right] / u_\omega(r)$ and $b = (n-1)(n-3)/4$. By (ii), there exist $r_0(n,p) > 0$ and $\omega_1(n,p) > 0$ such that

$$q(r) + \frac{b}{r^2} \geq \frac{1}{2}$$

for any $r \geq r_0$ and $\omega \geq \omega_1$. Let $w_\omega = v_\omega^2$; then w_ω satisfies

$$\frac{1}{2}(w_\omega)_r r = (v_\omega)_r^2 + \Big[q(r) + \frac{b}{r^2}\Big]w_\omega.$$

Thus, for $r \geq r_0$ and $\omega \geq \omega_1$, one has $(w_\omega)_r r \geq w_\omega$ and $w_\omega \geq 0$.

Now let $z_\omega = e^{-r}((w_\omega)_r + w_\omega)$. We have $(z_\omega)_r = e^{-r}((w_\omega)_r r - w_\omega) \geq 0$ for $r \geq r_0$ and $\omega \geq \omega_1$. Hence z_ω is nondecreasing on $(r_0, +\infty)$. If there exists $r_1 > r_0$ such that $z_\omega(r_1) > 0$, then $z_\omega(r) \geq z_\omega(r_1) > 0$ for any $r \geq r_1$. This implies that

$$(w_\omega)_r + w_\omega \geq z_\omega(r_1)e^r, \qquad r \geq r_1, \qquad \omega \geq \omega_1;$$

that is, $(w_\omega)_r + w_\omega$ is not integrable on $(r_1, +\infty)$. This is a contradiction since $(w_\omega)_r$ and w_ω are integrable. Hence, $z_\omega(r) \leq 0$ for all $r \geq r_0$, which implies that

$$(e^r w_\omega)_r = e^{2r} z_\omega \leq 0, \qquad r \geq r_0, \qquad \omega \geq \omega_1.$$

Therefore, $w_\omega(r) \leq C_0 e^{-r}$ and in turn

$$|u_\omega(r)| \leq C_0 r^{-(n-1)/2} e^{-r/2}$$

for all $r \geq r_0$ and $\omega \geq \omega_1$.

(iv) First, we note that there exists $C_1 > 0$ such that

$$\|L_0 v\|_2 \geq C_1 \|v\|_2$$

for any $v \in H^2_{\mathrm{rad}}(\mathbb{R}^n)$, where $L_0 := -\Delta + 1 - p\psi_1^{p-1}$. Moreover, we have $\|\tilde{\phi}_\omega - \psi_1\|_\infty \to 0$ and $\|\omega^{-1}V(\cdot/\sqrt{\omega})\tilde{\phi}_\omega - \omega^{-1}V(\cdot/\sqrt{\omega})\psi_1\|_\infty \to 0$ as $\omega \to \infty$. Indeed, since $\|\tilde{\phi}_\omega\|_{H^1}$ is uniformly bounded in large ω by (i), it follows from the elliptic regularity argument that $\tilde{\phi_\omega}$ converges to ψ_1 in C^2_{loc} as $\omega \to \infty$. Using a proof similar to that of (ii), the one-dimensional Sobolev embedding makes it possible to have $\|\tilde{\phi_\omega} - \psi_1\|_\infty \to 0$ as $\omega \to \infty$ since $\tilde{\phi_\omega}$ and ψ_1 are radially symmetric functions and $\tilde{\phi_\omega}$ converges strongly to ψ_1 in H^1_{rad} as $\omega \to \infty$. Moreover, $\tilde{\phi_\omega}$ and ψ_1 decay exponentially for large $|x| = r > 0$, independently of ω as we proved (iii). Accordingly, from (V0), (V1-1), and (V2) we have $\|\omega^{-1}V(\cdot/\sqrt{\omega})\tilde{\phi_\omega} - \omega^{-1}V(\cdot/\sqrt{\omega})\psi_1\|_\infty \to 0$ as $\omega \to \infty$. Since $\tilde{L_\omega} = L_0 + \omega^{-1}V(x/\sqrt{\omega}) - p(\tilde{\phi_\omega}^{p-1} - \psi_1^{p-1})$, we have

$$\begin{aligned}\langle \tilde{L_\omega} v, \tilde{L_\omega} v\rangle &= \|L_0 v\|_2^2 + 2\langle L_0 v, \omega^{-1}V(x/\sqrt{\omega})v\rangle - 2p\langle L_0 v, (\tilde{\phi_\omega}^{p-1} - \psi_1^{p-1})v\rangle \\ &\quad - 2p\langle \omega^{-1}V(x/\sqrt{\omega})v, (\tilde{\phi_\omega}^{p-1} - \psi_1^{p-1})v\rangle \\ &\quad + 2p\langle (\tilde{\phi_\omega}^{p-1} - \psi_1^{p-1})v, (\tilde{\phi_\omega}^{p-1} - \psi_1^{p-1})v\rangle. \end{aligned} \tag{2.5}$$

Concerning the second term of (2.5), we have the following estimate:

$$\begin{aligned}&\langle L_0 v, \omega^{-1}V(x/\sqrt{\omega})v\rangle = \langle L_0 v, \omega^{-1}V_1(x/\sqrt{\omega})v\rangle + \langle L_0 v, \omega^{-1}V_2(x/\sqrt{\omega})v\rangle \\ &= \langle -\Delta v + v - p\psi_1^{p-1}v, \omega^{-1}V_1(x/\sqrt{\omega})v\rangle + \langle L_0 v, \omega^{-1}V_2(x/\sqrt{\omega})v\rangle \\ &= \mathrm{Re}\int_{\mathbb{R}^n} \nabla v \cdot \nabla\Big(\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)\bar{v}\Big) + \int_{\mathbb{R}^n} \omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 \\ &\quad - p\int_{\mathbb{R}^n} \psi_1^{p-1}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 + \langle L_0 v, \omega^{-1}V_2(x/\sqrt{\omega})v\rangle \\ &= \frac{\omega^{-1}}{2}\int_{\mathbb{R}^n} \nabla|v|^2 \cdot \nabla V_1\Big(\frac{x}{\sqrt{\omega}}\Big) + \int_{\mathbb{R}^n}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|\nabla v|^2 + \int_{\mathbb{R}^n}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 \\ &\quad - p\int_{\mathbb{R}^n} \psi_1^{p-1}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 + \langle L_0 v, \omega^{-1}V_2(x/\sqrt{\omega})v\rangle \\ &= -\frac{\omega^{-1}}{2}\int_{\mathbb{R}^n} \Delta V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 + \int_{\mathbb{R}^n}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|\nabla v|^2 + \int_{\mathbb{R}^n}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 \\ &\quad - p\int_{\mathbb{R}^n} \psi_1^{p-1}\omega^{-1}V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v|^2 + \langle L_0 v, \omega^{-1}V_2(x/\sqrt{\omega})v\rangle. \end{aligned}$$

Here, from assumptions (V0) and (V1-2), we have

$$\begin{aligned}&\frac{\omega^{-1}}{2}\int_{\mathbb{R}^n} \Delta V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v(x)|^2 dx \\ &= \frac{\omega^{n/2-2}}{2}\int_{|y|\le 1} \Delta V_1(y)|v(\sqrt{\omega}y)|^2 dy + \frac{\omega^{n/2-2}}{2}\int_{|y|\ge 1} \Delta V_1(y)|v(\sqrt{\omega}y)|^2 dy \\ &\le \frac{\omega^{n/2-2}}{2}C\int_{\mathbb{R}^n} |v(\sqrt{\omega}y)|^2 dy = \frac{\omega^{-2}}{2}C\|v\|_2^2. \end{aligned}$$

Also, it follows from (V0) and (V1-1) that

$$
\begin{aligned}
&p\omega^{-1}\int_{\mathbb{R}^n}\psi_1^{p-1}(x)V_1\Big(\frac{x}{\sqrt{\omega}}\Big)|v(x)|^2dx\\
&=p\omega^{n/2-1}\int_{|y|\le 1}\psi_1^{p-1}(\sqrt{\omega}y)V_1(y)|v(\sqrt{\omega}y)|^2dy\\
&\quad+p\omega^{n/2-1}\int_{|y|\ge 1}\psi_1^{p-1}(\sqrt{\omega}y)V_1(y)|v(\sqrt{\omega}y)|^2dy\\
&\le p\omega^{n/2-1}\sup_{|y|\le 1}|V_1(y)|\int_{|y|\le 1}\psi_1^{p-1}(\sqrt{\omega}y)|v(\sqrt{\omega}y)|^2dy\\
&\quad+p\omega^{n/2}\int_{|y|\ge 1}\psi_1^{p-1}(\sqrt{\omega}y)C_0(1+|y|^{m_0})|v(\sqrt{\omega}y)|^2dy\\
&\le p\omega^{n/2-1}C\int_{\mathbb{R}^n}\psi_1^{p-1}(\sqrt{\omega}y)|v(\sqrt{\omega}y)|^2dy\\
&\quad+p\omega^{-1}\int_{\mathbb{R}^n}\psi_1^{p-1}(x)C_0(1+\omega^{-m_0/2}|x|^{m_0})|v(x)|^2dx\\
&\le p\omega^{-1}C\int_{\mathbb{R}^n}\psi_1^{p-1}(x)|v(x)|^2dx+p\omega^{-(1+m_0/2)}C\int_{\mathbb{R}^n}\psi_1^{p-1}(x)|x|^{m_0}|v(x)|^2dx\\
&\le p\omega^{-1}C\|v\|_2^2+p\omega^{-(1+m_0/2)}C\|v\|_2^2,
\end{aligned}
$$

since $\psi_1\in L^\infty(\mathbb{R}^n)$ and decays exponentially at infinity. Moreover, we have

$$\langle L_0v,\omega^{-1}V_2(x/\sqrt{\omega})v\rangle\le\omega^{-1}C\|L_0v\|_2\|v\|_2.$$

Thus we have in consequence

$$
\begin{aligned}
\langle L_0v,\omega^{-1}V(x/\sqrt{\omega})v\rangle\ge&-\frac{\omega^{-2}}{2}C\|v\|_2^2-p\omega^{-1}C\|v\|_2^2\\
&-p\omega^{-(1+m_0/2)}C\|v\|_2^2-\omega^{-1}C\|L_0v\|_2\|v\|_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(2.5)\ge&\|L_0v\|_2^2-C\Big(\frac{\omega^{-2}}{2}+p\omega^{-1}+p\omega^{-(1+m_0/2)}\Big)\|v\|_2^2-\omega^{-1}C\|L_0v\|_2\|v\|_2\\
&-2p\|\tilde{\phi_\omega}^{p-1}-\psi_1^{p-1}\|_\infty\|L_0v\|_2\|v\|_2\\
&-2p\|\omega^{-1}V(\cdot/\sqrt{\omega})(\tilde{\phi_\omega}^{p-1}-\psi_1^{p-1})\|_\infty\|v\|_2^2-2p\|\tilde{\phi_\omega}^{p-1}-\psi_1^{p-1}\|_\infty^2\|v\|_2^2\\
\ge&\frac{C_1}{2}\|L_0v\|_2\|v\|_2-\{C(\omega^{-2}/2+p\omega^{-1}+p\omega^{-(1+m_0/2)})\\
&+2p\|\omega^{-1}V(\cdot/\sqrt{\omega})(\tilde{\phi_\omega}^{p-1}-\psi_1^{p-1})\|_\infty+2p\|\tilde{\phi_\omega}^{p-1}-\psi_1^{p-1}\|_\infty\}\|v\|_2^2
\end{aligned}
$$

$$\geq \frac{C_1^2}{4}\|v\|_2^2$$

for sufficiently large ω. We remark that $C_2 (= C_1^2/4)$ is independent of ω.
(v) Let $\tilde{\phi_{1,\omega}}$ and $\tilde{\phi_{2,\omega}}$ be two distinct solutions of (2.2). Then $\tilde{\phi_{1,\omega}}$ and $\tilde{\phi_{2,\omega}}$ satisfy

$$L_\omega^*(\tilde{\phi_{1,\omega}} - \tilde{\phi_{2,\omega}}) = 0,$$

where

$$L_\omega^* := -\Delta + 1 + \omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big) - \frac{\tilde{\phi_{1,\omega}}^p - \tilde{\phi_{2,\omega}}^p}{\tilde{\phi_{1,\omega}} - \tilde{\phi_{2,\omega}}}.$$

Since $\tilde{\phi_{1,\omega}}$ and $\tilde{\phi_{2,\omega}}$ converge to ψ_1 in L^∞ as $\omega \to \infty$, we have

$$\frac{\tilde{\phi_{1,\omega}}^p - \tilde{\phi_{2,\omega}}^p}{\tilde{\phi_{1,\omega}} - \tilde{\phi_{2,\omega}}} \to p\psi_1^{p-1}, \qquad \omega \to \infty.$$

Thus, there exist $C_3 > 0$ and $\omega_3 > 0$ such that for any $\omega \geq \omega_3$, we have $\|L_\omega^* v\|_2 \geq C_3\|v\|_2$ for any $v \in H^2_{\rm rad}$ and $V(x)v \in L^2$ by the same method as (iv). Therefore we have the uniqueness of the positive solution.
(vi) We may prove (vi) in the same way as the proof of Theorem 18 in Shatah and Strauss [22] since we have proved (iv) and (v). □

Remark 2.2. The assumption $(V1-2)$ is needed only for the parts (iv), (v), and (vi).

3. Proof of Theorem 1

In this section, we verify the sufficient condition for stability for large ω. First, we need the following lemma.

Lemma 3.1. *Assume $1 < p < 2^* - 1$, $n \geq 2$, (A2), (V0), $(V1-1)$, and (V2). Let $\phi_\omega \in \mathcal{G}_\omega$. Then we have*

$$\text{(i)} \qquad \tilde{L_\omega}\Big(\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\Big) = -\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega},$$

$$\text{(ii)} \qquad \int_{\mathbb{R}^n}\tilde{\phi_\omega}^p(x)\frac{\partial\tilde{\phi_\omega}}{\partial\omega}(x)dx = \frac{1}{p-1}\int_{\mathbb{R}^n}\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}^2(x)dx.$$

Proof. Differentiating (2.2) with respect to ω gives

$$-\Delta\Big(\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\Big) + \frac{\partial\tilde{\phi_\omega}}{\partial\omega} + \omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\frac{\partial\tilde{\phi_\omega}}{\partial\omega} + \frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega} - p\tilde{\phi_\omega}^{p-1}\frac{\partial\tilde{\phi_\omega}}{\partial\omega} = 0. \tag{3.1}$$

This implies (i). If we multiply (3.1) by $\tilde{\phi_\omega}$ and integrate, we have

$$\int_{\mathbb{R}^n} \nabla\Big(\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\Big)\nabla\tilde{\phi_\omega} + \int_{\mathbb{R}^n} \frac{\partial\tilde{\phi_\omega}}{\partial\omega}\tilde{\phi_\omega} + \int_{\mathbb{R}^n} \omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\tilde{\phi_\omega} + \int_{\mathbb{R}^n} \frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}^2 - p\int_{\mathbb{R}^n} \tilde{\phi_\omega}^p\frac{\partial\tilde{\phi_\omega}}{\partial\omega} = 0.$$

If we multiply (2.2) by $\partial\tilde{\phi_\omega}/\partial\omega$ and integrate,

$$\int_{\mathbb{R}^n} \nabla\tilde{\phi_\omega}\nabla\Big(\frac{\partial\tilde{\phi_\omega}}{\partial\omega}\Big) + \int_{\mathbb{R}^n} \omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\tilde{\phi_\omega}\frac{\partial\tilde{\phi_\omega}}{\partial\omega} + \int_{\mathbb{R}^n} \tilde{\phi_\omega}\frac{\partial\tilde{\phi_\omega}}{\partial\omega} - \int_{\mathbb{R}^n} \tilde{\phi_\omega}^p\frac{\partial\tilde{\phi_\omega}}{\partial\omega} = 0.$$

The difference of the last two equations gives (ii). □

In the following Lemma, we prove the sufficient condition for stability $d''(\omega) > 0$ for sufficiently large ω.

Lemma 3.2. *Assume $p = 1 + 4/n$, $n \geq 2$, $(A2)$ $(V0)$ $(V1-1)$ $(V1-2)$ $(V2)$ $(V3-1)$ and $(V3-2)$. Let $\phi_\omega \in \mathcal{G}_\omega$. Then there exists $\omega_* > 0$ such that $d''(\omega) > 0$ for any $\omega \in (\omega_*, \infty)$.*

Proof. It follows from (2.3) that

$$d(\omega) = \frac{p-1}{2(p+1)}\omega^\alpha\|\tilde{\phi_\omega}\|_{p+1}^{p+1}, \quad \text{where} \quad \alpha = \frac{p+1}{p-1} - \frac{n}{2}.$$

Differentiating with respect to ω and using the relation (ii) of Lemma 3.1, we have

$$\begin{aligned} d'(\omega) &= \frac{p-1}{2}\omega^\alpha\int_{\mathbb{R}^n} \tilde{\phi_\omega}^p\frac{\partial\tilde{\phi_\omega}}{\partial\omega} + \frac{p-1}{2(p+1)}\alpha\omega^{\alpha-1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1} \\ &= \frac{\omega^\alpha}{2}\int_{\mathbb{R}^n} \frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}^2 + \frac{p-1}{2(p+1)}\alpha\omega^{\alpha-1}\|\tilde{\phi_\omega}\|_{p+1}^{p+1}. \end{aligned}$$

Furthermore, we differentiate once more with respect to ω and use (i) and (ii) of Lemma 3.1. Letting $p = 1 + 4/n$ (i.e., $\alpha = 1$), we have

$$d''(\omega) = \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\} + \frac{\omega}{2}\frac{\partial^2}{\partial\omega^2}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\Big]\tilde{\phi_\omega}^2 - \omega\int_{\mathbb{R}^n} \frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\tilde{L_\omega}^{-1}\Big[\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\Big].$$

Here, by Proposition 2 (iv), we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\tilde{L_\omega}^{-1}\Big[\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\Big] \\ &\leq \Big\|\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\Big\|_2\Big\|\tilde{L_\omega}^{-1}\Big[\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\Big]\Big\|_2 \end{aligned}$$

$$\leq \frac{1}{C_2}\Big\| \frac{\partial}{\partial \omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}\Big\|_2^2$$

for sufficiently large $\omega > 0$. Therefore,

$$\begin{aligned}
d''(\omega) &\geq \int_{\mathbb{R}^n} \Big[\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\} + \frac{\omega}{2}\frac{\partial^2}{\partial\omega^2}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\} \\
&\qquad - \frac{\omega}{C_2}\Big(\frac{\partial}{\partial\omega}\Big\{\omega^{-1}V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\Big)^2\Big]\tilde{\phi_\omega}^2 \\
&= \frac{\omega^{-2}}{8}\int_{\mathbb{R}^n}\Big\{3\frac{x}{\sqrt{\omega}}\cdot\nabla V\Big(\frac{x}{\sqrt{\omega}}\Big) + \Sigma_{j,k=1}^n \frac{x_j}{\sqrt{\omega}}\frac{x_k}{\sqrt{\omega}}\partial_j\partial_k V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}\tilde{\phi_\omega}^2(x)dx \\
&\qquad - 8C\omega^{-3}\int_{\mathbb{R}^n}\Big\{V\Big(\frac{x}{\sqrt{\omega}}\Big) + \frac{1}{2}\frac{x}{\sqrt{\omega}}\cdot\nabla V\Big(\frac{x}{\sqrt{\omega}}\Big)\Big\}^2\tilde{\phi_\omega}^2(x)dx \\
&= \frac{\omega^{n/2-2}}{8}\int_{\mathbb{R}^n}\Big\{3y\cdot\nabla V(y) + \Sigma_{j,k=1}^n y_j y_k \partial_j\partial_k V(y)\Big\}\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy \\
&\qquad - 8C\omega^{n/2-3}\int_{\mathbb{R}^n}\Big\{V(y) + \frac{1}{2}y\cdot\nabla V(y)\Big\}^2\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy \\
&= \text{(I)} - \text{(II)} + \text{(III)},
\end{aligned}$$

where

$$\begin{aligned}
\text{(I)} &= \frac{\omega^{n/2-2}}{8}\int_{|y|\leq 1}\Big\{3y\cdot\nabla V(y) + \Sigma_{j,k=1}^n y_j y_k\partial_j\partial_k V(y)\Big\}\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy, \\
\text{(II)} &= 8C\omega^{n/2-3}\int_{|y|\leq 1}\Big\{V(y) + \frac{1}{2}y\cdot\nabla V(y)\Big\}^2\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy, \\
\text{(III)} &= \frac{\omega^{n/2-2}}{8}\int_{|y|\geq 1}\Big\{3y\cdot\nabla V(y) + \Sigma_{j,k=1}^n y_j y_k\partial_j\partial_k V(y)\Big\}\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy \\
&\qquad - 8C\omega^{n/2-3}\int_{|y|\geq 1}\Big\{V(y) + \frac{1}{2}y\cdot\nabla V(y)\Big\}^2\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy.
\end{aligned}$$

It follows from Proposition 2 (iii) that (III) is very small for sufficiently large ω. Indeed, by (V1-1) and (V2), there exist $m, M > 0$ and $C > 0$ such that

$$|\text{(III)}| \leq C\omega^{\frac{n}{2}-2}\int_1^\infty r^m\tilde{\phi_\omega}^2(\sqrt{\omega}r)r^{n-1}dr + C\omega^{\frac{n}{2}-3}\int_1^\infty r^M\tilde{\phi_\omega}^2(\sqrt{\omega}r)r^{n-1}dr. \tag{3.2}$$

We choose $\omega > 0$ such that $\omega \geq \max\{r_0^2, \omega_1\}$. From Proposition 2 (iii), we have

$$|\tilde{\phi_\omega}(\sqrt{\omega}r)|^2 \leq C_0^2\omega^{-(n-1)/2}r^{-(n-1)}e^{-\sqrt{\omega}r}$$

for any $r \geq 1$ and $\omega \geq \omega_4$, where we put $\omega_4 = \max\{r_0^2, \omega_1\}$. Therefore,

$$(3.2) \leq C\omega^{-3/2}\int_1^\infty r^m e^{-\sqrt{\omega}r}dr + C\omega^{-5/2}\int_1^\infty r^M e^{-\sqrt{\omega}r}dr.$$

Thus, we have

$$|(\mathrm{III})| \leq C\omega^{-2}e^{-\sqrt{\omega}} \tag{3.3}$$

for sufficiently large ω since

$$\int_1^\infty r^m e^{-\sqrt{\omega}r}dr \leq C\omega^{-1/2}e^{-\sqrt{\omega}}$$

for large ω. Next, we observe that (I) is bounded from below for sufficiently large ω. By the assumption (V3-1),

$$\begin{aligned}(\mathrm{I}) &\geq \frac{\omega^{n/2-2}}{8}\delta_1\int_{|y|\leq 1}|y|^\beta\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy = \frac{\omega^{-(2+\beta/2)}}{8}\delta_1\int_{|x|\leq\sqrt{\omega}}|x|^\beta\tilde{\phi_\omega}^2(x)dx \\ &\geq \frac{\omega^{-(2+\beta/2)}}{8}\delta_1\int_{1\leq|x|\leq\sqrt{\omega}}|x|^\beta\tilde{\phi_\omega}^2(x)dx \geq \frac{\omega^{-(2+\beta/2)}}{8}\delta_1\int_{1\leq|x|\leq\sqrt{\omega}}\tilde{\phi_\omega}^2(x)dx \\ &\geq \frac{\omega^{-(2+\beta/2)}}{16}\delta_1\int_{1\leq|x|}\psi_1^2(x)dx \end{aligned} \tag{3.4}$$

for sufficiently large ω. In the last inequality, we used the convergence property of $\tilde{\phi_\omega}(x)$ which was discussed in Proposition 2 (i) of Section 2. Now, we estimate (II) from above. It follows from the assumption (V3-2) that

$$\begin{aligned}&|(\mathrm{II})| \leq 8C\omega^{n/2-3}\int_{|y|\leq 1}\left|V(y) + \frac{1}{2}y\cdot\nabla V(y)\right|^2\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy \\ &\leq 8C\omega^{\frac{n}{2}-3}\delta_2^2\int_{|y|\leq 1}|y|^{2\varepsilon}\tilde{\phi_\omega}^2(\sqrt{\omega}y)dy = 8C\omega^{-(3+\varepsilon)}\delta_2^2\int_{0\leq|x|\leq\sqrt{\omega}}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx \\ &\leq 8C\delta_2^2\omega^{-(3+\varepsilon)}\end{aligned} \tag{3.5}$$

since $\int_{0\leq|x|\leq\sqrt{\omega}}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx$ is finite and independent of ω. Indeed,

$$\int_{0\leq|x|\leq\sqrt{\omega}}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx = \int_{0\leq|x|\leq r_0}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx + \int_{r_0\leq|x|\leq\sqrt{\omega}}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx$$

for sufficiently large ω, where r_0 is defined in Proposition 2 (iii). Here,

$$\int_{0\leq|x|\leq r_0}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx \leq r_0^{2\varepsilon}\int_{\mathbb{R}^n}\tilde{\phi_\omega}^2(x)dx \leq Cr_0^{2\varepsilon}\int_{\mathbb{R}^n}\psi_1^2(x)dx$$

by Proposition 2 (i). Also, it follows from Proposition 2 (iii) that

$$\int_{r_0\le|x|\le\sqrt{\omega}}|x|^{2\varepsilon}\tilde{\phi_\omega}^2(x)dx \le C_0\int_{r_0\le|x|}|x|^{2\varepsilon-(n-1)}e^{-|x|}dx.$$

Combining (3.4) and (3.5) with (3.3), we have

$$d''(\omega) \ge \frac{\delta_1}{16}\omega^{-(2+\beta/2)}\int_{1\le|x|}\psi_1^2(x)dx - 8C\delta_2^2\omega^{-(3+\varepsilon)} - C\omega^{-2}e^{-\sqrt{\omega}}.$$

The right-hand side is positive for sufficiently large ω since $\beta < 2(1+\varepsilon)$. Thus, there exists $\omega_* > 0$ such that $d''(\omega) > 0$ for any $\omega \in (\omega_*, \infty)$. □

4. Orbital stability

We prove Proposition 1 in this section, following Fibich and Wang [8].

Lemma 4.1. *Assume $1 < p < 2^* - 1$, $n \ge 2$, (A2), (V0), $(V1-1)$, $(V1-2)$, and (V2). Let $\phi_\omega \in \mathcal{G}_\omega$ and $d''(\omega) > 0$ at $\omega = \omega_0$. Then there exists $\delta = \delta(\omega_0) > 0$ such that for all ω with $|\omega - \omega_0| < \delta$,*

$$d(\omega) \ge d(\omega_0) + d'(\omega_0)(\omega-\omega_0) + \frac{1}{4}d''(\omega_0)(\omega-\omega_0)^2.$$

Proof. We apply a Taylor expansion at $\omega = \omega_0$ to $d(\omega)$. Regularity of $d(\omega)$ with respect to large ω is ensured by Proposition 2 (vi). (Namely, we consider in practice ω_0 as a point which satisfies the statements of Proposition 2.) □

We define the C^1 map $\omega(\cdot) : U_\delta(\phi_\omega) \to \mathbb{R}$ defined by

$$\omega(u) = d^{-1}\Big(\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}\Big). \tag{4.1}$$

Here, let us denote ϕ_{ω_0} by ϕ_0 for simplicity.

Lemma 4.2. *Assume $1 < p < 2^* - 1$, $n \ge 2$, (A2), (V0), $(V1-1)$, $(V1-2)$, and (V2). Let $\phi_\omega \in \mathcal{G}_\omega$ and $d''(\omega) > 0$ at $\omega = \omega_0$. Then there exists $\delta = \delta(\omega_0) > 0$ such that for all $u \in U_\delta(\phi_0)$,*

$$E(u) - E(\phi_0) + \omega(u)\{Q(u) - Q(\phi_0)\} \ge \frac{1}{4}d''(\omega_0)(\omega(u)-\omega_0)^2.$$

Proof. We have the relation

$$E(u) + \omega(u)Q(u) = S_{\omega(u)}(u), \tag{4.2}$$

and we note that we have the following variational expression of $S_\omega(\phi_\omega)$ (see Section 3 of [10]):

$$S_\omega(\phi_\omega) = \inf\{S_\omega(v) :\ v \in X_{\mathrm{rad}},\ \|v\|_{p+1}^{p+1} = \|\phi_\omega\|_{p+1}^{p+1}\}.$$

In addition, from (4.1) and (2.1), we have for all $u \in U_\delta(\phi_0)$,

$$\frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} = d(\omega(u)) = \frac{p-1}{2(p+1)}\|\phi_{\omega(u)}\|_{p+1}^{p+1}.$$

Since $\phi_{\omega(u)}$ is a minimizer of $S_{\omega(u)}(u)$ subject to the constraint $\|u\|_{p+1}^{p+1} = \|\phi_{\omega(u)}\|_{p+1}^{p+1}$, we have $S_{\omega(u)}(u) \geq S_{\omega(u)}(\phi_{\omega(u)})$. Therefore, using Lemma 4.1 and the fact that $d'(\omega_0) = Q(\phi_0)$,

$$\begin{aligned}
&E(u) + \omega(u)Q(u) \geq S_{\omega(u)}(\phi_{\omega(u)}) = d(\omega(u)) \\
&\geq d(\omega_0) + d'(\omega_0)(\omega(u) - \omega_0) + \frac{1}{4}d''(\omega_0)(\omega(u) - \omega_0)^2 \\
&= E(\phi_0) + \omega_0 Q(\phi_0) + Q(\phi_0)(\omega(u) - \omega_0) + \frac{1}{4}d''(\omega_0)(\omega(u) - \omega_0)^2 \\
&= E(\phi_0) + \omega(u)Q(\phi_0) + \frac{1}{4}d''(\omega_0)(\omega(u) - \omega_0)^2. \qquad \square
\end{aligned}$$

Proof of Proposition 1. Assume that $e^{i\omega_0 t}\phi_0(x)$ is unstable in X_{rad}. From the definition of stability, there exist $\varepsilon_0 > 0$ and initial data $u_k(0) \in U_{1/k}(\phi_0)$ such that

$$\sup_{t\geq 0}\inf_{\theta\in\mathbb{R}} \|u_k(t) - e^{i\theta}\phi_0\|_X \geq \varepsilon_0,$$

where $u_k(t)$ is the solution of (1.1) with initial data $u_k(0)$. Let t_k be the first time at which

$$\inf_{\theta\in\mathbb{R}} \|u_k(t_k) - e^{i\theta}\phi_0\|_X = \frac{\varepsilon_0}{2}. \tag{4.3}$$

We denote $u_k(t_k)$ by v_k. Since E and Q are conserved in t, we have

$$|E(v_k) - E(\phi_0)| = |E(u_k(0)) - E(\phi_0)| \to 0, \tag{4.4}$$

$$|Q(v_k) - Q(\phi_0)| = |Q(u_k(0)) - Q(\phi_0)| \to 0 \tag{4.5}$$

as $k \to \infty$. From (4.3), we have $\|v_k\|_X \leq C$ uniformly in k. Also we note that $\omega_k = \omega(v_k)$ is uniformly bounded in k since $\omega(u)$ is a continuous map. Here, we take δ small enough so that Lemma 4.2 applies. Then we have

$$E(v_k) - E(\phi_0) + \omega_k\{Q(v_k) - Q(\phi_0)\} \geq \frac{1}{4}d''(\omega_0)(\omega_k - \omega_0)^2. \tag{4.6}$$

Since $d''(\omega_0) > 0$, this implies that $\lim_{k\to\infty}\omega_k = \omega_0$. Using (2.1) and the fact that $d(\cdot)$ is continuous, it follows that

$$\lim_{k\to\infty}\frac{p-1}{2(p+1)}\|v_k\|_{p+1}^{p+1} = \lim_{k\to\infty} d(\omega_k) = d(\omega_0) = \frac{p-1}{2(p+1)}\|\phi_0\|_{p+1}^{p+1}. \tag{4.7}$$

From (4.4) and (4.5), we have

$$S_{\omega_0}(v_k) = S_{\omega_0}(v_k) - S_{\omega_0}(\phi_0) + S_{\omega_0}(\phi_0)$$

$$= E(v_k) - E(\phi_0) + \omega_0(Q(v_k) - Q(\phi_0)) + d(\omega_0) \to d(\omega_0), \qquad (4.8)$$

as $k \to \infty$. Let $w_k = (\|\phi_0\|_{p+1}/\|v_k\|_{p+1})v_k$. Then, w_k satisfies $\|w_k\|_{p+1} = \|\phi_0\|_{p+1}$ and $\|w_k - v_k\|_X \to 0$ as $k \to \infty$. Furthermore, by (4.7) and (4.8), $S_{\omega_0}(w_k) \to d(\omega_0)$ as $k \to \infty$. Therefore, $\{w_k\}$ is a minimizing sequence for $d(\omega_0)$. By the compactness of the radial space and the uniqueness of the minimizer ((v) of Proposition 2), there exists a sequence $\{\theta_k\}$ such that $\lim_{k\to\infty} \|w_k - e^{i\theta_k}\phi_0\|_X = 0$. Namely, we get

$$\lim_{k\to\infty} \|v_k - e^{i\theta_k}\phi_0\|_X = 0,$$

which is a contradiction to (4.3). □

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