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# HYPERGEOMETRIC FUNCTIONS AND THE TRICOMI OPERATOR

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**Abstract.** In this paper we obtain for a generalized Tricomi operator fundamental solutions entirely supported in the hyperbolic region. Our method is based upon the notion of hypergeometric distributions introduced by S. Delache and J. Leray in [5].

## 1. Introduction

Consider the operator

$$\mathcal{T} = y\Delta_x + \frac{\partial^2}{\partial y^2}, \tag{1.1}$$

with $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $n \geq 1$, a natural generalization of the classical Tricomi operator in $\mathbb{R}^2$. In a previous article [2] we obtained via partial Fourier transform in $x$ explicit expressions for fundamental solutions of $\mathcal{T}$, relative to points on the hyperplane $y = 0$. Such a method led us to calculate inverse Fourier transforms of Bessel functions which, in turn, revealed the importance of certain hypergeometric functions (depending on the "space dimension" $n$) that are intimately related to the operator $\mathcal{T}$.

In the present article we look for fundamental solutions of $\mathcal{T}$ relative to an arbitrary point $(x_0, y_0)$, located in the hyperbolic region ($y < 0$) of the operator, and which are supported by the "forward" characteristic conoid of $\mathcal{T}$ with vertex at $(x_0, y_0)$. We follow the method of S. Delache and J. Leray

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in [5] where they introduced *hypergeometric distributions*, a notion also considered by I. M. Gelfand and G. E. Shilov in [7].

The plan of this article is the following. In Section 2 we deal with preliminary material that is needed throughout the paper. Hypergeometric distributions are introduced in Section 3 where we obtain the basic formula (3.19) which is used in Sections 4 and 5 to obtain fundamental solutions respectively in the cases $n = 1$ or $n$ even, and $n$ odd $\geq 3$. The latter case differs from the previous one by the fact that the fundamental solution is then a sum of two terms, one supported by the "forward" conoid (as in the cases $n = 1$ or $n$ even) and another supported by the boundary of the conoid. In Section 4 we also show how to derive from the methods used in this paper the results obtained previously by Barros-Neto and Gelfand in [4] and Barros-Neto and Cardoso in [2]. Finally, in the Appendix we prove or indicate the proof of results mentioned and utilized in Section 4.

## 2. Preliminaries

Let $\mathcal{T}$ be the operator given by (1.1) and let $\dot{\nabla} u$ be the "modified" gradient

$$\dot{\nabla} u = \langle y u_{x_1}, \ldots, y u_{x_n}, u_y \rangle, \tag{2.1}$$

One clearly has $\mathcal{T} u = \operatorname{div}(\dot{\nabla} u)$ and, in addition,

$$\iint_D (u \mathcal{T} v - v \mathcal{T} u)\, dV = \int_{\partial D} (u \dot{\nabla} v - v \dot{\nabla} u) \cdot \vec{n}\, dS \tag{2.2}$$

for all smooth $u$ and $v$ on the closure of an open bounded domain $D$ with smooth boundary $\partial D$.

Suppose $y < 0$ and set $t = 2(-y)^{3/2}/3 > 0$. The change of variables

$$x = x,\ t = 2(-y)^{3/2}/3 \quad \Leftrightarrow \quad x = x,\ y = -(2/3)^{-2/3} t^{2/3}, \tag{2.3}$$

whose Jacobian is

$$\frac{\partial(x, y)}{\partial(x, t)} = -(2/3)^{1/3} t^{-1/3}, \tag{2.4}$$

transforms $\mathcal{T}$ into the operator

$$2(\frac{3t}{2})^{2/3}\, \mathcal{T}_h, \tag{2.5}$$

where

$$\mathcal{T}_h = \frac{1}{2}(\frac{\partial}{\partial t^2} - \Delta_x) + \frac{1}{6t}\frac{\partial}{\partial t}. \tag{2.6}$$

We call $\mathcal{T}_h$ the *reduced hyperbolic* Tricomi operator. Its formal adjoint is

$$\mathcal{T}_h^* = \frac{1}{2}(\frac{\partial^2}{\partial t^2} - \Delta_x) - \frac{1}{6t}\frac{\partial}{\partial t} + \frac{1}{6t^2}. \tag{2.7}$$

It is a matter of verification that

$$\mathcal{T}_h^*(t^{1/3}u) = t^{1/3}\mathcal{T}_h(u).$$

Thus, if $u$ is a solution of $\mathcal{T}_h(u) = 0$, then $v = t^{1/3}u$ is a solution of $\mathcal{T}_h^* v = 0$, and conversely. Moreover, suppose that $E(x,t;0,t_0)$, with $t_0 \neq 0$, is a fundamental solution of $\mathcal{T}_h$ relative to the point $(0,t_0)$, that is,

$$\mathcal{T}_h E = \delta(x, t - t_0),$$

then $(t/t_0)^{1/3}E$ is a fundamental solution of $\mathcal{T}_h^*$ relative to the same point.

We now recall the definition of the distribution $\chi_q(s)$ (see [5, 7, 8]). Let $q \in \mathbb{C}$ be such that $\operatorname{Re} q > -1$. The locally integrable function

$$\chi_q(s) = \frac{s^q}{\Gamma(q+1)} \text{ if } s > 0, \quad \chi_q(s) = 0 \text{ if } s \leq 0 \tag{2.8}$$

defines a distribution in $\mathbb{R}$ that depends analytically on $q$ and it is such that

$$\chi_q(s) = \frac{d}{ds}\chi_{q+1}(s).$$

When $\operatorname{Re} q \leq -1$, and (2.8) is no longer locally integrable, one associates to $\chi_q$ a distribution as a principal value integral [8] (or regularization in the sense of [7]). $\chi_q$ extends then to an entire function of $q$ and we have

$$\chi_q(s) = \delta^{(-q-1)}(s) \qquad \forall q \in \mathbb{Z}^- \tag{2.9}$$

(see [7]). Moreover, $\chi_q(s)$ is positive homogeneous of degree $q$ and Euler's formula holds

$$s\chi_{q-1}(s) = q\chi_q(s). \tag{2.10}$$

Consider now the function

$$k(x, t - t_0) = \begin{cases} (t - t_0)^2 - |x|^2 & \text{if } t - t_0 > |x| \\ \\ 0 & \text{if } t - t_0 \leq |x|, \end{cases} \tag{2.11}$$

defined in the whole of $\mathbb{R}^{n+1}$. Since $k(x, t - t_0)$ is positive in the semi-cone $C = \{(x,t) \in \mathbb{R}^{n+1} : t - t_0 > |x|\}$, and identically zero outside of $C$, it follows that $\chi_q(k(x, t - t_0))$ (which, for simplicity and when no confusion is possible, we denote by $\chi_q(k(\cdot))$) is a distribution in $\mathbb{R}^{n+1}$ which is an entire analytic function of $q \in \mathbb{C}$. In particular, if $q$ is an integer $< 0$, then

$$\chi_q(k(\cdot)) = \delta^{(-q-1)}((k(\cdot)))$$

is a distribution concentrated on the boundary of $C$ (see [7]).

## 3. Hypergeometric distributions

Our aim is to find fundamental solutions for the Tricomi operator $\mathcal{T}$ relative to an arbitrary point $(0,b), b<0$, in $\mathbb{R}^{n+1}$, that is, a distribution $E(x,y;0,b)$ defined in $\mathbb{R}^{n+1}$ so that $\mathcal{T}E=\delta(x,y-b)$. In the guise of motivation, suppose that $E(x,y;0,b)$ is a locally integrable function. Then in view of formulas (2.3), (2.4), (2.5), and (2.6), we get

$$\begin{aligned}\phi(0,b) &= \langle \mathcal{T}E,\phi\rangle=\int_{\mathbb{R}^{n+1}} E(x,y;0,b)\mathcal{T}\phi\,dx\,dy \\ &= 2(\frac{3}{2})^{1/3}\int_{\mathbb{R}^{n+1}} t^{1/3}E^{\sharp}(x,t;0,t_0)\mathcal{T}_h\psi(x,t)\,dx\,dt.\end{aligned} \tag{3.1}$$

In the last formula $E^{\sharp}(x,t;0,t_0)$ and $\psi(x,t)$ denote, respectively, $E(x,y;0,b)$ and $\phi$ in the variables $x$ and $t$, and we have set $t_0=2(-b)^{3/2}/3$. Thus, our problem reduces to finding fundamental solutions relative to $(0,t_0)$ for the adjoint operator $\mathcal{T}_h^*$, which according to our remark in Section 1 is equivalent to finding fundamental solutions for $\mathcal{T}_h$ relative to the same point.

$\mathcal{T}_h$ belongs to a class of operators, called Euler–Poisson–Darboux operators, studied by Delache and Leray in [5], where they obtained explicit formulas for fundamental solutions of those operators. For the sake of completeness, we outline Delache and Leray's method in [5] relative to the reduced hyperbolic Tricomi operator $\mathcal{T}_h$, or more generally, the operator

$$\mathcal{P}_\alpha=\frac{1}{2}(\frac{\partial^2}{\partial t^2}-\Delta_x)+\frac{\alpha}{t}\frac{\partial}{\partial t}, \tag{3.2}$$

where $\alpha\in\mathbb{C}$. Note that $\mathcal{P}_\alpha$ remains invariant under the action of the group that leaves $t$ unchanged and transforms $(x_1,\ldots,x_n)$ by translations. Since $\mathcal{P}_\alpha$ is homogeneous of degree $-2$ and $\delta(x,t-t_0)$ is homogeneous of degree $-(n+1)$, a fundamental solution $E_\alpha$ of $\mathcal{P}_\alpha$ must be homogeneous of degree $1-n$. Monomials of the type

$$t_0^{\alpha-j}t^{-\alpha-j}\chi_{j+1/2-n/2}(k(x,t-t_0)),$$

have the desired homogeneity degree. If we consider the wave operator $\Box_{(x,t)}=(\partial^2/\partial t^2-\Delta_x)$ in $\mathbb{R}^{n+1}$, it is shown in [5] that

$$\Box_{(x,t)}(\frac{1}{2}\pi^{1/2-n/2}\chi_{1/2-n/2}(k(x,t)))=\delta(x,t), \tag{3.3}$$

in other words, the distribution $\pi^{1/2-n/2}\chi_{1/2-n/2}(k(x,t))/2$ is a fundamental solution of the wave operator.

As a consequence, it is natural to look for a fundamental solution to $\mathcal{P}_\alpha$ as a formal series

$$E_\alpha(x,t;0,t_0) = \pi^{1/2-n/2}(\frac{t_0}{t})^\alpha \sum_{j=0}^{\infty} c_j (t_0 t)^{-j} \chi_{j+1/2-n/2}(k(\cdot)), \tag{3.4}$$

with a suitable choice of the coefficients $c_j$. By applying $\mathcal{P}_\alpha$ to both sides of (3.4) one obtains, after routine calculations where the two identities

$$\Box_{(x,t)}\chi_{j+1/2-n/2}(k(\cdot)) = 4j\chi_{j-1/2-n/2}(k(\cdot))$$

and

$$\frac{\partial}{\partial t}[\chi_{j+1/2-n/2}(k(\cdot))] = 2(t-t_0)\chi_{j-1/2-n/2}(k(\cdot))$$

are used, the following result:

$$\mathcal{P}_\alpha E_\alpha = c_0\delta(x,t-t_0) + \pi^{1/2-n/2}\sum_{j=1}^{\infty}\{\frac{1}{2}(j-1+\alpha)(j-\alpha)c_{j-1} + 2jc_j\} \\ \times t_0^{\alpha-j+1}t^{-\alpha-j-1}\chi_{j-1/2-n/2}(k(\cdot)). \tag{3.5}$$

If we choose the coefficients $c_j$ so that

$$c_0 = 1 \quad \text{and} \quad \frac{1}{2}(j-1+\alpha)(j-\alpha)c_{j-1} + 2jc_j = 0,\ j \geq 1, \tag{3.6}$$

then (3.5) reduces to

$$\mathcal{P}_\alpha E_\alpha = \delta(x,t-t_0), \tag{3.7}$$

that is, $E_\alpha$ is a fundamental solution of $\mathcal{P}_\alpha$. Now recalling the notations

$$(a)_0 = 1, \quad (a)_j = a(a+1)\cdots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)},\ j \geq 1, \tag{3.8}$$

it follows from (3.6) that

$$c_j = (-\frac{1}{4})^j\frac{(\alpha)_j(1-\alpha)_j}{j!},\ j \geq 0. \tag{3.9}$$

Hence, we may rewrite (3.4) as

$$E_\alpha(x,t;0,t_0) = \pi^{1/2-n/2}(\frac{t_0}{t})^\alpha\Phi_\alpha(x,t), \tag{3.10}$$

where

$$\Phi_\alpha(x,t) = \sum_{j=0}^{\infty}\frac{(\alpha)_j(1-\alpha)_j}{j!}(-\frac{1}{4t_0t})^j\chi_{j+1/2-n/2}(k(\cdot)). \tag{3.11}$$

This series converges for $|k(\cdot)/4t_0t| < 1$. $\Phi_\alpha(x,t)$ is the *hypergeometric distribution* introduced by Delache and Leray in [5]. Hypergeometric distributions were also considered by Gelfand and Shilov in [7].The reason for that name is, as we will see, because $\Phi_\alpha$ can be represented by a hypergeometric function. We briefly review, in the Appendix, the definition of hypergeometric functions and some of their properties.

Clearly $\Phi_\alpha$ depends on the space dimension $n$ and we must consider two cases.

**Case I:** $n = 1$ or $n$ *even*. We rewrite (3.11) as

$$\Phi_\alpha(x,t) = \chi_{1/2-n/2}(k(\cdot)) + \sum_{j=1}^{\infty} \frac{(\alpha)_j(1-\alpha)_j}{j!}(-\frac{1}{4t_0t})^j \chi_{j+1/2-n/2}(k(\cdot)). \tag{3.12}$$

From Euler's formula (2.10) it follows by induction that

$$s^j \chi_q(s) = (q+1)_j \chi_{q+j}(s),$$

for each integer $j \geq 0$. Inserting this formula with $q = 1/2 - n/2$ into (3.12) we obtain

$$\begin{aligned}\Phi_\alpha(x,t) &= \chi_{1/2-n/2}(k(\cdot)) \sum_{j=0}^{\infty} \frac{(\alpha)_j(1-\alpha)_j}{(3/2-n/2)_j\, j!}(\frac{(t-t_0)^2 - |x|^2}{-4t_0t})^j \qquad (3.13)\\ &= \chi_{1/2-n/2}(k(\cdot)) F(\alpha, 1-\alpha, \frac{3}{2} - \frac{n}{2}; \frac{(t-t_0)^2 - |x|^2}{-4t_0t}).\end{aligned}$$

From now on, $F(\alpha, \beta, \gamma; \zeta)$ denotes a hypergeometric function (see the Appendix).

**Case II:** $n$ odd $> 1$. Let $n = 2m+1, m \geq 1$. Note that in this case $1/2 - n/2 = -m$, a negative integer. We split $\Phi_\alpha$ into two terms:

$$\begin{aligned}\Phi_\alpha(x,t) &= \sum_{j=0}^{m-1} \frac{(\alpha)_j(1-\alpha)_j}{j!}(-\frac{1}{4t_0t})^j \chi_{j-m}(k(\cdot)) \qquad (3.14)\\ &+ \sum_{j=m}^{\infty} \frac{(\alpha)_j(1-\alpha)_j}{j!}(-\frac{1}{4t_0t})^j \chi_{j-m}(k(\cdot)).\end{aligned}$$

Whenever $j - m < 0$, $\chi_{j-m}(k(\cdot)) = \delta^{(m-j-1)}(k(\cdot))$ is a distribution concentrated on the surface of the semi-cone $C$. Thus the first term in (3.14) corresponds to a finite sum of distributions supported by the boundary of $C$.

Recalling that $\chi_j(s) = s^j \chi_0(s)/j!$ and setting $j' = j - m$, rewrite the second term in (3.14) as

$$S = \chi_0(k(\cdot))(\frac{1}{-4t_0 t})^m \sum_{j'=0}^{\infty} \frac{(\alpha)_{j'+m}(1-\alpha)_{j'+m}}{(j'+m)!\, j'!}(\frac{k(\cdot)}{-4t_0 t})^{j'}.$$

Now $(\alpha)_{j'+m} = (\alpha)_m(\alpha+m)_{j'}$, $(1-\alpha)_{j'+m} = (1-\alpha)_m(1-\alpha+m)_{j'}$, and $(j'+m)! = m!(m+1)_{j'}$. Therefore,

$$S = \chi_0(k(\cdot))c_m(\frac{1}{-4t_0 t})^m F(\alpha+m, 1-\alpha+m, m+1, \frac{(t-t_0)^2 - |x|^2}{-4t_0 t}),$$

where $c_m = (\alpha)_m(1-\alpha)_m/m!$. Thus, the expression (3.14) for $\Phi_\alpha$ becomes

$$\Phi_\alpha(x,t) = \sum_{j=0}^{m-1} \frac{(\alpha)_j(1-\alpha)_j}{j!}(-\frac{1}{4t_0 t})^j \delta^{(m-j-1)}(k(\cdot)) \tag{3.15}$$
$$+ \chi_0(k(\cdot))c_m(\frac{1}{-4t_0 t})^m F(\alpha+m, 1-\alpha+m, m+1, \frac{(t-t_0)^2-|x|^2}{-4t_0 t}).$$

**Remarks.** 1) The support of all fundamental solutions above described is the closure of the semi-cone $C$ defined at the end of Section 2. In the case where $n$ is an odd integer $> 1$, besides the term that contains the hypergeometric function whose support is the closure of $C$ there are a finite number of terms whose support is the boundary of $C$.

2) Formula (3.15) can be viewed as a derivative with respect to $k(\cdot)$ of a certain hypergeometric distribution. More precisely, consider the hypergeometric distribution $\chi_0(s)F(a,b,1;rs)$ where $r$ is a real or complex parameter. The following formula holds

$$\frac{d^m}{ds^m}[\chi_0(s)F(a,b,1;rs)] = \sum_{j=0}^{m-1} \frac{(a)_j(b)_j}{j!} r^j \delta^{(m-j-1)}(s) \tag{3.16}$$
$$+ \chi_0(s)c_m r^m F(a+m, b+m, m+1; rs).$$

Indeed, just note that if $f(s)$ is a smooth function defined near $s=0$, then $f(s)\delta(s) = f(0)\delta(s)$, and whenever $c \neq 0, -1, -2, \cdots$ one has

$$\frac{d}{dz}F(a,b,c;z) = \frac{ab}{c}F(a+1,b+1,c+1;z).$$

Thus, we may rewrite (3.15) as a derivative:

$$\Phi_\alpha(x,t) = \frac{d^m}{d(k(\cdot))^m}[\chi_0(k(\cdot))F(\alpha, 1-\alpha, 1, \frac{k(\cdot)}{-4t_0 t})]. \tag{3.17}$$

Formulas (3.16) and (3.17) are analogous to formulas considered by Gelfand and Shilov in [7] and involving complex order derivatives of hypergeometric distributions of the type $\chi_0(s)F(a,b,c;s)$.

So far we have shown that whenever $|k(\cdot)/4t_0t| < 1$

$$E_{1/6}(x,t;0,t_0) = \pi^{1/2-n/2}(\frac{t_0}{t})^{1/6}\Phi_{1/6}(x,t) \tag{3.18}$$

is a fundamental solution of $\mathcal{T}_h$ relative to the point $(0,t_0)$. From our remarks at the end of Section 2, it follows that the distribution

$$(t/t_0)^{1/3}E_{1/6}(x,t;0,t_0) = \pi^{1/2-n/2}(\frac{t}{t_0})^{1/6}\Phi_{1/6}(x,t)$$

is a fundamental solution of $\mathcal{T}_h^*$ relative to the same point. In view of formula (3.1) we define the distribution $E^\sharp$ by

$$2(\frac{3}{2})^{1/3}t^{1/3}E^\sharp(x,t;0,t_0) = \pi^{1/2-n/2}(\frac{t}{t_0})^{1/6}\Phi_{1/6}(x,t),$$

or,

$$E^\sharp(x,t;0,t_0) = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3}}(\frac{1}{4t_0t})^{1/6}\Phi_{1/6}(x,t), \tag{3.19}$$

which written in terms of the variables $x$ and $y$ gives us a fundamental solution of the Tricomi operator $\mathcal{T}$ relative to the point $(0,b), b<0$. Moreover, in the next two sections, by using known formulas involving hypergeometric functions, we will recoup the fundamental solutions described in the papers [3, 4, 2].

## 4. Fundamental solutions, $n=1$ or $n$ even

We return to formula (3.19) and rewrite $\Phi_{1/6}$, given by (3.13), as

$$\Phi_{1/6}(x,t) = \chi_{1/2-n/2}(k(\cdot))F(\frac{5}{6},\frac{1}{6},\frac{3}{2}-\frac{n}{2};-\frac{k(\cdot)}{4t_0t}) \tag{4.1}$$

after recalling that $F(a,b,c;z) = F(b,a,c;z)$. Note that $\chi_{1/2-n/2}(k(\cdot))$ is supported by the cone $C$. It is well known that

$$F(a,b,c;z) = (1-z)^{-b}F(c-a,b,c;\frac{z}{z-1}). \tag{4.2}$$

If we set $z = (t-t_0)^2 - |x|^2/(-4t_0t)$, then

$$1-z = \frac{(t+t_0)^2-|x|^2}{4t_0t} \quad \text{and} \quad \frac{z}{z-1} = \frac{(t-t_0)^2-|x|^2}{(t+t_0)^2-|x|^2}, \tag{4.3}$$

hence from (4.2) we obtain

$$F(\frac{5}{6},\frac{1}{6},\frac{3}{2}-\frac{n}{2};-\frac{k(\cdot)}{4t_0t})$$
$$=(\frac{(t+t_0)^2-|x|^2}{4t_0t})^{-1/6}\times F(\frac{2}{3}-\frac{n}{2},\frac{1}{6},\frac{3}{2}-\frac{n}{2};\frac{(t-t_0)^2-|x|^2}{(t+t_0)^2-|x|^2}), \tag{4.4}$$

and so we may rewrite $E^\sharp$ as

$$\begin{aligned} E^\sharp(x,t;0,t_0) &= \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3}}\chi_{1/2-n/2}(k(\cdot))\,[(t+t_0)^2-|x|^2]^{-1/6} \\ &\times F(\frac{2}{3}-\frac{n}{2},\frac{1}{6},\frac{3}{2}-\frac{n}{2};\frac{(t-t_0)^2-|x|^2}{(t+t_0)^2-|x|^2}). \end{aligned} \tag{4.5}$$

Since $E^\sharp$ is supported by the closure of $C$, the last factor in formula (4.5) represents the hypergeometric series because the absolute value of its argument, in $C$, is $< 1$.

If we now set $t_0 = a$ and

$$u=9(|x|^2-a^2)+12a(-y)^{3/2}+4y^3,\quad v=9(|x|^2-a^2)-12a(-y)^{3/2}+4y^3, \tag{4.6}$$

then

$$(t-t_0)^2-|x|^2=-\frac{1}{9}u \quad\text{and}\quad (t+t_0)^2-|x|^2=-\frac{1}{9}v. \tag{4.7}$$

Define the region

$$D^n_{b,-}=\{(x,y)\in\mathbb{R}^{n+1}:9(|x|^2-a^2)+12a(-y)^{3/2}+4y^3<0,y<b\} \tag{4.8}$$

which corresponds to the semi-cone $C$, and let $\chi_{D^n_{b,-}}$ be its characteristic function. In terms of $x$ and $y$ the distribution (4.5) becomes

$$E_-(x,y;0,b)=c(n)\chi_{D^n_{b,-}}(x,y)(-u)^{1/2-n/2}(-v)^{-1/6}F(\frac{2}{3}-\frac{n}{2},\frac{1}{6},\frac{3}{2}-\frac{n}{2};\frac{u}{v}), \tag{4.9}$$

where

$$c(n)=\frac{\pi^{1/2-n/2}}{2^{1/3}3^{1-n}\Gamma(\frac{3}{2}-\frac{n}{2})}. \tag{4.10}$$

We observe that the distribution $E_-$ is real valued because in $D^n_{b,-}$ both $u$ and $v$ are $< 0$. Thus we have proved the following:

**Theorem 4.1.** *$E_-(x,y;0,b)$ is a fundamental solution of $\mathcal{T}$ relative to $(0,b)$ whose support is the closure of the region $D^n_{b,-}$.*

**Remarks.** 1) $E_-$ is the unique fundamental solution of $\mathcal{T}$, relative to $(0,b)$ whose support is $\bar{D}^n_{b,-}$. Indeed, any other such fundamental solution is of the form $E_- + f$, with $\mathcal{T}f = 0$ and $y \le b$ on $\operatorname{supp} f$. Since the convolution $E_- * f$ is well defined because the map

$$\operatorname{supp} E_- \times \operatorname{supp} f \ni ((x,y),(x',y')) \to (x+x', y+y')$$

is proper, we have

$$f = \mathcal{T}E_- * f = E_- * \mathcal{T}f = 0.$$

2) In [4], when $n = 1$, $E_-$ was obtained by a method different from the one here described and based upon the existence of the Riemann function for the reduced hyperbolic operator $\mathcal{T}_h$.

If we let $b \to 0$, we obtain a fundamental solution of $\mathcal{T}$ relative to the origin, namely:

**Corollary 4.1.** *The limit, in the sense of distributions, of $E_-(x,y;0,b)$ as $(0,b) \to (0,0)$ is*

$$F_-(x,y) = \tag{4.11}$$
$$= \begin{cases} \dfrac{\pi^{1/2-n/2}}{2^{1/3}3^{1-n}\Gamma(\frac{3}{2}-\frac{n}{2})} F(\frac{2}{3}-\frac{n}{2},\frac{1}{6},\frac{3}{2}-\frac{n}{2};1)\,|9|x|^2+4y^3|^{\frac{1}{3}-\frac{n}{2}} & \text{in } D^n_- \\ 0 & \text{elsewhere,} \end{cases}$$

*a fundamental solution of $\mathcal{T}$ relative to the origin whose support is the closure of the region $D^n_- = \{(x,y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\}$.*

It is a matter of verification that the fundamental solution given by formula (4.11) coincides with the fundamental solution given by formula (4.2) in Theorem 4.1 of [2]. The only apparent discrepancy between these two formulas is the multiplying constants. However, one can show that the constant in (4.11) coincides with the constant

$$C_- = \frac{3^n\Gamma(4/3)}{2^{2/3}\pi^{n/2}\Gamma(\frac{4}{3}-\frac{n}{2})}$$

on page 490 of [2].

An important observation, still in the case $n = 1$, is that it is possible to obtain another fundamental solution supported by the closure of $D^1_{b,+}$, the complement (in $\mathbb{R}^2$) of $\bar{D}^1_{b,-}$. Indeed, consider, as explained in the Appendix, $F(1/6, 1/6, 1; \zeta)$, the *principal branch* of the analytic continuation of the corresponding hypergeometric series, and define in the whole of $\mathbb{R}^2$ the function

$$\tilde{E}(x, y; 0, b) = \frac{(-v)^{-1/6}}{2^{1/3}} F(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{v}). \tag{4.12}$$

We show in the Appendix that $\tilde{E}(x, y; 0, b)$ is locally integrable in $\mathbb{R}^2$, singular when $v = 0$, real analytic in $\mathbb{R}^2 \setminus (r_a \cup r_{-a})$, and a solution of $\mathcal{T}u = 0$ in the sense of distributions. It then follows that the distribution $E_- - \tilde{E}$, identically zero in the region $D^1_{b,-}$, is also a fundamental solution of $\mathcal{T}$. Define the distribution

$$E_+(x, y; 0, b) = \begin{cases} -\tilde{E}(x, y; 0, b) & \text{in } D^1_{b,+} \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{4.13}$$

We have

**Theorem 4.2.** *$E_+$ is a fundamental solution of $\mathcal{T}$ relative to $(0, b)$ whose support is the closure of the region $D^1_{b,+}$.*

From this we obtain (see [3])

**Corollary 4.2.** *As $(0, b) \to (0, 0)$, a suitable linear combination of fundamental solutions of the type $E_+$ converges, in the sense of distributions, to the fundamental solution*

$$F_+(x, y) = \begin{cases} -\frac{1}{2^{1/3}3^{1/2}} F(\frac{1}{6}, \frac{1}{6}, 1; 1)(9x^2 + 4y^3)^{-1/6} & \textit{in } D_+ \\ \\ 0 & \textit{elsewhere,} \end{cases} \tag{4.14}$$

*where $D_+ = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 > 0\}$.*

**Remark.** It is plausible to expect, in the case of $n$ even, the existence of fundamental solutions that correspond to $E_+$ given by formula (4.13). However, if one follows the method used in this paper and derives from (4.9) a function that corresponds to $\tilde{E}$, in formula (4.12), that function is not locally integrable and it is not clear that it is a solution of $\mathcal{T}u = 0$. This is certainly a question that should be further investigated. In a forthcoming paper, we expect to describe, by using hypergeometric distributions, a complete family of fundamental solutions with poles in the elliptic region of the $\mathcal{T}$.

## 5. Fundamental solutions, $n$ odd $> 1$

Let $n = 2m+1$ with $m \geq 1$. Again from formula (3.19) we have

$$E^\sharp(x,t;0,t_0) = A_m(\frac{1}{4t_0t})^{1/6}\Phi_{1/6}(x,t), \tag{5.1}$$

where $A_m = 1/(2^{1/3}3^{1/3}\pi^m)$. From formula (3.15) $\Phi_{1/6}$ is given by

$$\begin{aligned}\Phi_{1/6}(x,t) &= \sum_{j=0}^{m-1} c_j(-\frac{1}{4t_0t})^j\delta^{(m-j-1)}(k(\cdot)) \\ &+ \chi_0(k(\cdot))c_m(\frac{1}{-4t_0t})^m F(m+\frac{5}{6}, m+\frac{1}{6}, m+1; \frac{k(\cdot)}{-4t_0t}),\end{aligned} \tag{5.2}$$

with

$$c_j = \frac{\Gamma(j+5/6)\Gamma(j+1/6)}{\Gamma(5/6)\Gamma(1/6)\Gamma(j+1)}, \quad 0 \leq j \leq m. \tag{5.3}$$

In view of (4.2) and (4.3), the hypergeometric function in (5.2) is equal to

$$(\frac{(t+t_0)^2-|x|^2}{4t_0t})^{-m-1/6}F(\frac{1}{6}, m+\frac{1}{6}, m+1; \frac{(t-t_0)^2-|x|^2}{(t+t_0)^2-|x|^2})$$

and we may rewrite $E^\sharp(x,t;0,t_0)$ as

$$\begin{aligned}E^\sharp(x,t;0,t_0) &= A_m\sum_{j=0}^{m-1}(-1)^j c_j(4t_0t)^{-j-1/6}\delta^{(m-j-1)}(k(\cdot)) \\ &+ (-1)^m A_m c_m((t+t_0)^2-|x|^2)^{-m-\frac{1}{6}}F(\tfrac{1}{6}, m+\tfrac{1}{6}, m+1, \frac{(t-t_0)^2-|x|^2}{(t+t_0)^2-|x|^2})\chi_0(k(\cdot)).\end{aligned} \tag{5.4}$$

Note that all terms in the sum contain distributions of the form $\delta^{(q)}(k(\cdot))$ which are supported by the surface of the semi-cone $C$. However, the support of the last term in (5.4) is the closure of $C$.

In [7] Gelfand and Shilov introduced the distribution $\delta(P)$ supported by the surface $S$ given by $P = 0$, where $P$ is a smooth function such that $\nabla P \neq 0$ on $S$. In particular, they proved that if $a(\cdot)$ is a nonvanishing function, then

$$\delta^{(q)}(aP) = a^{-(q+1)}\delta^{(q)}(P). \tag{5.5}$$

These results extend to our case, where $P = k(\cdot)$ has a singular point at $(0, t_0)$. We have the following

**Lemma 5.1.** *For all* $0 \leq j \leq m-1$,

$$(4t_0t)^{-j-1/6}\delta^{(m-j-1)}(k(\cdot)) = (4t_0t)^{5/6}\delta^{(m-j-1)}((4t_0t)^{(j+1)/(m-j)}k(\cdot)).$$

**Proof.** Indeed,

$$\begin{aligned}(4t_0t)^{-j-1/6}\delta^{(m-j-1)}(k(\cdot)) &= (4t_0t)^{5/6}(4t_0t)^{-(j+1)}\delta^{(m-j-1)}(k(\cdot)) \\ &= (4t_0t)^{5/6}[(4t_0t)^{(j+1)/(m-j)}]^{-(m-j)}\delta^{(m-j-1)}(k(\cdot)) \\ &= (4t_0t)^{5/6}\delta^{(m-j-1)}((4t_0t)^{(j+1)/(m-j)}k(\cdot)),\end{aligned}$$

by virtue of (5.5) and the fact that $4t_0t \neq 0$ in the region $t - t_0 > |x|$. □

As a consequence of this lemma, all terms that contain derivatives of $\delta$ in (5.4) tend to zero, as $t_0 \to 0$. By taking limits, it follows that the distribution

$$E^\sharp(x,t;0,0) \tag{5.6}$$
$$= \frac{(-1)^m}{2^{1/3}3^{1/3}\pi^m}\frac{\Gamma(m+5/6)\Gamma(m+1/6)}{\Gamma(5/6)\Gamma(1/6)\Gamma(m+1)}F(\tfrac{1}{6}, m+\tfrac{1}{6}, m+1; 1)(t^2-|x|^2)^{-m-\frac{1}{6}}$$

is a fundamental solution (relative to the origin) of $\mathcal{T}_h$ supported by the closure of the semi-cone $\{(x,t) : t > |x|\}$.

As we did in the previous sections, we rewrite $E^\sharp(x,t;0,t_0)$ in terms of the variables $x$ and $y$. Formulas (4.6) and (4.7) imply that $4t_0t = (v-u)/9$ and, following Gelfand and Shilov's notations, we replace $\delta^{(q)}(k(\cdot))$ by $\delta^{(q)}(u(\cdot))$, with the understanding that $u(\cdot)$ now means $u(x,y)$, with $y \leq b$. Thus (5.4) becomes

$$E_-(x,y;0,b) = A_m\sum_{j=0}^{m-1}(-1)^j c_j(\frac{v-u}{9})^{-j-1/6}\delta^{(m-j-1)}(u(\cdot)) \tag{5.7}$$
$$+ (-1)^m A_m c_m(-\frac{v}{9})^{-m-1/6}F(\frac{1}{6}, m+\frac{1}{6}, m+1, \frac{u}{v})\chi_{D^n_{b,-}}(x,y),$$

where $\chi_{D^n_{b,-}}$ is the characteristic function of the set (4.8).Then the following result holds

**Theorem 5.1.** *The distribution $E_-(x,y;0,b)$ is a fundamental solution of $\mathcal{T}$ relative to $(0,b)$ supported by the closure of the set $D^n_{b,-}$.*

Note that in (5.7) all terms inside the summation are supported by the boundary of $D^n_{b,-}$ while the last term is supported by the closure of $D^n_{b,-}$. If we let $b \to 0$ we obtain, at the limit, the fundamental solution $F_-(x,y)$ described in our previous paper [2], namely

**Theorem 5.2.** *The distribution*

$$F_-(x,y) = \begin{cases} \dfrac{3^n\Gamma(\frac{4}{3})}{2^{2/3}\pi^{n/2}\Gamma(\frac{4}{3}-\frac{n}{2})}\,|9|x|^2+4y^3|^{\frac{1}{3}-\frac{n}{2}} & \text{in } D^n_- \\ 0 & \text{elsewhere,} \end{cases} \tag{5.8}$$

*supported by the closure of the region* $D^n_- = \{(x,y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\}$, *is a fundamental solution of* $\mathcal{T}$.

**Proof.** Recall that $t = 2(-y)^{3/2}/3$ and that $t^2 - |x|^2 = \frac{1}{9}(-9|x|^2 - 4y^3)$. Hence, the right-hand side of (5.6) equals $A|9\,|x|^2 + 4y^3|^{-m-1/6}$, where

$$A = \frac{(-1)^m 3^{2m}}{2^{1/3}\pi^m} \frac{\Gamma(m+5/6)\Gamma(m+1/6)}{\Gamma(5/6)\Gamma(1/6)\Gamma(m+1)} F(\frac{1}{6}, m+\frac{1}{6}, m+1; 1). \tag{5.9}$$

Now the exponent $-m - 1/6$ equals $1/3 - n/2$ because $n = 2m+1$. On the other hand, it is a matter of verification that the constant

$$\frac{3^n \Gamma(\frac{4}{3})}{2^{2/3}\pi^{n/2}\Gamma(\frac{4}{3} - \frac{n}{2})} \tag{5.10}$$

(denoted by $C_-$ in [2]) which appears in (5.8) is the same as $A$. □

## 6. Appendix

We are going to prove that the function $\tilde{E}(x,y;0,b)$ defined by formula (4.12) in Section 4 is locally integrable in $\mathbb{R}^2$, singular when $v = 0$, real analytic in $\mathbb{R}^2 \backslash (r_a \cup r_{-a})$, and a solution of $\mathcal{T}w = 0$ in the sense of distributions. Recall that $r_a$ is the characteristic curve $3(x-a) - 2(-y)^{3/2} = 0$ originating from $(a,0)$, and $r_{-a}$ the characteristic curve $3(x+a) + 2(-y)^{3/2} = 0$ originating from $(-a,0)$.

The power series

$$F(\alpha,\beta,\gamma;\zeta) = \sum_{j=0}^{\infty} \frac{(\alpha)_j(\beta)_j}{(\gamma)_j j!}\zeta^j, \tag{6.1}$$

where $\alpha$, $\beta$, and $\gamma$ are complex numbers, $\gamma \neq 0, -1, -2, \cdots$, and $(\alpha)_j$ is given by (3.8), is called the *hypergeometric series*. The ratio test guarantees absolute convergence for $|\zeta| < 1$, [10]. If $\mathrm{Re}(\gamma - \alpha - \beta) > 0$, then the series converges for $|\zeta| \le 1$ and

$$F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{6.2}$$

*Barnes' contour integral* defines a single-valued analytic function of $\zeta$ in the region $|\arg(-\zeta)| < \pi$, that is, $\mathbb{C}$ minus the positive real axis, which gives the *principal branch* of the analytic continuation of the hypergeometric series $F(\alpha,\beta,\gamma;\zeta)$. More precisely we quote the following theorem whose proof is found in [10].

**Theorem 6.1. (Barnes)** *The integral*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)}(-\zeta)^s\,ds, \tag{6.3}$$

*where the contour of integration is curved (if necessary) to ensure that the poles of* $\Gamma(\alpha+s)\Gamma(\beta+s)$, *i.e.,* $s=-\alpha-n,-\beta-n,n=0,1,2,\cdots$, *lie on the left of the contour and the poles of* $\Gamma(-s)$ *lie on the right of the contour, defines a single-valued analytic function in the region* $|\arg(-\zeta)|<\pi$. *Moreover, in the unit disk* $|\zeta|<1$, *it coincides with the hypergeometric series*

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma;\zeta).$$

Following traditional practice we use the notation $F(\alpha,\beta,\gamma;\zeta)$ to denote either the hypergeometric series or the principal branch of its analytic continuation, and call it the hypergeometric function.

Barnes' integral may also be used to obtain a representation of the hypergeometric function in the form of a power series in $\zeta^{-1}$, convergent when $|\zeta|>1$. By choosing a suitable contour of integration one can prove (see [10]) that if $\alpha-\beta$ is not an integer or zero, then

$$\begin{aligned} F(\alpha,\beta,\gamma;\zeta) &= A(-\zeta)^{-\alpha}F(\alpha,1-\gamma+\alpha,1-\beta+\alpha;\zeta^{-1}) \\ &+ B(-\zeta)^{-\beta}F(\beta,1-\gamma+\beta,1-\alpha+\beta;\zeta^{-1}), \end{aligned} \tag{6.4}$$

where $A$ and $B$ are suitable constants and $|\arg(-\zeta)|<\pi$. This formula also describes the asymptotic behavior of the function $F(\alpha,\beta,\gamma;\zeta)$ near $|\zeta|=\infty$. If $\alpha-\beta$ is an integer or zero, formula (6.4) must be modified because some of the poles of $\Gamma(\alpha+s)\Gamma(\beta+s)$ are double poles. The reader should find the expression for $F(\alpha,\beta,\gamma;\zeta)$ in [6], Chapter II. In the case that interests us, that is, $\alpha=\beta$, that expression is

$$F(\alpha,\alpha,\gamma;\zeta)=(-\zeta)^{-\alpha}[\log(-\zeta)U(\zeta)+V(\zeta)], \tag{6.5}$$

where $|\arg(-\zeta)|<\pi$, and both $U(\zeta)$ and $V(\zeta)$ are power series in $\zeta^{-1}$ convergent for $|\zeta|>1$ (see [6] or [4]). We also mention that if $\mathrm{Re}(\gamma-\alpha-\beta)>0$, we have convergence for $|\zeta|\geq 1$.

From the above results, and in particular from (6.5), it follows that

$$\tilde{E}(x,y;0,b)=\frac{(-v)^{-1/6}}{2^{1/3}}F(\frac{1}{6},\frac{1}{6},1;\frac{u}{v}),$$

with $u$ and $v$ defined by (4.6), is locally integrable in $\mathbb{R}^2$, singular when $v=0$, and real analytic in $\mathbb{R}^2\backslash(r_a\cup r_{-a})$.

It remains to prove that $\tilde{E}(x,y;0,b)$ is a solution to $\mathcal{T}w = 0$ in the sense of distributions. For this we need several results proved in the paper [3]. In that paper we showed that the function

$$\mathcal{E}(\ell,m;\ell_0,m_0) = (\ell-m)^{-1/6}(\ell_0-m)^{-1/6}F(\frac{1}{6},\frac{1}{6},1;\frac{(\ell-\ell_0)(m-m_0)}{(\ell-m_0)(m-\ell_0})$$

is a classical solution of

$$\mathcal{T}_h w = \frac{\partial^2 w}{\partial\ell\partial m} - \frac{1/6}{\ell-m}(\frac{\partial w}{\partial\ell} - \frac{\partial w}{\partial m}) = 0,$$

the reduced hyperbolic Tricomi equation. Here $\ell = x + \frac{2}{3}(-y)^{3/2}$, $m = x - \frac{2}{3}(-y)^{3/2}$ are the characteristic coordinates. Except for the constant $1/2^{1/3}$, $\tilde{E}(x,y;0,b)$ is obtained from $\mathcal{E}(\ell,m;\ell_0,m_0)$ after replacement of $\ell$ and $m$ by their expressions above and by setting $\ell_0 = -m_0 = 2(-b)^{3/2}/3$. Thus, away from the set $\{v=0\} = r_a \cup r_{-a}$, $\tilde{E}(x,y;0,b)$ is a classical solution of $\mathcal{T}w = 0$.

To show that $\mathcal{T}\tilde{E} = 0$, in the sense of distributions, we have to contend with the fact that $\tilde{E}(x,y;0,b)$ has logarithmic singularities along the two characteristics $r_{-a}$ and $r_a$, or, equivalently, that $\mathcal{E}(\ell,m;\ell_0,m_0)$ has logarithmic singularities along the lines $\ell = -\ell_0$ and $m = \ell_0$. Since, as we have remarked, $\mathcal{T}\tilde{E} = 0$ away from the characteristics $r_{-a}$ and $r_a$,, in order to prove that $\mathcal{T}\tilde{E} = 0$ in the sense of distributions, it suffices to prove that

$$\langle\tilde{E},\mathcal{T}\phi\rangle = \int\int_{\mathbb{R}^2} \tilde{E}\mathcal{T}\phi dx\,dy = 0 \tag{6.6}$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ whose support intersects at least one of the characteristics $r_{-a}$ or $r_a$. If $\operatorname{supp}\phi$ does not intersect either of these characteristics, then (6.6) is automatically satisfied.

Suppose that $\operatorname{supp}\phi$ is contained in an open disk $D$ centered, say at $(a,0)$, and with radius $R$. Let $0 < r < R$ and denote by $D_\epsilon$ the set of points of $D$ at a distance $> \epsilon$ from the characteristic $r_a$. Then, from Green's formula for $\mathcal{T}$ (see [4], formula (4.5)) one gets

$$\begin{aligned}\int\int_D \tilde{E}\mathcal{T}\phi\,dx\,dy &= \lim_{\epsilon\to 0}\int\int_{D_\epsilon} \tilde{E}\mathcal{T}\phi\,dx\,dy \qquad (6.7)\\ &= \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon\cup\gamma_\epsilon\cup\Gamma'_\epsilon} \tilde{E}(y\phi_x\,dy - \phi_y\,dx) - \phi(y\tilde{E}_x\,dy - \tilde{E}_y\,dx),\end{aligned}$$

where $\Gamma_\epsilon$ is the characteristic $3(x-\alpha+\epsilon)-2(-y)^{3/2} = 0$, $\gamma_\epsilon$ the circumference centered at $\alpha$ with radius $\epsilon$, and $\Gamma'_\epsilon$ the characteristic $3(x-\alpha-\epsilon)-2(-y)^{3/2} = 0$. In order to prove (6.6) we must prove that the last limit in (6.7) is zero.

Most details of the proof are to be found in Section 4 of the paper [4]. We just point out that the integrand in (6.7) remains bounded along $\gamma_\epsilon$ thus, along this contour, the integral tends to zero with $\epsilon$. Along both $\Gamma_\epsilon$ and $\Gamma'_\epsilon$ we must take into account the asymptotic behavior of $F(1/6, 1/6, 1; \zeta)$ and its derivative $F(7/6, 7/6, 2; \zeta)$, at $\zeta = \infty$, according to (6.5). It turns out that at the limit, the values of these integrals cancel each other and this completes the proof.

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