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ON THE STOKES AND NAVIER-STOKES EQUATIONS IN A PERTURBED HALF-SPACE

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Abstract. We discuss L^p-L^q type estimates of the Stokes semigroup and their application to the Navier-Stokes equation in a perturbed half-space. Especially, we have the L^p-L^q type estimate of the gradient of the Stokes semigroup for any p and q with $1 < p \leqq q < \infty$, while the same estimate holds only for the exponents p and q with $1 < p \leqq q \leqq n$ in the exterior domain case, where n denotes the space dimension, and, therefore, we can get better results concerning the asymptotic behaviour of solutions to the Navier-Stokes equations compared with the exterior domain case.

Our proof of the L^p-L^q type estimate of the Stokes semigroup is based on the local energy decay estimate obtained by investigation of the asymptotic behavior of the Stokes resolvent near the origin. The order of asymptotic expansion of the Stokes resolvent near the origin is one half better compared with the exterior domain case, because we have the reflection principle on the boundary in the half-space case unlike the whole space case, and, then, such better asymptotics near the boundary are also obtained in the perturbed half-space by a perturbation argument. This is one of the reasons why the result in the perturbed half-space case is essentially better compared with the exterior domain case.

1. Introduction

Let Ω be a perturbed half-space with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$: to be precise we call an open set Ω the perturbed half-space if there is a positive number R such that $\Omega \backslash B_R = H \backslash B_R$ where $H := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

In $\Omega \times (0, \infty)$ we consider the nonstationary Stokes initial-boundary value problem concerning the velocity field $u(x, t)$ and the scalar pressure $\pi(x, t)$:

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$$\begin{cases} u_t - \Delta u + \nabla \pi = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

where $u_t = \partial u/\partial t$, Δ is the Laplacian, $\nabla = (\partial_1, \ \dots, \ \partial_n)$ with $\partial_j = \partial/\partial x_j$ is the gradient and $\nabla \cdot u = \mathrm{div} u = \sum_{j=1}^n \partial_j u_j$ is the divergence of u.

We shall introduce the known results concerning the half-space and the perturbed half-space Ω. To this end, we first define the spaces $J^p(\Omega)$ and $G^p(\Omega)$ by the relations:

$$J^p(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n : \ \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)^n}},$$
$$G^p(\Omega) = \{\nabla p \in L^p(\Omega)^n : p \in L^p_{loc}(\overline{\Omega})\}.$$

For the half-space and the perturbed half-space, R. Farwig and H. Sohr [13] proved that the Banach space $L^p(\Omega)^n$ $(1 < p < \infty)$ admits the Helmholtz decomposition : $L^p(\Omega)^n = J^p(\Omega) \oplus G^p(\Omega)$, where $\oplus$ denotes the direct sum. Let P be a continuous projection from $L^p(\Omega)^n$ to $J^p(\Omega)$. The Stokes operator A is defined by $A = -P\Delta$ with domain $D(A) = \{u \in J^p(\Omega) \cap W^{2,p}(\Omega)^n : u|_{\partial\Omega} = 0\}$. It is proved by R. Farwig and H. Sohr that $-A$ generates an analytic semigroup $T(t)$ on $J^p(\Omega)$.

The purpose of this paper is to prove the L^p-L^q estimate:

$$\|T(t)a\|_{L^q(\Omega)^n} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L^p(\Omega)^n}, \tag{1.2}$$

$$\|\nabla T(t)a\|_{L^q(\Omega)^{n^2}} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L^p(\Omega)^n} \tag{1.3}$$

where $a \in J^p(\Omega)$ and $t > 0$.

The L^p-L^q estimates of the Stokes semigroup have been already studied by many authors for different cases of the domains. In fact, when $\Omega = \mathbb{R}^n$, applying the Young inequality to the concrete solution formula, we have (1.2) and (1.3) for $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty, \ q \neq 1)$. When $\Omega = H$, applying the Fourier multiplier theorem to the concrete solution formula obtained by Ukai [24] and using the Sobolev inequality, we have:

$$\|\nabla^j u\|_{L_q(H)} \leqq C \|\nabla^m u\|^a_{L_p(H)} \|u\|^{1-a}_{L_p(H)} \tag{1.4}$$

provided that $0 \leqq j < m$, $1 \leqq p < \infty$, $m - j - n/p$ are not nonnegative integers, $j/m < a \leqq 1$ and $1/q = j/n + 1/p - am/n \geqq 0$, and we have (1.2) and (1.3) for $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty, \ q \neq 1)$ (cf. Borchers and Miyakawa [3] and Desch, Hieber and Prüss [10]).

When Ω is an exterior domain, (1.2) holds for $1 \leqq p \leqq q \leqq \infty$ ($p \neq \infty$, $q \neq 1$) but (1.3) holds only for $1 \leqq p \leqq q \leqq n$ ($q \neq 1$). This result was first proved by Iwashita [15] for $1 < p \leqq q < \infty$ in (1.2) and $1 < p \leqq q \leqq n$ in (1.3) when $n \geqq 3$. The refinement of his result was done by the following authors: Chen [6] ($n = 3$, $q = \infty$), Shibata [21] ($n = 3$, $q = \infty$), Borchers and Varnhorn [5] ($n = 2$, (1.2) for $p = q$), Dan and Shibata [7], [8] ($n = 2$), Dan, Kobayashi and Shibata [9] ($n = 2, 3$), and Maremonti and Solonnikov [19] ($n \geqq 2$). Especially, that Iwashita's restriction, $q \leqq n$ in (1.3), is unavoidable was shown by Maremonti and Solonnikov [19].

When Ω is an aperture domain, Abels [1] proved (1.2) for $1 < p \leqq q < \infty$ and (1.3) for $1 < p \leqq q < n$ when $n \geqq 3$; and Hishida [14] proved (1.2) for $1 \leqq p \leqq q \leqq \infty$ and (1.3) for $1 \leqq p \leqq q \leqq n$ ($q \neq 1$) and $1 \leqq p < n < q < \infty$ when $n \geqq 3$.

To discuss our results more precisely, at first we outline at this point our notation used throughout the paper. We fix R_0 such that $\Omega \backslash B_{R_0} = H \backslash B_{R_0}$. Given $R \geq R_0$, we take (and fix) the cut-off function ψ_R satisfying

$$\psi_R \in C^\infty(\mathbb{R}^n; [0,1]), \quad \psi_R = \begin{cases} 1 & \text{for } |x| > R+1, \\ 0 & \text{for } |x| < R. \end{cases} \tag{1.5}$$

To denote the special sets, we use the following symbols:

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}, \;\; \Omega_R = \Omega \cap B_R, \;\; B_R^+ = H \cap B_R,$$
$$D_R = \{x \in \mathbb{R}^n : R < |x| < R+1\}, \;\; D_R^+ = H \cap D_R.$$

We will use the standard symbol $L^q(\Omega)$ with norm $\| \cdot \|_{L^q(\Omega)}$. We set

$$\begin{aligned}
L^p_R(\Omega)^n &= \{u \in L^p(\Omega)^n : u(x) = 0 \text{ for } |x| > R\}, \\
W_0^{N,p}(D) &= \{f \in W^{N,p}(D) : \partial_x^\alpha f|_{\partial D} = 0 \quad \text{for} \quad |\alpha| \leq N-1\}, \qquad N \geq 1, \\
\dot{W}^{N,p}(D) &= \{f \in W_0^{N,p}(D) : \int_D f dx = 0\}, \qquad N \geq 1, \\
\dot{W}^{0,p}(D) &= \{f \in L^p(D) : \int_D f dx = 0\}, \\
\widehat{W}^{1,p}(D) &= \{f \in L^p_{loc}(D) : \nabla f \in L^p(D)\}.
\end{aligned}$$

For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $\mathcal{L}(X) = \mathcal{L}(X,X)$. $\mathcal{B}(U;X)$ defines the set of all X-valued bounded holomorphic functions on U, and $BC([0,T);X)$ denotes the class of X-valued bounded continuous functions on $[0,T)$. To denote various constants we use the same letter C, and by $C_{A,B,\cdots}$, we denote a constant depending on the quantities $A, B, \cdots$. The constants C and $C_{A,B,\cdots}$ may change from line to line.

Now we state our main results.

Theorem 1.1 (Local energy decay)**.** *Let $n \geq 2$. Let $1 < p < \infty$ and let m be a nonnegative integer. R is any positive number such that $\Omega \backslash B_R = H \backslash B_R$. Then there exists a positive constant $C_{p,m}$ such that*

$$\|\partial_t^m T(t) P f\|_{W^{2,p}(\Omega_R)^n} \leq C_{p,m} t^{-\frac{n+1}{2}-m} \|f\|_{L^p(\Omega)^n} \tag{1.6}$$

for any $t \geq 1$ and $f \in L^p_R(\Omega)^n$.

Theorem 1.2 (L^p-L^q estimates)**.** *Let $n \geq 2$.*

(1) *For all $t > 0$, $f \in J^p(\Omega)$, and $1 \leq p \leq q \leq \infty$ $(p \neq \infty,\ q \neq 1)$, there holds the estimate:*

$$\|T(t) f\|_{L^q(\Omega)^n} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)^n}. \tag{1.7}$$

(2) *For all $t > 0$, $f \in J^p(\Omega)$, and $1 < p \leq q < \infty$, there holds the estimate:*

$$\|\nabla T(t) f\|_{L^q(\Omega)^{n^2}} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^p(\Omega)^n}. \tag{1.8}$$

Next we apply the L^p-L^q estimate to the Navier-Stokes initial-value problem. To this end we consider the Navier-Stokes equation in a perturbed half-space:

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \pi = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega. \end{cases} \tag{NS}$$

Applying the solenoidal projection P to (NS), we can rewrite (NS) as follows:

$$u_t + Au + P(u \cdot \nabla u) = 0, \quad u(0) = a \tag{P-NS}$$

where $A = -P\Delta$ is the Stokes operator.

For given $a \in J^n(\Omega)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0,T)$ is called a strong solution of (NS) on $(0,T)$ if u belongs to

$$u \in C([0,T); J^n(\Omega)) \cap C((0,T); D(A)) \cap C^1((0,T); J^n(\Omega))$$

together with $\lim_{t\to 0} \|u(t) - a\|_{L^n} = 0$ and satisfies (P-NS) for $0 < t < T$ in $J^n(\Omega)$.

We can show the next theorem which tells us the global existence of a strong solution of (NS) that has several decay properties with small $\|a\|_{L^n}$:

Theorem 1.3. *Let $n \geq 2$. There is a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $a \in J^n(\Omega)$ satisfies $\|a\|_{L^n} \leq \delta$, the problem* (NS) *admits a unique strong solution $u(t)$ on $(0,\infty)$. Moreover, as $t \to \infty$,*

$$\|u(t)\|_{L^r} = o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad \text{for} \quad n \leq r \leq \infty, \tag{1.9}$$

$$\|\nabla u(t)\|_{L^r} = o(t^{-1+\frac{n}{2r}}) \qquad \text{for} \quad n \leq r < \infty. \tag{1.10}$$

If we assume that $a \in L^1 \cap J^n(\Omega)$ has small $\|a\|_{L^n}$, then we can show better decay properties of the solutions as in the following theorem:

Theorem 1.4. *Let $n \geq 2$. There is a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $a \in L^1 \cap J^n(\Omega)$ and a satisfies $\|a\|_{L^n} \leq \eta$, then the solution $u(t)$ obtained in Theorem* 1.3 *enjoys, as $t \to \infty$,*

$$\|u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \qquad \text{for} \quad 1 < r \leq \infty, \tag{1.11}$$

$$\|\nabla u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \qquad \text{for} \quad 1 < r < \infty. \tag{1.12}$$

2. PRELIMINARIES

We shall consider the Stokes resolvent problem in H:

$$(\lambda - \Delta)u + \nabla\pi = f, \quad \nabla \cdot u = 0 \quad \text{in } H, \quad u(x', 0) = 0. \tag{2.1}$$

Let $R(\lambda)f$ and $\Pi(\lambda)f$ be defined as solutions of $R(\lambda)f = u$ and $\Pi(\lambda)f = \pi$ which satisfy (2.1). By Farwig-Sohr[13] we can construct $R(\lambda)f \in W^{2,p}(H)^n$ and $\Pi(\lambda)f \in \widehat{W}^{1,p}(H)$ by using partial Fourier transforms, which satisfy the estimate:

$$\|R(\lambda)f\|_{W^{2,p}(H)^n} + \|\nabla\Pi(\lambda)f\|_{L^p(H)} \leq C_{\varepsilon,\lambda_0}\|f\|_{L^p(H)^n}$$

provided that $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \pi - \varepsilon\}$ for $\varepsilon > 0$ and $|\lambda| \geq \lambda_0$. We shall investigate the property of $R(\lambda)f$ and $\Pi(\lambda)f$ near $\lambda = 0$ when f has compact support. In [17], we have the following expansion formula of the solution operator $(R(\lambda), \Pi(\lambda))$ to (2.1) near $\lambda = 0$:

Theorem 2.1. *Let $n \geq 2$. We set $U_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and*

$$B = \mathcal{L}(L^p_R(H), W^{2,p}(B^+_R)^n \times W^{1,p}(B^+_R)).$$

Then $(R(\lambda), \Pi(\lambda))$ has the following expansion formula with respect to $\lambda \in U_{1/2}\backslash(-\infty, 0]$:

$$(R(\lambda), \Pi(\lambda)) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}}\log\lambda + G_3(\lambda) & \text{where } n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}}\log\lambda + G_3(\lambda) & \text{where } n \text{ is odd} \end{cases} \tag{2.2}$$

where $G_1(\lambda), G_2(\lambda)$ and $G_3(\lambda)$ are B-valued holomorphic functions in $U_{1/2}$.

Also, we have proved some properties of $(R(\lambda), \Pi(\lambda))$ at $\lambda = 0$ in [17].

Theorem 2.2. *Let $n \geq 2$, $1 < p < \infty$ and $f = (f_1, \ldots, f_n) \in L^p(H)^n$ with supp $f \subset B_R$. Let $R(\lambda)$ and $\Pi(\lambda)$ be the solution operators to (2.1) for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. Then, there exist operators $R(0): L^p_R(H) \to W^{2,p}_{loc}(H)^n$ and $\Pi(0): L^p_R(H) \to W^{1,p}_{loc}(H)$ which have the following properties:*

(i) *If we set $u = R(0)f$ and $\pi = \Pi(0)f$, then (u, π) satisfies the equation:*

$$-\Delta u + \nabla \pi = f, \ \nabla \cdot u = 0 \quad in\ H, \qquad u(x', 0) = 0.$$

(ii) *(u, π) satisfies the estimates:*

$$\|u\|_{W^{2,p}(B_L^+)^n} + \|\pi\|_{W^{1,p}(B_L^+)} \leq C_{R,L}\|f\|_{L^p(H)^n} \quad for \quad L > 0,$$

$$\sup_{|x|\geq 1,\ x\in H} \left[|x|^{n-1}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\right] \leq C_R\|f\|_{L^p(H)^n}$$

and the formula: for any $L > 0$ and $\lambda \in U_{1/2}\backslash(-\infty, 0]$

$$\|R(\lambda)f - R(0)f\|_{W^{2,p}(B_L^+)^n} \leq C\, p(|\lambda|)\|f\|_{L^p(H)^n},$$

$$\|\Pi(\lambda)f - \Pi(0)f\|_{W^{1,p}(B_L^+)} \leq C\, p(|\lambda|)\|f\|_{L^p(H)^n}$$

where $p(t) = \max(t,\ t^{\frac{n-1}{2}}|\log t|)$ for $t \in [0, 1)$.

Lemma 2.3. *Let $n \geq 2$ and $1 < p < \infty$. Let $\Omega = H$ or a perturbed half-space. Let $u \in W^{2,p}_{loc}(\Omega)^n$ and $\pi \in W^{1,p}_{loc}(\Omega)$ enjoy*

$$-\Delta u + \nabla \pi = 0, \ \nabla \cdot u = 0 \quad in \quad \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.3}$$

Moreover,

$$\sup_{x\in B_{R+3}} \left[|x|^{n-1}|u(x)| + |x|^{n-1}|\nabla u(x)| + |x|^{n-1}|\pi(x)|\right] < \infty. \tag{2.4}$$

Then $u = 0$ and $\pi = 0$.

Proof. Considering local regularity, we may assume that $u \in W^{2,q}_{loc}(\Omega)^n$ and $\pi \in W^{1,q}_{loc}(\Omega)$ for any $q \in (1, \infty)$. In particular we may assume that $u \in W^{2,2}_{loc}(\Omega)^n$ and $\pi \in W^{1,2}_{loc}(\Omega)$. Now we choose $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| < 1$ and $\psi(x) = 0$ for $|x| > 2$. Set $\psi_L(x) = \psi(x/L)$. Since $u(x)$ and $\pi(x)$ satisfy (2.3), we have

$$\begin{aligned}
0 &= (-\Delta u + \nabla\pi, \psi_L u) \\
&= (\nabla u, (\nabla\psi_L)u) + (\nabla u, (\psi_L)\nabla u) - (\pi, (\nabla\psi_L)\cdot u) - (\pi, \psi_L(\nabla\cdot u)) \\
&= (\nabla u, (\nabla\psi_L)u) + (\nabla u, (\psi_L)\nabla u) - (\pi, (\nabla\psi_L)\cdot u).
\end{aligned}$$

Then (2.4) implies that the first and the third terms of the right-hand side tend to 0 as $L \to \infty$ and therefore we have $0 = \|\nabla u\|^2_{L^2}$ which implies that

$\nabla u = 0$. Therefore, u is constant. Since $u(x)|_{\partial\Omega} = 0$, we obtain $u = 0$. By the equation we have $\nabla\pi = 0$, which implies that π is constant. Since $\pi(x) = O(|x|^{-(n-1)})$ as $|x| \to \infty$, we have $\pi = 0$. This completes the proof. □

The following proposition plays an important role in this paper. This proposition is used when we construct a parametrix in Ω.

Lemma 2.4 (Bogovskiĭ lemma)**.** *Let $1 < p < \infty$.For any integer $N \geq 0$ there is a linear operator $\mathbb{B}$ from $\dot{W}^{N,p}(D_R^+)$ into $\dot{W}^{N+1,p}(\mathbb{R}^n)^n$ such that*

$$\nabla \cdot \mathbb{B}f = f, \quad \|\mathbb{B}f\|_{W^{N+1,p}(\mathbb{R}^n)^n} \leq C_{N,p,R}\|f\|_{W^{N,p}(D_R^+)}, \quad supp\ \mathbb{B}f \subset \overline{D_R^+}$$

for any $f \in \dot{W}^{N,p}(D_R^+)$.

Lemma 2.5. *Let $1 < p < \infty$ and N be a nonnegative integer. Let D be a C^2-domain such that $D \supset D_R^+$ and $D_R \cap \partial H \subset \partial D$. Let ψ_R be the same function as in (1.5). Assume that $u \in W^{N,p}(D)^n$, $\nabla \cdot u = 0$ in D and $u|_{\partial D} = 0$. Then $(\nabla\psi_R) \cdot u \in \dot{W}^{N,p}(D_R^+)$. As a result we can operate $\mathbb{B}$ to $(\nabla\psi_R) \cdot u$ and we have $\mathbb{B}[(\nabla\psi_R) \cdot u] \in \dot{W}^{N+1,p}(\mathbb{R}^n)^n$. Moreover $\mathbb{B}[(\nabla\psi_R) \cdot u]$ satisfies the following properties:*

$$\nabla \cdot \mathbb{B}[(\nabla\psi_R) \cdot u] = (\nabla\psi_R) \cdot u \quad \text{in } \mathbb{R}^n, \quad supp\ \mathbb{B}[(\nabla\psi_R) \cdot u] \subset \overline{D_R^+},$$
$$\|\mathbb{B}[(\nabla\psi_R) \cdot u]\|_{W^{N+1,p}(\mathbb{R}^n)^n} \leq C_{N,p,R}\|\nabla\psi_R \cdot u\|_{W^{N,p}(D_R^+)}.$$

Proof. Since

$$\int_{D_R^+} (\nabla\psi_R) \cdot u dx = -\int_{D \cap \mathrm{supp}\ (1-\psi_R)} \nabla(1-\psi_R) \cdot u dx$$
$$= -\int_{D \cap \mathrm{supp}\ (1-\psi_R)} \nabla \cdot (1-\psi_R) u dx$$
$$= -\int_{\partial(D \cap \mathrm{supp}\ (1-\psi_R))} (1-\psi_R) u \cdot \nu d\sigma = 0,$$

Lemma 2.5 follows from Lemma 2.4. □

3. Analysis in the perturbed half-space Ω

The aim of this section is to show Theorem 3.1 which gives us the expansion formula of the solution operator in the perturbed half-space.

Theorem 3.1. *Let $1 < p < \infty$ and $R > R_0$. Set $B_\Omega = \mathcal{L}(L_R^p(\Omega); W^{2,p}(\Omega \cap B_R)^n \times W^{1,p}(\Omega \cap B_R))$ and $\dot{U}_{\lambda_0} = U_{\lambda_0} \backslash (-\infty, 0]$. Then there exist a $\lambda_0 > 0$ and $(U(\lambda), \Theta(\lambda))$ such that*

$$U(\lambda)f = (\lambda + A)^{-1}Pf$$

for $f \in L^p_R(\Omega)$ *and* $\lambda \in U_{\lambda_0}$, *and*

$$(U(\lambda), \Theta(\lambda)) = \begin{cases} H_1(\lambda)\lambda^{\frac{n-1}{2}} + H_2(\lambda)\lambda^{\frac{n}{2}} \log\lambda + H_3(\lambda) & \text{where } n \text{ is even,} \\ H_1(\lambda)\lambda^{\frac{n}{2}} + H_2(\lambda)\lambda^{\frac{n-1}{2}} \log\lambda + H_3(\lambda) & \text{where } n \text{ is odd} \end{cases} \tag{3.1}$$

for any $\lambda \in \dot{U}_{\lambda_0}$ *where* $H_j \in \mathcal{B}(\dot{U}_{\lambda_0}; B_\Omega)$, $j = 1, 2$ *and* $H_3 \in \mathcal{B}(U_{\lambda_0}; B_\Omega)$.

In order to show Theorem 3.1, we shall introduce the notation which is used to construct a parametrix. Fix $R > 0$ such that $\Omega \backslash B_R = H \backslash B_R$. Let E_R be a bounded domain with smooth boundary ∂E_R such that $E_R \subset \Omega \cap B_{R+5}$ and $E_R \cap B_{R+4} = \Omega \cap B_{R+4}$. In particular, we have $D^+_{R+1} \subset \overline{\Omega} \cap B_{R+3} \subset E_R$.

Let $R(\lambda)f$ and $\Pi(\lambda)f$ be the operators given in section 2. Recall that $w = R(\lambda)f_0$ and $\theta = \Pi(\lambda)f_0$ satisfy the equation:

$$(\lambda - \Delta)w + \nabla\theta = f_0, \ \nabla \cdot w = 0 \quad \text{in} \quad H, \qquad w|_{x_n=0} = 0$$

where $f_0(x) = f(x)$ for $|x| > R$ and $f_0(x) = 0$ for $|x| \le R$. Moreover we have $R(\lambda)f \in W^{2,p}_{loc}(H)^n$, $\Pi(\lambda)f \in W^{1,p}_{loc}(H)$ and the following estimates:

$$\|R(\lambda)f\|_{W^{2,p}(D_R)^n} + \|\Pi(\lambda)f\|_{W^{1,p}(D_R)} \le C_R\|f\|_{L^p(\Omega)^n} \qquad \text{for } \lambda \in U_{\lambda_0}, \tag{3.2}$$

$$\sup_{|x|\ge R} \left(|x|^{n-1}|R(0)f| + |x|^{n-1}|\nabla R(0)f| + |x|^{n-1}|\Pi(0)f|\right) \le C_R\|f\|_{L^p(\Omega)^n}. \tag{3.3}$$

Given $f \in L^p_{R+3}(\Omega)^n$, we set $Af = w$ and $\Phi f = \theta$ where w and θ are the solutions to the equation:

$$-\Delta w + \nabla\theta = f, \qquad \nabla \cdot w = 0 \quad \text{in } E_R, \qquad w|_{\partial E_R} = 0. \tag{3.4}$$

We know the unique existence of $Af \in W^{2,p}(E_R)^n$ and $\Phi f \in W^{1,p}(E_R)$ satisfying the estimate:

$$\|Af\|_{W^{2,p}(E_R)^n} + \|\nabla\Phi f\|_{L^p(E_R)} \le C_{R,p}\|f\|_{L^p(E_R)^n}$$

(cf. Farwig and Sohr [13]). By addition of some constant to Φf we may assume that

$$\int_{D^+_R} (\Phi f - \Pi(0)f)\, dx = 0. \tag{3.5}$$

We choose $\phi(x) \in C^\infty_0(\mathbb{R}^n)$ so that $\phi(x) = 1$ for $|x| \le R+1$ and $\phi(x) = 0$ for $|x| \ge R+2$. We set

$$U(\lambda)f = (1-\phi)R(\lambda)f + \phi Af + \mathbb{B}[(\nabla\phi) \cdot (R(\lambda)f - Af)],$$
$$\Theta(\lambda)f = (1-\phi)\Pi(\lambda)f + \phi\Phi f.$$

By Lemma 2.5 we have $(\nabla\phi)\cdot(R(\lambda)f - Af) \in \dot{W}^{2,p}(D^+_{R+1})$ and $\mathbb{B}[(\nabla\phi)\cdot(R(\lambda)f - Af)] \in W^{3,p}(D^+_{R+1})^n$, which implies that $\nabla\cdot U(\lambda)f = 0$ and $U(\lambda)f \in W^{2,p}_{loc}(\Omega)^n$, and then we have

$$\begin{aligned} (\lambda-\Delta)U(\lambda)f + \nabla\Theta(\lambda)f &= f + S_\lambda f &&\text{in}\quad \Omega,\\ \nabla\cdot U(\lambda)f &= 0 &&\text{in}\quad \Omega,\\ U(\lambda)f|_{\partial\Omega} &= 0, \end{aligned}$$

where

$$\begin{aligned} S_\lambda f = 2(\nabla\phi) : (\nabla R(\lambda)f) + (\Delta\phi)R(\lambda)f + (\lambda-\Delta)\mathbb{B}[(\nabla\phi)\cdot(R(\lambda)f - Af)]\\ + \lambda\phi Af - 2(\nabla\phi) : (\nabla Af) - (\Delta\phi)(Af) - 2(\nabla\phi)\Pi f + 2(\nabla\phi)\Phi f. \end{aligned}$$

Since $(\nabla\phi)\cdot(R(\lambda)f - Af) \in W^{2,p}(D^+_{R+1})$, the map $f \mapsto (\nabla\phi)\cdot(R(\lambda)f - Af) \in \dot{W}^{1,p}(D^+_{R+1})$ is compact. Therefore, we see that $S_\lambda :\ L^p_{R+3}(\Omega)^n \to L^p_{R+3}(\Omega)^n$ is a compact operator (see [14]). Moreover we have the following two lemmas.

Lemma 3.2. *For $1 < p < \infty$ the following relation holds*

$$\|S_\lambda - S_0\|_{\mathcal{L}(L^p_{R+3}(\Omega)^n)} \le Cp(|\lambda|)$$

for any $\lambda \in U_{1/2}\backslash(-\infty, 0]$, where $p(t)$ is the same as in Theorem 2.2.

Proof. Lemma 3.2 follows immediately from Theorem 2.2. □

Lemma 3.3. *It holds that $(1 + S_0)^{-1} \in \mathcal{L}(L^p_{R+3}(\Omega)^n)$.*

Proof. From the formula:

$$\begin{aligned} S_0 f = 2(\nabla\phi) : (\nabla R(0)f - \nabla Af) + (\Delta\phi)(R(0)f - Af)\\ - \Delta\mathbb{B}[(\nabla\phi)\cdot(R(0)f - Af)] - (\nabla\varphi)\Pi f + (\nabla\varphi)\Phi f, \end{aligned}$$

we see that S_0 is a compact operator in $L^p_{R+3}(\Omega)^n$. In order to prove Lemma 3.3, it is sufficient to show that $1 + S_0$ is injective in $L^p_{R+3}(\Omega)^n$. Let $f \in L^p_{R+3}(\Omega)^n$ satisfy $(1 + S_0)f = 0$. Set $u = U(0)f$ and $\pi = \Theta(0)f$. Since (u, π) satisfies all the assumptions of Lemma 2.3 from (3.3), we see $u(x) = 0$ and $\pi(x) = 0$. Therefore,

$$\begin{cases} (1-\phi)R(0)f + \phi Af + \mathbb{B}[(\nabla\phi)\cdot(R(0)f - Af)] = 0 & \text{in } \Omega,\\ (1-\phi)\Pi(\lambda)f + \phi\Phi f = 0 & \text{in } \Omega. \end{cases} \tag{3.6}$$

Since $\phi(x) = 0$ for $|x| \ge R + 2$, we have

$$R(0)f = 0,\ \Pi(0)f = 0 \qquad \text{for } |x| \ge R + 2. \tag{3.7}$$

Since $\phi(x) = 1$ for $|x| \leq R+1$, we have $Af(x) = 0$ and $\Phi f(x) = 0$ for $|x| \leq R+1$. Set $\widetilde{E_R} = \{x \in E_R : |x| \geq R\} \cup B_R^+$. If we set

$$w(x) = \begin{cases} Af(x), & |x| \geq R, \ x \in E_R, \\ 0, & |x| < R, \end{cases}$$

$$\theta(x) = \begin{cases} \Phi f(x), & |x| \geq R, \ x \in E_R, \\ 0, & |x| < R, \end{cases}$$

then $w \in W^{2,p}(\widetilde{E_R})^n$ and $\theta \in W^{1,p}(\widetilde{E_R})$ and (w, θ) satisfies the equation:

$$-\Delta w + \nabla\theta = f_0, \ \nabla \cdot w = 0 \quad \text{in} \quad \widetilde{E_R}, \qquad w|_{\partial\widetilde{E_R}} = 0.$$

On the other hand, by (3.7) we have

$$-\Delta R(0)f + \nabla\Pi(0)f = f_0, \ \nabla \cdot R(0)f = 0 \quad \text{in } \widetilde{E_R}, \qquad R(0)f|_{\partial\widetilde{E_R}} = 0.$$

By uniqueness we obtain $R(0)f = w$ in $\widetilde{E_R}$, and also we have $\nabla(\Pi(0)f - \theta) = 0$ and therefore $\Pi(0)f - \theta$ is a constant c in $\widetilde{E_R}$. Since

$$0 = \int_{D_R^+} (\Pi(0)f - \theta)dx = \int_{D_R^+} c dx = c|D_R^+|$$

as follows from (3.5), we obtain $c = 0$. Therefore $\Pi(0)f = \theta$ in $\widetilde{E_R}$. As a result we have

$$R(0)f = w = Af, \ \Pi(0)f = \theta = \Phi f \qquad \text{in } \widetilde{E_R}.$$

In particular it holds that $(\nabla\phi) \cdot [R(0)f - Af] = 0$ in Ω. By (3.6) we have

$$0 = R(0)f + \phi(Af - R(0)f) = R(0)f,$$
$$0 = \Pi(0)f + \phi(\Phi f - \Pi(0)f) = \Pi(0)f$$

for $|x| \geq R+1$ and $x \in \Omega$. Consequently, we obtain

$$f = -\Delta R(0)f + \nabla\Pi(0)f = 0$$

for $|x| \geq R+1$ and $x \in \Omega$. Moreover, since $Af = 0$ and $\Phi f = 0$ for $x \in \Omega, \ |x| \leq R+1$,

$$0 = -\Delta Af(x) + \nabla\Phi f(x) = f$$

for $|x| \leq R+1$ and $x \in \Omega$. Summing up, we have proved that $f = 0$. □

We get the following lemma from Lemmas 3.2 and 3.3.

Lemma 3.4. *There exists a $\lambda_0 > 0$ such that for $\lambda \in \Sigma_\varepsilon \cup \{0\}$ with $|\lambda| \leq \lambda_0$ the following relations holds:*

$$(1 + S_\lambda)^{-1} \in \mathcal{L}(L^p_{R+3}(\Omega)^n), \qquad \|(1 + S_\lambda)^{-1}\|_{\mathcal{L}(L^p_{R+3}(\Omega)^n)} \leq C.$$

By Lemma 3.4 we can denote the solution (u, π) as follows:

$$\begin{aligned} u(x) &= U(\lambda)(1+S_\lambda)^{-1}f \\ &= (1-\phi)R(\lambda)(1+S_\lambda)^{-1}f + \phi A(1+S_\lambda)^{-1}f \\ &\quad + \mathbb{B}[(\nabla\phi)\cdot(R(\lambda)-A)(1+S_\lambda)^{-1}f], \\ \pi(x) &= \Theta(\lambda)(1+S_\lambda)^{-1}f = (1-\phi)\Pi(\lambda)(1+S_\lambda)^{-1}f + \phi\Phi(1+S_\lambda)^{-1}f, \end{aligned}$$

where

$$\begin{aligned} (1+S_\lambda)^{-1}f &= [1+S_0+(S_\lambda-S_0)]^{-1}f \\ &= (1+S_0)^{-1}[1+(1+S_0)^{-1}(S_\lambda-S_0)]^{-1}f \\ &= (1+S_0)^{-1}\sum_{j=0}^{\infty}[(1+S_0)^{-1}(S_\lambda-S_0)]^j f, \\ S_\lambda - S_0 &= 2(\nabla\phi):\nabla(R(\lambda)-R(0)) + (\Delta\phi)(R(\lambda)-R(0)) \\ &\quad + \lambda\phi A f + (\lambda-\Delta)\mathbb{B}[(\nabla\phi)\cdot(R(\lambda)-R(0))]. \end{aligned}$$

When n is even, by this relation and Theorem 2.1 we have

$$S_\lambda - S_0 = \widetilde{G_1}(\lambda)\lambda^{\frac{n-1}{2}} + \widetilde{G_2}(\lambda)\lambda^{\frac{n}{2}}\log\lambda + \widetilde{G_4}(\lambda)\lambda,$$

where $\widetilde{G_i}(\lambda)$ $(i=1,2,4)$ are in $\mathcal{B}(U_{\lambda_0}, B_\Omega)$. Therefore we obtain

$$(S_\lambda - S_0)^j = \widetilde{H_1^j}(\lambda)\lambda^{\frac{n-1}{2}} + \widetilde{H_2^j}(\lambda)\lambda^{\frac{n}{2}}\log\lambda + \widetilde{H_3^j}(\lambda)\lambda,$$

where

$$\begin{aligned} &\widetilde{H_3^j}(\lambda) = \{\widetilde{G_4}(\lambda)\}^j\lambda^{j-1}, \\ &\widetilde{H_1^j}(\lambda) = [(\widetilde{G_1}(\lambda)\lambda^{\frac{n-1}{2}} + \widetilde{G_4}(\lambda)\lambda)^j - \widetilde{H_3^j}(\lambda)\lambda]/\lambda^{\frac{n-1}{2}}, \\ &\widetilde{H_2^j}(\lambda) = [(S_\lambda-S_0)^j - \widetilde{H_1^j}(\lambda)\lambda^{\frac{n-1}{2}} - \widetilde{H_3^j}(\lambda)\lambda]/(\lambda^{\frac{n-1}{2}}\log\lambda). \end{aligned}$$

In fact, set $\max_{|\lambda|\le\lambda_0}\|\widetilde{G_k^j}(\lambda)\| \le M$ for $k=1,2,4$ and then we have

$$\begin{aligned} \|\widetilde{H_3^j}(\lambda)\| &\le |\lambda|^{j-1}M^j, \\ \|\widetilde{H_1^j}(\lambda)\| &\le \sum_{i=1}^{j}\frac{j!}{i!(j-i)!}\left(|\lambda|^{\frac{n-1}{2}}\|\widetilde{G_1}(\lambda)\|\right)^i\left(|\lambda|\|\widetilde{G_4}(\lambda)\|\right)^{j-i} \\ &\le M^j\sum_{i=1}^{j}\frac{j!}{i!(j-i)!}(|\lambda|^{\frac{n-1}{2}})^i|\lambda|^{j-i} \end{aligned}$$

$$\leq M^j|\lambda|^{\frac{j-1}{2}}\sum_{i=1}^{j}\frac{j!}{i!(j-i)!}|\lambda|^{\frac{n-1}{2}(i-1)}|\lambda|^{\frac{j-i}{2}}$$

$$\leq M^j|\lambda|^{\frac{j-1}{2}}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!} = (2M)^j|\lambda|^{\frac{j-1}{2}}$$

and

$$\begin{aligned}\|\widetilde{H_2^j}\| &= \left\|(S_\lambda - S_0)^j - \widetilde{H_1^j}(\lambda)\lambda^{\frac{n-1}{2}} - \widetilde{H_3^j}(\lambda)\lambda\right\|(\lambda^{\frac{n}{2}}\log\lambda)^{-1} \\ &\leq \sum_{k=1}^{j}\frac{j!}{k!(j-k)!}|\lambda^{\frac{n}{2}}\log\lambda|^{k-1}\|\widetilde{G_2}(\lambda)\|^k \\ &\qquad\qquad \times\left(|\lambda|^{\frac{n-1}{2}}\|\widetilde{G_1}(\lambda)\| + |\lambda|\|\widetilde{G_4}(\lambda)\|\right)^{j-k} \\ &\leq M^j\sum_{k=1}^{j}\frac{j!}{k!(j-k)!}|\lambda^{\frac{n}{2}}\log\lambda|^{k-1}(|\lambda|^{\frac{n-1}{2}} + |\lambda|)^{j-k} \\ &\leq M^j|\lambda|^{\frac{j-1}{2}}\sum_{k=1}^{j}\frac{j!}{k!(j-k)!}|\lambda^{\frac{n-1}{2}}\log\lambda|^{k-1}(|\lambda|^{\frac{n-2}{2}} + |\lambda|^{\frac{1}{2}})^{j-k} \\ &\leq M^j|\lambda|^{\frac{j-1}{2}}\sum_{k=1}^{j}\frac{j!}{k!(j-k)!}1^k 2^{j-k} \leq (3M)^j|\lambda|^{\frac{j-1}{2}},\end{aligned}$$

where $|\lambda^{\frac{n-1}{2}}\log\lambda| \leq 1$ and $|\lambda| \leq 1$.

We have

$$(1+S_\lambda)^{-1} = \widehat{H_1}(\lambda)\lambda^{\frac{n-1}{2}} + \widehat{H_2}(\lambda)\lambda^{\frac{n}{2}}\log\lambda + \widehat{H_3}(\lambda),$$

where $\widehat{H_j} \in \mathcal{B}(\dot{U}_{\lambda_0}; B_\Omega)(j = 1, 2)$ and $\widehat{H_3} \in \mathcal{B}(U_{\lambda_0}; B_\Omega)$. Since we can show the expansion formula when n is odd in the same manner as in the case where n is even, we have obtained Theorem 3.1.

4. LOCAL ENERGY DECAY ESTIMATE IN A PERTURBED HALF-SPACE

We shall prove Theorem 1.1 from what we showed in section 3 in the same manner as in Iwashita [15]. For this purpose, it suffices to prove the following theorem:

Theorem 4.1. *Let $n \geq 2$ and $1 < p < \infty$. Then there exists a positive constant $C = C(p)$ such that the inequality*

$$\|T(t)Pf\|_{L^p(\Omega_R)} \leq Ct^{-\frac{n+1}{2}}\|f\|_{L^p(\Omega)}, \qquad t \geq 1 \tag{4.1}$$

is valid for any $f \in L^p_R(\Omega)$.

Proof. Let $\frac{\pi}{2} < \delta_0 < \delta < 2\pi$ and $0 < \varepsilon < \lambda_0$, where λ_0 is the same constant as in Theorem 3.1. Let Γ be a contour as follows : $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon,\ \arg\lambda = \pm\delta\},$$
$$\Gamma_2 = \{\lambda \in \mathbb{C} : |\lambda| > \varepsilon,\ \arg\lambda = \pm\delta\}.$$

The semigroup is described as follows :

$$T(t)Pf = \frac{-1}{2\pi i}\int_{\Gamma_1} T(t)U(\lambda)Pf d\lambda + \frac{-1}{2\pi i}\int_{\Gamma_2} T(t)(A+\lambda)^{-1}Pf d\lambda. \tag{4.2}$$

The second term of the right-hand side of (4.2) is estimated as

$$\left\|\frac{-1}{2\pi i}\int_{\Gamma_2} T(t)(\lambda + A)^{-1}\, d\lambda\right\|_{\mathcal{L}(J^p(\Omega))} \le C\int_{\varepsilon}^{\infty} e^{-tr}\, dr \le Ce^{-ct} \tag{4.3}$$

for some $c, C > 0$. In order to estimate the first term of the right-hand side of (4.2), we need the following lemma that is a direct consequence of the formula for the gamma function $\Gamma(\sigma)$.

Lemma 4.2. (i) *For* $\sigma > 0$ *and* $t > 0$, *it holds that*

$$\frac{-1}{2\pi i}\int_{\Gamma} e^{-tz} z^{\sigma-1} dz = -\frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}t^{-\sigma}.$$

(ii) *For a nonnegative integer* j *and any* $t > 0$,

$$\frac{-1}{2\pi i}\int_{\Gamma} e^{-tz} z^{j}\log z dz = -\frac{d}{d\sigma}\left[\frac{\sin\sigma\pi}{\pi}\Gamma(\sigma)e^{i\pi\sigma}\right]\Big|_{\sigma=j+1} t^{-j-1}.$$

Since $U(\lambda)Pf$ is described as

$$U(\lambda)P = \begin{cases} H_1(\lambda)\lambda^{\frac{n-1}{2}} + H_2(\lambda)\lambda^{\frac{n}{2}}\log\lambda + H_3(\lambda) & \text{where } n \text{ is even,} \\ H_1(\lambda)\lambda^{\frac{n}{2}} + H_2(\lambda)\lambda^{\frac{n-1}{2}}\log\lambda + H_3(\lambda) & \text{where } n \text{ is odd} \end{cases} \tag{4.4}$$

we can apply Lemma 4.2 to obtain

$$\left\|\frac{-1}{2\pi i}\int_{\Gamma_1} T(t)H_1(\lambda)\lambda^{\frac{n-1}{2}} Pf\, d\lambda\right\|_{L^p(\Omega_R)} \le Ct^{-\frac{n}{2}-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \tag{4.5}$$

$$\left\|\frac{-1}{2\pi i}\int_{\Gamma_1} T(t)H_2(\lambda)\lambda^{\frac{n}{2}}(\log\lambda)\, Pf\, d\lambda\right\|_{L^p(\Omega_R)} \le Ct^{-\frac{n}{2}-1}\|f\|_{L^p(\Omega)}. \tag{4.6}$$

Finally the operator $H_3(\lambda)$ is holmorphic in U_{λ_0} so that we have

$$\left\|\frac{-1}{2\pi i}\int_{\Gamma_1} T(t)H_3(\lambda)Pf\, d\lambda\right\|_{L^p(\Omega_R)} \le Ce^{-ct}\|f\|_{L^p(\Omega)}. \tag{4.7}$$

Combining (4.3) and (4.5)-(4.7) completes the proof of Theorem 4.1 if n is even and similarly if n is odd. □

5. L^p-L^q ESTIMATE IN A PERTURBED HALF-SPACE

The aim of this section is to show the L^p-L^q estimate in a perturbed half-space Ω. In order to prove the L^p-L^q estimate, at first we shall prove the following lemma.

Lemma 5.1. *Let $n \geq 2$, $1 < p < \infty$, and $R \geq R_0$. Then there exists a positive number $C = C(\Omega, n, p, R)$ such that*

$$\|\partial_t T(t)f\|_{W^{1,p}(\Omega_R)} + \|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \qquad t \geq 2$$

for $f \in J^p(\Omega)$.

Proof. Fix $R \geq R_0 + 2$. For $f \in J^p(\Omega)$ we set $g = T(1)f$. Then $g \in D(A^N)$ for any $N \in \mathbb{N}$ and there holds the estimate:

$$\|A^N g\|_{L^p(\Omega)} \leq C_{N,p}\|f\|_{L^p(\Omega)}. \tag{5.1}$$

We set $u(t) = T(t)g = T(t+1)f$ for $f \in J^p(\Omega)$. Then $u(t)$ belongs to $C^1([0,\infty); J^p(\Omega)) \cap C^0([0,\infty); D(A))$ and u is the solution of the following Stokes problem with some pressure term $\pi(t)$:

$$\begin{cases} \partial_t u(t) - \Delta u(t) + \nabla\pi(t) = 0, & \text{in } \Omega \times (0,\infty), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = g. \end{cases} \tag{5.2}$$

Set $h = \psi_R g - \mathbb{B}[(\nabla\psi_R)\cdot g]$, and then by (1.5) we have $h = g$ for $|x| \geq R+1$. Moreover we can see that $h \in D(A_H)$ where A_H denotes the Stokes operator on H. In fact, by Lemma 2.5 we have $\nabla \cdot \mathbb{B}[(\nabla\psi_R)\cdot g] = (\nabla\psi_R)\cdot g$, supp $\mathbb{B}[(\nabla\psi_R)\cdot g] \subset \overline{D_R^+}$ and the following estimate holds:

$$\|\mathbb{B}[(\nabla\psi_R)\cdot g]\|_{W^{2,p}(H)} \leq C\|(\nabla\psi_R)\cdot g\|_{W^{1,p}(H)} \leq C\|f\|_{L^p(\Omega)}.$$

Therefore, we see that $h \in W^{2,p}(H)$ and that

$$\|h\|_{W^{2,p}(H)} \leq C\|f\|_{L^p(\Omega)}. \tag{5.3}$$

The facts that $h|_{x_n=0} = 0$ and $\nabla \cdot h = 0$ imply that $h \in D(A_H)$.

By the solvability of the Stokes equation in the half-space (cf. Ukai [24]) we know that there exists a (v, ρ) such that

$$v(t) \in C^1([0,\infty); J^p(H)) \cap C^0([0,\infty); D(A_H)),$$
$$\nabla\rho(t) \in C^0([0,\infty); L^p(H)),$$

and (v, ρ) solves the following equation:

$$\begin{cases} \partial_t v(t) - \Delta v(t) + \nabla \rho(t) = 0 & \text{in} \quad H \times (0, \infty), \\ \nabla \cdot v = 0 & \text{in} \quad H \times (0, \infty), \\ v|_{t=0} = h, \ v|_{x_n=0} = 0, \end{cases} \tag{5.4}$$

and

$$\int_{D_R^+} \rho(t, x) dx = 0. \tag{5.5}$$

Moreover, from the L^q-L^r type estimate in the half-space which is proved by Borchers and Miyakawa [3] we have

$$\|\nabla^j v(t, \cdot)\|_{L^r(H)} \leq C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{j}{2}} \|h\|_{W^{2,q}(H)} \tag{5.6}$$

for $j = 0, 1$, $t \geq 1$ and $1 \leq q \leq r \leq \infty$ with $(q, r) \neq (1, 1)$ and

$$\|\nabla^2 v(t, \cdot)\|_{L^r(H)} + \|\partial_t v(t, \cdot)\|_{L^r(H)} \leq C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-1} \|h\|_{W^{2,q}(H)} \tag{5.7}$$

for $t \geq 1$ and $1 < q \leq r < \infty$.

By (5.4) and (5.7) we have

$$\|\nabla \rho(t)\|_{L^r(H)} \leq C_{q,r}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-1} \|h\|_{W^{2,q}(H)} \tag{5.8}$$

for $t \geq 1$ and $1 < q \leq r < \infty$, and in the cylinder $C_R^+ = \{x \in H : |x'| \leq R, \ x_n \leq R\}$ we know the estimates:

$$\|\nabla^j v(t, \cdot)\|_{L^p(C_R^+)} \leq C_R \|\nabla^j v(t, \cdot)\|_{L^\infty(H)} \leq C_R(1+t)^{-\frac{n}{2p}-\frac{j}{2}} \|h\|_{W^{2,p}(H)} \tag{5.9}$$

for $j = 0, 1$, and

$$\begin{aligned} &\|\nabla^2 v(t, \cdot)\|_{L^p(C_R^+)} + \|\partial_t v(t, \cdot)\|_{L^p(C_R^+)} \\ &\leq C_{R,p,r} \left(\|\nabla^2 v(t, \cdot)\|_{L^r(C_R^+)} + \|\partial_t v(t, \cdot)\|_{L^r(C_R^+)} \right) \\ &\leq C_{R,p,r}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})-1} \|h\|_{W^{2,p}(H)} \leq C_{R,p}(1+t)^{-\frac{n}{2p}-\frac{1}{2}} \|h\|_{W^{2,p}(H)} \end{aligned} \tag{5.10}$$

for $t \geq 1$ and $1 < p < r < \infty$ where $\max(p, n) \leq r < \infty$. By (5.4), Poincare's inequality, (5.5), and (5.8), we have

$$\|\rho(t, \cdot)\|_{L^p(C_R^+)} \leq C \|\nabla \rho(t, \cdot)\|_{L^p(C_R^+)} \leq C(1+t)^{-\frac{n}{2p}-\frac{1}{2}} \|h\|_{W^{2,p}(H)}. \tag{5.11}$$

Since $v(t, x) = \int_0^{x_n} \partial_n v(t, x', y_n) dy_n$, as follows from the fact that $v|_{x_n=0} = 0$, we obtain

$$\|v(t, \cdot)\|_{L^p(C_R^+)} \leq R \|\nabla v(t, \cdot)\|_{L^p(C_R^+)}, \tag{5.12}$$

which combined with (5.9) with $j = 1$ implies that

$$\|v(t,\cdot)\|_{L^p(C_R^+)} \le C_R(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|h\|_{W^{2,p}(H)}. \tag{5.13}$$

Summing up (5.3), (5.9) - (5.11), and (5.13), we have shown that

$$\begin{aligned} &\|v(t)\|_{W^{2,p}(C_R^+)} + \|\partial_t v(t,\cdot)\|_{L^p(C_R^+)} + \|\rho(t,\cdot)\|_{L^p(C_R^+)} \\ &\le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \end{aligned} \tag{5.14}$$

In order to estimate (u,π) in (5.2) we set

$$w(t) = u(t) - \{\psi_{R-1}v(t) - \mathbb{B}[(\nabla\psi_{R-1}) : v(t)]\}, \quad \theta(t) = \pi(t) - \psi_{R-1}\rho(t). \tag{5.15}$$

It is easily observed that $(w(t),\theta(t))$ satisfies the equations:

$$\partial_t w(t) - \Delta w(t) + \nabla\theta(t) = K(t),\ \nabla\cdot w(t) = 0 \qquad \text{in } \Omega\times(0,\infty), \tag{5.16}$$

$$w|_{\partial\Omega} = 0, \tag{5.17}$$

$$\begin{aligned} w(0) &= u(0) - (\psi_{R-1}v(0) - \mathbb{B}[(\nabla\psi_{R-1})\cdot v(0)]) \\ &= g - (\psi_{R-1}h - \mathbb{B}[(\nabla\psi_{R-1})\cdot h]), \end{aligned} \tag{5.18}$$

where

$$K(t) = 2\nabla\psi_{R-1} : \nabla v + (\Delta\psi_{R-1})v - (\partial_t - \Delta)\mathbb{B}[(\nabla\psi_{R-1})\cdot v] - (\nabla\psi_{R-1})\rho.$$

Noticing that supp $w(0) \subset \overline{B_R}$ by (1.5) and $w \in W^{2,p}(\Omega)$, we obtain $w \in D(A)\cap L^p_R(\Omega)$. Since $w(t) \in C^0([0,\infty); D(A)\cap L^p_R(\Omega)) \cap C^1([0,\infty); J^p(\Omega))$, we can write

$$w(t) = T(t)w(0) - \int_0^t T(t-s)PK(s)ds. \tag{5.19}$$

Our aim is to estimate $w(t)$ with Theorem 1.1. (1.5) implies that supp $K(t) \subset \overline{D_R^+}$ and by (5.14) we see that

$$\begin{aligned} \|K(t)\|_{L^p} &\le C\left\{\|v(t)\|_{W^{1,p}(C_R^+)} + \|\partial_t v(t)\|_{L^p(C_R^+)} + \|\rho(t)\|_{L^p(C_R^+)}\right\} \\ &\le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}. \end{aligned} \tag{5.20}$$

For $t \ge 1$ we have the estimates:

$$\begin{aligned} &\|w(t,\cdot)\|_{W^{1,p}(\Omega_R)} \le C(1+t)^{-\frac{n+1}{2}}\|w(0)\|_{L^p} \\ &+ C\int_{t-1}^t (t-s)^{-\frac{1}{2}}\|PK(s)\|_{L^p}ds + C\int_0^{t-1}(t-s)^{-\frac{n+1}{2}}\|PK(s)\|_{L^p}ds \\ &=: w_1 + w_2 + w_3. \end{aligned} \tag{5.21}$$

Then we shall estimate w_1 in (5.21). Since $w(0) \in D(A)$ and supp $w(0) \subset \overline{B_R \cap \Omega}$, we see that

$$\|w(0)\|_{L^p(\Omega)} \le C\|f\|_{L^p}. \tag{5.22}$$

We have the estimate of w_2 as follows:

$$w_2 \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \tag{5.23}$$

In fact by (5.20) we have

$$\begin{aligned} w_2 &\le \int_{t-1}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\|f\|_{L^p(\Omega)} \\ &\le Ct^{-\frac{n}{2p}-\frac{1}{2}}\int_{t-1}^{t}(t-s)^{-\frac{1}{2}}ds\|f\|_{L^p(\Omega)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \end{aligned}$$

Finally we shall estimate w_3. By (5.20) we have

$$\begin{aligned} w_3 &\le \int_0^{t-1}(t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\|f\|_{L^p(\Omega)} \\ &\le C\int_0^{t}(1+t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\|f\|_{L^p(\Omega)} \end{aligned}$$

because $t-s \ge 1$ for $0 \le s \le t-1$. We proceed to estimate:

$$\begin{aligned} w_3 &\le \left\{\int_0^{t/2} + \int_{t/2}^{t}\right\}(1+t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}\,ds\,\|f\|_{L^p(\Omega)} \\ &\le \int_0^{t/2}(1+t-s)^{-\frac{n+1}{2}}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}\,ds \\ &\qquad + \int_0^{t/2}(1+s)^{-\frac{n+1}{2}}(1+t-s)^{-\frac{n}{2p}-\frac{1}{2}}\,ds\,\|f\|_{L^p(\Omega)}. \end{aligned}$$

Noticing that $(1+t-s)^{-1} \le (1+s)^{-1}$ for $0 \le s \le \frac{t}{2}$, we have

$$\begin{aligned} w_3 &\le C\Big(1+\frac{t}{2}\Big)^{-\frac{n}{2p}-\frac{1}{2}}\int_0^{t/2}(1+s)^{-\frac{n+1}{2}}\,ds\,\|f\|_{L^p(\Omega)} \\ &\le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\int_0^{\infty}(1+s)^{-\frac{n+1}{2}}\,ds\,\|f\|_{L^p(\Omega)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \end{aligned} \tag{5.24}$$

By (5.22)-(5.24) we obtain

$$\|w(t)\|_{W^{1,p}(\Omega_R)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{5.25}$$

for $t \ge 1$.

By (5.14)-(5.15) and (5.25) we obtain

$$\|u(t)\|_{W^{1,p}(\Omega_R)} \le \|w(t)\|_{W^{1,p}(\Omega_R)} + C\|v(t)\|_{W^{1,p}(\Omega_R)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p} \tag{5.26}$$

for $t \ge 1$. In particular, since $T(t)f = u(t-1)$, for $f \in J^p(\Omega)$ we have

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \le Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \qquad t \ge 2. \tag{5.27}$$

Since $\partial_t u(t) = \partial_t T(t)g = T(t)Ag$ and $Ag \in D(A)$, by (5.1) we also have

$$\|\partial_t u(t)\|_{W^{1,p}(\Omega_R)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)} \qquad \text{for } t \ge 1,$$

which implies that

$$\|\partial_t T(t)f\|_{W^{1,p}(\Omega_R)} \le Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)} \qquad \text{for } t \ge 2.$$

Summing up, we have completed the proof of the lemma. □

Remark 5.2. *We know that, in the exterior domain, there holds the estimate:*

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \le Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}.$$

The reason why the exponent in the perturbed half-space is one half better than the one in the exterior domain is that (5.12) *holds.*

Next, we shall prove the L^p-L^q estimate in $\Omega\backslash\Omega_R$.

Lemma 5.3. *Let $f \in J^p(\Omega)$. Then for $t \ge 2$ we have*

$$\|T(t)f\|_{L^q(\Omega\backslash\Omega_R)} \le C_{p,q}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)} \tag{5.28}$$

for $1 < p \le q \le \infty$ and $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) < 1$, and we have

$$\|\nabla T(t)f\|_{L^p(\Omega\backslash\Omega_R)} \le C_{p,R}t^{-\frac{1}{2}}\|f\|_{L^p(\Omega)} \tag{5.29}$$

for $1 < p < \infty$ and $t \ge 2$.

Proof. Fix $R \ge R_0 + 2$. Set $g = T(1)f \in D(A^N)$ for any $N \in \mathbb{N}$ and set $u(t) = T(t)g = T(t+1)f$. We set

$$z(t) = \psi_{R-1}u(t) - \mathbb{B}[(\nabla\psi_{R-1})\cdot u(t)], \quad \Phi(t) = \psi_{R-1}\pi(t)$$

where $u(t)$ and $\pi(t)$ are the same as in (5.2) and

$$\int_{D^+_{R-1}} \pi(t,x)dx = 0. \tag{5.30}$$

It is observed that $(z(t), \Phi(t))$ satisfies the equations:

$$\partial_t z(t) - \Delta z(t) + \nabla\Phi(t) = L(t), \quad \nabla\cdot z = 0 \qquad \text{in } H\times(0,\infty), \tag{5.31}$$

$$z|_{x_n=0} = 0, \tag{5.32}$$

$$z(0) = \psi_{R-1}u(0) - \mathbb{B}[(\nabla\psi_{R-1})\cdot u(0)] = \psi_{R-1}g - \mathbb{B}[(\nabla\psi_{R-1})\cdot g] =: z_0 \tag{5.33}$$

where

$$L(t) = -2\nabla\psi_{R-1} : \nabla u(t) - (\Delta\psi_{R-1})u(t) \\ + (\partial_t - \Delta)\mathbb{B}[(\nabla\psi_{R-1})\cdot u] + (\nabla\psi_{R-1})\pi(t). \tag{5.34}$$

Since $z(t) \in C^1([0,\infty); J^p(H)) \cap C^0([0,\infty); D(A_H))$, we can write $z(t)$ as follows:

$$z(t) = E(t)z_0 - \int_0^t E(t-s)PL(s)ds = z_1 + z_2 \tag{5.35}$$

where $E(t)$ is the solution operator for the half-space H.

Given $\phi \in C_0^\infty(H)$ we set

$$\Theta = (\nabla\psi_{R-1})\phi - \frac{1}{|D_{R-1}^+|}\int_{D_{R-1}^+}(\nabla\psi_{R-1})\phi dx \tag{5.36}$$

and then $\int_{D_{R-1}^+}\Theta dx = 0$. By Lemma 2.4 we can choose $\chi \in W^{1,p'}(D_{R-1}^+)$ such that $\nabla\cdot\chi = \Theta$, $\chi|_{\partial D_{R-1}^+} = 0$, and

$$\|\chi\|_{W^{1,p'}(D_{R-1}^+)} \le C\|\Theta\|_{L^{p'}(D_{R-1}^+)} \le C\|\phi\|_{L^{p'}(D_{R-1}^+)}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

On the other hand by (5.30) and (5.36) we have

$$((\nabla\psi_{R-1})\pi(t), \phi) = \int_{D_{R-1}^+}\pi(t)(\nabla\psi_{R-1})\phi\, dx = \int_{D_{R-1}^+}\pi(t)\Theta\, dx \\ = (\pi, \nabla\cdot\chi) = (\Delta u - \partial_t u, \chi) = -(\nabla u, \nabla\chi) - (\partial_t u, \chi).$$

By Lemma 5.1 we have

$$\begin{aligned} &|((\nabla\psi_{R-1})\pi(t), \phi)| \\ &\le \|\nabla u\|_{L^p(D_{R-1}^+)}\|\chi\|_{W^{1,p'}(D_{R-1}^+)} + \|\partial_t u\|_{L^p(D_{R-1}^+)}\|\chi\|_{W^{1,p'}(D_{R-1}^+)} \\ &\le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}\|\phi\|_{L^{p'}} \end{aligned}$$

for any $\phi \in C_0^\infty(H)$ and $t \ge 1$, which implies that

$$\|(\nabla\psi_{R-1})\pi(t)\|_{L^p(H)} \le C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}, \qquad t \ge 1. \tag{5.37}$$

By (5.34) we have supp $L(t) \subset \overline{D^+_{R-1}} \subset \Omega$. By (5.34), (5.37), and Lemma 5.1 we have

$$\|PL(t)\|_{L^r} \le C_r\|L(t)\|_{L^r} \le C_{p,r}\|L(t)\|_{L^p} \le C_{p,r}(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p} \quad (5.38)$$

for $t \ge 1$.

Next we shall estimate $z(t)$ by using (5.38). We can show the estimate of z_1 by using the following L^p-L^q estimate for the half-space H: For $1 \le p \le q \le \infty$, $(p,q) \ne (1,1), (\infty,\infty)$,

$$\|z_1\|_{L^q(H)} \le C_{p,q}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|z_0\|_{L^p(H)}, \quad (5.39)$$

$$\|\nabla z_1\|_{L^q(H)} \le C_{p,q}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|z_0\|_{L^p(H)}. \quad (5.40)$$

Now we shall estimate z_2. For $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) < 1$ with $1 < p \le q \le \infty$ and $1 < r < \min\{p, n\}$, we have

$$\begin{aligned}
&\|z_2\|_{L^q(H)} \\
&\le \int_{t-1}^{t}(t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|PL(s)\|_{L^p}ds + \int_0^{t-1}(t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|PL(s)\|_{L^r}\,ds \\
&\le Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p}\int_0^1 s^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\,ds \\
&\qquad + C\int_0^t(1+t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}\,ds\,\|f\|_{L^p} \\
&= (I_1+I_2)\|f\|_{L^p}.
\end{aligned}$$

It is sufficient to estimate the second term I_2. We have

$$\begin{aligned}
I_2 &= \int_0^{t/2}(1+t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds \\
&\qquad + \int_0^{t/2}(1+s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1+t-s)^{-\frac{n}{2p}-\frac{1}{2}}ds \\
&= \int_0^{t/2}(1+t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(1+t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds \\
&\qquad + \int_0^{t/2}(1+s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}(1+t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(1+t-s)^{-\frac{n}{2q}-\frac{1}{2}}ds.
\end{aligned}$$

Noticing that $(1+s)^{-1} \ge (1+t-s)^{-1}$ for $0 \le s \le t/2$, we have

$$I_2 \le 2\left(1+\frac{t}{2}\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\int_0^{t/2}(1+s)^{-\frac{n}{2r}-\frac{1}{2}}ds.$$

Summing up, we have obtained

$$\|z_2\|_{L^q(H)} \le C_{p,q}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}, \qquad t \ge 1. \tag{5.41}$$

By (5.39) and (5.41) we obtain

$$\|z(t)\|_{L^q(H)} \le C_{p,q}(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}, \qquad t \ge 1$$

provided that $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) < 1,\ 1 < p \le q \le \infty$.

Since $z(t) = u(t) = T(t+1)f$ for $|x| \ge R$, we have

$$\|T(t)f\|_{L^q(\Omega\setminus\Omega_R)} \le C_{p,q}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}, \qquad t \ge 2.$$

Therefore we have (5.28). Next we shall show the following estimate:

$$\|\nabla T(t)f\|_{L^p(\Omega)} \le C_p t^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}, \qquad t > 2 \tag{5.42}$$

for $1 < p < \infty$. We remember (5.35). We can estimate z_1 as follows:

$$\|\nabla z_1\|_{L^p(H)} \le C(1+t)^{-\frac{1}{2}}\|z_0\|_{L^p(H)} \le C(1+t)^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}. \tag{5.43}$$

Next we shall estimate z_2. Take q with $1 < q < \min\{\frac{n}{2}, p\}$ and then we have

$$\begin{aligned}
&\|\nabla z_2\|_{L^p(H)}\\
&\le \int_{t-1}^{t}\|\nabla E(t-s)PL(s)\|_{L^p(H)}ds + \int_0^{t-1}\|\nabla E(t-s)PL(s)\|_{L^p(H)}ds\\
&\le C\int_{t-1}^{t}(t-s)^{-\frac{1}{2}}\|L(s)\|_{L^p(H)}ds + C\int_0^{t-1}(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|L(s)\|_{L^q(H)}ds\\
&\le Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}\int_0^1 s^{-\frac{1}{2}}ds\\
&\qquad\qquad + C\|f\|_{L^p(\Omega)}\int_0^t(1+t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\\
&= (I_1+I_2)\|f\|_{L^p(\Omega)}.
\end{aligned}$$

To estimate z_2, it is sufficient that we consider the second term I_2. We have

$$\begin{aligned}
I_2 &= \int_0^t(1+t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\\
&= \int_0^{t/2}(1+t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}(1+s)^{-\frac{n}{2p}-\frac{1}{2}}ds\\
&\qquad + \int_0^{t/2}(1+s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}(1+t-s)^{-\frac{n}{2p}-\frac{1}{2}}ds
\end{aligned}$$

$$\leq 2(1+t/2)^{-\frac{1}{2}} \int_0^{t/2} (1+s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{n}{2p}-\frac{1}{2}} ds$$
$$= 2(1+t/2)^{-\frac{1}{2}} \int_0^{t/2} (1+s)^{-\frac{n}{2q}-\frac{1}{2}} ds.$$

Since $\frac{n}{2q}+\frac{1}{2} > 1$, we have

$$\|\nabla z_2\|_{L^p(H)} \leq C(1+t)^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}, \qquad t \geq 1. \tag{5.44}$$

By (5.43) and (5.44) we obtain

$$\|\nabla z(t)\|_{L^p(H)} \leq C_p(1+t)^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}, \qquad t \geq 1$$

for $1 < p < \infty$. Since $z(t) = T(t+1)f$ for $|x| \geq R$, in view of Lemma 5.1 we have

$$\|\nabla T(t)f\|_{L^p(\Omega)} \leq C_p t^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}, \qquad t \geq 2.$$

This completes the proof. □

Next we shall show the L^p-L^q estimate for $0 < t \leq 2$.

Lemma 5.4. *For $0 < t \leq 2$, there exists a positive number $C = C(p,q,\Omega)$ such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)}$$

for $1 < p \leq q \leq \infty$ and $\frac{n}{2}(\frac{1}{p}-\frac{1}{q}) < 1$, and we have

$$\|\nabla T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$.

Proof. By complex interpolation for $s \in (0,2)$ we have

$$W^{s,p}(\Omega) = (L^p(\Omega), W^{2,p}(\Omega))_{s/2},$$

and by using the resolvent estimate, we have

$$\|T(t)f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$
$$\|T(t)f\|_{W^{2,p}(\Omega)} \leq Ct^{-1}\|f\|_{L^p(\Omega)}, \qquad 0 < t \leq 2.$$

Therefore we have

$$\|T(t)f\|_{W^{s,p}(\Omega)} \leq (C\|f\|_{L^p(\Omega)})^{1-(s/2)} (Ct^{-1}\|f\|_{L^p(\Omega)})^{s/2} \leq C_s t^{-\frac{s}{2}} \|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$ and $0 < t < 2$. Set $s = n(\frac{1}{p}-\frac{1}{q})$ and then, by Sobolev's embedding theorem, we obtain

$$\|T(t)f\|_{L^q(\Omega)} \leq C\|T(t)f\|_{W^{s,p}(\Omega)} \leq C_s t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)}$$

for $1 < p \le q < \infty$ and $0 < \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. In view of the resolvent estimate, we have

$$\|\nabla T(t)f\|_{L^p(\Omega)} \le C_p t^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}$$

for $1 < p < \infty$ and $0 < t < 2$. Therefore, we see

$$\begin{aligned}\|\nabla T(t)f\|_{L^q(\Omega)} &= \|\nabla T(t/2)T(t/2)f\|_{L^q(\Omega)} \le Ct^{-\frac{1}{2}}\|T(t/2)f\|_{L^q(\Omega)} \\ &\le Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}\end{aligned}$$

for $1 < p \le q < \infty$. Let $B^s_{p,q}(\Omega)$ be the Besov space. For $1 < p < \infty$ we know

$$B^{n/p}_{p,1}(\Omega) = [L^p(\Omega),\ W^{2,p}(\Omega)]_{\theta,1}, \quad \frac{n}{p} = 2\theta, \qquad 0 < \frac{n}{p} < 2,$$

where $[\cdot,\cdot]_{\theta,1}$ is the real interpolation functor. We have

$$\|T(t)f\|_{B^{n/p}_{p,1}(\Omega)} \le C\|T(t)f\|^{1-\theta}_{L^p(\Omega)}\|T(t)f\|^{\theta}_{W^{2,p}(\Omega)} \le Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}$$

for $0 < t \le 2$ and $\frac{n}{2} < p < \infty$. Since $B^{n/p}_{p,1}(\Omega) \subset L^\infty(\Omega)$, we have

$$\|T(t)f\|_{L^\infty(\Omega)} \le Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}$$

for $p > \frac{n}{2}$(see [18]). We have completed the proof. □

Finally we shall show Theorem 1.2 by using Lemmas 5.3 and 5.4.

Proof of Theorem 1.2. By Lemmas 5.3 and 5.4 we have

$$\|T(t)f\|_{L^q(\Omega)} \le C_{p,q}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)} \tag{5.45}$$

for $1 < p \le q \le \infty$, $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$ and $t > 0$. We will remove the restriction of (5.45). To this end we choose $p_1,\ \dots,\ p_\ell$ in such a way that $p = p_1 < p_2 < \cdots < p_\ell = q$ and $\frac{n}{2}(\frac{1}{p_{j-1}} - \frac{1}{p_j}) < 1$ for $j = 2, 3, 4, \cdots, \ell$. Then by Lemma (5.4) we have

$$\begin{aligned}\|T(t)f\|_{L^q(\Omega)} &= \Big\|T\Big(\frac{t}{\ell-1}\Big)T\Big(\frac{t}{\ell-1}\Big)\cdots T\Big(\frac{t}{\ell-1}\Big)f\Big\|_{L^q(\Omega)} \\ &\le C\prod_{j=2}^{\ell}\Big(\frac{t}{\ell-1}\Big)^{-\frac{n}{2}(\frac{1}{p_{j-1}}-\frac{1}{p_j})}\|f\|_{L^p(\Omega)} = Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}.\end{aligned}$$

Summing up we have obtained

$$\|T(t)f\|_{L^q(\Omega)} \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)} \tag{5.46}$$

for $1 < p \le q \le \infty$, $f \in J^p(\Omega)$ and $t > 0$.

Since for $\phi, \psi \in C_{0,\sigma}^\infty(\Omega)$ we have

$$|(T(t)\phi, \psi)| = |(\phi, T(t)\psi)| \le \|\phi\|_{L^1}\|T(t)\psi\|_{L^\infty} \le Ct^{-\frac{n}{2p'}}\|\phi\|_{L^1}\|\psi\|_{L^{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\|T(t)\phi\|_{L^p} \le Ct^{-\frac{n}{2}(1-\frac{1}{p})}\|\phi\|_{L^1}, \qquad 1 < p < \infty,$$
$$\|T(t)\phi\|_{L^\infty} \le Ct^{-\frac{n}{2p}}\|T(t/2)\phi\|_{L^p} \le Ct^{-\frac{n}{2p}}t^{-\frac{n}{2}(1-\frac{1}{p})}\|\phi\|_{L^1} \le Ct^{-\frac{n}{2}}\|\phi\|_{L^1}.$$

Summing up we have obtained (1.7) for $1 \le p \le q \le \infty$, $f \in J^p(\Omega)$, and $t > 0$.

By using (5.42) and (1.8) we have

$$\begin{aligned}\|\nabla T(t)f\|_{L^q(\Omega)} &= \Big\|\nabla T\Big(\frac{t}{2}\Big)T\Big(\frac{t}{2}\Big)f\Big\|_{L^q(\Omega)} \\ &\le C_q t^{-\frac{1}{2}}\Big\|T\Big(\frac{t}{2}\Big)f\Big\|_{L^q(\Omega)} \le C_q t^{-\frac{1}{2}}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}\end{aligned}$$

for $t > 0$ and $1 < p \le q < \infty$.

We have completed the proof of Theorem 1.2. □

6. The Navier-Stokes flow in a perturbed half-space

In this section we shall consider the application of L^p-L^q estimate to the Navier-Stokes equation. At first we begin to show Theorem 1.3.

Proof of Theorem 1.3. Employing the argument due to Kato [16] we can construct a unique global solution $u(t)$ of the integral equation

$$u(t) = T(t)a - \int_0^t T(t-\tau)P(u(\tau)\cdot\nabla u(\tau))\,d\tau \tag{IE}$$

provided that $\|a\|_{L^n} \le \delta_0$, where $\delta_0 = \delta_0(\Omega, n)$ is some small positive constant. The solution $u(t)$ enjoys the estimates:

$$\|u(t)\|_{L^r} \le Ct^{-\frac{1}{2}+\frac{n}{2r}}\|a\|_n \qquad \text{for } n \le r \le \infty, \tag{6.1}$$
$$\|\nabla u(t)\|_{L^r} \le Ct^{-1+\frac{n}{2r}}\|a\|_n \qquad \text{for } n \le r < \infty \tag{6.2}$$

for $t > 0$ together with the singular behavior

$$\|u(t)\|_{L^r} = o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad \text{for} \quad n < r \le \infty,$$
$$\|\nabla u(t)\|_{L^r} = o(t^{-1+\frac{n}{2r}}) \qquad \text{for} \quad n \le r < \infty$$

as $t \to 0$. (6.1) and (6.2) imply the Hölder estimate:

$$\|u(t) - u(\tau)\|_{L^\infty} + \|\nabla u(t) - \nabla u(\tau)\|_{L^n} \le C(t-\tau)^\theta \tau^{-\frac{1}{2}-\theta}\|a\|_{L^n} \tag{6.3}$$

for $0 < \tau < t$ and $0 < \theta < \frac{1}{2}$. Due to the Hölder estimate the solution $u(t)$ becomes actually a strong one of (NS) (see [23]). Furthermore, in the same way as in Hishida[14] we can obtain the decay properties (1.9) and (1.10) with $r = n$. Then we also have (1.10) for $n < r < \infty$. In fact for $n < r < \infty$ we have

$$\|\nabla u(t)\|_{L^r} \le \left\|\nabla T\Big(\frac{t}{2}\Big)u\Big(\frac{t}{2}\Big)\right\|_{L^r} + \int_{\frac{t}{2}}^{t} \|\nabla T(t-\tau)P(u\cdot\nabla u)(\tau)\|_{L^r}\,d\tau$$

$$\le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})-\frac{1}{2}}\left\|u\Big(\frac{t}{2}\Big)\right\|_{L^n} + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{r})-\frac{1}{2}}\,\|P(u\cdot\nabla u)(\tau)\|_{L^n}\,d\tau$$

$$\le Ct^{-1+\frac{n}{2r}}\left\|u\Big(\frac{t}{2}\Big)\right\|_{L^n} + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-1+\frac{n}{2r}}\|u(\tau)\|_{L^\infty}\|\nabla u(\tau)\|_{L^n}d\tau$$

$$\le Ct^{-1+\frac{n}{2r}}\left\|u\Big(\frac{t}{2}\Big)\right\|_{L^n} + C\int_{\frac{t}{2}}^{t}(t-\tau)^{-1+\frac{n}{2r}}\tau^{-1}\,d\tau\|a\|_{L^n}\sup_{\frac{t}{2}\le\tau\le t}\tau^{\frac{1}{2}}\|u(\tau)\|_{L^\infty}$$

$$\le Ct^{-1+\frac{n}{2r}}\left\|u\Big(\frac{t}{2}\Big)\right\|_{L^n} + Ct^{-1+\frac{n}{2r}}\|a\|_{L^n}\sup_{\frac{t}{2}\le\tau\le t}\tau^{\frac{1}{2}}\|u(\tau)\|_{L^\infty}.$$

Therefore, by (1.9) we obtain (1.10). The proof is complete. □

In the same manner as in Hishida [14] we can prove Theorem 1.4. The key to his proof is to show the following Lemma 6.1. According to Hishida's argument [14] we can also prove Lemma 6.1:

Lemma 6.1. *Let $n \ge 2$ and $a \in L^1(\Omega)\cap J^n(\Omega)$. For any small $\varepsilon > 0$ there is a constant $\eta_* = \eta_*(\Omega, n, \varepsilon)$ such that the solution $u(t)$ obtained in Theorem* 1.3 *satisfies, for $t > 0$,*

$$\|u(t)\|_{L^{\frac{n}{n-1}}} \le C(1+t)^{-\frac{1}{2}+\varepsilon}, \tag{6.4}$$

$$\|u(t)\|_{L^{2n}} \le Ct^{-\frac{1}{4}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon}, \tag{6.5}$$

$$\|\nabla u(t)\|_{L^n} \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon}. \tag{6.6}$$

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