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RADIALLY SYMMETRIC MINIMIZERS OF VARIATIONAL PROBLEMS IN THE PLANE

Arie Leizarowitz
Department of Mathematics, Technion, Haifa 32000, Israel

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Abstract. The functional $J_\Omega = \frac{1}{|\Omega|}\int_\Omega [(\Delta u)^2 - b|\nabla|^2 + \psi(u)]dxdy$ is considered on bounded domains $\Omega \subset R^2$ and configurations u, which are functions belonging to $H^2_{loc}(R^2)$. For such u the value of $J_\Omega(u)$ is studied in the limit where Ω extends to the whole plane R^2, which we denote $J(u)$. The integrand defining J_Ω is radially symmetric, and we raise the question whether $J(u)$ has a minimizer $u^\star$ which is radially symmetric, namely has the form $u^\star(x,y) = v(\sqrt{x^2+y^2})$. It is shown that the minimal value of $J(u)$ over the radially symmetric configurations is equal to its minimal value over 1-dimensional configurations, which are functions of the form $u(x,y) = v(x)$. Restricting to 1-dimensional configurations yields the 1-dimensional model, hence $\lambda_2 \le \lambda_1$, where λ_1 and λ_2 are the values of the 1 and 2-dimensional models respectively. Then the question which we address is whether strict inequality holds in $\lambda_2 \le \lambda_1$. We establish a sufficient condition for a configuration u to satisfy $J(u) \ge \lambda_1$, and use it to conclude that the minimal value of $J(u)$ on the class of quasi-convex configurations is λ_1.

1. Introduction

For a bounded domain Ω in the plane with piecewise smooth boundary, we consider the functional

$$J_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega [B(u) - b|\nabla u|^2 + \psi(u)]dxdy, \tag{1.1}$$

where $b > 0$ is a constant,

$$B(u) = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2,$$

and $\psi(\cdot)$ is continuous and satisfies $\psi(u)/u^{2+\epsilon} \to \infty$ as $|u| \to \infty$ for some $\epsilon > 0$. A typical function $\psi(\cdot)$ is $\psi(u) = (|u|^p - 1)^2$ for $p > 1$. We note that $J_\Omega(\cdot)$ is coercive on $H^2(\Omega)$.

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Replacing the second-order quadratic term $B(u)$ in (1.1) by the more familiar term $(\Delta u)^2 = u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2$ would yield a noncoercive functional. We will remark below on the relation between $B(u)$ and $(\Delta u)^2$, and in particular, it will be established that

$$\int_\Omega B(u)dxdy = \int_\Omega (\Delta u)^2 dxdy$$

for every smooth Ω and every C^2 function u on Ω such that $\nabla u|_{\partial\Omega} = 0$.

For $u \in H^2(\Omega)$, the value $J_\Omega(u)$ may be thought of as the mean energy associated with a "state" u of a body whose extent is the domain Ω. The state u may describe the concentration, or the mass density of the body. An equilibrium state for Ω is a state for which J_Ω obtains its infimum. As Ω expands to the whole plane, limits of $J_\Omega(u)$ represent the average energy of u on R^2, and whenever this limit has a meaning we define

$$J(u) = \lim_{|\Omega|\to\infty} J_\Omega(u), \tag{1.2}$$

which is the mean energy on the plane associated with u.

When considering the functional $J(\cdot)$ in (1.2) it will always be defined by using domains Ω which belong to a normal family, which we define next. Moreover, in such situations, we will verify that the value $J(u)$ thus obtained does not depend on the normal family used.

Definition 1.1. *Let $\mathcal{D}$ be a collection of bounded C^1 domains Ω in R^2. We say that $\mathcal{D}$ is* a normal family of domains *if*

$$\frac{l(\Omega)}{|\Omega|} \to 0 \text{ as } |\Omega| \to \infty \text{ and } \Omega \in \mathcal{D}, \tag{1.3}$$

where we denote by $l(\Omega)$ the length of the boundary $\partial\Omega$ of Ω and $|\Omega|$ denotes its area. If $\mathcal{D} = \{\Omega_n\}_{n=1}^\infty$ then we say that $\mathcal{D}$ is a normal sequence.

The functional $J(\cdot)$ in (1.2) may be interpreted in two different ways. One is to consider the minimization of $J(\cdot)$ as an approximation of the minimization of $J_\Omega(\cdot)$, for large domains Ω. This interpretation is useful when one has to minimize $J_\Omega(\cdot)$ on a domain Ω which is not precisely determined, but is known to be large.

Another interpretation is to consider $J(u)$ as a limiting functional of the singularly perturbed functional

$$J^\epsilon(u) = \frac{1}{|\Omega_0|}\int_{\Omega_0} [\epsilon^2 B(u) - \epsilon b|\nabla u|^2 + \psi(u)]dxdy, \quad \epsilon > 0, \tag{1.4}$$

for a certain fixed domain Ω_0. Then the minimization of $J(\cdot)$ over the whole plane provides the limit problem of minimizing J^ϵ when $\epsilon \to 0+$. This point of view proved to be very fruitful while studying the analogous one-dimensional problem, where the goal is to minimize the functional

$$j(v) = \lim_{L\to\infty} \frac{1}{L} \int_0^L [(v''(x))^2 - b(v'(x))^2 + \psi(v(x))]dx. \tag{1.5}$$

This problem is considered as the limit problem of minimizing

$$j^\epsilon(v) = \int_0^1 [\epsilon^2(v''(x))^2 - \epsilon b(v'(x))^2 + \psi(v(x))]dx,\ \epsilon > 0. \tag{1.6}$$

A prototype of the one-dimensional model was formulated by Bernard Coleman in [1], [2] in order to describe the structure of long polymeric fibers of a viscoelastic material under tension. The one-dimensional model was studied by Leizarowitz and Mizel in [6] for unconstrained problems, while the model with additional mass constraint was studied by Coleman, Marcus and Mizel [3] and Marcus [8]. For recent studies of this model in various settings see [9], [10] and [11] .

A *configuration* for the functional $J(\cdot)$ in (1.2) is a function $u \in H^2_{loc}(R^2)$ such that there exists a constant $K > 0$, which may depend on u, such that

$$||u||_{H^2(Q)} \leq K \tag{1.7}$$

for every square Q of edge 1 in the plane.

The functionals $J(\cdot)$ which we study are rotation invariant. A configuration u is called *radially symmetric* if $u(x, y) = v(r)$ for some function $v(\cdot)$ of $r = \sqrt{x^2 + y^2}$. The basic question which we address in this study is whether the infimum of $J(\cdot)$ over the radially symmetric configurations is equal to its infimum over all the configurations. This problem was first presented in [7].

When the answer to the above basic question is positive, then the study is reduced to that of a one-dimensional problem. Another one-dimensional problem is obtained while restricting to *one-dimensional* configurations u, which are configurations of the form $u(x, y) = v(x)$ for some function $v(\cdot)$ of x. Restricting $J(\cdot)$ to one-dimensional configurations we obtain the functional $v \mapsto j(v)$ in (1.5). We will establish that the infimum of $J(\cdot)$ over radially symmetric configurations is equal to the infimum of $j(\cdot)$ over one-dimensional configurations. This common infimum is denoted λ throughout the paper.

We remark that the first order analogous problem is not challenging, and the situation there is quite obvious. Namely, consider the functional

$$J^{(1)}_\Omega(u) = \frac{1}{|\Omega|} \int_\Omega L(u, |\nabla u|)dxdy \tag{1.8}$$

and study its limits as $|\Omega| \to \infty$. Assume that L is continuous, coercive and such that $v \mapsto L(u, v)$ is convex. It then follows easily that the minimum is attained by one-dimensional configurations, and consequently by radially symmetric configurations. This is described in Remark 3.2 below.

As a step towards resolving the question concerning the existence of radially symmetric equilibria for (1.2), our main result will establish a sufficient condition for a configuration u to satisfy

$$J(u) \geq \lambda. \tag{1.9}$$

This will be used to identify families $\mathcal{F}$ of configurations such that the infimum of $J(u)$ taken over $\mathcal{F}$ is equal to λ.

We introduce the following optimality notion.

Definition 1.2. *We say that $u^\star$ is* an equilibrium configuration *if for every configuration u, every normal sequence $\{\Omega_n\}_{n=1}^{\infty}$, and every $\epsilon > 0$,*

$$J_{\Omega_n}(u^\star) < J_{\Omega_n}(u) + \epsilon$$

holds for all large enough integers n. An equilibrium for a bounded domain Ω is a minimizer of J_Ω in $H^2(\Omega)$ subject to prescribed boundary conditions.

We finally comment on the relation between $B(u)$ and $(\Delta u)^2$, which are both quadratic expressions in the second-order derivatives u_{xx}, u_{yy}, and u_{xy} which are rotation invariant. For a $C^2(R^2)$ configuration u we compute the difference

$$\int_\Omega (\Delta u)^2 dxdy - \int_\Omega B(u) dxdy. \tag{1.10}$$

For such u we define the vector field

$$\vec{F}(x, y) = \begin{pmatrix} u_x u_{yy} \\ -u_x u_{xy} \end{pmatrix}, \tag{1.11}$$

and applying the divergence theorem to $\vec{F}$ on Ω yields

$$\int_\Omega [u_{xx}u_{yy} - u_{xy}^2] dxdy = \int_{\partial\Omega} u_x u_{yy} dx - u_x u_{xy} dy, \tag{1.12}$$

assuming that $\partial\Omega$, the boundary of Ω, is sufficiently smooth. Since the difference in (1.10) is half the left-hand side of (1.12), we obtain

$$\int_\Omega (\Delta u)^2 dxdy - \int_\Omega B(u) dxdy = \frac{1}{2} \int_{\partial\Omega} u_x u_{xx} dx - u_x u_{xy} dy, \tag{1.13}$$

so that the difference in (1.10) is a second-order boundary operator. In particular, this difference vanishes if $\nabla u|_{\partial\Omega} = 0$.

It follows that if we define, analogous to (1.1),

$$\tilde{J}_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega [(\Delta u)^2 - b|\nabla u|^2 + \psi(u)]dxdy, \tag{1.14}$$

we obtain

$$\tilde{J}_\Omega(u) - J_\Omega(u) = \frac{1}{2|\Omega|}\int_{\partial\Omega} u_x(u_{xx}dx - u_{xy}dy). \tag{1.15}$$

We then define, in analogy to (1.2)

$$\tilde{J}(u) = \lim_{|\Omega|\to\infty} \tilde{J}_\Omega(u). \tag{1.16}$$

We denote by $C_b^2(R^2)$ the space of all configurations u such that u and the first and second-order derivatives of u are all bounded on R^2. We obtain the following result:

Proposition 1.3. *Let $\{\Omega_n\}_{n=1}^\infty$ be a normal sequence of C^1 domains. Then $\tilde{J}(u) = J(u)$ for every $u \in C_b^2(R^2)$.*

The paper is organized as follows. In the next section we establish that the variational problems concerning radially symmetric configurations and one-dimensional configurations have the same infimal value. In section 3 we describe our approach, which is to express the integrand in (1.1) by using at each noncritical point $(x, y) \in \Omega$ a rectangular coordinate system which depends on that point. This coordinate system is such that one of the axes is parallel to the level curve of u through (x, y), while the other one is parallel to the gradient curve through that point. In section 4 we use this expression for the energy integrand and a result invoked from the study of the one-dimensional model to establish a sufficient condition for (1.9) to hold. In section 5 we employ this result and consider the family $\mathcal{F}$ of quasi-convex configurations, which are configurations of the form $u(x, y) = f(v(x, y))$, where v is convex on R^2 and $f(\cdot)$ is C^2 from R^1 to R^1 with bounded derivatives up to second order. We establish that the infimum of $J(\cdot)$ over $\mathcal{F}$ is equal to λ.

2. Radially symmetric configurations

In this section we restrict attention to radially symmetric configurations, and will show that the infimum over such configurations is equal to the infimum over configurations which depend only on one variable, say only on x. This latter infimum, λ, is the minimal value of $v \mapsto j(v)$ in (1.5).

In polar coordinates we have for a configuration $u(x, y) = v(r)$ that

$$u_{xx} = v''(r)\cos^2\theta + \frac{v'(r)}{r}\sin^2\theta,$$

$$u_{yy} = v''(r)\sin^2\theta + \frac{v'(r)}{r}\cos^2\theta$$

and

$$u_{xy} = \Big(v''(r) - \frac{v'(r)}{r}\Big)\sin\theta\cos\theta.$$

Therefore, for such a configuration, the value of the operator $B(u)$ is given by

$$B(u) = (v''(r))^2 + \Big(\frac{v'(r)}{r}\Big)^2,$$

and the square of the gradient is given by

$$|\bigtriangledown u|^2 = (v'(r))^2.$$

It follows that for a radially symmetric configuration $u = v(r)$, if Ω is the annulus centered at the origin with radii 1 and R, then the value of the functional (1.1) of u on Ω is given by

$$J_R(v) = \frac{2}{R^2}\int_1^R \Big[(v'')^2 + \big(\frac{v'}{r}\big)^2 - bv'^2 + \psi(v(r))\Big] rdr.$$

Since v' is bounded on $[1,\infty)$ it follows that the limit of $J_R(v)$ as $R \to \infty$ is the same as the limit of

$$i_R(v) = \frac{2}{R^2}\int_1^R [v''^2 - bv'^2 + \psi(v(r))] rdr \tag{2.1}$$

as $R \to \infty$. Similarly to (1.5) we denote

$$i(v) = \lim_{R\to\infty} \frac{2}{R^2}\int_1^R [v''^2 - bv'^2 + \psi(v(r))] rdr. \tag{2.2}$$

The following proposition will be useful in establishing our assertion concerning radially symmetric configurations of the two-dimensional problem and configurations of the analogous one-dimensional problem.

Proposition 2.1. *Let $q(\cdot)$ be defined for $r \geq 1$, and periodic of period P for some $P > 0$. Then*

$$\lim_{R\to\infty} \frac{2}{R^2}\int_1^R q(r) rdr = \lim_{R\to\infty} \frac{1}{R}\int_1^R q(r)dr.$$

Proof. Every $R > P$ can be written in the form $R = kP + L$ for some $0 \leq L < P$ and some integer $k \geq 1$. Then

$$\frac{2}{R^2}\int_P^R q(r)rdr = \frac{2}{R^2}\Big[\sum_{j=1}^{k-1}\int_{jP}^{(j+1)P} q(r)rdr + \int_{kP}^R q(r)rdr\Big]$$

$$= \frac{2}{R^2}\Big[\int_{kP}^{R} q(r)r\,dr + \sum_{j=1}^{k-1}\int_0^P (r+jP)q(r)dr\Big].$$

As $R \to \infty$ the first term tends to zero. We compute the second term:

$$\frac{2}{R^2}\Big[(k-1)\int_0^P q(r)r\,dr + \frac{k(k-1)P}{2}\int_0^P q(r)dr\Big],$$

and since $k/R^2 \to 0$ as $R \to \infty$, the limit of this expression as $R \to \infty$ is equal to

$$\lim_{R\to\infty} \frac{k(k-1)P^2}{R^2}\Big(\frac{1}{P}\int_0^P q(r)dr\Big). \tag{2.3}$$

In view of $R = kP + L$ and $0 \le L < P$, it follows that the expression in (2.3) tends to $\frac{1}{P}\int_0^P q(r)dr$ as $R \to \infty$. □

Clearly the value of the functional $j(v)$ in (1.5) remains unchanged when we replace zero in the lower limit of the integral there by any other fixed positive number.

Theorem 2.2. *The infimum over all the radially symmetric configurations is equal to the infimum of the analogous one-dimensional problem. Namely, the functionals $j(\cdot)$ and $i(\cdot)$ which are defined in* (1.5) *and* (2.2) *respectively on functions with two bounded derivatives in* $[1,\infty)$*, have the same minimal value.*

Proof. Denote by α the minimal value of $j(\cdot)$ and we'll prove that the minimal value of $i(\cdot)$ is at least α. In view of Proposition 2.1 and the existence of a periodic minimizer for $j(\cdot)$, this will establish the result. We fix a number $L > 0$ and let $T = mL$, where m is a large integer, and let R be much larger than T. We consider $R = r_0 + NT$ for a fixed $0 \le r_0 < T$:

$$\int_{r_0}^{R}[v''^2 - bv'^2 + \psi(v(r))]r\,dr = \sum_{k=0}^{N-1}\int_{r_0+kT}^{r_0+(k+1)T}[v''^2 - bv'^2 + \psi(v)]r\,dr.$$

We have that

$$\Big|\int_{r_0+kT}^{r_0+(k+1)T}[v''^2 - bv'^2 + \psi(v)][r-(r_0+kT)]dr\Big| \le MT^2,$$

where M is a bound on $|v''^2 - bv'^2 + \psi(v)|$. Dividing by R^2 we get an expression which is smaller than $M(T/R)^2$. Since there are N such terms and $N = (R - r_0)/T$, it follows that altogether these differences are of order

MT/R, which tends to zero as $R \to \infty$. Therefore we may replace the integral defining $i(\cdot)$ on the interval $[r_0 + kT, r_0 + (k+1)T]$ by the integral

$$(r_0 + kT) \int_{r_0+kT}^{r_0+(k+1)T} [v''^2 - bv'^2 + \psi(v)]dr. \tag{2.4}$$

For any given $\epsilon > 0$, the expression in (2.4) will exceed $(r_0 + kT)(\alpha T - \epsilon)$, provided we chose T sufficiently large in the beginning. When we sum over k in the last expression and include the coefficient $2/R^2$ in (2.2) we obtain $(N(N-1)/2)T^2(\alpha - \epsilon/T)(2/R^2)$ which tends to $\alpha - \epsilon/T$ as $N \to \infty$. This concludes the proof of the theorem. □

3. The energy in tangent coordinate system

We consider a coordinate system (ξ, η) rotated by angle θ counterclockwise with respect to (x, y). The transformation is

$$\xi = x\cos\theta + y\sin\theta, \ \eta = -x\sin\theta + y\cos\theta$$

and the function $u(x, y)$ is transformed into $v(\xi, \eta)$ satisfying

$$v(\xi, \eta) = u(x, y).$$

Then the following relations hold:

$$u_x = v_\xi \cos\theta - v_\eta \sin\theta, \ u_y = v_\xi \sin\theta + v_\eta \cos\theta,$$

implying

$$u_{xx} = v_{\xi\xi}\cos^2\theta - 2v_{\xi\eta}\sin\theta\cos\theta + v_{\eta\eta}\sin^2\theta$$
$$u_{yy} = v_{\xi\xi}\sin^2\theta + 2v_{\xi\eta}\sin\theta\cos\theta + v_{\eta\eta}\cos^2\theta$$
$$u_{xy} = v_{\xi\xi}\sin\theta\cos\theta + v_{\xi\eta}(\cos^2\theta - \sin^2\theta) - v_{\eta\eta}\sin\theta\cos\theta,$$

which can be written in the form

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_{\xi\xi} & v_{\xi\eta} \\ v_{\xi\eta} & v_{\eta\eta} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.1}$$

Denoting

$$H_{xy} = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}, \ H_{\xi\eta} = \begin{pmatrix} v_{\xi\xi} & v_{\xi\eta} \\ v_{\xi\eta} & v_{\eta\eta} \end{pmatrix}, \ R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then (3.1) is written as

$$H_{xy} = R_\theta^{-1} H_{\xi\eta} R_\theta. \tag{3.2}$$

It follows from (3.2) that the determinants of the Hessian matrices in the two coordinate systems H_{xy} and $H_{\xi\eta}$ are equal so that

$$u_{xx}u_{yy} - u_{xy}^2 = v_{\xi\xi}v_{\eta\eta} - v_{\xi\eta}^2. \tag{3.3}$$

We consider now a configuration $u(x,y)$ and its gradient curves, which are solutions of

$$z'(\tau) = \frac{\nabla u(z)}{|\nabla u(z)|} \tag{3.4}$$

where we denote by z a point (x,y) in R^2. Moreover, the independent variable τ in (3.4) is the arclength parameter of the gradient curve $z(\cdot)$ which is the solution of (3.4). Let γ be such a gradient curve, let P be a point on γ, and choose at P a coordinate system (s,t) such that the t-axis is tangent to l, and the s-axis is perpendicular to it. We employ this coordinate system to compute the integrand of the two-dimensional functional, and for every point P in R^2 we usa coordinate system (s,t) which depends on P, and in general varies with P. In the above notation we have $t=\xi$ and $s=\eta$. Since s is the variable perpendicular to γ, we have $v_s = 0$ and moreover,

$$u_x = v_t \cos\theta,\ u_y = v_t \sin\theta,$$

so that

$$|\nabla u(x,y)|^2 = (v_t(s,t))^2.$$

We now focus on the restriction of $u(\cdot,\cdot)$ to the gradient line γ parametrized by arclength, which we denote $u(\tau)$.

Proposition 3.1. *The following relations hold*

$$\frac{du(\tau)}{d\tau} = v_t(P), \tag{3.5}$$

$$\frac{d^2u(\tau)}{d\tau^2} = v_{tt}(P), \tag{3.6}$$

and

$$v_{st}(P) = \pm|\nabla u|\frac{\partial\varphi}{\partial\tau}, \tag{3.7}$$

where φ is the angle which the gradient makes with a fixed axis, and the sign in (3.7) is the sign of v_t, so that $v_{st}(P) = v_t\varphi_\tau$.

Proof. Let T be the straight line tangent to γ at P, let Q be of distance Δt from P along T and let Q' be of distance Δt from P along γ. Then clearly $d(Q,Q') = o(\Delta t)$, which implies (3.5).

To see (3.6) we observe that we have, in addition to $d(Q,Q') = o(\Delta t)$, that $|\nabla u(Q) - \nabla u(Q')| = O(\Delta t)$, and that the angle φ which ∇u makes at Q' with the t-axis is bounded by $K\Delta t$ for some constant K. To compute v_{tt} we need the projection of ∇u onto the t-axis at the point Q. The absolute value of this projection is equal to $|\nabla u(Q)|\cos\varphi$. Since $d(Q,Q') = O(\Delta t)$, $|\varphi| \le K\Delta t$ and $\cos\varphi = 1 + o(\varphi)$, it follows that $|v_t(Q)| = |\nabla u(Q')| + o(\Delta t)$, establishing (3.6), in view of (3.5).

The proof of (3.7) is similar, using the fact that $u_s(P) = 0$ at every point P, where the differentiation is in the perpendicular direction to the gradient curve at that point. At the point Q the gradient makes an angle φ with γ hence the projection of ∇u on the s-axis at this point is $|\nabla u| \sin\varphi$. Since at P this projection is zero, we obtain v_{st} by dividing the above projection by Δt and letting it tend to zero. This yields the expression in (3.7), concluding the proof of the proposition. □

We now express the energy functional

$$\int_\Omega [B(u) - b|\bigtriangledown u|^2 + \psi(u)]dxdy \tag{3.8}$$

by employing at every noncritical point P (namely one where $\bigtriangledown u(P) \neq 0$) the tangential and perpendicular coordinates at that point P. We emphasize that we are not performing a change of variables from (x, y) to (s, t) in (3.8), but rather use at every noncritical point P the rectangular coordinates (s, t) at P to compute the integrand of (3.8) at P. It is clear from $v_s = 0$ that $|\bigtriangledown u(P)| = |v_t|$, and in view of (3.5) $|\bigtriangledown u(P)| = |u'(\tau)|$. Since

$$(\Delta u)^2 - B(u) = 2[u_{xx}u_{yy} - u_{xy}^2],$$

it follows from (3.3) that the expression $B(u)$ is rotation invariant, so that

$$\int_\Omega B(u)dxdy = \int_\Omega [v_{tt}^2 + 2v_{st}^2 + v_{ss}^2]dxdy,$$

hence, in view of (3.6)

$$\int_\Omega B(u)dxdy \geq \int_\Omega (u''(\tau))^2 dxdy. \tag{3.9}$$

It follows from (3.9) that the energy functional is bounded below by

$$\int_\Omega [v_{tt}^2 - bv_t^2 + \psi(v)]dxdy,$$

which in view of (3.5) and (3.6) is equal to

$$\int_\Omega [(u''(\tau))^2 - b(u'(\tau))^2 + \psi(u(\tau))]dxdy. \tag{3.10}$$

We recall that $u(\tau)$ is the value of $u(\cdot,\cdot)$ along a gradient curve γ, and that τ is the arclength parameter along γ. The integration in (3.10) is with respect to the Lebesgue measure on the plane, where the parameter τ is used just to compute the value of the energy integrand at every noncritical point of u.

Remark 3.2. We consider in this remark the first-order functinal (1.8). The one-dimensional analog is the functional

$$\Phi_T(z(\cdot)) = \frac{1}{2T}\int_{-T}^{T} L(z(t), z'(t))dt, \tag{3.11}$$

and the result described here is related to the discussion of the scalar case in section 6 of [$L2$]. It follows that the infinite horizon problem obtained when letting $T \to \infty$ in (3.11) has a constant function minimizer $z(t) = z_0$ for some real number z_0 and for all $-\infty < t < \infty$, and the minimal value of the problem is

$$\mu = \min_z L(z, 0). \tag{3.12}$$

Analogous to (3.10) we obtain that

$$J^{(1)}(u) = \frac{1}{|\Omega|}\int_\Omega L(u(\tau), u'(\tau))dxdy, \tag{3.13}$$

where $u(\tau)$ is the value of $u(\cdot,\cdot)$ along a gradient curve γ, and where τ is the arclength parameter along γ. Moreover, taking σ to be an arclength parametrization along the direction opposite to that of τ, we obtain

$$J^{(1)}(u) = \frac{1}{|\Omega|}\int_\Omega L(u(\tau), -u'(\tau))dxdy. \tag{3.14}$$

Adding (3.13) and (3.14) and dividing by 2, the convexity of $v \mapsto L(u, v)$ implies that

$$J^{(1)}(u) = \frac{1}{|\Omega|}\int_\Omega L(u(\tau), 0)dxdy,$$

and in view of (3.12) we obtain

$$J^{(1)}(u) \geq \mu.$$

Since we have a constant minimizer (which is obviously radially symmetric), it follows that the minimum is attained by a radially symmetric configuration.

4. The comparison with 1-dimensional configurations

In this section we employ (3.10) to compare the average energy of two-dimensional configurations

$$\frac{1}{|\Omega|}\int_\Omega [(u''(\tau))^2 - b(u'(\tau))^2 + \psi(u(\tau))]dxdy$$

with the one-dimensional average energy

$$\frac{1}{T}\int_0^T [v''^2 - bv'^2 + \psi(v)]dt.$$

We will establish a sufficient condition for a configuration to have average energy which is bounded below by λ, the one-dimensional minimal average energy. To this end, we use a representation formula for the one-dimensional integrand. It follows from the representation formula of [4] and from Proposition 5.7 in [6] that for every interval $[0, T]$ and every one-dimensional configuration $v(\cdot)$ the following holds:

$$\int_0^T [(v''(t))^2 - b(v'(t))^2 + \psi(v(t))]dt \geq \lambda T + p(v(0), v'(0)) - p(v(T), v'(T)) \tag{4.1}$$

where λ is a constant, and where $p(\cdot, \cdot)$ is a Lipschitz-continuous function, independent of T. It follows from (4.1) that

$$\int_0^T [(v''(t))^2 - b(v'(t))^2 + \psi(v(t)) - \lambda - \frac{d}{dt}p(v(t), v'(t))]dt \geq 0 \tag{4.2}$$

for every configuration $v(\cdot)$ and for every interval $[0, T]$. The function $p(\cdot, \cdot)$ is Lipschitz continuous so that the derivative $\frac{d}{dt}p(v(t), v'(t))$ is defined for almost every t, for every C^2 configuration $v(\cdot)$. It follows that the integrand in (4.2) is nonnegative for almost every t, and this for every C^2 function $v(\cdot)$. We denote

$$(\Theta[v])(t) = (v''(t))^2 - b(v'(t))^2 + \psi(v(t)) - \lambda - \frac{d}{dt}p(v(t), v'(t))$$

for a C^2 configuration $v(\cdot)$. It follows that the integrand in (4.2) is $(\Theta[v])(t)$, and in view of (4.2) we conclude that

$$(\Theta[v])(t) \geq 0 \tag{4.3}$$

for almost every t and for every C^2 function $v(\cdot)$. We have thus the following representation formula

$$(v''(t))^2 - b(v'(t))^2 + \psi(v(t)) = \lambda + \frac{d}{dt}p(v(t), v'(t)) + (\Theta[v])(t) \tag{4.4}$$

which holds for almost every t, where λ is a constant and Θ satisfies (4.3).

We substitute (4.4) in the integrand (3.10) obtaining

$$\int_\Omega \left[\lambda + \frac{\partial}{\partial t}p(u(t, 0), u_t(t, 0)) + (\Theta[u(\cdot, 0)])(t = 0)\right] dxdy. \tag{4.5}$$

The integration in (4.5) is with respect to the two-dimensional Lebesgue measure in the plane. When dividing the energy expressions in (4.5) by $|\Omega|$, the first term in the integrand yields λ, the third term gives a nonnegative contribution, and it remains to deal with the contribution of the second term, which we henceforth denote by I.

In the rest of this section we describe one approach to deal with I. This approach will be employed to quasi-convex configurations in the next section, and then extended to more general configurations.

We introduce a vector field

$$\vec{F} = p(u(t,s), u_t(t,s))\hat{\nu} \tag{4.6}$$

where $\hat{\nu}$ is a unit vector in the direction of ∇u at a noncritical point labeled (s,t). We note that $\vec{F}$ may be noncontinuous near critical points of u. However, this does not influence our development, since $\vec{F}$ is bounded, and it is assumed that the set of critical points of u has measure zero in the plane. We compute the line integral

$$\int_{\partial\Omega} \vec{F} \cdot \hat{n} dl \tag{4.7}$$

of $\vec{F}$ over the boundary of Ω, and by the divergence theorem this integral is equal to the integral of $\nabla \cdot \vec{F}$ over Ω.

At a certain point P in Ω we compute $\nabla \cdot \vec{F}(P)$ using a rectangular coordinate system (s,t), where the origin is located at P, the t-axis is tangent to the gradient curve at P, and the s-axis is tangent to the level curve at P. *Thus the rectangular coordinate system employed to compute* $\nabla \cdot \vec{F}(P)$ *depends on* P. The derivative with respect to t of the projection of $\vec{F}$ onto the t-axis is

$$\frac{\partial}{\partial t} p(u(t,0), u_t(t,0))|_{t=0},$$

which is the second term in the integrand in (4.5). It thus remains to deal with the integral of the derivative with respect to s of the projection of $\vec{F}$ on the s-axis.

The computation of the projection of $\vec{F}$ onto the s-axis is similar to the one made in the proof of (3.7). We thus consider a point Q of distance Δs from P on the s-axis and let the gradient curve at Q make an angle $\pi/2 - \varphi$ with the s-axis. Then the projection of F on the s-axis at Q is, up to $o(\Delta s)$, given by $p(u, u_t)\sin\varphi$, hence since $\varphi \approx 0$ the derivative of this projection with respect to s is

$$p(u(0,0), u_t(0,0))\frac{\partial \varphi}{\partial s}.$$

It thus remains to deal with the following integral

$$\int_{\Omega} p(u(P), u_t(P))\varphi_s(P) dxdy \tag{4.8}$$

where $\varphi_s(P)$ in (4.8) is the derivative of the angle of the gradient at a certain point P with respect to changes along the level set at P. The integration in (4.8) is with respect to the area Lebesgue measure in the plane.

Remark 4.1. In this remark we make the integral in (4.8) somewhat more explicit. The derivative $\varphi_s(P)$ has a simple geometric meaning, so that

$$\frac{\partial \varphi}{\partial s} = \kappa(P) \tag{4.9}$$

is the *curvature of the level curve* at the point P, and

$$R(P) = \frac{1}{|\kappa(P)|}$$

is the *radius of curvature* at this point.

We have two choices of directions for the t-axis: either parallel or anti-parallel to the gradient at the point P. In the parallel case we have $u_t(P) = |\nabla u(P)|$, so that in this choice the integral (4.8) is

$$\int_\Omega p(u(P), |\nabla u(P)|)\kappa(P) dxdy. \tag{4.10}$$

Analogously, in the anti-parallel choice, the integral (4.8) is

$$-\int_\Omega p(u(P), -|\nabla u(P)|)\kappa(P) dxdy, \tag{4.11}$$

and clearly the two integrals in (4.10) and (4.11) *are not* simply related.

5. Quasi-convex configurations

In this section we employ the result we obtained above to the family $\mathcal{F}_Q$ of configurations which are of the form $u(x,y) = f(v(x,y))$, where v is convex on R^2 and f is C^2 from R^1 to R^1 with bounded derivatives up to second order. We call such u *quasi-convex configuration*. We will then extend the result to obtain a sufficient condition for the minimization over a general class of configurations to have a radially symmetric solution.

We consider quasi-convex configurations, and suppose first that the level curves of v are bounded, closed curves, and that they don't contain any straight line segments. We introduce a family of curves σ_θ defined as follows. Let us fix an axis e and measure angles of straight lines with respect to e. Since the level curves don't contain straight line segments it follows that for $0 \le \theta < 2\pi$ and a level curve l there is a unique point on l where the outward normal makes an angle θ with e, and we denote this point by $P(l,\theta)$. We then define the curve σ_θ by

$$\sigma_\theta = \{P(l,\theta) : l \text{ is a level curve}\},$$

which we call *a direction line*.

We next estimate the integral in (4.8). The function $p(\cdot,\cdot)$ is bounded everywhere, say by M. Moreover, the convexity of $v(\cdot,\cdot)$ implies the special property that $\varphi_s(P)$ has a constant sign when P varies along a level curve, and then

$$\int_l \varphi_s(P)ds = \pm 2\pi,$$

the sign being the constant sign of $\varphi_s(P)$ on l. It follows that the integral in (4.8) is bounded in absolute value by

$$M\int_\Omega |\varphi_s(P)|dsdt. \tag{5.1}$$

Recall that $u = f(v)$ where v is convex and coercive on R^2, and we suppose that the origin is located at the minimal point of v. Let Ω be a convex domain containing the origin, and we consider the collection of direction curves $\{\sigma_\theta\}_{0\le\theta<2\pi}$. Let $\{\theta_j\}_{j=1}^N$ be a partition of $[0, 2\pi)$ with mesh δ, and consider the collection of lines $\{\sigma_{\theta_j}\}_{1\le j\le N}$, denoting by Ω_j the domain bounded by σ_{θ_j}, $\sigma_{\theta_{j+1}}$ and $\partial\Omega$. Then $\Omega = \bigcup_j \Omega_j$, and the integral in (5.1) is

$$\int_\Omega |\varphi_s(P)|dsdt = \sum_{j=1}^{N-1}\int_{\Omega_j} |\varphi_s(P)|dsdt.$$

We consider sequence of partitions such that the mesh δ satisfies $\delta \to 0$. It is then easy to see that

$$\int_{\Omega_j} |\varphi_s(P)|dsdt = L_\Omega(\theta_j)(\theta_{j+1} - \theta_j) + o(\delta), \tag{5.2}$$

where $L_\Omega(\theta)$ is the length of the projection of $\sigma_\theta \cap \Omega$ on the line e_θ, which is the line obtained by rotating e by the angle θ. It thus follows that the integral in (4.8) is bounded in absolute value by

$$M\int_0^{2\pi} L_\Omega(\theta)d\theta. \tag{5.3}$$

We consider now a convex domain Ω. We denote by Π_θ the orthogonal projection from R^2 onto the line e_θ, and by $l(\Pi_\theta(\Omega))$ the length of the projection of Ω onto e_θ. We then clearly have the relation

$$L_\Omega(\theta) \le l(\Pi_\theta(\Omega)). \tag{5.4}$$

It is easy to see that

$$|\Omega| \ge \frac{1}{2}ab \tag{5.5}$$

where

$$a = \min_\theta l(\Pi_\theta(\Omega)) \text{ and } b = \max_\theta l(\Pi_\theta(\Omega)).$$

We have the relation

$$a \to \infty \text{ as } |\Omega| \to \infty \text{ along a normal family,} \tag{5.6}$$

and since (5.4) implies $L_\Omega(\theta) \le b$, it follows from (5.5) and (5.6) that

$$\frac{1}{|\Omega|} \int_0^{2\pi} L_\Omega(\theta) d\theta \to 0 \text{ as } |\Omega| \to \infty \text{ along a normal family.} \tag{5.7}$$

Since the expression (5.3) is an upper bound for the integral in (4.8), it follows that

$$\frac{1}{|\Omega|} \int_\Omega p(u(P), u_t(P))\varphi_s(P) dxdy \to 0 \text{ as } |\Omega| \to \infty \text{ along a normal family.} \tag{5.8}$$

The above computation was performed for u with strictly convex level curves. If the level curves are just convex in D, then we can approximate them with strictly convex level curves. We note that we have thus a formula for the bound of the integrals (4.8), which is given by (5.3), uniformly for all the approximations, yielding in the limit a similar bound for the function u. The above discussion implies the following resut:

Theorem 5.1. *The minimization of $J(\cdot)$ on the class of quasi-convex configurations $\mathcal{F}_Q$ is attained by a radially symmetric configuration.*

Proof. Any radially symmetric configuration $u = f(|r|)$, where $f : R^1_+ \mapsto R^1$ has continuous, bounded derivatives up to second order, is quasi-convex, since $(x, y) \mapsto |r|$ is strictly convex. Thus the assertion of the theorem follows from (5.8). □

The above argument concerning quasi-convex configurations may be extended to more general situations as follows. Analogous to the direction curves we define *direction sets* for arbitrary configurations. As above we fix an axis e and measure angles of straight lines with respect to e. We also keep the notation e_θ for the line obtained from rotating e by the angle θ. We first consider configurations u whose level curves don't contain any straight line segments. For $0 \le \theta < 2\pi$ we denote by $\Gamma(\theta)$ the set of all noncritical points $(x, y) \in R^2$ such that $\nabla u(x, y)$ makes an angle θ with e. Moreover, for a bounded, closed domain D we denote

$$\Gamma_D(\theta) = \Gamma(\theta) \cap D. \tag{5.9}$$

We call the sets $\Gamma(\theta)$ and $\Gamma_D(\theta)$ *direction sets.*

Proposition 5.2. *For a generic configuration u in $C^2(R^2)$, for every closed, bounded domain D and every level curve l the set*

$$\Gamma_{D,l}(\theta) = \Gamma_D(\theta) \cap l$$

is finite. Consequently the set $\Gamma_D(\theta)$ is composed of a finite union of continuous lines.

Proof. Since level curves do not contain straight lines the points in $P_{D,l}(\theta)$ are isolated, and it follows that $P_{D,l}(\theta)$ is a finite set. Let $A = (x_0, y_0)$ be a point in $\Gamma_{D,l}(\theta)$, and U a neighborhood of A.

For a configuration $u(\cdot,\cdot)$, a direction set $\Gamma(\theta)$ is given by

$$\Gamma(\theta) = \{(x, y) : u_y - (\tan\theta)u_x = 0\},$$

namely it is the set of points (x, y) which satisfy

$$\Phi_c(x, y) = 0, \tag{5.10}$$

where we denote $\Phi_c(x, y) = u_y - cu_x$ and take $c = \tan\theta$. If $A = (x_0, y_0)$ satisfies (5.10) then $\Gamma(\theta) \cap U$ is a curve for a sufficiently small neighborhood U of A, provided that $\nabla\Phi_c(x_0, y_0) \neq 0$, namely that (x_0, y_0) doesn't satisfy both

$$u_{xy}(x, y) - cu_{xx}(x, y) = 0 \tag{5.11}$$

and

$$u_{yy}(x, y) - cu_{xy}(x, y) = 0, \tag{5.12}$$

where $c = \tan\theta$. Let $Z \subset R^3$ denote the set of triples (c, x, y) which satisfy (5.10), (5.11), and (5.12), and for a generic function u, for every $R > 0$, the set $Z \cap B(R)$ is finite, so that Z is countable. It follows that for such u, for every θ except for a countable set of values, the set $\Gamma_D(\theta)$ is the union of finitely many curves. □

Of course, for a specific configuration u, the set Z of zeros of (5.10), (5.11), and (5.12) may be more complicated than what is described in Proposition 5.2.

Suppose that

$$\Gamma_D(\theta) = \bigcup_{j=1}^{N} d_j$$

where $\{d_j\}_{j=1}^N$ are the direction curves which compose $P_D(\theta)$. Analogous to the quasi-convex case, we denote

$$L(\theta) = \sum_{j=1}^{N} L_\theta(d_j),$$

where $L_\theta(d_j)$ is the length of the projection of d_j on e_θ. Then the same argument leading to (5.2) in the quasi-convex case implies that (5.2) holds for a generic function as well.

Theorem 5.3. *Let u be any configuration such that the level curves of u don't contain straight line segments. Then the integral in* (4.8) *is bounded by the expression in* (5.3), *namely the following relation holds:*

$$\int_\Omega p(u(P), u_t(P))\varphi_s(P)dxdy \le M \int_0^{2\pi} L(\theta)d\theta. \tag{5.13}$$

Corollary 5.4. *Let $N_R(\theta)$ denote the number of curves composing the direction set $\Gamma_{Q(R)}(\theta)$, where $Q(R)$ is the square of side R centered at the origin, with sides parallel to the axes. If*

$$\frac{1}{R} \max_{0\le\theta\le 2\pi} N_R(\theta) \to 0 \text{ as } R \to \infty, \tag{5.14}$$

then $J(u) \ge \mu$.

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