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CHAOTIC BEHAVIOR OF SOLUTIONS OF THE NAVIER-STOKES SYSTEM IN $\mathbb{R}^N$

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Abstract. In this paper, we study the relationship between the long time behavior of a solution $u(t,x)$ of the Navier-Stokes system with no external force in $\mathbb{R}^N$ and the asymptotic behavior as $|x| \to \infty$ of its initial value u_0. In particular, for initial values u_0 small in a certain sense, we show that if the sequence of dilations $\lambda_n u_0(\lambda_n \cdot)$ converges weakly to $z(\cdot)$ as $\lambda_n \to \infty$, then the rescaled solution $\sqrt{t}u(t, \cdot\sqrt{t})$ converges uniformly on $\mathbb{R}^N$ to $\boldsymbol{\mathcal{S}}(1)z$ along the subsequence $t_n = \lambda_n^2$, where $\boldsymbol{\mathcal{S}}(t)$ is the Navier-Stokes flow. If $N = 2$ or 3, we show there exists an initial value U_0 such that the set of all possible z attainable in this fashion is a closed ball B of an infinite-dimensional space. The resulting "universal" solution is therefore asymptotically close along appropriate subsequences to all solutions with initial values in B. Moreover, for a fixed $t_0 > 0$, $\boldsymbol{\mathcal{S}}(t_0)$ followed by an appropriate dilation generates a chaotic discrete dynamical system.

1. Introduction

In this paper, we consider the Navier-Stokes system in $\mathbb{R}^N$, $N \ge 2$

$$\begin{cases} u_t - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = 0, \\ \nabla \cdot u = 0, \qquad u(0) = u_0, \end{cases} \tag{1.1}$$

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which we study in the form

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds. \tag{1.2}$$

Here, $u = (u_1, \dots, u_N) : [0,\infty)\times\mathbb{R}^N \to \mathbb{R}^N$, $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ satisfies $\nabla\cdot u_0 = 0$ and $\mathbb{P} = I - \nabla(\Delta^{-1})\mathrm{div}$ is the projection onto the solenoidal (divergence-free) vector fields. We are primarily interested in the cases $N = 2$ and $N = 3$.

The purpose of this paper is to study the long time asymptotic behavior of global solutions of equation (1.2). There is, of course, a substantial literature on this topic dating back more than 20 years. See [1, 10, 11, 12, 14, 15, 16, 18] for a sample of recent results. The present paper is best understoood in the context of self-similar solutions. Recall that the set of solutions of equation (1.1) (or (1.2)) is invariant under the scaling

$$u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x), \tag{1.3}$$

for all $\lambda > 0$. A solution u is self-similar if and only if it is a fixed point of this group of transformations. It is easily seen that a solution u is self-similar if and only if it has the form $u(t,x) = t^{-\frac{1}{2}}f(x/\sqrt{t})$ where $f(\cdot) = u(1,\cdot)$ is called the profile of u. Furthermore, if a self-similar solution u has an initial value u_0 in the sense of $\mathcal{S}'(\mathbb{R}^N)$, then u_0 is homogeneous of degree -1 (independent of the dimension N). Giga and Miyakawa [13] and Cannone and Planchon [4] have constructed self-similar solutions of (1.2) in $\mathbb{R}^3$ by solving the Cauchy problem with homogeneous initial values. More recently, Planchon [18] proved that if the initial value $u_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is asymptotically homogeneous as $|x| \to \infty$; i.e., $u_0(x) \approx |x|^{-1}\zeta(x/|x|)$, then the resulting solution of (1.2) is asymptotic as $t \to \infty$ to the self-similar solution of (1.2) whose initial value is $|x|^{-1}\zeta(x/|x|)$.

In this paper, we study solutions of (1.2) whose initial values u_0 are bounded by a multiple of $|x|^{-1}$, but for which $|x|u_0(x)$ does not necessarily have a specified behavior as $|x| \to \infty$. This class includes, of course, homogeneous functions $u_0(x)$ of degree -1. In contrast to [18] we find solutions $u(t)$ which exhibit different asymptotic behaviors along different sequences $t_n \to \infty$. In fact, the set of possible time-asymptotic limits of $u(t_n)$ as $t_n \to \infty$ is directly related to the set of space-asymptotic limits of $\lambda_n u_0(\lambda_n x)$ as $\lambda_n \to \infty$.

In order to describe our results more precisely, we need to establish notation and introduce various spaces of functions. We denote by $\boldsymbol{L}^p$ or $\boldsymbol{L}^p(\mathbb{R}^N)$ the space $(L^p(\mathbb{R}^N))^N$, for $1 \le p \le \infty$. Similarly, we denote by $\boldsymbol{C}_0(\mathbb{R}^N)$ the space $(C_0(\mathbb{R}^N))^N$. We consider the Banach space

$$\boldsymbol{\mathcal{W}} = \{u \in (L^1_{\mathrm{loc}}(\mathbb{R}^N))^N;\ |\cdot|u(\cdot) \in \boldsymbol{L}^\infty(\mathbb{R}^N)\}, \tag{1.4}$$

endowed with the norm $\|u\|_{\boldsymbol{\mathcal{W}}} = \|\,|\cdot|u(\cdot)\|_{\boldsymbol{L}^\infty}$. In other words, $\boldsymbol{\mathcal{W}} = (\mathcal{W})^N$, where $\mathcal{W}$ is the Banach space

$$\mathcal{W} = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N);\ |\cdot|u(\cdot) \in L^\infty(\mathbb{R}^N)\}, \tag{1.5}$$

endowed with the norm $\|u\|_{\mathcal{W}} = \|\,|\cdot|u(\cdot)\|_{L^\infty}$. Given $\delta > 0$, we let

$$\boldsymbol{\mathcal{B}}_\delta = \{u \in \boldsymbol{\mathcal{W}};\ \|u\|_{\boldsymbol{\mathcal{W}}} \le \delta\}, \tag{1.6}$$

and we set

$$\boldsymbol{\mathcal{C}}_\delta = \{u \in \boldsymbol{\mathcal{B}}_\delta;\ \nabla \cdot u = 0\}, \tag{1.7}$$

which is a closed subset of $\boldsymbol{\mathcal{B}}_\delta$. Finally, we consider the set

$$E_K = \{u \in (L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^N))^N;\ \sup_{t>0, x\in\mathbb{R}^N} (t + |x|^2)^{\frac{1}{2}} |u(t,x)| \le K\}, \tag{1.8}$$

for $K > 0$.

The following theorem describes the class of solutions of (1.2) studied in this paper. While its precise formulation seems to be new, it is very close in spirit to many of the known existence theorems (see [3, 4, 14]).

Theorem 1.1. *There exist $M \ge 1$ and $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then for every $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, there exists a unique solution $u \in E_{M\delta}$ of (1.2).*

Remark 1.2. Let $\delta, K > 0$, let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, and $u \in E_K$.

(i) Since $|u(t,x)| \le K(t + |x|^2)^{-\frac{1}{2}}$, we see that, given any $\gamma, T > 0$,

$$\begin{aligned}\sup_{x_0\in\mathbb{R}^N} \int_{\{|x-x_0|<\gamma\}} \int_0^T |u(t,x)|^2 dtdx &\le K \int_{\{|x|<\gamma\}} \int_0^T \frac{dtdx}{t + |x|^2} \\ &\le K \int_{\{|x|<\gamma\}} \log\Big(\frac{T + |x|^2}{|x|^2}\Big) dx < \infty.\end{aligned}$$

In particular, u is *uniformly locally square integrable* as defined in Definition 1.2 page 5 in [14]. Therefore, it follows from Theorem 11.2 page 112 in [14] that u satisfies (1.1) if and only if it satisfies (1.2).

(ii) If u satisfies (1.1) (or, equivalently, (1.2)), then it follows from Proposition 15.1 page 147 in [14] that $u \in C^\infty((0,\infty) \times \mathbb{R}^N)^N$. In particular, $u \in C((0,\infty), \boldsymbol{C}_0(\mathbb{R}^N))$.

(iii) If u satisfies (1.1) (or, equivalently, (1.2)), then it follows from (2.4) and (2.13) below that $u(t) - e^{t\Delta}u_0 \to 0$ in $\boldsymbol{L}^p(\mathbb{R}^N)$ as $t \downarrow 0$, for all $N/2 < p < N$. Note that $e^{t\Delta}u_0 \to u_0$ as $t \downarrow 0$ in $\boldsymbol{L}^p(\{|x| > 1\})$ for all

$N < p < \infty$ and in $\boldsymbol{L}^p(\{|x| < 1\})$ for all $1 \le p < N$. In particular, $u(t) \to u_0$ in $\boldsymbol{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \downarrow 0$.

Definition 1.3. *Let $M \ge 1$ and $\delta_0 > 0$ be as in Theorem* 1.1*. Given $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ for some $0 < \delta < \delta_0$, we define $\boldsymbol{\mathcal{S}}(t)u_0 = u(t)$ for $t \ge 0$, where $u \in E_{M\delta}$ is the unique solution of* (1.2) *given by Theorem* 1.1*.*

Remark 1.4. Note that, given any $0 < \delta < \delta_0$ and $t > 0$, $\boldsymbol{\mathcal{S}}(t) : \boldsymbol{\mathcal{C}}_\delta \to \boldsymbol{\mathcal{C}}_{M\delta} \cap C^\infty(\mathbb{R}^N)$.

The formulation of our next result requires the weak* topology of $\boldsymbol{\mathcal{W}}$. More precisely, since the Banach space $\boldsymbol{\mathcal{W}}$ is isometrically isomorphic to $\boldsymbol{L}^\infty(\mathbb{R}^N)$, it is the dual of a Banach space isometrically isomorphic to $\boldsymbol{L}^1(\mathbb{R}^N)$. It follows that for any $\delta > 0$, the set $\boldsymbol{\mathcal{B}}_\delta$ endowed with the weak* topology of $\boldsymbol{\mathcal{W}}$ is compact. Since $\boldsymbol{L}^1(\mathbb{R}^N)$ is separable, the weak* topology on $\boldsymbol{\mathcal{B}}_\delta$ is metrizable. We denote by d^*_δ a corresponding metric, so that $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$ is a compact metric space (hence complete and separable), for all $\delta > 0$. By Proposition 2.1 in [6], convergence of a sequence in $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$ is equivalent to convergence in $\mathcal{D}'(\mathbb{R}^N \setminus \{0\})$ and to convergence in $\mathcal{D}'(\mathbb{R}^N)$. It follows in particular that $\boldsymbol{\mathcal{C}}_\delta$ is closed, therefore compact, in $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$.

We consider the dilations D_λ defined for all $\lambda > 0$ by

$$D_\lambda u(x) = \lambda u(\lambda x). \tag{1.9}$$

To say that the $u_0(x)$ is asymptotically homogeneous as $|x| \to \infty$ means that $D_\lambda u_0 \to z$ as $\lambda \to \infty$ for some homogeneous $z \in \boldsymbol{\mathcal{W}}$. We generalize this notion with the following definition.

Definition 1.5. For $\delta > 0$ and $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, we set

$$\Omega(u_0) = \{z \in \boldsymbol{\mathcal{C}}_\delta;\ \exists \lambda_n \to \infty, \mathrm{d}^*_\delta(D_{\lambda_n} u_0, z) \to 0\} = \bigcap_{\mu > 0} \overline{\bigcup_{\lambda > \mu} \{D_\lambda u_0\}}, \tag{1.10}$$

where the closure is in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$. In particular, $\Omega(u_0)$ is a nonempty, compact, connected subset of $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$. (Note that $z \in \Omega(u_0)$ is not necessarily homogeneous.)

To say that the solution $u(t,x)$ is asymptotic as $t \to \infty$ to the self-similar solution with profile f means that $D_{\sqrt{t}} u(t) \to f$. This motivates the following definition.

Definition 1.6. Let $\delta, K > 0$, let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and let $u \in E_K$ be a solution of (1.2). We set

$$\begin{aligned} \omega(u) &= \{f \in \boldsymbol{C}_0(\mathbb{R}^N);\ \exists t_n \to \infty, \|D_{\sqrt{t_n}} u(t_n) - f\|_{\boldsymbol{L}^\infty} \to 0\} \\ &= \bigcap_{T > 0} \overline{\bigcup_{t > T} \{D_{\sqrt{t}} u(t)\}} \end{aligned} \tag{1.11}$$

where the closure is in $\boldsymbol{C}_0(\mathbb{R}^N)$.

The following theorem is our first result on the asymptotic behavior of solutions.

Theorem 1.7. *Let* $0 < \delta < \delta_0$ *with* δ_0 *as in Theorem* 1.1. *Given any* $u_0 \in \boldsymbol{\mathcal{C}}_\delta$*, let* u *be the corresponding solution of* (1.2) *given by Theorem* 1.1. *It follows that* $\omega(u) = \boldsymbol{\mathcal{S}}(1)\Omega(u_0)$*, and that* $\omega(u)$ *is a nonempty, compact, connected subset of* $\boldsymbol{C}_0(\mathbb{R}^N)$.

Remark 1.8. Let $0 < \delta < \delta_0$ with δ_0 as in Theorem 1.1. Let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, and let $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$ be the corresponding solution of (1.2) given by Theorem 1.1. If $z \in \Omega(u_0)$ so that $\boldsymbol{\mathcal{S}}(1)z \in \omega(u)$, then by a rescaled version of formula (1.11), using the relation (3.1), there exists a sequence $t_n \to \infty$ such that

$$\|\boldsymbol{\mathcal{S}}(t_n)u_0 - \boldsymbol{\mathcal{S}}(t_n)D_{\frac{1}{\sqrt{t_n}}}z\|_{\boldsymbol{L}^\infty} = \|\boldsymbol{\mathcal{S}}(t_n)u_0 - D_{\frac{1}{\sqrt{t_n}}}\boldsymbol{\mathcal{S}}(1)z\|_{\boldsymbol{L}^\infty} = o(t_n^{-\frac{1}{2}}), \quad (1.12)$$

as $n \to \infty$. In particular, if z is homogeneous of degree -1, then

$$\|\boldsymbol{\mathcal{S}}(t_n)u_0 - \boldsymbol{\mathcal{S}}(t_n)z\|_{\boldsymbol{L}^\infty} = o(t_n^{-\frac{1}{2}}) \quad (1.13)$$

as $n \to \infty$. In other words, $u(t)$ is asymptotic along the sequence t_n to the self-similar solution $\boldsymbol{\mathcal{S}}(t)z$.

Remark 1.9. Intuitively, the set $\omega(u)$ describes all possible asymptotic forms of the solution u which are visible under the scaling (1.9) (as expressed through (1.11)). Of course, there exist solutions of (1.2) whose asymptotic behavior under this scaling is trivial, i.e., $\omega(u) = \Omega(u_0) = \{0\}$. This is the case, for example, with the solutions described in Brandolese [1], Gallay and Wayne [11, 12], and Miyakawa [15], where initial values which decay faster than $|x|^{-1}$ are considered. In this situation, a finer description of the asymptotic behavior can be studied by using a different scaling, better adaptated to the faster decay of those solutions. See Gallay and Wayne [11, 12]. Note that none of those solutions are self-similar solutions of (1.2) with respect to the scaling (1.3) (but they are all asymptotically self-similar to the trivial solution 0). On the other hand, their faster decay tends to make the nonlinear term in (1.2) negligible, so they may be asymptotic to a self-similar solution of the heat equation (with respect to a different scaling than (1.3)), as follows from [11, 12]. In the present paper, we are only concerned with properties of solutions under the scaling (1.3).

The larger the set $\omega(u)$, the more complex the asymptotic behavior of u. By Theorem 1.7, the size of $\omega(u)$ is determined by the size of $\Omega(u_0)$. At one extreme, if $\Omega(u_0)$ and therefore $\omega(u)$ are singletons, then the element

$z \in \Omega(u_0)$ is homogeneous and the solution u is asymptotic as $t \to \infty$ to the self-similar solution with profile $f = \boldsymbol{\mathcal{S}}(1)z$ (see Propositions 4.3 and 5.4). At the other extreme, we would like to determine the largest possible set $\Omega(u_0)$. For example, is it possible to have $\Omega(u_0) = \boldsymbol{\mathcal{C}}_\delta$? Without the divergence condition (i.e., using (1.10) with $\boldsymbol{\mathcal{B}}_\delta$ instead of $\boldsymbol{\mathcal{C}}_\delta$), the answer is known to be yes (see Theorem 1.2 in [6]). In the present context, we can produce a "large" set $\Omega(u_0)$ by imposing an additional constraint, described by the following definition. For every $\delta > 0$, let

$$\widetilde{\boldsymbol{\mathcal{C}}}_\delta = \{u \in \boldsymbol{\mathcal{C}}_\delta;\, x \cdot u(x) = 0 \text{ a.e.}\}, \tag{1.14}$$

where $\boldsymbol{\mathcal{C}}_\delta$ is defined by (1.7). In particular, $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is a closed subset of $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$.

Theorem 1.10. *Given $\delta > 0$, there exists $U_0 \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ such that $\Omega(U_0) = \widetilde{\boldsymbol{\mathcal{C}}}_\delta$. If, in addition, $\delta < \delta_0$ with δ_0 as in Theorem* 1.1*, then $\omega(U) = \boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ where $U(t) \equiv \boldsymbol{\mathcal{S}}(t)U_0$.*

As a consequence, Theorem 1.10 implies that the solution $\boldsymbol{\mathcal{S}}(\cdot)U_0$ is asymptotic along some sequence $t_n \to \infty$ to any given self-similar solution with initial value in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$. See Corollary 3.6. It is not too hard to see that if $N = 3$, the set of homogeneous functions (i.e., initial values of self-similar solutions) in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ contains a ball in an infinite dimensional Banach space (see Proposition 3.7). Thus the long time asymptotic behavior of the solution of equation (1.2) constructed in Theorem 1.10 cannot be reduced to the study of a finite-dimensional dynamical system (see Corollary 3.8).

If $N = 2$, Theorem 1.10 is not sufficient to construct a solution with infinite-dimensional asymptotic behavior. Nonetheless, we are able to construct such a solution by a different method. Since the details are rather technical, we include them in Appendix B. See Theorem 8.1.

Another way to study the asymptotic behavior of solutions of (1.2) is via the discrete dynamical system generated by the flow at a fixed time. To prevent the iterated maps from converging trivially to 0, we need to compensate with a dilation. Let $0 < \delta < \delta_0$ with δ_0 as in Theorem 1.1. For every $\lambda > 1$, we define the mapping $F_\lambda : \boldsymbol{\mathcal{C}}_\delta \to \boldsymbol{\mathcal{C}}_{M\delta} \cap \boldsymbol{C}_0(\mathbb{R}^N)$ by

$$F_\lambda = D_\lambda \boldsymbol{\mathcal{S}}(\lambda^2 - 1). \tag{1.15}$$

The mapping F_λ is essentially the same as what Bricmont, Kupiainen, and Lin [2] call the "renormalization group map". It follows from the scaling invariance (3.2) that

$$F_\lambda \boldsymbol{\mathcal{S}}(1) = D_\lambda \boldsymbol{\mathcal{S}}(\lambda^2) = \boldsymbol{\mathcal{S}}(1) D_\lambda. \tag{1.16}$$

In other words, F_λ is the image of D_λ under the mapping $\boldsymbol{\mathcal{S}}(1)$. It turns out that F_λ generates a chaotic dynamical system.

Proposition 1.11. *For any fixed $\lambda > 1$ and $\delta > 0$, the mapping D_λ generates a chaotic, discrete dynamical system on $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}^*_\delta)$. Furthermore, if $\delta < \delta_0$ with δ_0 as in Theorem 1.1, then F_λ defined by (1.15) leaves invariant the compact subset of $\boldsymbol{C}_0(\mathbb{R}^N)$ given by $\boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$. Moreover, F_λ generates a chaotic, discrete dynamical system on $\boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$.*

In other words (Devaney [9]), periodic points of F_λ are dense in $\boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$, F_λ is topologically transitive and F_λ has sensitive dependence on initial conditions.

The major disadvantage of imposing the condition $x \cdot u_0 = 0$ in Theorem 1.10 and Proposition 1.11 is that it seems artificially restrictive. One way to avoid this problem in dimension $N = 3$ is to use the vorticity to construct initial values via the Biot-Savart law

$$u(x) = \boldsymbol{\mathcal{T}}\nu(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \nu(y)\,dy, \tag{1.17}$$

as done in Miyakawa [15] and Gallay and Wayne [12]. We thank Professor Ben-Artzi for bringing to our attention the relevance of the vorticity in studying certain aspects of the Navier-Stokes system. In the present context, use of the vorticity operator requires the introduction of a space with scaling properties different from $\boldsymbol{\mathcal{W}}$.

In general, given $0 \le \sigma < N$, we define the Banach space

$$\boldsymbol{\mathcal{V}}_\sigma = \{u \in (L^1_{\mathrm{loc}}(\mathbb{R}^N))^N;\ |\cdot|^\sigma u(\cdot) \in \boldsymbol{L}^\infty(\mathbb{R}^N)\}, \tag{1.18}$$

endowed with the norm $\|u\|_{\boldsymbol{\mathcal{V}}_\sigma} = \|\,|\cdot|^\sigma u(\cdot)\|_{\boldsymbol{L}^\infty}$. In particular, $\boldsymbol{\mathcal{V}}_\sigma = (\mathcal{V}_\sigma)^N$, where $\mathcal{V}_\sigma$ is defined in an obvious way. In the specific case $N = 3$, we set

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_2, \tag{1.19}$$

and, given $\delta > 0$, we set

$$\boldsymbol{\mathcal{A}}_\delta = \{u \in \boldsymbol{\mathcal{V}};\ \|u\|_{\boldsymbol{\mathcal{V}}} \le \delta\}. \tag{1.20}$$

It is clear (see Lemma 3.10) that $\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{W}}$ is a bounded operator.

The following result is analogous to Theorem 1.10 without imposing the orthogonality condition of (1.14).

Theorem 1.12. *Assume $N = 3$. Let $\delta > 0$ and set $\delta' = \delta/\|\boldsymbol{\mathcal{T}}\|_{\mathcal{L}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{W}})}$. There exists $U_0 \in \boldsymbol{\mathcal{C}}_\delta$ such that $\Omega(U_0) = \boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{A}}_{\delta'}$. If, in addition, $\delta < \delta_0$ with δ_0 as in Theorem 1.1, then $\omega(U) = \boldsymbol{\mathcal{S}}(1)\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{A}}_{\delta'}$ where $U(t) \equiv \boldsymbol{\mathcal{S}}(t)U_0$.*

Theorem 1.12 is subsumed by Proposition 3.11 in Section 3 below. Also, we refer the reader to Corollary 3.12 and Remark 3.13 for a reformulation of this result explicitly in terms of the asymptotic behavior of the solution $\boldsymbol{\mathcal{S}}(\cdot)U_0$. In particular, the long time asymptotic behavior of the solution of equation (1.2) constructed in Theorem 1.12 cannot be reduced to the study of a finite dimensional dynamical system. In addition, using the vorticity, we have the following result concerning the chaotic behavior of the mapping F_λ.

Proposition 1.13. *Assume $N=3$. Let $\delta>0$ such that $\delta\|\boldsymbol{\mathcal{T}}\|_{\mathcal{L}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{W}})}<\delta_0$ with δ_0 as in Theorem 1.1. For any fixed $\lambda>1$ the mapping F_λ defined by (1.15) leaves invariant the compact subset of $\boldsymbol{C}_0(\mathbb{R}^N)$ given by $\boldsymbol{\mathcal{S}}(1)\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{A}}_\delta$. Moreover, F_λ generates a chaotic, discrete dynamical system on $\boldsymbol{\mathcal{S}}(1)\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{A}}_\delta$.*

In the results described so far, we have used the norm of $\boldsymbol{C}_0(\mathbb{R}^N)$ to determine the asymptotic limits of the rescaled solutions of (1.2). Another natural notion of convergence is the stronger norm of the Banach space $\boldsymbol{\mathcal{W}}\cap\boldsymbol{C}_0(\mathbb{R}^N)$. Indeed, if $u\in E_{M\delta}$ is a solution constructed by Theorem 1.1, then

$$\|D_{\sqrt{t}}u(t)\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty}\le M\delta,$$

where

$$\|u\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty}:=\|(1+|\cdot|^2)^{\frac{1}{2}}u\|_{\boldsymbol{L}^\infty}.$$

This motivates the following definition.

Definition 1.14. Let $\delta,K>0$, let $u_0\in\boldsymbol{\mathcal{C}}_\delta$, and let $u\in E_K$ be a solution of (1.2). We set

$$\begin{aligned}\omega_1(u)=\{f\in\boldsymbol{\mathcal{W}}\cap\boldsymbol{C}_0(\mathbb{R}^N);\,\exists t_n\to\infty,\|D_{\sqrt{t_n}}u(t_n)-f\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty}\to0\}\\ =\bigcap_{T>0}\overline{\bigcup_{t>T}\{D_{\sqrt{t}}u(t)\}},\end{aligned}\tag{1.21}$$

where the closure is in $\boldsymbol{\mathcal{W}}\cap\boldsymbol{C}_0(\mathbb{R}^N)$.

To obtain asymptotic limits of $D_{\sqrt{t}}u(t)$ in $\boldsymbol{\mathcal{W}}\cap\boldsymbol{C}_0(\mathbb{R}^N)$, one needs a stronger notion of asymptotic limit of the initial values $D_\lambda u_0$. Thus, given $u_0\in\boldsymbol{\mathcal{W}}$, we set

$$|||u_0|||_{\boldsymbol{\mathcal{W}}}=\sum_{n\ge1}2^{-n}\operatorname*{ess\,sup}_{|x|>\frac{1}{n}}|x||u_0(x)|\le\|u_0\|_{\boldsymbol{\mathcal{W}}}.\tag{1.22}$$

This norm is not complete on $\boldsymbol{\mathcal{W}}$ (since it is not equivalent to the norm $\|\cdot\|_{\boldsymbol{\mathcal{W}}}$). On the other hand, if d_M is the metric on $\boldsymbol{\mathcal{B}}_M$ induced by $|||\cdot|||_{\boldsymbol{\mathcal{W}}}$, then $(\boldsymbol{\mathcal{B}}_M,\mathrm{d}_M)$ is a complete metric space, which is not compact. By

Proposition 2.2 in [6], convergence of a sequence $(u_n)_{n\geq 0}$ in $(\boldsymbol{\mathcal{B}}_M, \mathrm{d}_M)$ is equivalent to convergence of $(|\cdot|u_n)_{n\geq 0}$ uniformly on the complement of every neighborhood of the origin in $\mathbb{R}^N$. This topology is clearly stronger than the weak* topology and weaker than the norm topology on $\boldsymbol{\mathcal{B}}_M$.

Definition 1.15. Given $u_0 \in \boldsymbol{\mathcal{W}}$, we consider $M \geq \|u_0\|_{\boldsymbol{\mathcal{W}}}$ and we set

$$\begin{aligned} \Omega_1(u_0) &= \{z \in \boldsymbol{\mathcal{B}}_M;\ \exists \lambda_n \to \infty \text{ such that } \mathrm{d}_M(D_{\lambda_n}u_0, z) \to 0\} \\ &= \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{D_\lambda u_0\}}, \end{aligned} \tag{1.23}$$

where the closure is in $(\boldsymbol{\mathcal{B}}_M, \mathrm{d}_M)$.

Remark 1.16. Note that the sets $\omega_1(u)$ defined by (1.21) and $\Omega_1(u_0)$ defined by (1.23) may be empty.

The relationship between $\Omega_1(u_0)$ and $\omega_1(u_0)$ given in the next theorem is not as simple as the result given in Theorem 1.7.

Theorem 1.17. *Let $\delta > 0$ be sufficiently small. Let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition* 1.3*, let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, set $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$, and let $\omega_1(u)$ be defined by* (1.21) *and $\Omega_1(u_0)$ be defined by* (1.23)*. If $\Omega_1(u_0) \neq \emptyset$, then $\omega_1(u) = \overline{\boldsymbol{\mathcal{S}}(1)\Omega_1(u_0)}$, where the closure is in $\boldsymbol{\mathcal{W}} \cap C_0(\mathbb{R}^N)$.*

See Theorem 5.6 below for a more precise description of the smallness condition on δ. Theorem 1.17 is not completely satisfactory in that we do not know if in fact $\omega_1(u) = \boldsymbol{\mathcal{S}}(1)\Omega_1(u_0)$. On the other hand, the hypothesis $\Omega_1(u_0) \neq \emptyset$ is necessary, since there exists u_0 for which $\Omega_1(u_0) = \emptyset$ and $\omega_1(u) \neq \emptyset$ (see Remark refeFurp). Furthermore, we know that results analogous to Theorems 1.10 and 1.12 but for $\Omega_1(u_0)$ and $\omega_1(u)$ cannot be true. Indeed, by Proposition 5.4, $\omega_1(u)$ can contain at most one profile of a self-similar solution.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 and some additional continuity properties of the solutions. Section 3 contains the main results concerning $\omega(u)$ and the chaotic behavior of F_λ. Section 4 concerns asymptotically self-similar solutions. The results concerning $\omega_1(u)$ are contained in Sections 5 and 6.

Results analogous to those in the present paper for the linear and nonlinear heat equations are given in [6] and [7]. Schrödinger equations are treated in a forthcoming paper [8]. See Vázquez and Zuazua [19] for similar results in a more abstract setting which includes the porous medium and p-Laplacian equations.

The authors thank the referee for several pertinent remarks.

2. Existence and continuity properties of solutions

The purpose of this section is to prove Theorem 1.1. Lemma 2.2 and Remark 2.3 give the estimates needed for the contraction mapping argument. We also prove the continuity of the fixed-time flow map $\boldsymbol{\mathcal{S}}(t)$ with respect to the weak* topology.

Remark 2.1. Consider the dilations D_λ defined by (1.9). We note that $\mathbb{P}$ is a pseudo-differential operator of order zero, so that $\mathbb{P}D_\lambda = D_\lambda\mathbb{P}$. Next, it is well-known that $e^{t\Delta}D_\lambda = D_\lambda e^{\lambda^2 t\Delta}$, and in particular $e^{\Delta}D_\lambda = D_\lambda e^{\lambda^2\Delta}$. Furthermore, it is trivial to verify that $D_\lambda\nabla = \lambda^{-1}\nabla D_\lambda$ and $D_\lambda(u\otimes v) = \lambda^{-1}(D_\lambda u\otimes D_\lambda v)$.

Lemma 2.2. *Given $0\le\sigma<N$, there exists a constant C such that*

$$\|e^{t\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\|_{\boldsymbol{\mathcal{W}}} \le Ct^{-\frac12}\|u\|_{\boldsymbol{\mathcal{W}}}\|v\|_{\boldsymbol{L}^\infty}, \tag{2.1}$$

$$\|e^{t\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\|_{\boldsymbol{L}^\infty} \le Ct^{-\frac34}\|u\|_{\boldsymbol{\mathcal{W}}}^{\frac12}\|u\|_{\boldsymbol{L}^\infty}^{\frac12}\|v\|_{\boldsymbol{L}^\infty}, \tag{2.2}$$

$$\|e^{t\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\|_{\boldsymbol{\mathcal{V}}_\sigma} \le Ct^{-\frac12}\|u\|_{\boldsymbol{\mathcal{V}}_{\frac{\sigma}{2}}}\|v\|_{\boldsymbol{\mathcal{V}}_{\frac{\sigma}{2}}}, \tag{2.3}$$

for all $t>0$, where $\boldsymbol{\mathcal{V}}_\sigma$ is defined by (1.18)*. Moreover, given any $1\le p\le\infty$, there exists a constant C such that*

$$\|e^{t\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\|_{\boldsymbol{L}^p} \le Ct^{-\frac12}\|u\|_{\boldsymbol{L}^{2p}}\|v\|_{\boldsymbol{L}^{2p}}, \tag{2.4}$$

for all $t>0$.

Proof. Observe that (2.1)–(2.3) follow once we show the stronger inequality

$$|e^{t\Delta}\mathbb{P}\nabla\cdot(u\otimes v)|(x) \le Ct^{-\frac12}(t+|x|^2)^{-\frac{\sigma}{2}}\Big(\sup_{y\in\mathbb{R}^N}|y|^\sigma|u(y)||v(y)|\Big), \tag{2.5}$$

where $0\le\sigma<N$ (take $\sigma=1/2$ to prove (2.2)). We first prove (2.4) and (2.5) for $t=1$; i.e.,

$$|e^{\Delta}\mathbb{P}\nabla\cdot(u\otimes v)|(x) \le C(1+|x|^2)^{-\frac{\sigma}{2}}\Big(\sup_{y\in\mathbb{R}^N}|y|^\sigma|u(y)||v(y)|\Big), \tag{2.6}$$

$$\|e^{\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\|_{\boldsymbol{L}^p} \le C\|u\|_{\boldsymbol{L}^{2p}}\|v\|_{\boldsymbol{L}^{2p}}. \tag{2.7}$$

Since the operator $e^{\Delta}\mathbb{P}\nabla$ is a vector-valued convolution with functions $G_{i,j,k}$ verifying $|G_{i,j,k}(x)|\le C(1+|x|)^{-N-1}$ (see for example Proposition 11.1 page 107 in [14] or Lemma 2.1 in [15]), this amounts to showing the estimates

$$|G\star(fg)|(x) \le C(1+|x|^2)^{-\frac{\sigma}{2}}\Big(\sup_{x\in\mathbb{R}^N}|x|^\sigma|f(x)||g(x)|\Big), \tag{2.8}$$

$$\|G\star(fg)\|_{L^p} \le C\|f\|_{L^{2p}}\|g\|_{L^{2p}}, \tag{2.9}$$

where $|G(x)| \le C(1+|x|)^{-N-1}$. We observe that (2.8)–(2.9) are in turn implied by the inequalities

$$|G \star \varphi|(x) \le C(1+|x|^2)^{-\frac{\sigma}{2}} \Big(\sup_{x\in\mathbb{R}^N} |x|^\sigma |\varphi(x)| \Big), \tag{2.10}$$

$$\|G \star \varphi\|_{L^p} \le C \|\varphi\|_{L^p}. \tag{2.11}$$

Formula (2.11) is a consequence of Young's inequality, since $G \in L^1(\mathbb{R}^N)$, and (2.10) follows from the elementary inequality

$$\int_{\mathbb{R}^N} (1+|y|)^{-\gamma} |x-y|^{-\sigma}\, dy \le C(1+|x|^2)^{-\frac{\sigma}{2}},$$

valid for $0 \le \sigma < N < \gamma$ (see Lemma 7.1). This proves (2.6) and (2.7).

We easily deduce from Remark 2.1 that

$$D_\lambda e^{\lambda^2\Delta} \mathbb{P}\nabla \cdot (u \otimes v) = \lambda^{-2} e^{\Delta} \mathbb{P}\nabla \cdot (D_\lambda u \otimes D_\lambda v),$$

and so, letting $\lambda = \sqrt{t}$, we obtain

$$e^{t\Delta} \mathbb{P}\nabla \cdot (u \otimes v) = t^{-1} D_{\frac{1}{\sqrt{t}}} e^{\Delta} \mathbb{P}\nabla \cdot (D_{\sqrt{t}} u \otimes D_{\sqrt{t}} v).$$

Formulas (2.5) and (2.4) follow immediately from the above identity by applying (2.6) and (2.7) with u and v replaced by $D_{\sqrt{t}} u$ and $D_{\sqrt{t}} v$. □

Remark 2.3. We observe that if $u \in E_K$, then with the notation (1.18), if $0 \le \theta \le 1$,

$$\|u(t)\|_{\boldsymbol{\mathcal{V}}_\theta} \le K t^{-\frac{1-\theta}{2}}, \tag{2.12}$$

for all $t > 0$ and all $u \in E_K$. Moreover, if $p > N$, then there exists a constant C such that

$$\|u(t)\|_{\boldsymbol{L}^p} \le C K t^{-\frac{1}{2}\left(1-\frac{N}{p}\right)}, \tag{2.13}$$

for all $t > 0$ and all $u \in E_K$. We note also that

$$\begin{aligned} \frac{1}{2} \sup_{t>0} \{ \|u(t)\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}} \|u(t)\|_{\boldsymbol{L}^\infty} \} &\le \sup_{\substack{t>0\\ x\in\mathbb{R}^N}} (t+|x|^2)^{\frac{1}{2}} |u(t,x)| \\ &\le \sup_{t>0} \{ \|u(t)\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}} \|u(t)\|_{\boldsymbol{L}^\infty} \}, \end{aligned} \tag{2.14}$$

for all $u \in (L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N))^N$.

Proof of Theorem 1.1. We let $K > 0$ to be chosen later, and we equip the set E_K with the distance

$$\mathrm{d}(u,v) = \sup_{s>0} \|u(s) - v(s)\|_{\boldsymbol{\mathcal{W}}}, \tag{2.15}$$

so that (E_K, d) is a complete metric space. For $u, v \in E_K$, we set

$$\Phi(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)\,ds, \tag{2.16}$$

and

$$\Psi(u)(t) = e^{t\Delta}u_0 - \Phi(u,u)(t). \tag{2.17}$$

Given $u \in E_K$, it follows from (2.2) and (2.14) that

$$\begin{aligned}\|\Phi(u,u)(t)\|_{\boldsymbol{L}^\infty} &\le C\int_0^t (t-s)^{-\frac{3}{4}}\|u(s)\|_{\boldsymbol{\mathcal{W}}}^{\frac{1}{2}}\|u(s)\|_{\boldsymbol{L}^\infty}^{\frac{3}{2}} \\ &\le CK^2\int_0^t (t-s)^{-\frac{3}{4}}s^{-\frac{3}{4}} \le CK^2 t^{-\frac{1}{2}}.\end{aligned}$$

Next, it follows from (2.1) and (2.14) that

$$\begin{aligned}\|\Phi(u,u)(t)\|_{\boldsymbol{\mathcal{W}}} &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{\boldsymbol{\mathcal{W}}}\|u(s)\|_{\boldsymbol{L}^\infty} \\ &\le CK^2\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}} \le CK^2.\end{aligned}$$

Therefore,

$$\sup_{t>0}\{\|\Phi(u,u)(t)\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}}\|\Phi(u,u)(t)\|_{\boldsymbol{L}^\infty}\} \le C_1K^2, \tag{2.18}$$

for some constant $C_1 > 0$ independent of K and $u \in E_K$. We next observe that, by Corollary 8.3 in [5], there exists a constant M such that

$$\sup_{t>0}\{\|e^{t\Delta}u_0\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}}\|e^{t\Delta}u_0\|_{\boldsymbol{L}^\infty}\} \le \frac{M}{2}\|u_0\|_{\boldsymbol{\mathcal{W}}}. \tag{2.19}$$

We deduce from (2.18), (2.19), and (2.14) that

$$\sup_{\substack{t>0\\ x\in\mathbb{R}^N}} (t+|x|^2)^{\frac{1}{2}}|\Psi(u)(t,x)| \le C_1K^2 + \frac{M}{2}\|u_0\|_{\boldsymbol{\mathcal{W}}}. \tag{2.20}$$

Next, using again (2.1) and (2.14), one shows that there exists a constant C_2 such that

$$\begin{aligned}\sup_{s>0}\|\Psi(u)(s) - \Psi(v)(s)\|_{\boldsymbol{\mathcal{W}}} &= \sup_{s>0}\|\Phi(u,u)(s) - \Phi(v,v)(s)\|_{\boldsymbol{\mathcal{W}}} \\ &\le C_2K\sup_{s>0}\|u(s)-v(s)\|_{\boldsymbol{\mathcal{W}}},\end{aligned} \tag{2.21}$$

for all $u, v \in E_K$. We now let

$$\delta_0 = \min\Big\{\frac{1}{2C_1M}, \frac{1}{C_2M}\Big\}.$$

It follows from (2.20) and (2.21) that if $\|u_0\|_{\boldsymbol{\mathcal{W}}} \le \delta$ for some $0 < \delta < \delta_0$, then Ψ is a strict contraction of $E_{M\delta}$, and the result follows from Banach's fixed point theorem. □

Remark 2.4. Here are some comments on Theorem 1.1 and its proof.

(i) It follows easily from the proof of Theorem 1.1 (and in particular (2.19) and (2.21)) that if $0 < \delta < \delta_0$, then there exists C such that

$$\sup_{t>0} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{\mathcal{W}}} \le C\|u_0 - v_0\|_{\boldsymbol{\mathcal{W}}}, \tag{2.22}$$

for all $u_0, v_0 \in \boldsymbol{\mathcal{C}}_\delta$.

(ii) Let δ_0 be as in Theorem 1.1. There exists $0 < \delta_1 \le \delta_0$ such that if $0 < \delta < \delta_1$, then there is a constant C such that

$$\sup_{t>0} t^{\frac{1}{2}} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{L}^\infty} \le C\|u_0 - v_0\|_{\boldsymbol{\mathcal{W}}}, \tag{2.23}$$

for all $u_0, v_0 \in \boldsymbol{\mathcal{C}}_\delta$. Indeed, it follows from (2.13) that $t^{\frac{1}{2}}\|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{L}^\infty} \le C\delta < \infty$. Next, using (2.19) and (2.2), one shows easily that

$$\sup_{t>0} t^{\frac{1}{2}} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{L}^\infty} \le C\|u_0 - v_0\|_{\boldsymbol{\mathcal{W}}} + CM\delta \sup_{t>0} t^{\frac{1}{2}} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{L}^\infty},$$

and the result follows.

(iii) In the same spirit as (ii) above, we see that the proof of Theorem 1.1 could have been carried out (for δ_0 possibly smaller) by using the metric

$$\mathrm{d}'(u, v) = \sup_{s>0} \{\|u(s) - v(s)\|_{\boldsymbol{\mathcal{W}}} + s^{\frac{1}{2}} \|u(s) - v(s)\|_{\boldsymbol{L}^\infty}\}, \tag{2.24}$$

instead of (2.15).

(iv) Let $M, \delta > 0$ be as in Theorem 1.1 and let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition 1.3. Let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and suppose, in addition, that $u_0 \in (C(\mathbb{R}^N \setminus \{0\})^N$. It follows that $\boldsymbol{\mathcal{S}}(t)u_0 - u_0 \to 0$ as $t \downarrow 0$, uniformly on $\{|x| \ge \varepsilon\}$ for every $\varepsilon > 0$. To see this, we first observe that $|\boldsymbol{\mathcal{S}}(t)u_0|^2 \le M^2\delta^2(t + |x|^2)^{-1}$. In particular, $|\boldsymbol{\mathcal{S}}(t)u_0 - u_0| \le 2M\delta/|x|$, so that we need only show uniform convergence on bounded subsets of $\mathbb{R}^N \setminus \{0\}$. We deduce from (2.5) that, for a given $1 < \sigma < N$, $\sigma \le 2$,

$$|e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\boldsymbol{\mathcal{S}}(s)u_0 \otimes \boldsymbol{\mathcal{S}}(s)u_0)|(x) \le C(t-s)^{-\frac{1}{2}}|x|^{-\sigma}s^{-\frac{2-\sigma}{2}}.$$

Thus

$$\Big| \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (\boldsymbol{\mathcal{S}}(s)u_0 \otimes \boldsymbol{\mathcal{S}}(s)u_0)(x)\, ds \Big| \le C t^{\frac{\sigma-1}{2}} |x|^{-\sigma},$$

so that $|||\boldsymbol{\mathcal{S}}(t)u_0 - e^{t\Delta}u_0|||_{\boldsymbol{\mathcal{W}}} \le Ct^{\frac{\sigma-1}{2}} \to 0$ as $t \downarrow 0$. Since $e^{t\Delta}u_0 - u_0 \to 0$ as $t \downarrow 0$, uniformly on bounded subsets of $\mathbb{R}^N \setminus \{0\}$ by Lemma 2.2 in [5], the result follows.

Proposition 2.5. *Let $M, \delta > 0$ be as in Theorem* 1.1 *and let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition* 1.3. *It follows that for every fixed $t_0 > 0$, $\boldsymbol{\mathcal{S}}(t_0) : (\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*) \to \boldsymbol{C}_0(\mathbb{R}^N)$ is continuous. In particular, $\boldsymbol{\mathcal{S}}(t_0)\boldsymbol{\mathcal{C}}_\delta$ is a compact subset of $\boldsymbol{C}_0(\mathbb{R}^N)$.*

Proof. Assume $u_0^n \to u_0$ in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*)$ and set $u^n(t) = \boldsymbol{\mathcal{S}}(t)u_0^n$. Since $u^n \in E_{M\delta}$, it follows that for every $m \ge 1$, the sequence u^n is bounded in $L^\infty((1/m, m)\times\mathbb{R}^N)^N$. By well-known regularity (see e.g. the proof of Proposition 15.1 page 147 in [14]), the sequence u^n is bounded in $C^1([1/m, m] \times \{|x| \le m\})^N$. By the Ascoli theorem and a standard diagonal procedure, we obtain the existence of a subsequence n_k and a function $u \in E_{M\delta}$ such that $u^{n_k} \to u$ in $C([1/m, m] \times \{|x| \le m\})^N$ for all $m \in \mathbb{N}$. Furthermore, since the sequence u^n is bounded in $L^\infty((0, \infty), \boldsymbol{\mathcal{W}})$, the convergence also takes place in $C([1/m, m], \boldsymbol{C}_0(\mathbb{R}^N))$. To finish the proof, it suffices to show that $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$ for all $t > 0$. To see this, we note first that $e^{t\Delta}u_0^{n_k} \to e^{t\Delta}u_0$ in $\boldsymbol{L}^\infty(\mathbb{R}^N)$ by Proposition 3.8 in [6]. Next, it follows easily from (2.1) that for any $\varepsilon > 0$,

$$\int_\varepsilon^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u^{n_k} \otimes u^{n_k})\, ds \underset{k\to\infty}{\longrightarrow} \int_\varepsilon^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)\, ds,$$

in $\boldsymbol{\mathcal{W}}$. Furthermore, (2.4) and (2.13) imply that

$$\Big\| \int_0^\varepsilon e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u^{n_k} \otimes u^{n_k})\, ds \Big\|_{\boldsymbol{L}^{\frac{2N}{3}}} + \Big\| \int_0^\varepsilon e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)\, ds \Big\|_{\boldsymbol{L}^{\frac{2N}{3}}} \le C \int_0^\varepsilon (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

uniformly in $k \ge 0$. The result now follows since a standard $\varepsilon/2$ argument shows that $\langle \Phi(u^{n_k}, u^{n_k}) - \Phi(u, u), \varphi \rangle_{\mathcal{S}', \mathcal{S}} \to 0$ as $k \to \infty$ for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$. □

3. Universal solutions and chaos

In this section we establish the main results of this paper: the existence of solutions with infinite-dimensional asymptotic behavior and the chaotic nature of the fixed-time flow map. As will be clear from the proofs, these results depend in a fundamental way on the scaling properties of the equation (1.2).

Lemma 3.1. *Let M, δ_0 be as in Theorem 1.1, let $0 < \delta < \delta_0$, and let D_λ be defined by (1.9). Given any $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and $\lambda > 0$, it follows that*

$$D_\lambda \boldsymbol{\mathcal{S}}(\lambda^2 t) u_0 = \boldsymbol{\mathcal{S}}(t) D_\lambda u_0, \tag{3.1}$$

for all $\lambda > 0$ and $t \ge 0$. In particular,

$$D_\lambda \boldsymbol{\mathcal{S}}(\lambda^2) u_0 = \boldsymbol{\mathcal{S}}(1) D_\lambda u_0, \tag{3.2}$$

for all $\lambda > 0$.

Proof. Let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and let $u \in E_{M\delta}$ be the solution of (1.2) given by Theorem 1.1. Let $\lambda > 0$ and set $v_0 = D_\lambda u_0$. It follows that $v_0 \in \boldsymbol{\mathcal{C}}_\delta$ and we denote by $v \in E_{M\delta}$ the corresponding solution of (1.2) given by Theorem 1.1. Finally, we set $w(t) = D_\lambda u(\lambda^2 t)$ for all $t \ge 0$. Since $u \in E_{M\delta}$, we see that $w \in E_{M\delta}$. Formula (3.1) means that $w = v$. Thus, by uniqueness, it suffices to show that w is a solution of (1.2) with the initial value v_0. This is an easy calculation using the integral equation (1.2) and the commutation relations of Remark 2.1. □

Lemma 3.2. *Let $M, \delta > 0$ be as in Theorem 1.1 and let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition 1.3. If $\varphi \in \boldsymbol{\mathcal{C}}_\delta$ is homogeneous of degree -1 and $f = \boldsymbol{\mathcal{S}}(1)\varphi$, then $u(t) = \boldsymbol{\mathcal{S}}(t)\varphi$ is a self-similar solution of (1.1) with profile f; i.e., $u(t) = D_{\frac{1}{\sqrt{t}}} f$ for all $t > 0$. If, in addition, $\varphi \in C(\mathbb{R}^N \setminus \{0\})$, then $|x| f(x) - \varphi(x/|x|) \to 0$ as $|x| \to \infty$.*

Proof. Given any $\lambda > 0$, $D_\lambda \varphi = \varphi$ so that $u(t) = \boldsymbol{\mathcal{S}}(t) D_\lambda \varphi$. Applying (3.1), we deduce that $u(t) = D_\lambda \boldsymbol{\mathcal{S}}(\lambda^2 t)\varphi = D_\lambda u(\lambda^2 t)$. Letting $\lambda = 1/\sqrt{t}$, we see that $u(t) = D_{\frac{1}{\sqrt{t}}} \boldsymbol{\mathcal{S}}(1)\varphi = D_{\frac{1}{\sqrt{t}}} f$. Finally, since

$$|x| f(x) = u\Big(\frac{1}{|x|^2}, \frac{x}{|x|}\Big),$$

the last statement follows from Remark 2.4 (iv). □

The following proposition shows backwards uniqueness of self-similar solutions of (1.2) as constructed by Theorem 1.1. This property is an essential ingredient to our analysis. (See the proofs of Corollary 3.8, Propositions 1.11

and 1.13 and Theorem 8.1.) Note that this does not follow from the standard results since, by the second part of Lemma 3.2, these self-similar solutions are not L^2 solutions (at least those with initial value in $C(\mathbb{R}^N \setminus \{0\})$).

Proposition 3.3. *Under the hypotheses of Lemma* 3.2*, if* $\varphi_1, \varphi_2 \in \boldsymbol{\mathcal{C}}_\delta$ *are two homogeneous functions and* $\varphi_1 \neq \varphi_2$*, then* $\boldsymbol{\mathcal{S}}(t)\varphi_1 \neq \boldsymbol{\mathcal{S}}(t)\varphi_2$ *for all* $t > 0$*.*

Proof. Suppose by contradiction that there exists $t_0 > 0$ such that $\boldsymbol{\mathcal{S}}(t_0)\varphi_1 = \boldsymbol{\mathcal{S}}(t_0)\varphi_2$. It follows from Lemma 3.2 that $\boldsymbol{\mathcal{S}}(t)\varphi_1 = \boldsymbol{\mathcal{S}}(t)\varphi_2$ for all $t > 0$, so that $\varphi_1 = \varphi_2$ by Remark 1.2 (iii). □

The following proposition is an immediate consequence of formula (3.2).

Proposition 3.4. *Let* M, δ_0 *be as in Theorem* 1.1*, let* $0 < \delta < \delta_0$*, and let* $u_0 \in \boldsymbol{\mathcal{C}}_\delta$*. Let* $u = \boldsymbol{\mathcal{S}}(\cdot)u_0$ *be the solution of* (1.2) *given by Theorem* 1.1 *and let* $\omega(u)$ *be given by Definition* 1.6*. It follows that*

$$\begin{aligned} \omega(u) &= \{f \in \boldsymbol{C}_0(\mathbb{R}^N);\, \exists \lambda_n \to \infty, \|\boldsymbol{\mathcal{S}}(1)D_{\lambda_n}u_0 - f\|_{\boldsymbol{L}^\infty} \to 0\} \\ &= \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{\boldsymbol{\mathcal{S}}(1)D_\lambda u_0\}}, \end{aligned} \tag{3.3}$$

where the closure is in $\boldsymbol{C}_0(\mathbb{R}^N)$*.*

Proof of Theorem 1.7. Let $z \in \Omega(u_0)$. It follows that there exists $\lambda_n \to \infty$ such that $D_{\lambda_n}u_0 \to z$ in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*)$. Since $\boldsymbol{\mathcal{S}}(1)$ is continuous from $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*) \to \boldsymbol{C}_0(\mathbb{R}^N)$ (by Proposition 2.5), we deduce that $\boldsymbol{\mathcal{S}}(1)D_{\lambda_n}u_0 \to \boldsymbol{\mathcal{S}}(1)z$ in $\boldsymbol{C}_0(\mathbb{R}^N)$; i.e., $\boldsymbol{\mathcal{S}}(1)z \in \omega(u)$ by Proposition 3.4. Thus $\boldsymbol{\mathcal{S}}(1)\Omega(u_0) \subset \omega(u)$. Conversely, let $f \in \omega(u)$. Again by Proposition 3.4, it follows that there exists $\lambda_n \to \infty$ such that $\boldsymbol{\mathcal{S}}(1)D_{\lambda_n}u_0 \to f$ uniformly. Since $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*)$ is compact, there exists a subsequence $(\lambda_{n_k})_{k\geq 0}$ and $z \in \boldsymbol{\mathcal{C}}_\delta$ such that $D_{\lambda_{n_k}}u_0 \to z$ in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*)$. It follows that $z \in \Omega(u_0)$ and, by the previous argument, we see that $\boldsymbol{\mathcal{S}}(1)D_{\lambda_{n_k}}u_0 \to \boldsymbol{\mathcal{S}}(1)z$ uniformly. Therefore, $f = \boldsymbol{\mathcal{S}}(1)z \in \boldsymbol{\mathcal{S}}(1)\Omega(u_0)$. This shows that $\omega(u) = \boldsymbol{\mathcal{S}}(1)\Omega(u_0)$. The last statement follows from the continuity of $\boldsymbol{\mathcal{S}}(1) : (\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta^*) \to \boldsymbol{C}_0(\mathbb{R}^N)$. □

Remark 3.5. Here are some observations on the set $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ defined by (1.14).

(i) If $u(x) = |x|^{-2}Ax$ where A is a skew-symmetric matrix, then $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ for $\delta \geq \|A\|_{\mathcal{L}(\mathbb{R}^N)}$.

(ii) It is easy to check that if $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ and if $\varphi \in C^1(\mathbb{R}^N)$ is radially symmetric with $\|\varphi\|_{L^\infty} \leq 1$, then $\varphi u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$. This is the key property used in the proof of Theorem 1.10. The analogous statement for $u \in \boldsymbol{\mathcal{C}}_\delta$ instead of $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is false.

(iii) It follows from (ii) above and a standard mollifier argument (see for example the proof of Proposition 2.1 (iv) in [6]) that $\widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C_{\rm c}^\infty(\mathbb{R}^N \setminus \{0\}))^N$ is dense in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, {\rm d}_\delta^*)$.

Proof of Theorem 1.10. We proceed in three steps.
Step 1. We claim that, for any fixed sequence $(\psi_n)_{n\ge 0} \subset \widetilde{\boldsymbol{\mathcal{C}}}_\delta$, there exists $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$ with the following property. For every $n \ge 0$, there exists a sequence $(\lambda_{n,j})_{j\ge 0}$ such that $\lambda_{n,j} \to \infty$ as $j \to \infty$ and, given any compact set $K \subset \mathbb{R}^N \setminus \{0\}$, $D_{\lambda_{n,j}} u = \psi_n$ on K for j large enough. In particular, $(\psi_n)_{n\ge 0} \subset \Omega(u)$. Moreover, if $(\psi_n)_{n\ge 0} \subset (C^\infty(\mathbb{R}^N \setminus \{0\}))^N$, then one can choose $u \in (C^\infty(\mathbb{R}^N))^N$.

To see this, consider two sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ such that

$$0 < a_0 < b_0 < a_1 < b_1 < a_2 < b_2 < \cdots, \tag{3.4}$$

and

$$\frac{b_n}{a_n} \underset{n\to\infty}{\longrightarrow} \infty. \tag{3.5}$$

(For example, we may let $a_n = (2n+2)!$ and $b_n = (2n+3)!$.) Using a bijection $\mathbb{N} \to \mathbb{N}^2$, we rename the sequences as $(a_{k,\ell})_{\substack{k\ge 0\\ \ell\ge 0}}$ and $(b_{k,\ell})_{\substack{k\ge 0\\ \ell\ge 0}}$ (we use the same bijection for the two sequences). It follows from (3.4) that

$$(a_{k,\ell}, b_{k,\ell}) \cap (a_{m,n}, b_{m,n}) = \emptyset \quad \text{if} \quad (k,\ell) \ne (m,n). \tag{3.6}$$

We now set

$$\lambda_{k,\ell} = (a_{k,\ell} b_{k,\ell})^{\frac{1}{2}} \underset{\ell\to\infty}{\longrightarrow} \infty,$$

so that by (3.5)

$$\frac{a_{k,\ell}}{\lambda_{k,\ell}} \underset{\ell\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{b_{k,\ell}}{\lambda_{k,\ell}} \underset{\ell\to\infty}{\longrightarrow} \infty \tag{3.7}$$

for all $k \ge 0$. It follows from (3.4) and (3.6) that there exist $\varepsilon > 0$ and a sequence $(\varphi_{k,\ell})_{\substack{k\ge 0\\ \ell\ge 0}} \subset C_{\rm c}^\infty(\varepsilon, \infty)$ such that $0 \le \varphi_{k,\ell} \le 1$,

$$\varphi_{k,\ell}(r) = 1 \quad \text{for} \quad a_{k,\ell} < r < b_{k,\ell}, \tag{3.8}$$

and

$$\operatorname{supp} \varphi_{k,\ell} \cap \operatorname{supp} \varphi_{m,n} = \emptyset \quad \text{if} \quad (k,\ell) \ne (m,n). \tag{3.9}$$

Finally, let u be defined by

$$u(x) = \sum_{k\ge 0} \sum_{\ell \ge 0} \varphi_{k,\ell}(|x|) D_{\lambda_{k,\ell}^{-1}} \psi_k(x). \tag{3.10}$$

It follows from (3.9) that all the functions in the sum (3.10) have disjoint support. Therefore, given $x \in \mathbb{R}^N \setminus \{0\}$, the sum (3.10) reduces to one element and $\|u\|_{\boldsymbol{\mathcal{W}}} \le \delta$. Moreover, it is clear that $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ (see Remark 3.5 (ii)). In addition, $u(x) = 0$ for $|x| < \varepsilon$, so that $u \in L^\infty(\mathbb{R}^N)$. Moreover we deduce from (3.8)-(3.9) that, given any $n \ge 0$, $D_{\lambda_{n,j}} u(x) = \psi_n(x)$ for $a_{n,j}/\lambda_{n,j} < |x| < b_{n,j}/\lambda_{n,j}$. Applying (3.7), we see that for every compact set $K \subset \mathbb{R}^N \setminus \{0\}$, $D_{\lambda_{n,j}} u = \psi_n$ on K if j is large enough. The fact that $u \in (C^\infty(\mathbb{R}^N))^N$ if $(\psi_n)_{n \ge 0} \subset (C^\infty(\mathbb{R}^N \setminus \{0\}))^N$ follows immediately from (3.9) and the fact that u vanishes in a neighborhood of the origin.

Step 2. Proof of the first statement in the theorem. Since $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}_\delta^*)$ is separable and $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is a closed subset of $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}_\delta^*)$, we see that $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$ is separable. Since $\widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ is dense in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$ (see Remark 3.5 (iii)), there exists a sequence $(\psi_n)_{n \ge 0} \subset \widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ which is dense in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$. Let $U_0 \in \boldsymbol{\mathcal{B}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ be (as constructed in Step 1) such that $(\psi_n)_{n \ge 0} \subset \Omega(U_0) \subset \widetilde{\boldsymbol{\mathcal{C}}}_\delta$. On the other hand, since $\Omega(U_0)$ is closed in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$, we see that $\widetilde{\boldsymbol{\mathcal{C}}}_\delta \subset \Omega(U_0)$.

Step 3. Proof of the second statement. This follows from Step 2 and from Theorem 1.7. □

The following corollary translates Theorem 1.10 into an explicit description of the asymptotic behavior of the solution $\boldsymbol{\mathcal{S}}(t)U_0$.

Corollary 3.6. *Let $\delta < \delta_0$ and $U_0 \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ be as in Theorem 1.10. Given $u_0 \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$, there exist $t_n \to \infty$ such that*

$$\|\boldsymbol{\mathcal{S}}(t_n)U_0 - \boldsymbol{\mathcal{S}}(t_n) D_{\frac{1}{\sqrt{t_n}}} u_0\|_{\boldsymbol{L}^\infty} = \|\boldsymbol{\mathcal{S}}(t_n)U_0 - D_{\frac{1}{\sqrt{t_n}}} \boldsymbol{\mathcal{S}}(1) u_0\|_{\boldsymbol{L}^\infty} = o(t_n^{-\frac{1}{2}})$$

as $n \to \infty$. In particular, if u_0 is homogeneous of degree -1, then

$$\|\boldsymbol{\mathcal{S}}(t_n)U_0 - \boldsymbol{\mathcal{S}}(t_n) u_0\|_{\boldsymbol{L}^\infty} = o(t_n^{-\frac{1}{2}})$$

as $n \to \infty$. In other words, $\boldsymbol{\mathcal{S}}(t)U_0$ is asymptotic, along some appropriate subsequence, to every possible self-similar solution of (1.2) *with initial value in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$.*

The following two results show, if $N = 3$, that the set of self-similar solutions in the previous corollary is very large.

Proposition 3.7. *Assume $N = 3$ and $\delta > 0$. The closed subspace of $\boldsymbol{\mathcal{W}}$ given by $\{u \in \boldsymbol{\mathcal{W}};\ u \text{ is homogeneous of degree } -1, \nabla \cdot u = x \cdot u = 0\}$ is infinite dimensional.*

Proof. Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be C^2 and homogeneous of degree 0 and set

$$u(x) = \frac{x}{|x|} \times \nabla f = \frac{1}{|x|} \begin{pmatrix} x_2\partial_3 f - x_3\partial_2 f \\ x_3\partial_1 f - x_1\partial_3 f \\ x_1\partial_2 f - x_2\partial_1 f \end{pmatrix},$$

for $x \neq 0$. It is easy to check that u is homogeneous of degree -1 and that $x \cdot u(x) = \nabla \cdot u(x) = 0$ for all $x \neq 0$. Since $\nabla \cdot u$ is homogeneous of degree -2, and therefore integrable at 0, we deduce that $\nabla \cdot u(x) = 0$ in $(\mathcal{D}'(\mathbb{R}^3))^3$. Furthermore, the kernel of the linear mapping $f \mapsto u$ is the set of all f such that $\nabla f(x) \parallel x$ for all $x \neq 0$; i.e., the set of radially symmetric (and therefore constant) functions f. This concludes the proof. □

Corollary 3.8. *Assume $N = 3$ and $\delta < \delta_0$ with δ_0 as in Theorem 1.1, and let $\widetilde{\boldsymbol{\mathcal{C}}}_\delta^H$ be the set of homogeneous functions in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$. Let $U_0 \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta \cap (C^\infty(\mathbb{R}^N))^N$ be as in Theorem 1.10 and set $U(t) \equiv \boldsymbol{\mathcal{S}}(t)U_0$. For every $n \in \mathbb{N}$, there exists a subset $B_n \subset \boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta^H \subset \omega(U)$ which is homeomorphic, as a subset of $\boldsymbol{C}_0(\mathbb{R}^N)$, to a ball of dimension n.*

Proof. It is clear that $\widetilde{\boldsymbol{\mathcal{C}}}_\delta^H$ is a closed, therefore compact, subset of $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$. Since $\boldsymbol{\mathcal{S}}(1) : (\widetilde{\boldsymbol{\mathcal{C}}}_\delta^H, \mathrm{d}_\delta^*) \to \boldsymbol{C}_0(\mathbb{R}^N)$ is injective by Proposition 3.3 and continuous by Proposition 2.5, it is a homeomorphism onto its image. The result follows from Proposition 3.7. □

Remark 3.9. Assume $N = 2$ and $\delta > 0$. Every homogeneous function (of degree -1) in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is of the form $\alpha|x|^{-2}x^\perp$ where $x^\perp = (-x_2, x_1)$.

Proof of Proposition 1.11. We give the proof for $\lambda = 2$ and we proceed in two steps.
Step 1. Proof of the first statement. We consider a sequence $(\varphi_k)_{k\geq 1} \subset C_c^\infty(\mathbb{R}^N)$ of radially symmetric functions such that

$$\begin{cases} 0 \leq \varphi_k \leq 1, \\ \varphi_k(x) = 1 \quad \text{for} \quad 2^{-k+1} \leq |x| \leq 2^{k-1}, \\ \varphi_k(x) = 0 \quad \text{for} \quad |x| \leq 2^{-k} \quad \text{and} \quad |x| \geq 2^k. \end{cases}$$

To see that the periodic points are dense, let $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$, and define $v_k \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ by

(i) $v_k(x) = \varphi_k(x)u(x)$ for $2^{-k} \leq |x| < 2^k$,
(ii) $v_k(2^{2k}x) = 2^{-2k}v_k(x)$, for $x \neq 0$.

It is clear that ${D_\lambda}^{2k}v_k = v_k$, so that v_k is a periodic point, and that the sequence $v_k \to u$ in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$ as $k \to \infty$ since it converges in $\mathcal{D}'(\mathbb{R}^N \setminus \{0\})$.

Next we show that D_λ is topologically transitive. In Step 1 of the proof of Theorem 1.10, choose $a_n = 4^{(2n+2)!}$ and $b_n = 4^{(2n+3)!}$, so that $\lambda_{k,\ell}$ is always a power of 2. If in addition $(\psi_n)_{n\geq 0} \subset \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is dense in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$, then the orbit $({D_\lambda}^k u)_{k\geq 0}$ is dense in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$, where $u \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is as constructed in Step 1 of the proof of Theorem 1.10. This implies that D_λ is topologically transitive since the dense orbit has to intersect every open set in $(\widetilde{\boldsymbol{\mathcal{C}}}_\delta, \mathrm{d}_\delta^*)$.

Finally we show that D_λ satisfies the hypotheses of Lemma 4.4 in [6]. Let $z(x) = |x|^{-2}Ax \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ and $\widetilde{z}(x) = |x|^{-2}\widetilde{A}x \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ where $A \neq \widetilde{A}$ are two skew-symmetric matrices. We consider two sequences $(\alpha_k)_{k\geq 1} \subset C_\mathrm{c}^\infty(\mathbb{R}^N)$ and $(\beta_k)_{k\geq 1} \subset C_\mathrm{c}^\infty(\mathbb{R}^N)$ of radially symmetric functions such that

$$\begin{cases} 0 \leq \alpha_k \leq 1, \\ \alpha_k(x) = 1 \quad \text{for} \quad |x| \leq 2^{k-1}, \\ \alpha_k(x) = 0 \quad \text{for} \quad |x| \geq 2^k, \end{cases} \qquad \begin{cases} 0 \leq \beta_k \leq 1, \\ \beta_k(x) = 0 \quad \text{for} \quad |x| \leq 2^k, \\ \beta_k(x) = 1 \quad \text{for} \quad |x| \geq 2^{k+1}. \end{cases}$$

Given $w \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$, we define the sequences $(v_k)_{k\geq 0}, (\widetilde{v}_k)_{k\geq 0} \subset \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ by $v_k = \alpha_k w + \beta_k z$ and $\widetilde{v}_k = \alpha_k w + \beta_k \widetilde{z}$. It is clear that $\mathrm{d}_\delta^*(v_k, w) \to 0$ as $k \to \infty$ but that for each k, $\mathrm{d}_\delta^*({D_\lambda}^n v_k, z) \to 0$ as $n \to \infty$, and similarly for $(\widetilde{v}_k)_{k\geq 0}$. The result now follows from Proposition 4.5 (i) in [6].

Step 2. Proof of the second and third statements. Note that $\boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is compact by Proposition 2.5. If $f \in \boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$, then $f = \boldsymbol{\mathcal{S}}(1)z$ for some $z \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$. By formula (1.16), $F_\lambda f = \boldsymbol{\mathcal{S}}(1)D_\lambda z \in \boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$. This proves the invariance of $\boldsymbol{\mathcal{S}}(1)\widetilde{\boldsymbol{\mathcal{C}}}_\delta$. Finally, if $z \neq \widetilde{z}$ are two homogeneous functions in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ as in Step 1, then $\boldsymbol{\mathcal{S}}(1)z \neq \boldsymbol{\mathcal{S}}(1)\widetilde{z}$ by Proposition 3.3. The result then follows from Step 1 and from Proposition 4.5 in [6]. □

The rest of this section is devoted to the results formulated in terms of the initial vorticity (in particular, $N = 3$). In order to do so, we need definitions related to the space $\boldsymbol{\mathcal{V}}$ given by (1.19) and the set $\boldsymbol{\mathcal{A}}_\delta$ given by (1.20). Given $\nu \in \boldsymbol{\mathcal{V}}$ and $\lambda > 0$, we define $\widetilde{D}_\lambda \nu$ by

$$\widetilde{D}_\lambda \nu(x) = \lambda^2 \nu(\lambda x).$$

Also, the set $\boldsymbol{\mathcal{A}}_\delta$ endowed with the weak* topology of $\boldsymbol{\mathcal{V}}$ is compact and metrizable. We denote by $\mathrm{d}_\delta^{A,*}$ a corresponding metric and by $\widetilde{\Omega}$ the corresponding ω-limit set analogous to formula (1.10).

Lemma 3.10. *Let $N = 3$ and let $\boldsymbol{\mathcal{T}}$ be given by formula* (1.17).

(i) *$\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{W}}$ is continuous.*

(ii) *$D_\lambda \boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}} \widetilde{D}_\lambda$ for all $\lambda > 0$.*

(iii) *$\boldsymbol{\mathcal{T}} : (\boldsymbol{\mathcal{A}}_{\delta'}, \mathrm{d}_{\delta'}^{A,*}) \to (\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}_\delta^*)$ is continuous whenever $\delta \geq \delta' \|\boldsymbol{\mathcal{T}}\|_{\mathcal{L}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{W}})}$.*

(iv) $\nabla \cdot \boldsymbol{\mathcal{T}}\nu = 0$ *for all* $\nu \in \boldsymbol{\mathcal{V}}$. *In particular,* $\boldsymbol{\mathcal{T}}\boldsymbol{\mathcal{A}}_{\delta'} \subset \boldsymbol{\mathcal{C}}_\delta$ *whenever* $\delta \geq \delta' \|\boldsymbol{\mathcal{T}}\|_{\mathcal{L}(\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{W}})}$.

(v) *If* $\nu \in \boldsymbol{\mathcal{V}}$ *satisfies* $\nabla \cdot \nu = 0$, *then* $\nabla \times \boldsymbol{\mathcal{T}}\nu = \nu$. *In particular,* $\boldsymbol{\mathcal{T}}$ *is injective on* $\{\nu \in \boldsymbol{\mathcal{V}};\ \nabla \cdot \nu = 0\}$.

Proof. (i) It is clear that if $x \neq 0$, then

$$I(x) := \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \frac{1}{|y|^2}\, dy < \infty.$$

Moreover, a simple scaling argument shows that $I(\lambda x) = \lambda^{-1} I(x)$. Since I is radially symmetric, thus bounded on $\{|x| = 1\}$, we deduce that $I(x) = C|x|^{-1}$ for all $x \neq 0$. The result easily follows.

(ii) This follows from an elementary scaling argument.

(iii) Let $(\nu_n)_{n\geq 0} \subset \boldsymbol{\mathcal{A}}_\delta$ converge to ν in $(\boldsymbol{\mathcal{A}}_{\delta'}, \mathrm{d}^{A,*}_{\delta'})$. To show that $\boldsymbol{\mathcal{T}}\nu_n \to \boldsymbol{\mathcal{T}}\nu$ in $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$ it suffices, by Proposition 2.1 (i) in [6], to show that $\boldsymbol{\mathcal{T}}\nu_n \to \boldsymbol{\mathcal{T}}\nu$ in $(\mathcal{D}'(\mathbb{R}^3))^3$. Fix $\varphi \in (C^\infty_{\mathrm{c}}(\mathbb{R}^3))^3$.

$$\begin{aligned}\langle \boldsymbol{\mathcal{T}}\nu_n, \varphi\rangle &= \int_{\mathbb{R}^3} \boldsymbol{\mathcal{T}}\nu_n \varphi = -\frac{1}{4\pi}\int\int \frac{x-y}{|x-y|^3} \times \nu_n(y)\, dy\, \varphi(x)\, dx \\ &= \int_{\mathbb{R}^3} \psi(y) \times \nu_n(y)\, dy,\end{aligned}$$

where

$$\psi(y) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\varphi(x)\, dx.$$

We observe that by Lemma 7.1, $|\psi(y)| \leq C(1+|y|^2)^{-1}$. In particular, $|\cdot|^{-2}\psi(\cdot) \in L^1(\mathbb{R}^3)$, so that ψ belongs to the predual of $\boldsymbol{\mathcal{V}}$ (see the proof of Proposition 2.1 in [6]). It follows that $\langle \boldsymbol{\mathcal{T}}\nu_n, \varphi\rangle \to \langle \boldsymbol{\mathcal{T}}\nu, \varphi\rangle$.

(iv) It follows easily from Proposition 2.1 (i) in [6] that $\boldsymbol{\mathcal{A}}_{\delta'} \cap (C^\infty_{\mathrm{c}}(\mathbb{R}^3))^3$ is dense in $(\boldsymbol{\mathcal{A}}_{\delta'}, \mathrm{d}^{A,*}_{\delta'})$. Thus, given $\nu \in \boldsymbol{\mathcal{A}}_{\delta'}$, let $(\nu_n)_{n\geq 0} \subset \boldsymbol{\mathcal{A}}_{\delta'} \cap (C^\infty_{\mathrm{c}}(\mathbb{R}^3))^3$ converge to ν in $(\boldsymbol{\mathcal{A}}_{\delta'}, \mathrm{d}^{A,*}_{\delta'})$. It is well-known (see Lemma 2.1 in [12]) that $\nabla \cdot \boldsymbol{\mathcal{T}}\nu_n = 0$. By statement (iii), $\boldsymbol{\mathcal{T}}\nu_n \to \boldsymbol{\mathcal{T}}\nu$ in $(\boldsymbol{\mathcal{B}}_\delta, \mathrm{d}^*_\delta)$, hence in $(\mathcal{D}'(\mathbb{R}^3))^3$. Therefore, $\nabla \cdot \boldsymbol{\mathcal{T}}\nu = 0$.

(v) Let $\nu \in \boldsymbol{\mathcal{V}}$ satisfy $\nabla \cdot \nu = 0$. Let (for example) $\nu_n = e^{\frac{1}{n}\Delta}\nu$. It follows from Corollary 8.3 in [5] that there exists a constant M such that $\|\nu_n\|_{\boldsymbol{\mathcal{V}}} \leq M$. Moreover, $\nu_n \in (C^\infty(\mathbb{R}^3))^3$ and $\nabla \cdot \nu_n = 0$. Since ν_n is smooth, it is well-known (see for example Lemma 2.1 in [12]) that

$$\nabla \times \boldsymbol{\mathcal{T}}\nu_n = \nu_n. \tag{3.11}$$

We observe that $\nu_n \to \nu$ in $\mathcal{D}'(\mathbb{R}^3)$, so that (see Proposition 2.1 (i) in [6]) $\nu_n \to \nu$ in $(\boldsymbol{\mathcal{A}}_M, \mathrm{d}^{A,*}_M)$. Applying property (iii) above, we deduce that $\boldsymbol{\mathcal{T}}\nu_n \to$

$\mathcal{T}\nu$ in $(\mathcal{B}_K, \mathrm{d}^*_K)$ provided $K \ge M\|\mathcal{T}\|_{\mathcal{L}(\mathcal{V},\mathcal{W})}$. In particular, $\mathcal{T}\nu_n \to \mathcal{T}\nu$ in $\mathcal{D}'(\mathbb{R}^3)$ and the result follows by letting $n \to \infty$ in (3.11). □

Proposition 3.11. *Assume $N = 3$. Let $\delta > 0$, let $V_0 \in \mathcal{A}_\delta \cap (C^\infty(\mathbb{R}^3))^3$ be such that $\widetilde{\Omega}(V_0) = \mathcal{A}_\delta$ as in Theorem 1.2 of [6] and let $U_0 = \mathcal{T}V_0$. It follows that $\Omega(U_0) = \mathcal{T}\mathcal{A}_\delta$. If, in addition, $\delta\|\mathcal{T}\|_{\mathcal{L}(\mathcal{V},\mathcal{W})} < \delta_0$ with δ_0 as in Theorem 1.1, then $\omega(U) = \mathcal{S}(1)\mathcal{T}\mathcal{A}_\delta$ where $U(t) \equiv \mathcal{S}(t)U_0$.*

Proof. The first statement follows from Lemma 3.10 and the compactness of $(\mathcal{A}_\delta, \mathrm{d}^{A,*}_\delta)$, and the second from Theorem 1.7. □

Corollary 3.12. *Assume $N = 3$. Let $\delta > 0$ satisfy $\delta\|\mathcal{T}\|_{\mathcal{L}(\mathcal{V},\mathcal{W})} < \delta_0$ with δ_0 as in Theorem 1.1 and let $V_0 \in \mathcal{A}_\delta \cap (C^\infty(\mathbb{R}^3))^3$ be as in Proposition 3.11. Given $\nu_0 \in \mathcal{A}_\delta$, there exist $t_n \to \infty$ such that*

$$\|\mathcal{S}(t_n)\mathcal{T}V_0 - \mathcal{S}(t_n)D_{\frac{1}{\sqrt{t_n}}}\mathcal{T}\nu_0\|_{\boldsymbol{L}^\infty} = \|\mathcal{S}(t_n)\mathcal{T}V_0 - D_{\frac{1}{\sqrt{t_n}}}\mathcal{S}(1)\mathcal{T}\nu_0\|_{\boldsymbol{L}^\infty}$$
$$= o(t_n^{-\frac{1}{2}})$$

as $n \to \infty$. In particular, if ν_0 is homogeneous of degree -2, then $\mathcal{T}\nu_0$ is homogeneous of degree -1 and

$$\|\mathcal{S}(t_n)\mathcal{T}V_0 - \mathcal{S}(t_n)\mathcal{T}\nu_0\|_{\boldsymbol{L}^\infty} = o(t_n^{-\frac{1}{2}})$$

as $n \to \infty$. Thus, $U(t) \equiv \mathcal{S}(t)\mathcal{T}V_0$ is asymptotic, along some appropriate subsequence, to every possible self-similar solution of (1.2) with initial vorticity in $\mathcal{A}_\delta$.

Remark 3.13. Let ${\mathcal{E}_\delta}^H$ be the set of divergence free, homogeneous of degree -2, elements of $\mathcal{A}_\delta$. By Lemma 3.10 (v) and Proposition 3.3, $\mathcal{S}(1)\mathcal{T}$ is injective on ${\mathcal{E}_\delta}^H$. By a reasoning analogous to the proof of Corollary 3.8, it follows that, for every $n \in \mathbb{N}$, there exists a subset $B_n \subset \mathcal{S}(1)\mathcal{T}{\mathcal{E}_\delta}^H \subset \omega(U)$ which is homeomorphic, as a subset of $\boldsymbol{C}_0(\mathbb{R}^N)$, to a ball of dimension n.

Proof of Proposition 1.13. We first remark that $\mathcal{S}(1)\mathcal{T}\mathcal{A}_\delta$ is compact by Lemma 3.10 (iii) and Proposition 2.5. We next observe that Proposition 2.11 in [6] trivially applies to vector-valued functions. In particular, $\widetilde{D}_\lambda$ generates a chaotic, discrete dynamical system on $(\mathcal{A}_\delta, \mathrm{d}^{A,*}_\delta)$. We deduce from Lemma 3.10 (ii) and from (1.16) that $F_\lambda\mathcal{S}(1)\mathcal{T} = \mathcal{S}(1)\mathcal{T}\widetilde{D}_\lambda$. Furthermore, if $\nu(x) = |x|^{-3}Ax \in \mathcal{A}_\delta$ and $\widetilde{\nu}(x) = |x|^{-3}\widetilde{A}x \in \mathcal{A}_\delta$ where $A \ne \widetilde{A}$ are two skew-symmetric matrices, then it follows from Lemma 3.10 (v) that $\mathcal{T}\nu \ne \mathcal{T}\widetilde{\nu}$ are two homogeneous functions in $\mathcal{T}\mathcal{A}_\delta$. Thus, $\mathcal{S}(1)\mathcal{T}\nu \ne \mathcal{S}(1)\mathcal{T}\widetilde{\nu}$ by Proposition 3.3. The result then follows from Proposition 4.5 in [6]. □

Remark 3.14. The solutions described in Corollaries 3.6 and 3.12 have encoded in them the long time asymptotic behavior of every other solution with initial value in $\widetilde{\boldsymbol{\mathcal{C}}}_\delta$ or $\boldsymbol{\mathcal{TA}}_\delta$, respectively. Consequently, we refer to these solutions as "universal solutions". In particular, universal solutions are the opposite of asymptotically self-similar solutions. The latter ultimately settle down into a unique asymptotic form, and the former approach all possible asymptotic forms in a certain class.

Remark 3.15. Let $M, \delta > 0$ be as in Theorem 1.1, let $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, and let u be the corresponding solution of (1.2) constructed in Theorem 1.1. If, in addition, $u_0 \in \boldsymbol{\mathcal{V}}_\sigma$ for some $1 < \sigma < N$ with $\boldsymbol{\mathcal{V}}_\sigma$ defined by (1.18), then clearly $\Omega(u_0) = \Omega_1(u_0) = \{0\}$. It follows from Theorems 1.7 and 1.17 that $\omega(u) = \{0\}$ and, if δ is sufficiently small, $\omega_1(u) = \{0\}$ where $u(t) \equiv \boldsymbol{\mathcal{S}}(t)u_0$. In this case, a more precise asymptotic behavior of the solution u can be obtained using dilation operators D^σ_λ leaving invariant the $\boldsymbol{\mathcal{V}}_\sigma$ norm and the corresponding ω-limit sets $\Omega^\sigma(u_0)$, $\Omega^\sigma_1(u_0)$, $\omega^\sigma(u)$, and $\omega^\sigma_1(u)$. In fact, $\omega^\sigma(u) = e^\Delta \Omega^\sigma(u_0)$ and, if $\Omega^\sigma_1(u_0) \neq \emptyset$, $\omega^\sigma_1(u) = \overline{e^\Delta \Omega^\sigma_1(u_0)}$. To prove this, one can use estimates similar to those in Lemma 2.2 to show that (if δ is sufficiently small) $\|D^\sigma_{\sqrt{t}}(u(t) - e^{t\Delta}u_0)\|_{\boldsymbol{\mathcal{V}}_\sigma \cap \boldsymbol{L}^\infty} \to 0$ as $t \to \infty$. Thus the asymptotic form is the same as that of $e^{t\Delta}u_0$, which is described in [6].

4. Asymptotically self-similar solutions

In this section we characterize asymptotically self-similar solutions u in terms of $\omega(u)$. First, however, we need to strengthen some of the continuity properties proved in Section 2.

Proposition 4.1. *Let M, δ_0 be as in Theorem 1.1. There exists $0 < \delta_2 \le \delta_0$ such that if $0 < \delta < \delta_2$ and $u_0 \in \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$, then*

$$\sup_{t>0} \|\boldsymbol{\mathcal{S}}(t)u_0\|_{\boldsymbol{L}^\infty} \le 2\|u_0\|_{\boldsymbol{L}^\infty}, \tag{4.1}$$

where $\boldsymbol{\mathcal{S}}(t)$ is given by Definition 1.3. Moreover, there exists a constant C such that

$$\sup_{t>0} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{L}^\infty} \le C\|u_0 - v_0\|_{\boldsymbol{L}^\infty}, \tag{4.2}$$

for all $u_0, v_0 \in \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$.

Proof. We first show (4.1). To do this, we slightly modify the proof of Theorem 1.1. Instead of E_K, we consider the set

$$\widetilde{E}_K = \{u \in E_K; \sup_{t>0} \|u(t)\|_{\boldsymbol{L}^\infty} \le 2\|u_0\|_{\boldsymbol{L}^\infty}\}. \tag{4.3}$$

equipped with the distance (2.15), so that $(\widetilde{E}_K, \mathrm{d})$ is a complete metric space. We need only show that if $u \in \widetilde{E}_{M\delta}$ with $\delta > 0$ sufficiently small, then $\Psi(u) \in \widetilde{E}_{M\delta}$, where Ψ is defined by (2.17). Using the estimates of the proof of Theorem 1.1, it remains to estimate $\|\Psi(u)(t)\|_{\boldsymbol{L}^\infty}$. Applying (2.4) with $p = \infty$, we obtain

$$\|\Psi(u)(t)\|_{\boldsymbol{L}^\infty} \le \|u_0\|_{\boldsymbol{L}^\infty} + C\int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{\boldsymbol{L}^\infty}^2 \, ds. \tag{4.4}$$

On the one hand, since $u \in E_{M\delta}$, $\|u(s)\|_{\boldsymbol{L}^\infty} \le M\delta s^{-\frac{1}{2}}$. On the other hand, it follows from (4.3) that $\|u(s)\|_{\boldsymbol{L}^\infty} \le 2\|u_0\|_{\boldsymbol{L}^\infty}$. Therefore, we deduce from (4.4) that

$$\begin{aligned}\|\Psi(u)(t)\|_{\boldsymbol{L}^\infty} &\le \|u_0\|_{\boldsymbol{L}^\infty} + 2CM\delta\|u_0\|_{\boldsymbol{L}^\infty}\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}\, ds \\ &\le \Big(1 + 2CM\delta\int_0^1 (1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\, d\sigma\Big)\|u_0\|_{\boldsymbol{L}^\infty} \le 2\|u_0\|_{\boldsymbol{L}^\infty},\end{aligned}$$

provided

$$2CM\delta_2 \int_0^1 (1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\, d\sigma \le 1. \tag{4.5}$$

This shows (4.1). Consider now $u_0, v_0 \in \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$ with $\delta < \delta_2$ and let $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$ and $v(t) = \boldsymbol{\mathcal{S}}(t)v_0$. Arguing as above, we see that

$$\begin{aligned}\|u(t) - v(t)\|_{\boldsymbol{L}^\infty} &\le \|u_0 - v_0\|_{\boldsymbol{L}^\infty} \\ &\quad + C\int_0^t (t-s)^{-\frac{1}{2}} (\|u(s)\|_{\boldsymbol{L}^\infty} + \|v(s)\|_{\boldsymbol{L}^\infty})\|u(s) - v(s)\|_{\boldsymbol{L}^\infty}\, ds \\ &\le \|u_0 - v_0\|_{\boldsymbol{L}^\infty} + 2CM\delta\Big(\int_0^1 (1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\, d\sigma\Big) \sup_{s>0}\|u(s) - v(s)\|_{\boldsymbol{L}^\infty};\end{aligned}$$

and so,

$$\begin{aligned}\sup_{s>0}\|u(s) - v(s)\|_{\boldsymbol{L}^\infty} &\le \|u_0 - v_0\|_{\boldsymbol{L}^\infty} \\ &\quad + 2CM\delta\Big(\int_0^1 (1-\sigma)^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\, d\sigma\Big) \sup_{s>0}\|u(s) - v(s)\|_{\boldsymbol{L}^\infty}.\end{aligned}$$

(4.2) now follows, by using the smallness condition (4.5). □

Corollary 4.2. *Let δ_2 be as in Proposition* 4.1 *and assume $0 < \delta < \delta_2$. It follows that there exists a constant C such that*

$$\sup_{t>0} \|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(t)v_0\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \le C\|u_0 - v_0\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty}, \tag{4.6}$$

for all $u_0, v_0 \in \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$, where $\boldsymbol{\mathcal{S}}(t)$ is given by Definition 1.3.

Proof. This follows from (2.22) and (4.2). □

Proposition 4.3. *Suppose $\delta > 0$ satisfies $M\delta < \delta_2$, where M is as in Theorem* 1.1 *and δ_2 is as in Proposition* 4.1. *Given $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, set $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$ where $\boldsymbol{\mathcal{S}}(t)$ is given by Definition* 1.3, *and let $\omega(u)$ be defined by* (1.11).

(i) *If $f \in \omega(u)$, then $D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f \in \omega(u)$ for all $s \ge 0$.*

(ii) *If $\omega(u) = \{f\}$, then $f = \boldsymbol{\mathcal{S}}(1)\varphi$ with $\varphi \in \boldsymbol{\mathcal{C}}_\delta$ homogeneous of degree -1. Moreover, $D_{\sqrt{t}}u(t) = \boldsymbol{\mathcal{S}}(1)D_{\sqrt{t}}u_0 \to f$ in $\boldsymbol{C}_0(\mathbb{R}^N)$ as $t \to \infty$, or equivalently $\|u(t) - \boldsymbol{\mathcal{S}}(t)\varphi\|_{\boldsymbol{L}^\infty} = o(t^{-\frac{1}{2}})$ as $t \to \infty$.*

Proof. (i) We note that $u(t) \in \boldsymbol{\mathcal{C}}_{M\delta}$ for all $t \ge 0$. Suppose $D_{\sqrt{t_n}}u(t_n) \to f$ in $\boldsymbol{C}_0(\mathbb{R}^N)$. It follows from (3.1) that

$$D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)[D_{\sqrt{t_n}}u(t_n)] = D_{\sqrt{(1+s)t_n}}u((1+s)t_n). \tag{4.7}$$

Applying (4.2), we deduce that $D_{\sqrt{(1+s)t_n}}u((1+s)t_n) \to D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f$ in $\boldsymbol{C}_0(\mathbb{R}^N)$ and (i) follows.

(ii) It follows from (i), setting $s = \tau/t$ and applying (3.1) with t replaced by τ and $\lambda = 1/\sqrt{t}$, that

$$\boldsymbol{\mathcal{S}}(\tau)D_{\frac{1}{\sqrt{t}}}f = D_{\frac{1}{\sqrt{t+\tau}}}f, \tag{4.8}$$

for all $t, \tau > 0$. By compactness of $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$, there exist $\varphi \in \boldsymbol{\mathcal{C}}_\delta$ and $t_n \downarrow 0$ such that $D_{\frac{1}{\sqrt{t_n}}}f \to \varphi$ in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$. Applying (4.8) and Proposition 2.5, we deduce that

$$D_{\frac{1}{\sqrt{t}}}f = \lim_{n\to\infty} D_{\frac{1}{\sqrt{t+t_n}}}f = \lim_{n\to\infty} \boldsymbol{\mathcal{S}}(t)D_{\frac{1}{\sqrt{t_n}}}f = \boldsymbol{\mathcal{S}}(t)\varphi, \tag{4.9}$$

where the limits are in $\boldsymbol{C}_0(\mathbb{R}^N)$. In particular, $f = \boldsymbol{\mathcal{S}}(1)\varphi$. Set $v(t) = \boldsymbol{\mathcal{S}}(t)\varphi$. It follows from (4.9) that for all $t, \lambda > 0$,

$$v(t) = D_{\frac{1}{\sqrt{t}}}f = D_\lambda D_{\frac{1}{\sqrt{\lambda^2 t}}}f = D_\lambda v(\lambda^2 t).$$

Since $v(t) \to \varphi$ in $\boldsymbol{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \downarrow 0$ by Remark 1.2 (iii), we deduce that $\varphi = D_\lambda\varphi$ for all $\lambda > 0$; i.e., φ is homogeneous of degree -1. We finally show that $D_{\sqrt{t}}u(t) = \boldsymbol{\mathcal{S}}(1)D_{\sqrt{t}}u_0 \to f$ in $\boldsymbol{C}_0(\mathbb{R}^N)$ as $t \to \infty$. Indeed, assume by contradiction that there exists $\varepsilon > 0$ such that $\|D_{\sqrt{t_n}} u(t_n) - f\|_{\boldsymbol{L}^\infty} = \|\boldsymbol{\mathcal{S}}(1)D_{\sqrt{t_n}} u_0 - f\|_{\boldsymbol{L}^\infty} \geq \varepsilon$ for some sequence $t_n \to \infty$. By compactness of $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$, there exist $w \in \boldsymbol{\mathcal{C}}_\delta$ and a subsequence, which we still denote $(t_n)_{n\geq 0}$ such that $D_{\sqrt{t_n}} u_0 \to w$ in $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}^*_\delta)$. Applying Proposition 2.5, we deduce that $\|\boldsymbol{\mathcal{S}}(1)D_{\sqrt{t_n}} u_0 - \boldsymbol{\mathcal{S}}(1)w\|_{\boldsymbol{L}^\infty} \to 0$. Thus $\boldsymbol{\mathcal{S}}(1)w \in \omega(u)$, so that $\boldsymbol{\mathcal{S}}(1)w = f$, which yields a contradiction. □

5. Further continuity properties of the flow

In this section we study the finer structure of the set $\omega_1(u)$ and in particular prove the relationship between $\Omega_1(u)$ and $\omega_1(u)$ given in Theorem 1.17.

Proposition 5.1. *Let δ_1 be as in Remark* 2.4 (ii) *and δ_2 be as in Proposition* 4.1*, and assume $0 < \delta < \min\{\delta_1, \delta_2\}$. It follows that there exists a constant C such that*

$$\sup_{s\geq 0} \|D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)f - D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \leq C\|f-g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty},$$

for all $f, g \in \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$ and $s \geq 0$, where $\boldsymbol{\mathcal{S}}(t)$ is given by Definition 1.3.

Proof. It follows from an elementary calculation that

$$\sup_{0\leq s\leq 1} \|D_{\sqrt{1+s}}\varphi\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \leq C\|\varphi\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty}, \tag{5.1}$$

for all $\varphi \in \boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$. Therefore,

$$\begin{aligned}\sup_{0\leq s\leq 1} &\|D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)f - D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \\ &\leq C \sup_{0\leq s\leq 1} \|\boldsymbol{\mathcal{S}}(s)f - \boldsymbol{\mathcal{S}}(s)g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \leq C\|f-g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty},\end{aligned} \tag{5.2}$$

where the last inequality follows from (4.6). We now consider $s \geq 1$. Since $D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s) = D_{\sqrt{1+\frac{1}{s}}}\boldsymbol{\mathcal{S}}(1)D_{\sqrt{s}}$ by (3.2) and since $D_{\sqrt{1+\frac{1}{s}}}$ is bounded on $\boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$, uniformly in $s \geq 1$ by (5.1), we see that

$$\begin{aligned}\sup_{s\geq 1} &\|D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)f - D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s)g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \\ &\leq C \sup_{s\geq 1} \|\boldsymbol{\mathcal{S}}(1)D_{\sqrt{s}}f - \boldsymbol{\mathcal{S}}(1)D_{\sqrt{s}}g\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \\ &\leq C\|D_{\sqrt{s}}f - D_{\sqrt{s}}g\|_{\boldsymbol{\mathcal{W}}} \leq C\|f-g\|_{\boldsymbol{\mathcal{W}}},\end{aligned} \tag{5.3}$$

where the last two inequalities follow from (2.22)-(2.23) and the fact that $D_{\sqrt{s}}$ is an isometry on $\boldsymbol{\mathcal{W}}$, respectively. The result is an immediate consequence of (5.2)-(5.3). □

Remark 5.2. Since $D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s) = F_\lambda$ with $\lambda = \sqrt{1+s}$, where F_λ is defined by (1.15), the previous proposition shows that the F_λ, $\lambda \geq 1$ are uniformly Lipschitz on a ball in $\boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty(\mathbb{R}^N)$. Thus, the chaotic behavior of these maps (Propositions 1.11 and 1.13) on certain compact subsets of $\boldsymbol{C}_0(\mathbb{R}^N)$, which are also closed subsets of $\boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$, does not persist in the stronger topology of $\boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$.

Remark 5.3. If δ is as in Theorem 1.1, then it follows from the scaling relation (3.2) that

$$\begin{aligned}\omega_1(u) &= \{f \in \boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N);\ \exists \lambda_n \to \infty, \|\boldsymbol{\mathcal{S}}(1)D_{\lambda_n}u_0 - f\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \to 0\} \\ &= \bigcap_{\mu>0} \overline{\bigcup_{\lambda>\mu} \{\boldsymbol{\mathcal{S}}(1)D_\lambda u_0\}}, \quad (5.4)\end{aligned}$$

where the closure is in $\boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$.

Proposition 5.4. *Assume* $0 < M\delta < \min\{\delta_1, \delta_2\}$, *where* $M > 0$ *is as in Theorem* 1.1, δ_1 *is as in Remark* 2.4 (ii), *and* δ_2 *is as in Proposition* 4.1. *Let* $\boldsymbol{\mathcal{S}}(t)$ *be given by Definition* 1.3, *let* $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, *set* $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$, *let* $\omega(u)$ *be defined by* (1.11), *and let* $\omega_1(u)$ *be defined by* (1.21).

(i) *If* $\omega_1(u) \neq \emptyset$, *then*

$$\omega_1(u) = \overline{\bigcup_{s\geq 0}\{D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f\}} = \bigcap_{s_0\geq 0}\overline{\bigcup_{s\geq s_0}\{D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f\}},$$

for every $f \in \omega_1(u)$, *where the closures are in* $\boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$. *Moreover,* $\inf_{w\in\omega_1(u)} \|D_{\sqrt{t}}u(t) - w\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \underset{t\to\infty}{\longrightarrow} 0$.

(ii) *If* $\omega_1(u) = \{f\}$, *then* $f = \boldsymbol{\mathcal{S}}(1)\varphi$ *with* $\varphi \in \boldsymbol{\mathcal{C}}_\delta$ *homogeneous of degree* -1.

(iii) *If* $f \in \omega_1(u)$ *and* $f = \boldsymbol{\mathcal{S}}(1)\varphi$ *with* $\varphi \in \boldsymbol{\mathcal{C}}_\delta$ *homogeneous of degree* -1, *then* $\|D_{\sqrt{t}}u(t) - f\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \to 0$ *as* $t \to \infty$. *In particular,* $\omega_1(u) = \omega(u) = \{f\}$.

(iv) *If* $\omega_1(u) \neq \emptyset$, *then* $\omega(u) = \overline{\omega_1(u)}$, *where the closure is in* $\boldsymbol{C}_0(\mathbb{R}^N)$.

Proof. (i) Let $f \in \omega_1(u)$ and set $E(s_0) = \bigcup_{s\geq s_0}\{D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f\}$. We first show that $\omega_1(u) = \overline{E(s_0)}$, for all $s_0 \geq 0$. We deduce from (1.21), (4.7) and the continuity of $D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s) : \boldsymbol{\mathcal{C}}_\delta \cap \boldsymbol{L}^\infty(\mathbb{R}^N) \to \boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$ (see (4.6)) that $E(s_0) \subset E(0) \subset \omega_1(u)$ and, since $\omega_1(u)$ is closed in $\boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$, $\overline{E(s_0)} \subset \omega_1(u)$. To show the reverse inclusion, let $s_0 \geq 0$ and consider a sequence $t_n \to \infty$ such that

$$\|D_{\sqrt{t_n}}u(t_n) - f\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \underset{n\to\infty}{\longrightarrow} 0.$$

Applying (4.7) and Proposition 5.1, we deduce that

$$\sup_{s\geq 0} \|D_{\sqrt{(1+s)t_n}} u((1+s)t_n) - D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s) f\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} = \sup_{s\geq 0} \|D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s) D_{\sqrt{t_n}} u(t_n) - D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s) f\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \underset{n\to\infty}{\longrightarrow} 0.$$

In particular,

$$\sup_{s\geq s_0} \inf_{w\in E(s_0)} \|D_{\sqrt{(1+s)t_n}} u((1+s)t_n) - w\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \underset{n\to\infty}{\longrightarrow} 0,$$

so that $\omega_1(u) \subset \overline{E(s_0)}$. The last statement in (iv) easily follows from the previous estimate.

(ii) This follows from exactly the same argument for the analogous property in Proposition 4.3 (ii), except using property (i) above instead of Proposition 4.3 (i).

(iii) Given $s \geq 0$, it follows from (3.2) that

$$D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(s) f = D_{\sqrt{1+s}} \boldsymbol{\mathcal{S}}(1+s)\varphi = \boldsymbol{\mathcal{S}}(1) D_{\sqrt{1+s}}\varphi = \boldsymbol{\mathcal{S}}(1)\varphi = f.$$

Therefore, we deduce from property (i) above that $\omega_1(u) = \{f\}$ and that $\|D_{\sqrt{t}} u(t) - f\|_{\boldsymbol{\mathcal{W}}\cap \boldsymbol{L}^\infty} \to 0$ as $t\to\infty$.

(iv) It follows from the last property in (i) that $\omega(u) \subset \overline{\omega_1(u)}$. On the other hand, $\omega_1(u) \subset \omega(u)$ and, since $\omega(u)$ is closed, $\overline{\omega_1(u)} \subset \omega(u)$. □

Proposition 5.5. *Let $M, \delta > 0$ be as in Theorem* 1.1 *and let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition* 1.3. *It follows that for every fixed $t_0 > 0$, $\boldsymbol{\mathcal{S}}(t_0)$ is continuous from $(\boldsymbol{\mathcal{C}}_\delta, \mathrm{d}_\delta) \to \boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$.*

Proof. Assume $\mathrm{d}_\delta(u_0^n, u_0) \to 0$ and set $u^n(t) = \boldsymbol{\mathcal{S}}(t)u_0^n$ and $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$. We fix $1 < \sigma \leq 2$, $\sigma < N$. With the notation (2.16), we deduce from (2.3) and (2.12) that

$$\|\Phi(u,u)(t)\|_{\boldsymbol{\mathcal{V}}_\sigma} + \|\Phi(u^n,u^n)(t)\|_{\boldsymbol{\mathcal{V}}_\sigma} \leq CM^2\delta^2 t^{\frac{\sigma-1}{2}}, \tag{5.5}$$

uniformly in $n \geq 0$. Given any $R > 0$

$$\begin{aligned} &\sup_{|x|>R} |x|\,|u^n(t_0,x) - u(t_0,x)| \\ &\quad\leq \|e^{t_0\Delta}(u_0^n - u_0)\|_{\boldsymbol{\mathcal{W}}} + R^{-(\sigma-1)}\|\Phi(u^n,u_n)(t_0) - \Phi(u,u)(t_0)\|_{\boldsymbol{\mathcal{V}}_\sigma} \\ &\quad\leq \|e^{t_0\Delta}(u_0^n - u_0)\|_{\boldsymbol{\mathcal{W}}} + CR^{-(\sigma-1)}, \end{aligned}$$

where we use (5.5) in the last inequality. Given $\varepsilon > 0$, we fix R large enough so that $CR^{-(\sigma-1)} < \varepsilon/2$, then we deduce from Proposition 3.8 in [6] that

$\|e^{t_0\Delta}(u_0^n - u_0)\|_{\boldsymbol{\mathcal{W}}} \le \varepsilon/2$ for n sufficiently large. Thus

$$\sup_{|x|>R} |x|\, |u^n(t_0,x) - u(t_0,x)| \le \varepsilon,$$

for n sufficiently large. On the other hand, it follows from Proposition 2.5 that

$$\sup_{|x|<R} |u^n(t_0,x) - u(t_0,x)| \le \frac{\varepsilon}{R},$$

for n sufficiently large, and this completes the proof. □

Theorem 5.6. *Assume* $0 < M\delta < \min\{\delta_1, \delta_2\}$, *where* $M > 0$ *is as in Theorem* 1.1, δ_1 *is as in Remark* 2.4 (ii), *and* δ_2 *is as in Proposition* 4.1. *Let* $\boldsymbol{\mathcal{S}}(t)$ *be given by Definition* 1.3, *let* $u_0 \in \boldsymbol{\mathcal{C}}_\delta$, *set* $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$, *and let* $\omega_1(u)$ *be defined by* (1.21) *and* $\Omega_1(u_0)$ *be defined by* (1.23). *If* $\Omega_1(u_0) \neq \emptyset$, *then* $\omega_1(u) = \overline{\boldsymbol{\mathcal{S}}(1)\Omega_1(u_0)}$, *where the closure is in* $\boldsymbol{\mathcal{W}} \cap C_0(\mathbb{R}^N)$.

Proof. Given $z \in \Omega_1(u_0)$, there exists $\lambda_n \to \infty$ such that $\mathrm{d}_\delta(D_{\lambda_n}u_0, z) \to 0$. It follows from Proposition 5.5 that $\boldsymbol{\mathcal{S}}(1)D_{\lambda_n}u_0 \to \boldsymbol{\mathcal{S}}(1)z$ in $\boldsymbol{\mathcal{W}}\cap\boldsymbol{C}_0(\mathbb{R}^N)$; i.e., $\boldsymbol{\mathcal{S}}(1)z \in \omega_1(u)$. Since $\omega_1(u)$ is closed, $\overline{\boldsymbol{\mathcal{S}}(1)\Omega_1(u_0)} \subset \omega_1(u)$. We now show the reverse inclusion. Given $\varphi \in \Omega_1(u_0)$, it follows from what precedes that $f = \boldsymbol{\mathcal{S}}(1)\varphi \in \omega_1(u)$. Proposition 5.4 (i) implies that $\omega_1(u) = \overline{\underset{s\ge 0}{\cup}\{D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(s)f\}}$, and thus

$$\omega_1(u) = \overline{\underset{s\ge 0}{\cup}\{D_{\sqrt{1+s}}\boldsymbol{\mathcal{S}}(1+s)\varphi\}} = \overline{\underset{s\ge 0}{\cup}\{\boldsymbol{\mathcal{S}}(1)D_{\sqrt{1+s}}\varphi\}} = \overline{\boldsymbol{\mathcal{S}}(1)\underset{s\ge 0}{\cup}\{D_{\sqrt{1+s}}\varphi\}}.$$

By Proposition 2.5 (iv) in [6], $D_{\sqrt{1+s}}\varphi \in \Omega_1(u_0)$ for all $s \ge 0$. Thus $\overline{\boldsymbol{\mathcal{S}}(1)\underset{s\ge 0}{\cup}\{D_{\sqrt{1+s}}\varphi\}} \subset \overline{\boldsymbol{\mathcal{S}}(1)\Omega_1(u_0)}$ and the result follows. □

Remark 5.7. The hypothesis $\Omega_1(u_0) \neq \emptyset$ is necessary in the last part of Theorem 5.6. More precisely, there exists $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ such that $\Omega_1(u_0) = \emptyset$ and $\omega_1(u) = \{0\}$. To see this, we proceed as follows. We set

$$E_{K,0} = \{u \in E_K;\ \|u(t)\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}}\|u(t)\|_{\boldsymbol{L}^\infty} \to 0 \text{ as } t \to \infty\},$$

and observe that $E_{K,0}$ is a closed subset of (E_K, d'), where d' is given by (2.24). It is a straightforward consequence of (2.1) and (2.2) that if $u \in E_{K,0}$, then $\Phi(u,u) \in E_{K,0}$ where Φ is defined by (2.16). Next, using the radially symmetric function constructed in Remark 3.17 of [6], one can easily show that there exists $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ such that $\Omega_1(u_0) = \emptyset$ and $\|e^{t\Delta}u_0\|_{\boldsymbol{\mathcal{W}}} + t^{\frac{1}{2}}\|e^{t\Delta}u_0\|_{\boldsymbol{L}^\infty} \to 0$ as $t \to \infty$. This, along with formula (2.19), implies that $e^{\cdot\Delta}u_0 \in E_{\frac{M\delta}{2},0}$. It now follows from Remark 2.4 (iii) that $u = \boldsymbol{\mathcal{S}}(\cdot)u_0$ belongs to $E_{M\delta,0}$ provided δ is sufficiently small. In particular, $\omega_1(u) = \{0\}$.

Remark 5.8. It turns out that there is a large class of initial values u_0 for which $\Omega_1(u_0) = \Omega(u_0)$ and $\omega_1(u) = \omega(u)$. For example, let

$$u_0(x) = g(\log|x|)\zeta(x),$$

where $\zeta \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ is homogeneous (of degree -1) and $g \in C^1(\mathbb{R})$ satisfies $\|g\|_{L^\infty} \le 1$ (so that $u_0 \in \widetilde{\boldsymbol{\mathcal{C}}}_\delta$ by Remark 3.5 (ii)). Set $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$ for $t \ge 0$ and $v(s) = \boldsymbol{\mathcal{S}}(1)D_{e^s}u_0 = D_{\sqrt{t}}u(t)$ where $t = e^{2s}$, $s \in \mathbb{R}$. If $g(t)$ is almost periodic (respectively, asymptotically almost periodic as $t \to \infty$), then $v(s)$ is almost periodic (respectively, asymptotically almost periodic as $t \to \infty$) as a function from $\mathbb{R} \to \boldsymbol{\mathcal{W}} \cap \boldsymbol{C}_0(\mathbb{R}^N)$. In these cases, $\Omega_1(u_0) = \Omega(u_0)$ and $\omega_1(u) = \omega(u)$. The proof is completely analogous to the proof of Proposition 3.19 in [7], where the reader can also find the definition of an almost periodic function. Note that one may replace "almost periodic" by "periodic" in the above statements.

6. Translation invariance of the ω-limit sets

In this section of the paper, we prove the "intuitively obvious" result that the sets $\omega(u)$ and $\omega_1(u)$ are time-translation invariant. Since the definitions of $\omega(u)$ and $\omega_1(u)$ involve a time-dependent dilation, this in fact is not so obvious.

Lemma 6.1. *Let M, δ_0 be as in Theorem 1.1 and assume $0 < \delta < \delta_0$. It follows that there exists a constant C such that*

$$\|\boldsymbol{\mathcal{S}}(t)u_0 - \boldsymbol{\mathcal{S}}(s)u_0\|_{\boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty} \le C(t-s)^{\frac{1}{4}}, \tag{6.1}$$

for all $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and all $1 \le s \le t \le 2$, where $\boldsymbol{\mathcal{S}}(t)$ is given by Definition 1.3.

Proof. Let $u(t) = \boldsymbol{\mathcal{S}}(t)u_0$. Given $1 \le s < t \le 2$ and $1/2 \le \tau < s$, we write

$$u(t) - u(s) = (e^{(t-\tau)\Delta} - e^{(s-\tau)\Delta})u(\tau) - \int_\tau^t e^{(t-\sigma)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\sigma)\,d\sigma + \int_\tau^s e^{(s-\sigma)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\sigma)\,d\sigma = \Lambda_1 - \Lambda_2 + \Lambda_3,$$

and we estimate separately the Λ_j's. Since $u \in E_{M\delta}$ and $\tau \ge 1/2$, we see that there exists a constant C such that

$$\|u(\theta)\|_{\boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty} \le C, \tag{6.2}$$

for all $\tau \le \theta < \infty$. Therefore, it follows from (6.2) with $\theta = \tau$ and formula (4.2) in [6] that

$$\|\Lambda_1\|_{\boldsymbol{\mathcal{W}} \cap \boldsymbol{L}^\infty} \le C\frac{t-s}{s-\tau}. \tag{6.3}$$

Next, we deduce from (6.2) with $\theta = \sigma \ge \tau$ and formulas (2.1) and (2.4) (with $p = \infty$) that

$$\|e^{(t-\sigma)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\sigma)\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \le C(t-\sigma)^{-\frac{1}{2}};$$

and so,

$$\|\Lambda_2\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \le C\int_\tau^t (t-\sigma)^{-\frac{1}{2}}\,d\sigma \le C(t-\tau)^{\frac{1}{2}}. \tag{6.4}$$

Similarly,

$$\|\Lambda_3\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \le C(s-\tau)^{\frac{1}{2}}. \tag{6.5}$$

The result follows from (6.3), (6.4), and (6.5) by letting, for example, $s-\tau = \sqrt{t-s}/4$. □

Proposition 6.2. *Let M, δ_0 be as in Theorem 1.1, assume $0 < \delta < \delta_0$, and let $\boldsymbol{\mathcal{S}}(t)$ be given by Definition 1.3. It follows that there exists a constant C such that*

$$\|D_{\sqrt{t}}u(t+\tau) - D_{\sqrt{t}}u(t)\|_{\boldsymbol{\mathcal{W}}\cap\boldsymbol{L}^\infty} \le C(\tau/t)^{\frac{1}{4}} \underset{t\to\infty}{\longrightarrow} 0,$$

for all $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ and $\tau \ge 0$, $t \ge 1$, where $u(\cdot) = \boldsymbol{\mathcal{S}}(\cdot)u_0$. In particular, $\omega(u_\tau) = \omega(u)$ and $\omega_1(u_\tau) = \omega_1(u)$, where $u_\tau(t) \equiv u(\tau+t)$, ω is defined by (1.11), and ω_1 is defined by (1.21).

Proof. Let $\tau \ge 0$ and $t \ge 1$, $t \ge \tau$. It follows from (3.1) that

$$\begin{aligned} D_{\sqrt{t}}u(t+\tau) - D_{\sqrt{t}}u(t) &= D_{\sqrt{t}}\boldsymbol{\mathcal{S}}(t+\tau)\varphi - D_{\sqrt{t}}\boldsymbol{\mathcal{S}}(t)\varphi \\ &= \boldsymbol{\mathcal{S}}\Big(1+\frac{\tau}{t}\Big)D_{\sqrt{t}}\varphi - \boldsymbol{\mathcal{S}}(1)D_{\sqrt{t}}\varphi. \end{aligned}$$

D_λ being an isometry on $\boldsymbol{\mathcal{W}}$, we see that $D_{\sqrt{t}}\varphi \in \boldsymbol{\mathcal{C}}_\delta$ for all $t > 0$ and the result follows from (6.1). □

7. Appendix A

We prove the following estimate (see Oru [17]).

Lemma 7.1. *Given $0 \le \sigma < N < \gamma$, there exists a constant C such that*

$$\int_{\mathbb{R}^N} (1+|x-y|)^{-\gamma}|y|^{-\sigma}\,dy \le C(1+|x|^2)^{-\frac{\sigma}{2}}, \tag{7.1}$$

for all $x \in \mathbb{R}^N$.

Proof. We denote by $I(x)$ the left-hand side of (7.1). We first note that

$$\int_{\{|y|<1\}} (1+|x-y|)^{-\gamma}|y|^{-\sigma}\,dy \le \int_{\{|y|<1\}} |y|^{-\sigma} < \infty, \tag{7.2}$$

and

$$\begin{aligned}\int_{\{|y|>1\}} (1+|x-y|)^{-\gamma}|y|^{-\sigma}\,dy &\le \int_{\mathbb{R}^N} (1+|x-y|)^{-\gamma}\,dy \\ &= \int_{\mathbb{R}^N} (1+|y|)^{-\gamma}\,dy < \infty.\end{aligned} \tag{7.3}$$

We thus deduce from (7.2)-(7.3) that $\sup I(x) < \infty$. Consider now $|x| \neq 0$. If $|y| < |x|/2$, then $|x-y| \ge |x|/2$, so that

$$\begin{aligned}\int_{\{|y|<|x|/2\}} (1+|x-y|)^{-\gamma}|y|^{-\sigma}\,dy &\le 2^{\gamma}|x|^{-\gamma}\int_{\{|y|<|x|/2\}} |y|^{-\sigma}\,dy \\ &\le C|x|^{-\gamma+N-\sigma}.\end{aligned} \tag{7.4}$$

Finally,

$$\begin{aligned}\int_{\{|y|>|x|/2\}} (1+|x-y|)^{-\gamma}|y|^{-\sigma}\,dy &\le 2^{\sigma}|x|^{-\sigma}\int_{\mathbb{R}^N} (1+|x-y|)^{-\gamma}\,dy \\ &\le C|x|^{-\sigma},\end{aligned} \tag{7.5}$$

and it follows from (7.4)-(7.5) that $\sup_{\{|x|\ge 1\}} |x|^{\sigma} I(x) < \infty$. This proves (7.1). □

8. Appendix B

In this appendix we show that, for $N = 2$, there exist initial values $u_0 \in \boldsymbol{\mathcal{C}}_\delta$ such that, if u is the resulting solution of (1.2) given by Theorem 1.1, then the asymptotic behavior of u, as described by the set $\omega(u) \subset \boldsymbol{C}_0(\mathbb{R}^2)$, is "infinite dimensional". More precisely, in light of Remark 1.8 (and in particular formula (1.13)), we wish to construct u_0 such that $\Omega(u_0)$ contains an infinite-dimensional ball of homogeneous functions. Recall that if $N = 3$, this was acomplished by Theorem 1.10 and Proposition 3.7. By Remark 3.9, this approach will not work for $N = 2$.

The method we employ, unfortunately, is somewhat technical. The basic idea can be described as follows. If $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is homogeneous of degree 2, $\varphi(x) \neq 0$ for $x \neq 0$, then

$$w = \frac{(-\partial_2\varphi, \partial_1\varphi)}{\varphi} \in C^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2),$$

is homogeneous of degree -1 and divergence free. It is not hard to construct a linearly independent infinite set of such functions $(w_i)_{i\geq 1}$. The main technical problem is to construct a divergence-free initial value which is equal to arbitrary linear combinations of the w_i (with rational coefficients) on successively larger rings around the origin. For this purpose, we use the rather complicated cut-off functions described in Proposition 8.3 below.

In fact, we prove the following result.

Theorem 8.1. *Let $(\varphi_i)_{i\geq 1} \subset C^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ and suppose there exist $0 < \alpha_i < \beta_i$ and $\gamma_i > 0$ such that*

$$\alpha_i|x|^2 \leq \varphi_i(x) \leq \beta_i|x|^2, \tag{8.1}$$

and

$$|\nabla\varphi_i(x)| \leq \gamma_i|x|, \tag{8.2}$$

for all $i \geq 1$ and $x \in \mathbb{R}^2$. Assume that

$$\sup_{i\geq 1} \frac{\gamma_i}{\alpha_i} = L < \infty, \tag{8.3}$$

and define $w_i \in C^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}^2)$ by

$$w_i(x) = \frac{(-\partial_2\varphi_i(x), \partial_1\varphi_i(x))}{\varphi_i(x)} = \nabla^\perp \log \varphi_i, \tag{8.4}$$

so that $(w_i)_{i\geq 1} \subset \boldsymbol{\mathcal{W}}$ and

$$\sup_{i\geq 1} \|w_i\|_{\boldsymbol{\mathcal{W}}} \leq L. \tag{8.5}$$

For every $r > 0$, there exists $u_0 \in \boldsymbol{\mathcal{C}}_{rL}$ such that $\boldsymbol{E}_r \subset \Omega(u_0)$, where

$$\boldsymbol{E}_r = \{z \in \boldsymbol{\mathcal{W}};\, z = \sum_{i=1}^\infty \ell_i w_i \text{ where } \sum_{i=1}^\infty |\ell_i| \leq r\} \subset \boldsymbol{\mathcal{C}}_{rL}. \tag{8.6}$$

Suppose, in addition, that $rL < \delta_0$ as in Theorem 1.1, that all the φ_i are homogeneous of degree 2 (so that all the elements of $\boldsymbol{E}_r$ are homogeneous of degree -1), and that the resulting w_i are linearly independent. If $u = \boldsymbol{\mathcal{S}}(\cdot)u_0$, then for every $n \in \mathbb{N}$, there exists a subset $B_n \subset \boldsymbol{\mathcal{S}}(1)\boldsymbol{E}_r \subset \omega(u)$ which is homeomorphic, as a subset of $\boldsymbol{C}_0(\mathbb{R}^2)$, to a ball of dimension n.

Remark 8.2. There exist functions φ_i that satisfy all the assumptions of Theorem 8.1. For example, we define the sequence $(\varphi_i)_{i\geq 1} \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ in polar coordinates by

$$\varphi_i(r, \theta) = [2 + 2^{-i}\cos(2^{i+1}\pi\theta)]r^2.$$

It is clear that the φ_i are homogeneous functions of degree 2 satisfying conditions (8.1), (8.2) and (8.3). We claim that the corresponding w_i defined by (8.4) are linearly independent. To see this, we assume that for $\ell_m, \dots, \ell_{m+n} \in \mathbb{R}$ with $\ell_m \neq 0$,

$$0 = \sum_{i=m}^{m+n} \ell_i w_i = \nabla^\perp \log\Big(\prod_{i=m}^{m+n} \varphi_i^{\ell_i} \Big);$$

and so, $\prod_{i=m}^{m+n} \varphi_i^{\ell_i} \equiv C$. This is impossible since $\prod_{i=m+1}^{m+n} \varphi_i^{\ell_i}$ has period 2^{-m-1} in θ, but $\varphi_m^{\ell_m}$ does not.

The rest of the appendix is devoted to the proof of Theorem 8.1. Denote by $\ell^\infty_{\mathrm{c}}(\mathbb{N}, \mathbb{Z})$ the set of all $\mathbb{Z}$-valued sequences with only finitely many nonzero elements. It is straightforward to check that if $\boldsymbol{k} = (k_1, k_2, \cdots) \in \ell^\infty_{\mathrm{c}}(\mathbb{N}, \mathbb{Z})$, then

$$\sum_{i=1}^{\infty} \frac{k_i}{\|\boldsymbol{k}\|_{\ell^1}} w_i = v, \tag{8.7}$$

where

$$v(x) = \frac{1}{\|\boldsymbol{k}\|_{\ell^1}} \frac{(-\partial_2 \psi(x), \partial_1 \psi(x))}{\psi(x)}, \tag{8.8}$$

with

$$\psi(x) = \prod_{i \ge 1}^{\infty} \varphi_i(x)^{k_i}, \tag{8.9}$$

for $x \neq 0$. Note that, by (8.5) and (8.7),

$$\|v\|_{\mathcal{W}} \le L. \tag{8.10}$$

Suppose, in addition, that $\boldsymbol{k}$ satisfies

$$\sum_{i=1}^{\infty} k_i \neq 0. \tag{8.11}$$

It follows that

$$c|x|^p \le \psi(x) \le d|x|^p, \tag{8.12}$$

where

$$p = 2 \sum_{i=1}^{\infty} k_i \neq 0, \tag{8.13}$$

and

$$c = \prod_{k_i > 0} \alpha_i^{k_i} \prod_{k_i < 0} \beta_i^{k_i} < d = \prod_{k_i < 0} \alpha_i^{k_i} \prod_{k_i > 0} \beta_i^{k_i}. \tag{8.14}$$

Let $0 < a < b$ and let $\eta_{\boldsymbol{k}} \in C^\infty([0,\infty))$ be such that

$$0 \le \eta_{\boldsymbol{k}} \le 1, \tag{8.15}$$

and

$$\eta_{\boldsymbol{k}}(t) = \begin{cases} 0 & t \le ca^p/2, \\ 1 & ca^p \le t \le db^p, \\ 0 & t \ge 2db^p, \end{cases} \tag{8.16}$$

if $p > 0$ and

$$\eta_{\boldsymbol{k}}(t) = \begin{cases} 0 & t \le cb^p/2, \\ 1 & cb^p \le t \le da^p, \\ 0 & t \ge 2da^p, \end{cases} \tag{8.17}$$

if $p < 0$ (p is given by (8.13) and c, d are given by (8.14)).

Proposition 8.3. *Let $0 < a < b$, let $\boldsymbol{k} \in \ell^\infty_c(\mathbb{N}, \mathbb{Z})$ satisfy (8.11) and let $\eta_{\boldsymbol{k}}$ be defined by (8.16)-(8.17). If $\widetilde{\psi} \in C(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ satisfies (8.12), then*

$$\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 1, \tag{8.18}$$

for $a \le |x| \le b$. Moreover,

$$\operatorname{supp}(\eta_{\boldsymbol{k}} \circ \widetilde{\psi}) \subset \mathcal{A}_{\boldsymbol{k}} = \{x \in \mathbb{R}^2;\ \Big(\frac{c}{2d}\Big)^{\frac{1}{|p|}} a \le |x| \le \Big(\frac{2d}{c}\Big)^{\frac{1}{|p|}} b\}. \tag{8.19}$$

Proof. Suppose first $p > 0$. If $a \le |x| \le b$, then

$$ca^p \le c|x|^p \le \widetilde{\psi}(x) \le d|x|^p \le db^p,$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 1$. Next, if $|x| \le (c/2d)^{\frac{1}{p}} a$; i.e., $|x|^p \le (c/2d)a^p$, then

$$\widetilde{\psi}(x) \le d|x|^p \le \frac{ca^p}{2},$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 0$. Similarly, if $|x| \ge (2d/c)^{\frac{1}{p}} b$; i.e., $|x|^p \ge (2d/c)b^p$, then

$$\widetilde{\psi}(x) \ge c|x|^p \ge 2db^p,$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 0$.

Suppose now $p < 0$. If $a \le |x| \le b$, then

$$cb^p \le c|x|^p \le \widetilde{\psi}(x) \le d|x|^p \le da^p,$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 1$. Next, if $|x| \le (2d/c)^{\frac{1}{p}} a$; i.e., $|x|^p \ge (2d/c)a^p$, then

$$\widetilde{\psi}(x) \ge c|x|^p \ge 2da^p,$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 0$. Similarly, if $|x| \ge (c/2d)^{\frac{1}{p}} b$; i.e., $|x|^p \le (c/2d) b^p$, then

$$\widetilde{\psi}(x) \le d|x|^p \le \frac{cb^p}{2},$$

and so $\eta_{\boldsymbol{k}}(\widetilde{\psi}(x)) = 0$. □

Remark 8.4. If ψ satisfies (8.12), then $\widetilde{\psi}$ defined by $\widetilde{\psi}(x) = \lambda^p \psi(x/\lambda)$ also satisfies (8.12).

Consider a sequence $(\boldsymbol{k}_n)_{n\ge1} \subset \boldsymbol{K}$, where

$$\boldsymbol{K} = \{\boldsymbol{k} \in \ell^\infty_{\mathrm{c}}(\mathbb{N}, \mathbb{Z});\ \boldsymbol{k} \text{ satisfies (8.11)}\}. \tag{8.20}$$

Let $v_n, \psi_n, p_n, c_n, d_n$ be as in (8.8), (8.9), (8.13), and (8.14), respectively. Let $\eta_n = \eta_{\boldsymbol{k}_n}$ as defined by (8.16)-(8.17). We define recursively the sequences

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots, \tag{8.21}$$

by

$$\begin{cases} a_1 = 1, \\ b_n = (n+1) a_n, \\ a_{n+1} = 2\Big(\frac{2d_{n+1}}{c_{n+1}}\Big)^{\frac{1}{|p_{n+1}|}} \Big(\frac{2d_n}{c_n}\Big)^{\frac{1}{|p_n|}} b_n. \end{cases} \tag{8.22}$$

It follows easily that, if $\mathcal{A}_n = \mathcal{A}_{\boldsymbol{k}_n}$ as defined in (8.19), then

$$\mathcal{A}_n \cap \mathcal{A}_m = \emptyset \quad \text{if} \quad n \ne m, \tag{8.23}$$

and

$$\{|x| < (c_1/2d_1)^{\frac{1}{|p_1|}}\} \subset \mathbb{R}^2 \setminus (\underset{n\ge1}{\cup} \mathcal{A}_n). \tag{8.24}$$

Set

$$\lambda_n = \sqrt{a_n b_n}, \tag{8.25}$$

so that

$$\frac{a_n}{\lambda_n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \frac{b_n}{\lambda_n} \underset{n\to\infty}{\longrightarrow} \infty. \tag{8.26}$$

Given a sequence $(\rho_n)_{n\ge1} \subset [0, r]$, let

$$u_0(x) = \sum_{n=1}^\infty \rho_n \eta_n(\lambda_n^{p_n} \psi_n(\lambda_n^{-1} x))(D_{\frac{1}{\lambda_n}} v_n)(x). \tag{8.27}$$

We deduce from Proposition 8.3 and Remark 8.4 that, for every $n \ge 1$, the function $\rho_n \eta_n(\lambda_n^{p_n} \psi_n(\lambda_n^{-1} x))(D_{\frac{1}{\lambda_n}} v_n)(x)$ is supported in $\mathcal{A}_n$. It follows from (8.23) that for any given $x \neq 0$, the sum (8.27) consists of one element and

$$\begin{aligned} \|u_0\|_{\boldsymbol{\mathcal{W}}} &\le \sup_{n\ge 1} \|\rho_n \eta_n(\lambda_n^{p_n}\psi_n(\lambda_n^{-1}\cdot)) D_{\frac{1}{\lambda_n}} v_n\|_{\boldsymbol{\mathcal{W}}} \\ &\le r\|D_{\frac{1}{\lambda_n}} v_n\|_{\boldsymbol{\mathcal{W}}} = r\|v_n\|_{\boldsymbol{\mathcal{W}}} \le rL, \end{aligned} \tag{8.28}$$

where we used (8.15) and (8.10). Furthermore, (8.18) implies that

$$(D_{\lambda_n} u_0)(x) = \rho_n v_n(x) \quad \text{for} \quad \frac{a_n}{\lambda_n} \le |x| \le \frac{b_n}{\lambda_n}. \tag{8.29}$$

Moreover, it follows from (8.24) that

$$u_0 \in C^\infty(\mathbb{R}^2, \mathbb{R}^2). \tag{8.30}$$

We claim that

$$\nabla \cdot u_0 = 0. \tag{8.31}$$

Indeed, it suffices to show that $\rho_n \eta_n(\lambda_n^{p_n}\psi_n(\lambda_n^{-1}x))(D_{\frac{1}{\lambda_n}} v_n)(x)$ is divergence free for each $n \ge 1$. Applying D_{λ_n}, we need only show that if ω is any function of $C_{\mathrm{c}}^\infty(0,\infty)$, then

$$\nabla \cdot (\omega(\psi) v) = 0.$$

To see this, we observe that we may write

$$\omega(\psi) v = \widetilde{\omega}(\psi)(-\partial_2\psi, \partial_1\psi),$$

with $\widetilde{\omega} \in C_{\mathrm{c}}^\infty(0,\infty)$. Thus we have

$$\nabla \cdot (\omega(\psi) v) = \widetilde{\omega}(\psi)\nabla \cdot (-\partial_2\psi, \partial_1\psi) + \widetilde{\omega}'(\psi)\nabla\psi \cdot (-\partial_2\psi, \partial_1\psi) = 0,$$

which proves the claim (8.31).

Finally, we observe that $[0,r] \cap \mathbb{Q}$ and $\boldsymbol{K}$ are two countable sets. Thus, there exist sequences $(\rho_n)_{n\ge 1} \subset [0,r] \cap \mathbb{Q}$ and $(\boldsymbol{k}_n)_{n\ge 1} \subset \boldsymbol{K}$ such that every element $(\rho, \boldsymbol{k}) \in ([0,r] \cap \mathbb{Q}) \times \boldsymbol{K}$ appears infinitely often in the sequence $(\rho_n, \boldsymbol{k}_n)$. We consider u_0 given by (8.27) for these particular sequences. It follows from (8.29) and (8.26) that $\Omega(u_0)$ contains every finite linear combination $\sum \ell_i w_i$ such that $\ell_i \in \mathbb{Q}$, $\sum |\ell_i| \le r$ and $\sum \ell_i \neq 0$. Since $\Omega(u_0)$ is closed in $(\boldsymbol{\mathcal{C}}_{rL}, \mathrm{d}_{rL}^*)$, it follows that $\boldsymbol{E}_r \subset \Omega(u_0)$.

To prove the last statement of the theorem, we observe that $\boldsymbol{\mathcal{S}}(1) : (\boldsymbol{E}_r, \mathrm{d}_{rL}^*) \to \boldsymbol{C}_0(\mathbb{R}^2)$ is injective by Proposition 3.3 and continuous by Proposition 2.5, and thus a homeomorphism onto its image.

References

[1] L. Brandolese, *Space-time decay of Navier-Stokes flows invariant under rotation,* Math. Ann., 329 (2004), 685–706.

[2] J. Bricmont, A. Kupiainen, and G. Lin, *Renormalization group and asymptotics of solutions of nonlinear parabolic equations,* Comm. Pure Appl. Math., 47 (1994), 893–922.

[3] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations,* Rev. Mat. Iberoamericana, 13 (1997), 515–541.

[4] M. Cannone and F. Planchon, *Self-similar solutions of the Navier-Stokes equations in* $\mathbb{R}^3$*,* Comm. Partial Differential Equations , 21 (1996), 179–193.

[5] T. Cazenave, F. Dickstein, M. Escobedo, and F.B. Weissler, *Self-similar solutions of a nonlinear heat equation,* J. Math. Sci. Univ. Tokyo, 8 (2001), 501–540.

[6] T. Cazenave, F. Dickstein, and F.B. Weissler, *Universal solutions of the heat equation,* Discrete Contin. Dynam. Systems, 9 (2003), 1105–1132.

[7] T. Cazenave, F. Dickstein, and F.B. Weissler, *Universal solutions of a nonlinear heat equation,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), 2 (2003), 77–117.

[8] T. Cazenave and F.B. Weissler, *Spatial decay and time-asymptotic profiles for solutions of Schrödinger equations,* preprint, 2004.

[9] R.L. Devaney, *Dynamics of simple maps,* in Chaos and fractals (Providence, RI, 1988), Proc. Sympos. Appl. Math., 39, Amer. Math. Soc., Providence, RI, 1989, 1–24.

[10] Y. Fujigaki and T. Miyakawa, *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the whole space,* SIAM J. Math. Anal., 33 (2001), 523–544.

[11] T. Gallay and C.E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbb{R}^2$*,* Arch. Ration. Mech. Anal., 163 (2002), 209–258.

[12] T. Gallay and C.E. Wayne, *Long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbb{R}^3$*,* in Recent developments in the mathematical theory of water waves (Oberwolfach, 2001), R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 360 (2002), 2155–2188.

[13] Y. Giga and T. Miyakawa, *Navier-Stokes flow in* $\mathbb{R}^3$ *with measures as initial vorticity and Morrey spaces,* Comm. Partial Differential Equations , 14 (1989), 577–618.

[14] P.-G. Lemarié-Rieusset, "Recent developments in the Navier-Stokes problem," Research Notes in Mathematics, 431, Chapman & Hall/CRC, London, 2002.

[15] T. Miyakawa, *On space-time decay properties of nonstationary incompressible Navier-Stokes flows in* $\mathbb{R}^N$*,* Funkcial. Ekvac., 43 (2000), 541–557.

[16] T. Miyakawa and M. Schonbek *On optimal decay rates for weak solutions to the Navier-Stokes equations in* $\mathbb{R}^N$*,* Proceedings of Partial Differential Equations and Applications (Olomouc, 1999), Math. Bohem., 126 (2001), 443–455.

[17] F. Oru, Rôle des oscillations dans quelques problèmes d'analyse non-linéaire, Doctoral thesis, ENS Cachan, 1998.

[18] F. Planchon, *Asymptotic behavior of global solutions to the Navier-Stokes equations in* $\mathbb{R}^N$*,* Rev. Mat. Iberoamericana, 14 (1998), 71–93.

[19] J.L. Vázquez and E. Zuazua, *Complexity of large time behaviour of evolution equations with bounded data,* Chinese Ann. Math. Ser. B, 23 (2002), 293–310.