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MULTIPLE SOLUTIONS FOR THE BREZIS-NIRENBERG PROBLEM

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Abstract. We establish the existence of multiple solutions to the Dirichlet problem for the equation

$$-\Delta u = \lambda u + |u|^{\frac{4}{N-2}} u$$

on a bounded domain Ω of $\mathbb{R}^N$, $N \geq 4$. We show that, if $\lambda > 0$ is not a Dirichlet eigenvalue of $-\Delta$ on Ω, this problem has at least $\frac{N+1}{2}$ pairs of nontrivial solutions. If λ is an eigenvalue of multiplicity m then it has at least $\frac{N+1-m}{2}$ pairs of nontrivial solutions.

1. Introduction

Consider the problem

$$(\wp) \qquad \begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $\lambda > 0$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

This problem has been extensively studied in the last twenty years. We briefly recall what is known about existence and multiplicity of solutions. Let λ_n be the n-th Dirichlet eigenvalue of $-\Delta$ on Ω (counted with multiplicity). In a celebrated paper [5] Brezis and Nirenberg showed that for $N \geq 4$ and $\lambda \in (0, \lambda_1)$ problem $(\wp)$ has at least one positive solution. The same is true for $N = 3$ if λ lies in some small left neighborhood of λ_1. If $N \geq 4$ and

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$\lambda \neq \lambda_n$ for every $n \geq 1$, Capozzi, Fortunato and Palmieri [7] showed that $(\wp)$ has a nontrivial solution (see also Zhang [26]). If $N \geq 5$ the same is true for every $\lambda > 0$ [7, 26].

The first multiplicity result was obtained by Cerami, Fortunato and Struwe [8]. They showed that the number of pairs of nontrivial solutions of $(\wp)$ is bounded below by the number of eigenvalues λ_j lying in the interval $(\lambda, \lambda + S\,|\Omega|^{-2/N})$, where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, and $|\Omega|$ is the Lebesgue measure of Ω. Note, however, that the interval $(\lambda, \lambda + S\,|\Omega|^{-2/N})$ might not contain any eigenvalue at all (cf. [15, p. 256]). Cerami, Solimini and Struwe [9] showed that for $N \geq 6$ and $\lambda \in (0, \lambda_1)$ problem $(\wp)$ has at least two pairs of nontrivial solutions, one of which changes sign (see also Tarantello [23]).

Quite recently, Devillanova and Solimini obtained new strong multiplicity results. In [12] they showed that, if $N \geq 7$, then $(\wp)$ has infinitely many solutions for every $\lambda > 0$. Moreover, in [13] they showed that, if $N \geq 4$ and $\lambda \in (0, \lambda_1)$, then $(\wp)$ has at least $\frac{N}{2} + 1$ pairs of nontrivial solutions. Here we extend this last result to all parameters $\lambda > 0$. Namely, we prove the following.

Theorem 1. *Let $N \geq 4$.*

(i) If $\lambda_n < \lambda < \lambda_{n+1}$ then problem $(\wp)$ has at least $\frac{N+1}{2}$ pairs of nontrivial solutions.

(ii) If $0 < \lambda < \lambda_1$ then $(\wp)$ has at least $\frac{N+2}{2}$ pairs of nontrivial solutions.

(iii) If $\lambda = \lambda_{n+1} = \cdots = \lambda_{n+m}$ is an eigenvalue of multiplicity $m < N+2$ then $(\wp)$ has at least $\frac{N+1-m}{2}$ pairs of nontrivial solutions.

These solutions satisfy

$$\int_\Omega |\nabla u|^2 < 2S^{N/2}.$$

The method used in [13] does not carry over to the case $\lambda \geq \lambda_1$. It also contains a gap. We shall come back to these questions in Section 3 below. In contrast, our method allows us to recover the result in [13] as a special case.

In Section 3 we also give a brief sketch of how our method applies to the corresponding critical biharmonic equation

$$\Delta^2 u = \lambda u + |u|^{\frac{8}{N-4}} u \qquad \text{in } \Omega \tag{1.1}$$

subject either to Dirichlet boundary conditions

$$u = \nabla u = 0 \qquad \text{on} \quad \partial\Omega \tag{1.2}$$

or to Navier boundary conditions

$$u = \Delta u = 0 \qquad \text{on} \quad \partial\Omega. \tag{1.3}$$

Boundary-value problems for equation (1.1) have generated much interest in recent years, see e.g. [4, 14, 15, 16, 17, 18, 20, 24].

2. Proof of the main theorem

We first fix some notation. The Hilbert space $D^{1,2}(\mathbb{R}^N)$ is the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|$ induced by the scalar product

$$\langle u, v\rangle := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v.$$

For $A \subset D^{1,2}(\mathbb{R}^N)$ and $u \in D^{1,2}(\mathbb{R}^N)$ we write

$$\operatorname{dist}(u, A) := \inf_{v\in A} \|u - v\|.$$

For $u \in L^p(\mathbb{R}^N)$, the usual L^p-norm of u will be denoted by $|u|_p$.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Set $H := H_0^1(\Omega) \subset D^{1,2}(\mathbb{R}^N)$. For $A \subset H$ and $\delta > 0$ we write

$$B_\delta(A) := \{u \in H : \operatorname{dist}(u, A) \le \delta\},$$

and we write $\operatorname{int}(A)$ for the interior of A in H. We choose a sequence of orthonormal eigenfunctions e_n corresponding to the Dirichlet eigenvalues λ_n, $n \in \mathbb{N}$, of $-\Delta$. Set $\lambda_0 := 0$. We fix $n, m \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$ such that

$$\lambda_n < \lambda < \lambda_{n+m+1},$$

where n is the greatest integer with $\lambda_n < \lambda$ and m is the smallest integer with $\lambda < \lambda_{n+m+1}$, and we set

$$V^- := span\{e_1, ..., e_n\}, \qquad V^+ := span\{e_j : j > n + m\}.$$

The solutions of problem $(\wp)$ are the critical points of the C^2-functional $J_\lambda : H \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*}.$$

We consider the negative gradient flow $\varphi : \mathcal{G} \to H$ of J_λ, defined by

$$\frac{\partial}{\partial t}\varphi(t, u) = -\nabla J_\lambda(\varphi(t, u)), \qquad \varphi(0, u) = u$$

where $\mathcal{G} = \{(t,u) : u \in H,\ 0 \le t < T(u)\}$ and $T(u) \in (0,\infty]$ is the maximal existence time for the trajectory $t \mapsto \varphi(t,u)$. A subset D of H is called *strictly positively invariant* if

$$\varphi(t,u) \in \operatorname{int}(D) \quad \text{for every } u \in D \text{ and every } t \in (0, T(u)).$$

If $d \in \mathbb{R}$ is a regular value of J_λ, then the sublevel set

$$J_\lambda^d := \{u \in H : J_\lambda(u) \le d\}$$

is strictly positively invariant. We write $P := \{u \in H : u \ge 0\}$ for the convex cone of nonnegative functions in H.

Lemma 2. *If $0 < \lambda < \lambda_1$, then there exists $\alpha_0 > 0$ such that the neighborhoods $B_\alpha(P)$ and $B_\alpha(-P)$ are strictly positively invariant for all $\alpha \le \alpha_0$.*

Proof. We only consider $B_\alpha(P)$. The gradient $\nabla J_\lambda : H \to H$ is given by $\nabla J_\lambda(u) = u - K(u)$, where $K(u) = Lu + G(u)$ and $Lu, G(u) \in H$ are the unique solutions of the equations

$$-\Delta(Lu) = \lambda u \qquad \text{and} \qquad -\Delta(G(u)) = |u|^{2^*-2}u.$$

In other words, Lu and $G(u)$ are uniquely determined by the relations

$$\langle Lu, v\rangle := \lambda \int_\Omega uv \quad \text{and} \quad \langle G(u), v\rangle := \int_\Omega |u|^{2^*-2}uv \quad \text{for all } v \in H. \tag{2.1}$$

By the maximum principle, $Lu \in P$ and $G(u) \in P$ if $u \in P$. Let $u \in H$ and $v \in P$ be such that $\operatorname{dist}(u,P) = \|u - v\|$. Then

$$\operatorname{dist}(Lu, P) \le \|Lu - Lv\| \le \frac{\lambda}{\lambda_1}\|u - v\| = \frac{\lambda}{\lambda_1}\operatorname{dist}(u,P). \tag{2.2}$$

Set $u^- := \min\{u, 0\}$. Note that

$$\left|u^-\right|_{2^*} = \min_{v \in P} |u - v|_{2^*} \le S^{-1/2} \min_{v\in P} \|u - v\| = S^{-1/2}\operatorname{dist}(u,P) \tag{2.3}$$

for every $u \in H$. Using (2.1) and (2.3) we obtain

$$\begin{aligned}
&\operatorname{dist}(G(u),P)\left\|G(u)^-\right\| \le \left\|G(u)^-\right\|^2 = \left\langle G(u), G(u)^-\right\rangle = \int_\Omega |u|^{2^*-2}uG(u)^- \\
&\le \int_\Omega |u^-|^{2^*-2}u^-G(u)^- \le |u^-|_{2^*}^{2^*-1}\left|G(u)^-\right|_{2^*} \\
&\le S^{-2^*/2}\operatorname{dist}(u,P)^{2^*-1}\left\|G(u)^-\right\|.
\end{aligned}$$

Hence,

$$\operatorname{dist}(G(u),P) \le S^{-2^*/2}\operatorname{dist}(u,P)^{2^*-1} \quad \text{for all } u \in H.$$

Choose $\frac{\lambda}{\lambda_1} < \nu < 1$. Then there exists $\alpha_0 > 0$ such that, if $\alpha \le \alpha_0$,

$$\operatorname{dist}(G(u), P) \le (\nu - \frac{\lambda}{\lambda_1})\operatorname{dist}(u, P) \quad \text{for all } u \in B_\alpha(P). \tag{2.4}$$

Fix $\alpha \le \alpha_0$. Inequalities (2.2) and (2.4) yield

$$\operatorname{dist}(K(u), P) \le \operatorname{dist}(Lu, P) + \operatorname{dist}(G(u), P) \le \nu \operatorname{dist}(u, P) \tag{2.5}$$

for all $u \in B_\alpha(P)$. Thus, $K(u) \in \operatorname{int}(B_\alpha(P))$ if $u \in B_\alpha(P)$. Since $B_\alpha(P)$ is closed and convex, Theorem 5.2 in [11] implies

$$u \in B_\alpha(P) \quad \Longrightarrow \quad \varphi(t, u) \in B_\alpha(P) \quad \text{for} \quad t \in [0, T(u)). \tag{2.6}$$

To conclude the proof, we suppose for the sake of contradiction that there is $u \in B_\alpha(P)$ and $t \in (0, T(u))$ such that $\varphi(t, u) \in \partial B_\alpha(P)$. By Mazur's separation theorem, there exists a continuous linear functional $\ell \in H^*$ and $\beta > 0$ such that $\ell(\varphi(t, u)) = \beta$ and $\ell(u) > \beta$ for $u \in \operatorname{int}(B_\alpha(P))$. It follows that

$$\frac{\partial}{\partial s}\bigg|_{s=t} \ell(\varphi(s, u)) = \ell(-\nabla J(\varphi(t, u))) = \ell(K(\varphi(t, u))) - \beta > 0.$$

Hence, there exists $\varepsilon > 0$ such that $\ell(\varphi(s, u)) < \beta$ for $s \in (t - \varepsilon, t)$. Thus, $\varphi(s, u) \notin B_\alpha(P)$ for $s \in (t - \varepsilon, t)$. This contradicts (2.6). The proof is finished. □

Now, if $\lambda \ge \lambda_1$, we fix a regular value $0 < d_\lambda < \frac{1}{N} S^{N/2}$ of J_λ and $r_\lambda > 0$ so that

$$J_\lambda(u) \ge 2d_\lambda \quad \text{for every } u \in V^+ \text{ with } \|u\| = r_\lambda. \tag{2.7}$$

If $\lambda < \lambda_1$, we set $d_\lambda = 0$. Fix $0 < \alpha < \alpha_0$, and set

$$D_\lambda := \begin{cases} B_\alpha(P) \cup B_\alpha(-P) \cup J_\lambda^0 & \text{if } 0 < \lambda < \lambda_1 \\ J_\lambda^{d_\lambda} & \text{if } \lambda \ge \lambda_1 \end{cases}$$

Then D_λ is symmetric (i.e. $u \in D_\lambda$ if and only if $-u \in D_\lambda$) and strictly positively invariant. We shall need the following quantitative deformation lemma.

Lemma 3. *Let $\varepsilon, \delta > 0$, $c \in \mathbb{R}$ and $C \subset H$ be a symmetric subset such that*

$$\|\nabla J_\lambda(u)\| \ge \frac{2\varepsilon}{\delta} \quad \text{for every } u \in J_\lambda^{-1}[c - \varepsilon, c + \varepsilon] \cap B_\delta(C). \tag{2.8}$$

Then there exists an odd continuous map $\vartheta : [J_\lambda^{c+\varepsilon} \cap C] \cup D_\lambda \to J_\lambda^{c-\varepsilon} \cup D_\lambda$ such that $\vartheta(u) = u$ for every $u \in D_\lambda$.

Proof. Let $u \in J_\lambda^{c+\varepsilon} \cap C$. We claim that $\varphi(t,u) \in J_\lambda^{c-\varepsilon}$ for some $t \in (0, T(u))$. Indeed, (2.8) immediately implies that the trajectory $t \mapsto \varphi(t,u)$ cannot stay in $J_\lambda^{-1}[c-\varepsilon, c+\varepsilon] \cap B_\delta(C)$ for all positive times t. Now assume that $\varphi(t,u) \notin B_\delta(C)$ for some $t > 0$, and put

$$t_0 := \inf\{t > 0 \,:\, \varphi(t,u) \notin B_\delta(C)\}.$$

If $J_\lambda(\varphi(t_0,u)) \geq c - \varepsilon$, then (2.8) yields

$$\delta \leq \int_0^{t_0} \|\frac{\partial \varphi(t,u)}{\partial t}\| dt \leq \frac{\delta}{2\varepsilon} \int_0^{t_0} \|\nabla J_\lambda(\varphi(t,u))\|^2 dt \leq \frac{\delta}{2\varepsilon}[J_\lambda(u) - J_\lambda(\varphi(t_0,u))],$$

and hence $J_\lambda(\varphi(t_0,u)) \leq J_\lambda(u) - 2\varepsilon \leq c - \varepsilon$. This proves our claim.
Now, for $u \in [J_\lambda^{c+\varepsilon} \cap C] \cup D_\lambda$, let $t_\lambda(u)$ be the smallest $t \in [0, T(u))$ such that $\varphi(t,u) \in J_\lambda^{c-\varepsilon} \cup D_\lambda$. Then the function $t_\lambda : [J_\lambda^{c+\varepsilon} \cap C] \cup D_\lambda \to [0, \infty)$ is even and lower semicontinuous (since $J_\lambda^{c-\varepsilon} \cup D_\lambda$ is closed). We show that t_λ is also upper semicontinuous. For this let $u \in [J_\lambda^{c+\varepsilon} \cap C] \cup D_\lambda$, and let $\tau > 0$. If $\varphi(t_\lambda(u), u) \in \partial D_\lambda$, then by the strict positive invariance of D_λ we have $\varphi(t_\lambda(u) + \tau, v) \in \operatorname{int}(D_\lambda)$ for v sufficiently close to u, hence $t_\lambda(v) \leq t_\lambda(u) + \tau$ for v sufficiently close to u. If $\varphi(t_\lambda(u), u) \in \partial J_\lambda^{c-\varepsilon}$, then the estimate from above shows $\varphi(t_\lambda(u), u) \in B_\delta(C) \cap J_\lambda^{-1}[c-\varepsilon, c+\varepsilon]$, and hence $\varphi(t_\lambda(u), u)$ is not a critical point of J_λ. As a consequence, $J_\lambda(\varphi(t_\lambda(u) + \tau, v)) < c - \varepsilon$ for v sufficiently close to u, and therefore $t_\lambda(v) \leq t_\lambda(u) + \tau$ for v sufficiently close to u. We conclude that t_λ is a continuous function. Now $\vartheta : [J_\lambda^{c+\varepsilon} \cap C] \cup D_\lambda \to J_\lambda^{c-\varepsilon} \cup D_\lambda$ defined by $\vartheta(u) = \varphi(t_\lambda(u), u)$ has the asserted properties. □

We recall the notion of relative equivariant Lusternik-Schnirelmann category.

Definition 4. *Let $D \subset Y$ be closed symmetric subsets of H. The equivariant category of Y relative to D, denoted $\gamma_D(Y)$, is the smallest number k such that Y can be covered by $k+1$ open symmetric subsets $U_0, U_1, ..., U_k$ of H which satisfy:*

(i) $D \subset U_0$ and there exists an odd continuous map $\chi_0 : U_0 \to D$ such that $\chi_0(u) = u$ for every $u \in D$.

(ii) There exists an odd continuous map $\chi_j : U_j \to \{-1, 1\}$ for every $j = 1, ..., k$.

If no such covering exists we set $\gamma_D(Y) := \infty$.

If $D = \varnothing$, the equivariant category of Y is nothing but its Krasnoselski genus [10, Proposition 2.4]. We write $\gamma(Y) := \gamma_\varnothing(Y)$. Since D_λ is a $\mathbb{Z}/2$-neighborhood retract in H and Y is closed, Tietze's theorem implies that

$\gamma_{D_\lambda}(Y)$ coincides with $\{\mathbb{Z}/2\}$-$\mathrm{cat}_H(Y, D_\lambda)$ as defined in [2]. The following properties are easily verified (cf. [10, Proposition 3.4]).

Lemma 5. *Let Y and Z be closed symmetric subsets of H with $D_\lambda \subset Y$.*
(a) $\gamma_{D_\lambda}(Y \cup Z) \leq \gamma_{D_\lambda}(Y) + \gamma(Z)$.
(b) If $D_\lambda \subset Z$, and if there exists an odd continuous map $\phi : Y \to Z$ with $\phi(u) = u$ for every $u \in D_\lambda$, then $\gamma_{D_\lambda}(Y) \leq \gamma_{D_\lambda}(Z)$.

Define

$$c_k := \inf\{c \in \mathbb{R} : \gamma_{D_\lambda}(J_\lambda^c \cup D_\lambda) \geq k\} \qquad \text{for} \quad k \in \mathbb{N}.$$

Note that $c_1 \geq d_\lambda$ and that (c_k) is a nondecreasing sequence. As usual, we say that a sequence (u_m) in H is a $(PS)_c$-sequence for J_λ if

$$J_\lambda(u_m) \to c, \quad \|\nabla J_\lambda(u_m)\| \to 0, \quad \text{as } m \to \infty.$$

Lemmas 3 and 5 yield the following.

Corollary 6. *For every $k \geq 1$ there exists a $(PS)_{c_k}$-sequence (u_m) for J_λ. Moreover, if $0 < \lambda < \lambda_1$, then $\mathrm{dist}(u_m, P \cup [-P]) \geq \alpha/2$ for all m.*

Proof. Let $0 < \lambda < \lambda_1$. If there is no $(PS)_{c_k}$-sequence (u_m) for J_λ with $\mathrm{dist}(u_m, P \cup -P) \geq \frac{\alpha}{2}$ for all m, then there exists $\varepsilon > 0$ such that $\|\nabla J_\lambda(u)\| \geq 4\varepsilon/\alpha$ for every $u \in J_\lambda^{-1}[c_k - \varepsilon, c_k + \varepsilon] \setminus \mathrm{int}(B_{\frac{\alpha}{2}}(P) \cup B_{\frac{\alpha}{2}}(-P))$. Applying Lemma 3 with $C := H \setminus \mathrm{int}(D_\lambda)$, $\delta = \frac{\alpha}{2}$, and Lemma 5, we get a contradiction of the definition of c_k. The proof for $\lambda \geq \lambda_1$ is similar. □

It is well known that $(PS)_c$ sequences for J_λ are bounded but not necessarily relatively compact, thus the values c_k might not be critical values of J_λ. Struwe [21, Theorem 3.1] gave a characterization of all Palais-Smale sequences for J_λ. In the following we only consider those with $c < \frac{2}{N} S^{N/2}$ and recall Struwe's result for this special case. For $\varepsilon > 0$ and $y \in \mathbb{R}^N$ we consider the Aubin-Talenti instanton [1, 22] $U_{\varepsilon,y} \in D^{1,2}(\mathbb{R}^N)$ defined by

$$U_{\varepsilon,y}(x) := [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2}{2}}. \tag{2.9}$$

The closed set

$$M := \left\{ U_{\varepsilon,y} : \varepsilon > 0,\ y \in \mathbb{R}^N \right\} \subset D^{1,2}(\mathbb{R}^N)$$

is an $(N+1)$-dimensional manifold which consists precisely of the positive solutions $u \in D^{1,2}(\mathbb{R}^N)$ of the equation

$$-\Delta u = |\, u\,|^{2^*-2}\, u.$$

Lemma 7. *Let (u_m) be a $(PS)_c$-sequence for J_λ.*

(a) If $c < \frac{1}{N}S^{N/2}$, then (u_m) is relatively compact in H.

(b) If $\frac{1}{N}S^{N/2} \le c < \frac{2}{N}S^{N/2}$, then a subsequence of (u_m) -still denoted (u_m)- satisfies one of the following two conditions:

(b.1) (u_m) converges strongly in H to a critical point of J_λ.

(b.2) There is a critical point u of J_λ with $J_\lambda(u) = c - \frac{1}{N}S^{N/2}$ such that

$$\operatorname{dist}(u_m - u, M) \to 0 \quad \text{or} \quad \operatorname{dist}(u_m - u, -M) \to 0.$$

This follows directly from [21, Theorem 3.1].

Corollary 8. *(a) If $c_k < \frac{1}{N}S^{N/2}$, then c_k is a critical value of J_λ.*

(b) If $\frac{1}{N}S^{N/2} \le c_k < \frac{2}{N}S^{N/2}$, then either c_k or $c_k - \frac{1}{N}S^{N/2}$ is a critical value of J_λ.

(c) If $0 < \lambda < \lambda_1$ and $c_k = \frac{1}{N}S^{N/2}$, then c_k is a critical value of J_λ.

Proof. (a) and (b) are immediate consequences of Corollary 6 and Lemma 7. We prove (c). Corollary 6 implies the existence of a $(PS)_c$-sequence (u_m) for $c = \frac{1}{N}S^{N/2}$ with $\operatorname{dist}(u_m, P \cup -P) \ge \frac{\alpha}{2}$. Passing to a subsequence, we may assume that either (b.1) or (b.2) of Lemma 7 holds. If (b.1) holds, then $\frac{1}{N}S^{N/2}$ is a critical value of J_λ, as claimed. If (b.2) holds, then we may assume that $\operatorname{dist}(u_m, M) \to 0$, hence there are $y_m \in \mathbb{R}^N$, $\varepsilon_m > 0$, $m \in \mathbb{N}$ such that

$$\tilde{u}_m := \varepsilon_m^{-\frac{N-2}{2}} u_m(\varepsilon_m(\cdot - y_m)) \to U_{1,0} \qquad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

Since $U_{1,0}$ is positive, we have $\|u_m^-\| = \|\tilde{u}_m^-\| \to 0$ as $m \to \infty$. This contradicts the fact that $\operatorname{dist}(u_m, P) \ge \frac{\alpha}{2}$ for all m. Hence (b.2) does not occur, and the proof is finished. □

Set

$$K_c := \{u \in H : J_\lambda(u) = c,\ \nabla J_\lambda(u) = 0\}, \qquad c \in \mathbb{R}.$$

Lemma 9. *If $c_k = c_{k+1} < \frac{2}{N}S^{N/2}$, then K_{c_k} is infinite.*

Proof. Let $c := c_k = c_{k+1}$. If $c < \frac{1}{N}S^{N/2}$, then a standard argument using Lemma 3 and Lemma 7(a) shows that $\gamma(K_c) > 1$. In particular, K_c is infinite. We now consider the more difficult case where

$$\frac{1}{N}S^{N/2} \le c < \frac{2}{N}S^{N/2}.$$

We put $c_* = c - \frac{1}{N}S^{N/2}$, and we consider the sets

$$\mathcal{U}_+(\delta) = \{v \in H : \operatorname{dist}(v - u, M) \le \delta \text{ for some } u \in K_{c_*}\},$$

$$\mathcal{U}_-(\delta) = \{v \in H : \operatorname{dist}(v-u, -M) \le \delta \text{ for some } u \in K_{c_*}\} = -\mathcal{U}_+(\delta),$$
$$\mathcal{U}(\delta) = \mathcal{U}_+(\delta) \cup \mathcal{U}_-(\delta)$$

for $\delta > 0$. We claim that

$$\mathcal{U}_+(\delta) \cap \mathcal{U}_-(\delta) = \varnothing \qquad \text{for } \delta > 0 \text{ sufficiently small.} \tag{2.10}$$

Indeed, suppose for the sake of contradiction that there exist $v_m \in \mathcal{U}_+(\frac{1}{m}) \cap \mathcal{U}_-(\frac{1}{m})$ for each $m \ge 1$. Choose $u_m^1, u_m^2 \in K_{c_*}$, $\omega_m^1 \in M$, and $\omega_m^2 \in -M$ such that

$$\left\|v_m - (u_m^1 + \omega_m^1)\right\| \le \frac{1}{m} \quad \text{and} \quad \left\|v_m - (u_m^2 + \omega_m^2)\right\| \le \frac{1}{m}.$$

Then $\omega_m^1, \omega_m^2 \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ and

$$\left\|(u_m^1 + \omega_m^1) - (u_m^2 + \omega_m^2)\right\| \le \frac{2}{m}. \tag{2.11}$$

It follows that $u_m^1 - u_m^2 \rightharpoonup 0$ weakly in H. Since K_{c_*} is compact, up to a subsequence, $\left(u_m^1\right)$ and $\left(u_m^2\right)$ converge strongly in H. Hence, $u_m^1 - u_m^2 \to 0$ strongly in H. Inequality (2.11) yields $\left\|\omega_m^1 - \omega_m^2\right\| \to 0$, and therefore

$$|\omega_m^1|_{2^*} \le |\omega_m^1 - \omega_m^2|_{2^*} \le S^{-1/2}\|\omega_m^1 - \omega_m^2\| \to 0.$$

This is a contradiction. Hence (2.10) holds.
To finish the proof, we now assume, for the sake of contradiction, that K_c is finite. Then $\gamma(K_c) \le 1$. We fix $\delta > 0$ such that $\mathcal{U}_+(\delta) \cap \mathcal{U}_-(\delta) = \varnothing$, $\gamma(B_\delta(K_c)) = \gamma(K_c)$ and $B_\delta(K_c) \cap \mathcal{U}(\delta) = \varnothing$. It follows that $\gamma(B_\delta(K_c) \cup \mathcal{U}(\delta)) \le 1$. By Lemma 7 there exists $\varepsilon > 0$ such that

$$\|\nabla J_\lambda(u)\| \ge \frac{4\varepsilon}{\delta} \quad \text{for every } u \in J_\lambda^{-1}[c-\varepsilon, c+\varepsilon] \setminus \operatorname{int}(B_{\delta/2}(K_c) \cup \mathcal{U}(\delta/2)).$$

Lemma 3 yields an odd continuous map $\vartheta : [J_\lambda^{c+\varepsilon} \setminus \operatorname{int}(B_\delta(K_c) \cap \mathcal{U}(\delta))] \cup D_\lambda \to J_\lambda^{c-\varepsilon} \cup D_\lambda$ with $\vartheta(u) = u$ for every $u \in D_\lambda$. Hence, by Lemma 5,

$$\begin{aligned} k+1 \le \gamma_{D_\lambda}(J_\lambda^{c+\varepsilon} \cup D_\lambda) &\le \gamma_{D_\lambda}(J_\lambda^{c-\varepsilon} \cup D_\lambda) + \gamma(B_\delta(K_c) \cup \mathcal{U}(\delta)) \\ &\le \gamma_{D_\lambda}(J_\lambda^{c-\varepsilon} \cup D_\lambda) + 1 \le k. \end{aligned}$$

This is a contradiction, and hence the lemma is proved. □

We shall now prove the following.

Proposition 10. *(i) If $\lambda_n < \lambda < \lambda_{n+1}$ for some $n \ge 1$, then $c_{N+2} < \frac{2}{N} S^{N/2}$.*

(ii) If $0 < \lambda < \lambda_1$, then $c_{N+1} < \frac{2}{N} S^{N/2}$.

(iii) If $\lambda_n < \lambda = \lambda_{n+1} = \cdots = \lambda_{n+m} < \lambda_{n+m+1}$, $m < N+2$, then $c_{N+2-m} < \frac{2}{N} S^{N/2}$.

We recall some notions which we need for the proof. We consider the Nehari set

$$\mathcal{N}_\lambda := \{u \in H \setminus \{0\} : \langle \nabla J_\lambda(u), u \rangle = 0\}.$$

This is not a closed set if $\lambda \geq \lambda_1$, but it has the property that

$$J_\lambda(u) = \max_{t \geq 0} J_\lambda(tu) \qquad \text{for every } u \in \mathcal{N}_\lambda.$$

We set $\mathcal{V}_\lambda := \{u \in H : \|u\|^2 - \lambda |u|_2^2 > 0\}$ and write

$$\rho_\lambda : \mathcal{V}_\lambda \to \mathcal{N}_\lambda, \qquad \rho_\lambda(u) := \left(\frac{\|u\|^2 - \lambda |u|_2^2}{|u|_{2^*}^{2^*}} \right)^{\frac{N-2}{4}} u$$

for the radial projection.

Given a bounded domain Θ of $\mathbb{R}^N$ and a subset K of Θ the *capacity of K with respect to Θ* is defined as

$$cap_\Theta K = \inf \left\{ \int_\Theta |\nabla u|^2 : u \in H_0^1(\Theta) \text{ and } u \geq 1 \text{ on } K \right\}.$$

If the closed convex set $\{u \in H_0^1(\Theta) : u \geq 1 \text{ on } K\}$ is nonempty, $cap_\Theta K$ is uniquely achieved at a $\psi \in H_0^1(\Theta)$ which satisfies $\psi \equiv 1$ on K [19].

We write

$$\mathbb{S}^k = \left\{ x \in \mathbb{R}^{k+1} : |x| = 1 \right\} \qquad \text{and} \qquad \mathbb{B}^k = \left\{ x \in \mathbb{R}^k : |x| \leq 1 \right\}$$

for $k \in \mathbb{N}$ and set

$$B(x,r) = \left\{ y \in \mathbb{R}^N : |y - x| < r \right\}.$$

As before, we consider $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ for $u \in D^{1,2}(\mathbb{R}^N)$. The following lemma was proved by Devillanova and Solimini [13, Proof of Lemma 2.3]. In fact, they considered only the case $\lambda < \lambda_1$ but the proof carries over to arbitrary $\lambda > 0$. We sketch it here for the reader's convenience.

Lemma 11. *For every ball $B(x_1, r_1) \subset \Omega$ there exists an odd continuous map*

$$h : \mathbb{S}^N \to H_0^1(B(x_1, r_1))$$

such that $h(\theta)^\pm \in \mathcal{N}_\lambda$ and $J_\lambda(h(\theta)^\pm) < \frac{1}{N} S^{N/2}$ for every $\theta \in \mathbb{S}^N$.

Proof. Let $r_2 := r_1/3$ and let $\eta \in C_c^\infty(B(0, r_2))$ be a radially symmetric cut-off function. Since $\lambda > 0$, following [5] we may choose $\varepsilon_0 > 0$ such that $u_0 := \rho_\lambda(\eta U_{\varepsilon_0,0}) \in \mathcal{N}_\lambda$ satisfies $J_\lambda(u_0) < \frac{1}{N} S^{N/2}$, with $U_{\varepsilon_0,0}$ as in (2.9). For

$0 < r < r_2$ let $\psi_r \in H_0^1(B(0,r_2))$ be the unique function with $\psi_r \equiv 1$ on $B(0,r)$ and

$$\|\psi_r\|^2 = cap_{B(0,r_2)}(B(0,r)).$$

Then $\|\psi_r\| \to 0$ as $r \to 0$. We fix $r \in (0,r_2)$ small enough so that

$$\max_{|z|\le r_2} J_\lambda(\rho_\lambda[(1-\psi_r)u_0(\cdot+z)]) < \frac{1}{N}S^{N/2}.$$

Our choice of u_0 allows us to modify it continuously to obtain a path of positive functions $u_s \in \mathcal{N}_\lambda$ with support in $B(0,(r-r_2)s+r_2)$ such that $J_\lambda(u_s) < \frac{1}{N}S^{N/2}$ for every $s \in [0,1]$. For $y \in \mathbb{B}^N$ we set $t = |y|$ and $\theta = \frac{y}{|y|}$, and define

$$\widetilde{h}(y) := \begin{cases} u_{2-2t}(\cdot - 2r_2(2t\theta - \theta)) - u_0(\cdot + 2r_2\theta) & \text{if } \frac{1}{2} \le t \le 1 \\ u_1 - \rho_\lambda[(1-\psi_r)u_0(\cdot + 4r_2t\theta)] & \text{if } 0 \le t \le \frac{1}{2} \end{cases}$$

Then $\widetilde{h}$ is continuous on $\mathbb{B}^N$ and satisfies $\widetilde{h}(y)^\pm \in \mathcal{N}_\lambda$ and $J_\lambda(\widetilde{h}(y)^\pm) < \frac{1}{N}S^{N/2}$. Since $\widetilde{h}$ is odd on $\mathbb{S}^{N-1}$, it induces an odd continuous map $h : \mathbb{S}^N \to H_0^1(B(x_1,r_1))$ given by

$$h(x_1,...,x_{N+1}) = \begin{cases} \widetilde{h}(x_1,...,x_N) & \text{if } x_{N+1} \ge 0 \\ -\widetilde{h}(-x_1,...,-x_N) & \text{if } x_{N+1} \le 0 \end{cases}$$

with the desired properties. □

Lemma 12. *If $\lambda_n < \lambda$ for some $n \in \mathbb{N} \cup \{0\}$, then there exists an odd continuous map $\overline{h} : \mathbb{R}^{n+N+2} \to H$ such that*

$$\lim_{|x|\to\infty} J_\lambda(\overline{h}(x)) = -\infty \qquad \text{and} \qquad \sup_{u\in\overline{h}(\mathbb{R}^{n+N+2})} J_\lambda(u) < \frac{2}{N}S^{N/2}.$$

Proof. If $n \ge 1$ put $S^- := \{u \in V^- : \|u\| = 1\}$ and choose $\delta > 0$ such that

$$\|u\|^2 - \lambda\,|u|_2^2 < 0 \quad \text{ for every } u \in B_\delta(S^-). \tag{2.12}$$

Choose $x_1 \in \Omega$ and $r_1 > 0$ such that $B(x_1,r_1) \subset \Omega$. For $r \in (0,r_1)$ let $\psi_r \in H_0^1(B(0,r_1))$ be the unique function with $\psi_r \equiv 1$ on $B(0,r)$ and

$$\|\psi_r\|^2 = cap_{B(0,r_1)}(B(0,r)).$$

Fix $r \in (0,r_1)$ small enough so that $(1-\psi_r)u \in B_\delta(S^-)$ for every $u \in S^-$, and consider the linear map

$$h_1 : \mathbb{R}^n \to H_0^1(\Omega \setminus B(x_1,r)), \qquad h_1(x_1,...,x_n) := (1-\psi_r)\sum_{j=1}^n x_j e_j.$$

Note that (2.12) yields

$$\sup_{u\in h_1(\mathbb{R}^n)} J_\lambda(u) \le 0.$$

On the other hand, by Lemma 11, for every $\lambda > 0$ there exists an odd continuous map

$$h : \mathbb{S}^N \to H_0^1(B(x_1, \frac{r}{2}))$$

such that $h(\theta)^\pm \in \mathcal{N}_\lambda$ and $J_\lambda(h(\theta)^\pm) < \frac{1}{N}S^{N/2}$ for every $\theta \in \mathbb{S}^N$. Fix a positive function $v_0 \in H_0^1(B(x_1,r) \setminus B(x_1, \frac{r}{2})) \cap \mathcal{N}_\lambda$ with $J_\lambda(v_0) < \frac{1}{N}S^{N/2}$. Let

$$Z := (\mathbb{S}^N \times [-1,1]) \cup (\mathbb{B}^{N+1} \times \{-1,1\}) \subset \mathbb{R}^{N+1} \times \mathbb{R} \equiv \mathbb{R}^{N+2},$$

and extend h to a map $\widetilde{h} : Z \to H_0^1(B(x_1,r))$ as follows: For $\theta \in \mathbb{S}^N$, $s \in [0,1]$, $t \in [-1,1]$ we set

$$\widetilde{h}(s\theta,t) := \begin{cases} (1-t)h(\theta)^- + (1+t)h(\theta)^+ & \text{if } \ s=1 \\ 2sh(\theta)^+ + (1-s)v_0 & \text{if } \ t=1 \\ 2sh(\theta)^- - (1-s)v_0 & \text{if } \ t=-1 \end{cases}$$

Next, we extend $\widetilde{h}$ radially to a map $h_2 : \mathbb{R}^{N+2} \to H_0^1(B(x_1,r))$ by

$$h_2(tz) := t\widetilde{h}(z) \qquad \text{for } z \in Z,\ t \in [0,\infty).$$

By construction, h_2 is odd and continuous and satisfies

$$\sup_{u\in h_2(\mathbb{R}^{N+2})} J_\lambda(u) < \frac{2}{N}S^{N/2}, \qquad \text{and} \qquad \lim_{|x|\to\infty} J_\lambda(h_2(x)) \to -\infty.$$

If $n=0$ we take $\overline{h} := h_2$. If $n \ge 1$, the map $\overline{h} : \mathbb{R}^{n+N+2} \to H$ given by

$$h(y,z) := h_1(y) + h_2(z), \qquad y \in \mathbb{R}^n,\ z \in \mathbb{R}^{N+2},$$

has the desired properties. □

Proof of Proposition 10. Let $n \in \mathbb{N} \cup \{0\}$ be the greatest integer such that $\lambda_n < \lambda$ and let $\overline{h} : \mathbb{R}^{n+N+2} \to H$ be as in Lemma 12. Set

$$\overline{c} := \sup_{u\in \overline{h}(\mathbb{R}^{n+N+2})} J_\lambda(u) < \frac{2}{N}S^{N/2} \tag{2.13}$$

and

$$k := \gamma_{D_\lambda}(J_\lambda^{\overline{c}} \cup D_\lambda).$$

By Definition 4 there exists an open covering of $J_\lambda^{\overline{c}} \cup D_\lambda$ by open symmetric subsets $U_0, U_1, ..., U_k$ of H, with $D_\lambda \subset U_0$, and odd continuous maps $\chi_0 : U_0 \to D_\lambda$ with $\chi_0(u) = u$ for $u \in D_\lambda$, and $\chi_j : U_j \to \{-e_{n+j}, e_{n+j}\}$ for

$j = 1, ..., k$. By Tietze's theorem we may assume that χ_0 is the restriction of an odd continuous function $\overline{\chi}_0 : H \to H$. We distinguish three cases.

Case (i): $\lambda_n < \lambda < \lambda_{n+1}$, $n \geq 1$. Let $r_\lambda > 0$ be as in (2.7) and set

$$\mathcal{O} := \{x \in \mathbb{R}^{n+N+2} : \left\|\overline{\chi}_0(\overline{h}(x))\right\| \leq r_\lambda\}.$$

Since $\lim_{|x|\to\infty} J_\lambda(\overline{h}(x)) = -\infty$, $\mathcal{O}$ is a bounded symmetric neighborhood of the origin. Set

$$V_j := (\overline{h}^{-1} U_j) \cap \partial\mathcal{O} \quad \text{for } j = 0, 1, ..., k.$$

Since

$$\overline{\chi}_0(\overline{h}(V_0)) \subset \{u \in H : \|u\| = r_\lambda\} \setminus V^+,$$

composing $\overline{\chi}_0 \circ \overline{h}\,|_{V_0}$ with the orthogonal projection $H \to V^-$ yields an odd continuous map $\widetilde{\chi}_0 : V_0 \to V^- \setminus \{0\}$.

Take a partition of unity $\{\pi_0, \pi_1, ..., \pi_k\}$ subordinated to the covering $\{V_0, V_1, ..., V_k\}$ of $\partial\mathcal{O}$ consisting of even functions, and define

$$\chi : \partial\mathcal{O} \to span\{e_1, ..., e_{n+k}\} \cong \mathbb{R}^{n+k}$$

$$\chi(\zeta) = \pi_0(\zeta)\widetilde{\chi}_0(\zeta) + \sum_{j=1}^{k} \pi_j(\zeta)\chi_j(\overline{h}(\zeta)).$$

This is an odd continuous map such that $\chi(\zeta) \neq 0$ for every $\zeta \in \partial\mathcal{O}$. The Borsuk-Ulam theorem yields $n+k \geq n+N+2$, that is, $\gamma_{D_\lambda}(J_\lambda^{\overline{c}} \cup D_\lambda) \geq N+2$. This, together with (2.13), proves assertion *(i)*.

Case (ii): $0 < \lambda < \lambda_1$. In this case $\mathcal{N}_\lambda$ is radially diffeomorphic to the unit sphere in H and there exists $c_0 > 0$ such that $J_\lambda(u) = \frac{1}{N}\,|u|_{2^*}^{2^*} \geq c_0$ for every $u \in \mathcal{N}_\lambda$. Let

$$\mathcal{E}_\lambda := \{u \in \mathcal{N}_\lambda : u^+ \in \mathcal{N}_\lambda \;\text{ and }\; u^- \in \mathcal{N}_\lambda\}.$$

Inequality (2.3), and the analogous one with u^+ instead of u^-, yield the existence of a constant $\alpha_1 > 0$ such that

$$\operatorname{dist}(u, P \cup [-P]) \geq \alpha_1 \qquad \text{for every } u \in \mathcal{E}_\lambda. \tag{2.14}$$

Thus, choosing $\alpha < \alpha_1$ in our definition of D_λ we obtain that $D_\lambda \cap \mathcal{N}_\lambda \subset \mathcal{N}_\lambda \setminus \mathcal{E}_\lambda$. It is well known that $\mathcal{N}_\lambda \setminus \mathcal{E}_\lambda$ consists of two connected components of the form W and $-W$ (see e.g. [6, Lemma 2.5]). Hence it admits an odd map $\phi : \mathcal{N}_\lambda \setminus \mathcal{E}_\lambda \to \{-e_{k+1}, e_{k+1}\}$. Let $\mathcal{C}_0$ be the connected component of $H \setminus \mathcal{N}_\lambda$ which contains 0, and set

$$\mathcal{O} := \{x \in \mathbb{R}^{N+2} : \overline{\chi}_0(\overline{h}(x)) \in \overline{\mathcal{C}_0}\}$$

Since $\lim_{|x|\to\infty} J_\lambda(\overline{h}(x)) = -\infty$, $\mathcal{O}$ is a bounded symmetric neighborhood of the origin. Set $V_j := (\overline{h}^{-1}U_j) \cap \partial\mathcal{O}$, and define

$$\widetilde{\chi}_0 := \phi \circ \overline{\chi}_0 \circ \overline{h} : V_0 \to \{-e_{k+1}, e_{k+1}\}\,.$$

Using a partition of unity, by the same formula as above, we obtain an odd continuous map

$$\chi : \partial\mathcal{O} \to span\{e_1, ..., e_{k+1}\} \cong \mathbb{R}^{k+1}$$

such that $\chi(\zeta) \neq 0$ for every $\zeta \in \partial\mathcal{O}$. The Borsuk-Ulam theorem yields $k+1 \geq N+2$, that is, $\gamma_{D_\lambda}(J_\lambda^{\overline{c}} \cup D_\lambda) \geq N+1$. This, together with (2.13), proves assertion *(ii)*.

Case (iii): $\lambda = \lambda_{n+1} = \cdots = \lambda_{n+m}$ is an eigenvalue of multiplicity $m < N+2$. The proof is analogous to that of case *(i)* except that now $\widetilde{\chi}_0 : V_0 \to span\{e_1, ..., e_{n+m}\} \setminus \{0\}$ and therefore

$$\chi : \partial\mathcal{O} \to span\{e_1, ..., e_{n+m+k}\} \setminus \{0\} \cong \mathbb{R}^{n+m+k} \setminus \{0\}$$

yielding $\gamma_{D_\lambda}(J_\lambda^{\overline{c}} \cup D_\lambda) \geq N+2-m$. □

Proof of Theorem 1. If K_c is infinite for some $c < \frac{2}{N}S^{N/2}$, we are done. So we assume that K_c is finite for all $c < \frac{2}{N}S^{N/2}$ and distinguish three cases:

Case (i): $\lambda_n < \lambda < \lambda_{n+1}$ for some $n \geq 1$. In this case Lemma 9 and Proposition 10 give

$$c_1 < c_2 < \cdots < c_{N+2} < \frac{2}{N}S^{N/2}.$$

Let k_0 be such that $c_{k_0} < \frac{1}{N}S^{N/2} \leq c_{k_0+1}$. By Corollary 8, J_λ has at least

$$k_1 := \max\{k_0, (N+2)-(k_0+1)\} = \max\{k_0, N+1-k_0\}$$

nontrivial critical points. Since $k_1 \geq \frac{N+1}{2}$, the proof in this case is finished.

Case (ii): $0 < \lambda < \lambda_1$. In this case

$$0 < c_0 := \inf_{\mathcal{N}_\lambda} J_\lambda = \inf\{J_\lambda(u) : u \in H \setminus \{0\},\ \nabla J_\lambda(u) = 0\} < \frac{1}{N}S^{N/2}$$

and c_0 is attained by J_λ [5]. Moreover, $K_{c_0} \subset P \cup (-P)$. Therefore, $c_0 < c_1$. Lemma 9 and Proposition 10 yield

$$c_0 < c_1 < c_2 < \cdots < c_{N+1} < \frac{2}{N}S^{N/2}.$$

Let j_0 be such that $c_{j_0} \leq \frac{1}{N}S^{N/2} < c_{j_0+1}$. By Corollary 8 J_λ has at least

$$j_1 := \max\{j_0+1, (N+2)-(j_0+1)\}$$

nontrivial critical points. Since $j_1 \geq \frac{N+2}{2}$, the proof of *(ii)* is finished.

Case (iii): $\lambda = \lambda_{n+1} = \cdots = \lambda_{n+m}$ is an eigenvalue of multiplicity $m < N + 2$. The proof is analogous to that of case *(i)*. $\square$

3. REMARKS AND EXTENSIONS TO CRITICAL BIHARMONIC PROBLEMS

As mentioned in the introduction, for $0 < \lambda < \lambda_1$ we obtain the same number of solutions as Devillanova and Solimini [13], but our proof is different. In [13], Devillanova and Solimini apply minimax arguments on the set

$$\mathcal{E}_\lambda := \{u \in H : u^+ \neq 0 \neq u^-, \int_\Omega (|\nabla u^\pm|^2 - \lambda |u^\pm|^2) = \int_\Omega |u^\pm|^{2^*}\}$$

However, for this one needs a deformation type lemma on $\mathcal{E}_\lambda$, which is not proved in [13]. We point out that $\mathcal{E}_\lambda$ is not a differentiable manifold. Indeed, the maps $u \mapsto \int_\Omega |\nabla u^\pm|^2$ are not differentiable on $H_0^1(\Omega)$, cf. [3, Section 3]. We also remark that for $\lambda \geq \lambda_1$ the set $\mathcal{E}_\lambda$ is not closed in H, hence minimax arguments on $\mathcal{E}_\lambda$ certainly do not apply in this case.

We now turn to a brief discussion of the biharmonic problems (1.1), (1.2) and (1.1), (1.3). For a detailed account of existence and nonexistence results for these problems depending on λ and Ω, see [16]. Solutions of (1.1), (1.2) (respectively (1.1), (1.3)) are critical points of the C^2-functional

$$u \mapsto I_\lambda(u) = \frac{1}{2}\int_\Omega [|\Delta u|^2 - \lambda u^2] - \frac{N-4}{2N}\int_\Omega |u|^{\frac{2N}{N-4}}$$

defined on $H_0^2(\Omega)$ (respectively $H^2(\Omega) \cap H_0^1(\Omega)$). Note that in general $u \in H^2(\Omega)$ does not imply that $u^\pm \in H^2(\Omega)$, hence it is not possible to work on a set similar to $\mathcal{E}_\lambda$ above. For the same reason we cannot prove that neighborhoods of the convex cone of positive functions are positively invariant under the negative gradient flow of I_λ (cf. Lemma 2). However, for any $\lambda > 0$ which is not a Dirichlet (respectively Navier) eigenvalue of Δ^2 on Ω, we may consider the positively invariant set $D_\lambda = I_\lambda^{d_\lambda}$, where $d_\lambda > 0$ is a regular value of I_λ close to zero. Consider the values

$$c_k := \inf\{c \in \mathbb{R} : \gamma_{D_\lambda}(I_\lambda^c \cup D_\lambda) \geq k\}.$$

For $N \geq 8$ we then have

$$c_{N+2} < \frac{4}{N}\mathcal{S}^{\frac{N}{4}},$$

where now $\mathcal{S}$ denotes the best constant for the Sobolev embedding $D^{2,2}(\mathbb{R}^N) \subset L^{\frac{2N}{N-4}}(\mathbb{R}^N)$. Indeed, this can be proved along the lines of the proof of

Proposition 10(i), now using the $D^{2,2}$-capacity [16, Definition 2] and standard estimates for biharmonic critical problems in the space dimensions $N \geq 8$ (these are known as *noncritical dimensions*, cf. [14, 20]). We also have the following partial classification of Palais Smale sequences.

Lemma 13. *Let* (u_m) *be a* $(PS)_c$*-sequence for* I_λ. *Then*

(*a*) *If* $c < \frac{2}{N}\mathcal{S}^{N/4}$, *then* (u_m) *is relatively compact in* H.

(*b*) *If* $\frac{2}{N}\mathcal{S}^{N/4} \leq c < \frac{4}{N}\mathcal{S}^{N/4}$, *then a subsequence of* (u_m) *-still denoted* (u_m)*- satisfies one of the following two conditions:*

(*b*.1) (u_m) *converges strongly to a critical point of* I_λ.

(*b*.2) *There is a critical point* u *of* I_λ *with* $I_\lambda(u) = c - \frac{2}{N}\mathcal{S}^{N/4}$ *such that*

$$\operatorname{dist}(u_m - u, \mathcal{M}) \to 0 \quad \text{or} \quad \operatorname{dist}(u_m - u, -\mathcal{M}) \to 0.$$

Here $\mathcal{M} \subset D^{2,2}(\mathbb{R}^N)$ is the $(N+1)$-dimensional manifold of positive solutions of the equation $\Delta^2 u = |u|^{\frac{8}{N-4}} u$, cf. [16, Lemma 1]. The proof of Lemma 13 is not completely straightforward, since a precise analogue of Struwe's compactness Lemma [21, Theorem 3.1] is *not* available in the biharmonic case. This is due to the fact that there is no general nonexistence result for problems (1.1), (1.2) (respectively (1.1),(1.3)) on a halfspace $\Omega = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \,:\, x_N > 0\}$. However, it is known that these special problems have no positive solutions [18], and that sign-changing solutions may occur only with energy values $I_\lambda(u) \geq \frac{4}{N}\mathcal{S}^{\frac{N}{4}}$ [16, Lemma 4]. Using these facts, Lemma 13 follows from [16, Lemma 8].

As a consequence we now see, as in the second order case, that I_λ has infinitely many critical points whenever $c_k = c_{k+1} < \frac{4}{N}\mathcal{S}^{\frac{N}{4}}$, cf. Lemma 9. By the same argument as in the end of the last section we therefore obtain the following multiplicity result.

Theorem 14. *Let* $N \geq 8$. *If* $\lambda > 0$ *is not a Dirichlet* (*respectively Navier*) *eigenvalue of* Δ^2 *on* Ω, *then problem* (1.1), (1.2) (*respectively* (1.1), (1.3)) *has at least* $\frac{N+1}{2}$ *pairs of nontrivial solutions. If* λ *is an eigenvalue of multiplicity* $m < N+2$, *then it has at least* $\frac{N+1-m}{2}$ *pairs of nontrivial solutions.*

So far, only the existence of one pair of nontrivial solutions was known if $N \geq 8$ and $\lambda > 0$ is not an eigenvalue or if $N \geq 10$ and λ is an eigenvalue; see [15, Corollary 2.2.].

References

[1] Th. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Diff. Geom. **11** (1976), 573–598.

[2] Th. Bartsch and M. Clapp, *Critical point theory for indefinite functionals with symmetries*, J. of Funct. Anal. **138** (1996), 107–136.
[3] Th. Bartsch and T. Weth, *A note on additional properties of sign changing solutions to superlinear elliptic equations*, **22** (2003), 1–14
[4] F. Bernis, J. García-Azorero, and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order*, Adv. in Differential Equations **1**, 219–240
[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Commun. Pure Appl. Math. **36** (1983), 437–477.
[6] A. Castro, J. Cossio, and J.M. Neuberger, *A sign changing solution for a superlinear Dirichlet problem,* Rocky Mountain J. Math. **27** (1997), 1041–1053.
[7] A. Capozzi, D. Fortunato, and G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent,* Ann. Inst. H. Poincaré, Anal. Non Linéaire **2** (1985), 463–440.
[8] G. Cerami, D. Fortunato, and M. Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents,* Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 341–350.
[9] G. Cerami, S. Solimini, and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents,* J. Funct. Anal. **69** (1986), 289–306.
[10] M. Clapp and D. Puppe, *Critical point theory with symmetries*, J. reine angew. Math. **418** (1991), 1–29.
[11] K. Deimling, *Ordinary differential equations in Banach spaces,* Lect. Notes. Math. **596**, Springer-Verlag, Berlin-Heidelberg-New York 1977.
[12] G. Devillanova and S. Solimini, *Concentration estimates and multiple solutions to elliptic problems at critical growth,* Adv. Diff. Eq. **7** (2002), 1257–1280.
[13] G. Devillanova and S. Solimini, *A multiplicity result for elliptic equations at critical growth in low dimension,* Comm. Contemp. Math. **5** (2003), 171–177.
[14] D.E. Edmunds, D. Fortunato, and E. Janelli, *Critical exponents, critical dimensions and the biharmonic operator*, Arch. Rational Mech. Anal. **112** (1990), 269–289
[15] F. Gazzola, *Critical growth problems for polyharmonic operators,* Proc. Roy. Soc. Edinburgh **128A** (1998), 251–263.
[16] F. Gazzola, H.-C. Grunau, and M. Squassina, *Existence and nonexistence result for critical growth biharmoic equations,* Calc. Var. Partial Differential Equations **18** (2003), 117–143.
[17] H. Grunau and G. Sweers, *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. **307** (1997), 588–626
[18] E. Mitidieri, *A Rellich type identity and applications,* Comm. Partial Differential Equations **18** (1993), 125–151.
[19] D. Passaseo, *The effect of the domain shape on the existence of positive solutions of the equation* $\Delta u + u^{2^*-1} = 0$, Top. Meth. Nonl. Anal. **3** (1994), 27–54.
[20] P. Pucci and J. Serrin, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. **69** (1990), 55–83
[21] M. Struwe, *Variational Methods. Applications to nonlinear partial differential equations and Hamiltonian systems,* Springer-Verlag, Berlin-Heidelberg 1990.

[22] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pure Appl. **110** (1976), 353–372.
[23] G. Tarantello, *Nodal solutions of semilinear elliptic equations with critical exponent,* Differential and Integral Equations **5** (1992), 25–42.
[24] R.C.A.M. van der Vorst, *Fourth order elliptic equations with critical growth*, C. R. Acad. Sci. Paris Sér. I Math **320** (1995), 295–299
[25] M. Willem, *Minimax theorems,* PNLDE **24**, Birkhäuser, Boston-Basel-Berlin 1996.
[26] D. Zhang, *On multiple solutions of* $\Delta u + \lambda u + |u|^{\frac{4}{n-2}} = 0$, Nonlinear Anal. TMA **13** (1989), 353–372.