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ON A YAMABE-TYPE PROBLEM ON A THREE-DIMENSIONAL THIN ANNULUS

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Abstract. We consider the problem: $(P_\varepsilon) : -\Delta u_\varepsilon = u_\varepsilon^5$, $u_\varepsilon > 0$ in A_ε; $u_\varepsilon = 0$ on ∂A_ε, where $\{A_\varepsilon \subset \mathbb{R}^3 : \varepsilon > 0\}$ is a family of bounded annulus-shaped domains such that A_ε becomes "thin" as $\varepsilon \to 0$. We show that, for any given constant $C > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the problem (P_ε) has no solution u_ε, whose energy, $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2$, is less than C. Such a result extends to dimension three a result previously known in higher dimensions. Although the strategy to prove this result is the same as in higher dimensions, we need a more careful and delicate blow up analysis of asymptotic profiles of solutions u_ε when $\varepsilon \to 0$.

1. Introduction

We consider the following nonlinear elliptic problem

$$(P_\Omega) \qquad \begin{cases} -\Delta u = u^5, \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$.

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The equation (P_Ω) arises in many mathematical and physical contexts (see [7]), but its greatest interest lies in its relation to the Yamabe problem. From this geometric point of view, we think of u as defining the conformal metric $g_{ij} = u^{\frac{4}{n-2}}\,\delta_{ij}$. Equation (P_Ω) then says that the metric g has constant scalar curvature.

It is well known that, if Ω is starshaped, (P_Ω) has no solution (see Pohozaev [16]) and if Ω has nontrivial topology, in the sense that $H_{2k-1}(\Omega; Q) \neq 0$ or $H_k(\Omega; Z/2Z) \neq 0$ for some $k \in \mathbb{N}$, Bahri and Coron [3] have shown that (P_Ω) has a solution. Nevertheless, Ding [12], Dancer [11], and Passaseo [15] gave the example of a contractible domain on which (P_Ω) has a solution. Then, the question related to existence or nonexistence of a solution of (P_Ω) remained open.

In this paper, we study the problem (P_Ω) when $\Omega = A_\varepsilon$ is an annulus-shaped domain in $\mathbb{R}^3$and ε is a small positive parameter. More precisely, let f be any smooth function: $f : \mathbb{R}^2 \longrightarrow [1, 2]$, $(\theta_1, \theta_2) \longrightarrow f(\theta_1, \theta_2)$, which is periodic of period π with respect to θ_1 and of period 2π with respect to θ_2.

We set $S_1(f) = \{x \in \mathbb{R}^3 : r = f(\theta_1, \theta_2\}$, where (r, θ_1, θ_2) are the polar coordinates of x.

For ε positive small enough, we introduce the following map

$$g_\varepsilon : S_1(f) \longrightarrow g_\varepsilon(S_1(f)) = S_2(f), \;\; x \longmapsto g_\varepsilon(x) = x + \varepsilon n_x,$$

where n_x is the outward normal to $S_1(f)$ at x. We denote by $(A_\varepsilon)_{\varepsilon>0}$ the family of annulus-shaped domains in $\mathbb{R}^3$ such that $\partial A_\varepsilon = S_1(f) \cup S_2(f)$.

We are mainly interested in the existence of finite energy solutions; our main result is the following theorem.

Theorem 1.1. *Let C be any positive constant. Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, the problem $(P_\varepsilon) : -\Delta u_\varepsilon = u_\varepsilon^5$, $u_\varepsilon > 0$ in A_ε, $u_\varepsilon = 0$ on ∂A_ε, has no solution such that $\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 \le C$.*

Such a nonexistence result of finite energy solutions to Yamabe-type problems on nontrivial domains is a new and interesting phenomenon, and it is a subject of current investigations by the authors. It turns out that such a nonexistence result of finite energy solutions is closely related to nonexistence results of solutions of finite Morse index, and has its explanation in the behavior of the first eigenvalue of the Laplace operator, or more generally, of Laplace Beltrami operators on complete manifolds. We hope that such results will be useful to find necessary and sufficient conditions on the manifold for the solvability of Yamabe problem on complete manifolds. The results of such investigations will appear elsewhere. We notice that the higher-dimensional analogue of our result has been recently proved by the first three authors [6].

Our strategy to prove Theorem 1.1 is the same as in higher dimensions, however, as usual in elliptic equations involving critical Sobolev exponent [8], we need more refined estimates of the asymptotic profiles of solutions u_ε when $\varepsilon \to 0$ to treat the three-dimensional case. Such refined estimates, which are of interest in themselves, are highly nontrivial and use in a crucial way the refined properties of blowing up solutions of Yamabe-type problems in the spirit of R. Schoen [19], [20], [21] and Y. Y. Li [13]. The input of such a refined blow up analysis enables us to rule out some *bad configurations* for which the higher-dimensional estimates cannot be improved.

Another ingredient of our proof is a careful expansion of the Euler Lagrange functional associated to (P_ε), and its gradient near a small neighborhood of highly concentrated functions. To perform such expansions we extensively make use of the techniques developed by A. Bahri [2] and O. Rey [17], [18] in the framework of the *theory of critical points at infinity.*

The organization of the paper is as follows. The next section is devoted to setting up some notation. In Section 3, we study the asymptotic behavior of bounded energy solutions of (P_ε). In Section 4, we prove Theorem 1.1. Lastly, we prove in Section 5 some useful facts and careful estimates needed for the previous sections.

2. NOTATION

We denote by G_ε the Green's function of the Laplace operator defined by

$$\forall\, x \in A_\varepsilon \quad -\Delta G_\varepsilon(x,.) = c'\delta_x \text{ in } A_\varepsilon \quad , \quad G_\varepsilon(x,.) = 0 \text{ on } \partial A_\varepsilon, \tag{2.1}$$

where δ_x is the Dirac mass at x and $c' = meas(S^2)$. We denote by H_ε the regular part of G_ε; that is,

$$H_\varepsilon(x_1, x_2) = |x_1 - x_2|^{-1} - G_\varepsilon(x_1, x_2), \text{ for } (x_1, x_2) \in A_\varepsilon \times A_\varepsilon. \tag{2.2}$$

For $p \in \mathbb{N}^*$ and $x = (x_1, ..., x_p) \in A_\varepsilon^p$, we denote by $M = M_\varepsilon(x)$ the matrix defined by

$$M = (m_{ij})_{1\le i,j\le p}, \text{ where } m_{ii} = H_\varepsilon(x_i, x_i), m_{ij} = -G_\varepsilon(x_i, x_j), i \neq j \tag{2.3}$$

and define $\rho_\varepsilon(x)$ as the smallest eigenvalue of M ($\rho_\varepsilon(x) = -\infty$ if $x_i = x_j$ for some $i \neq j$. For $a \in \mathbb{R}^3$ and $\lambda > 0$, $\delta_{(a,\lambda)}$ denotes the function

$$\delta_{(a,\lambda)}(x) = c_0 \frac{\lambda^{1/2}}{(1 + \lambda^2 |x - a|^2)^{1/2}}. \tag{2.4}$$

It is well known (see [9]) that if c_0 is suitably chosen ($c_0 = 3^{1/4}$), the $\delta_{(a,\lambda)}$ are the only solutions of

$$-\Delta u = u^5, u > 0 \text{ in } \mathbb{R}^3 \tag{2.5}$$

and they are also the only minimizers for the Sobolev inequality

$$S = \inf\{|\nabla u|^2_{L^2(\mathbb{R}^3)}|u|^{-2}_{L^6(\mathbb{R}^3)} : \nabla u \in L^2, u \in L^6, u \neq 0\}. \tag{2.6}$$

We also denote by $P_\varepsilon\delta_{(a,\lambda)}$ the projection of $\delta_{(a,\lambda)}$ on $H^1_0(A_\varepsilon)$; that is,

$$\Delta P_\varepsilon\delta_{(a,\lambda)} = \Delta\delta_{(a,\lambda)} \text{ in } A_\varepsilon, P_\varepsilon\delta_{(a,\lambda)} = 0 \text{ on } \partial A_\varepsilon,$$

and by $\theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P_\varepsilon\delta_{(a,\lambda)}$. We define on $H^1_0(A_\varepsilon) \setminus \{0\}$ the functional

$$J_\varepsilon(u) = \frac{\int_{A_\varepsilon} |\nabla u|^2}{\left(\int_{A_\varepsilon} u^6\right)^{1/3}} \tag{2.7}$$

whose positive critical points, up a multiplicative constant, are solutions of (P_ε). Lastly, let

$$\langle u, v\rangle = \int_{A_\varepsilon} \nabla u \nabla v, \quad ||u|| = \Big(\int_{A_\varepsilon} |\nabla u|^2\Big)^{1/2}, \quad u,\, v\, \in H^1_0\big(A_\varepsilon\big).$$

3. Asymptotic behavior of bounded energy solutions

This section is devoted to the study of the asymptotic behavior of bounded energy solutions of (P_ε). Such a precise description is a cornerstone in the proof of our results. It says, roughly speaking, that our solutions concentrate at a finite number of points such that the distance of one of them to the other is at least comparable to ε.

In the sequel to this paper we consider a solution u_ε of (P_ε) which satisfies

$$\int_{A_\varepsilon} \mid \nabla u_\varepsilon \mid^2 \leq C, \tag{3.1}$$

where C is a positive constant independent of ε. Our aim in this section is to prove the following result:

Theorem 3.1. *Let u_ε be a solution of problem (P_ε) which satisfies (3.1). Then, after passing to a subsequence, there exist $p \in \mathbb{N}^*, (x_{1,\varepsilon}, ..., x_{p,\varepsilon}) \in A^p_\varepsilon, (\lambda_{1,\varepsilon}, ..., \lambda_{p,\varepsilon}) \in (\mathbb{R}^*_+)^p$, and a positive constant $\alpha > 0$ such that:*

$$\Big|\Big|u_\varepsilon - \sum_{i=1}^p P_\varepsilon\delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}\Big|\Big| \to 0,\ \lambda_{i,\varepsilon}d_{i,\varepsilon} \to +\infty \; for\; 1 \leq i \leq p \quad as \quad \varepsilon \to 0,$$

$$\lambda_{i,\varepsilon} \mid x_{i,\varepsilon} - x_{j,\varepsilon} \mid \to \infty \;\, as\; \varepsilon \to 0, \mid x_{i,\varepsilon} - x_{j,\varepsilon} \mid \geq \alpha\, \varepsilon \quad for \quad i \neq j,$$

where $d_{i,\varepsilon} = d(x_{i,\varepsilon}, \partial A_\varepsilon)$ *and* $\lambda_{i,\varepsilon} = 3^{-1/2}\big(u_\varepsilon(x_{i,\varepsilon})\big)^2$.

Remark 3.2. The above theorem is true in all dimensions $n \geq 3$, however a weaker version used in [6] was enough to derive the equivalent of our result in dimension $n \geq 4$.

To prove Theorem 3.1, we start by establishing some useful facts. Let $x_{1,\varepsilon} \in A_\varepsilon$ be such that

$$u_\varepsilon(x_{1,\varepsilon}) = \max_{A_\varepsilon} u_\varepsilon := M_{1,\varepsilon}.$$

Let $\widetilde{A}_\varepsilon = M_{1,\varepsilon}^2(A_\varepsilon - x_{1,\varepsilon})$, and denote by v_ε the function defined on $\widetilde{A}_\varepsilon$ by

$$v_\varepsilon(y) = M_{1,\varepsilon}^{-1} u_\varepsilon(x_{1,\varepsilon} + M_{1,\varepsilon}^{-2} y). \tag{3.2}$$

By Lemma 2.3 of [6], we know that

$$M_{1,\varepsilon}^2 d(x_{1,\varepsilon}, \partial A_\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

Furthermore, $v_\varepsilon \to \delta_{(0,\alpha_0)}$ in $C^2_{loc}(\mathbb{R}^3)$ as $\varepsilon \to 0$, where $\alpha_0 = 3^{-1/2}$.

Now, we prove the following crucial lemma:

Lemma 3.3. *There exist positive constants* δ *and* $\bar{c}$ *such that*

$$\max_{|y| \leq \delta\varepsilon M_{1,\varepsilon}^2} \mid v_\varepsilon(y) - \delta_{(0,\alpha_0)}(y) \mid \leq \bar{c}\big(\varepsilon M_{1,\varepsilon}^2\big)^{-1}.$$

Proof. First, it follows from Lemma 3.2 of [10], that there exist positive constants δ and $\bar{c}$ such that

$$v_\varepsilon(y) \leq \bar{c}\delta_{(0,\alpha_0)}(y) \quad \text{for} \quad \mid y \mid \leq \delta\varepsilon M_{1,\varepsilon}^2. \tag{3.3}$$

Now, let

$$m_\varepsilon = \max_{|y| \leq \delta\varepsilon M_{1,\varepsilon}^2} \mid v_\varepsilon(y) - \delta_{(0,\alpha_0)}(y) \mid := \mid v_\varepsilon(y_\varepsilon) - \delta_{(0,\alpha_0)}(y_\varepsilon) \mid .$$

Arguing by contradiction, we assume that $m_\varepsilon \varepsilon M_{1,\varepsilon}^2 \to +\infty$ as $\varepsilon \to 0$.

Letting $w_\varepsilon(y) = m_\varepsilon^{-1}\big(v_\varepsilon(y) - \delta_{(0,\alpha_0)}(y)\big)$, w_ε satisfies

$$\Delta w_\varepsilon + f_\varepsilon w_\varepsilon = 0 \quad \text{with} \quad f_\varepsilon = \frac{v_\varepsilon^5 - \delta_{(0,\alpha_0)}^5}{v_\varepsilon - \delta_{(0,\alpha_0)}}.$$

By (3.3), we have

$$\mid f_\varepsilon \mid \leq c\big(1 + \mid y \mid\big)^{-4} \quad \text{for} \quad \mid y \mid \leq \delta\varepsilon M_{1,\varepsilon}^2. \tag{3.4}$$

Applying the Green's representation leads to

$$w_\varepsilon(y) = a\big(\int_{B_\varepsilon} G_{B_\varepsilon}(y,x) f_\varepsilon(x) w_\varepsilon(x)dx - \int_{\partial B_\varepsilon} \frac{\partial G_{B_\varepsilon}}{\partial \nu}(y,x) w_\varepsilon(x) d\sigma(x)\big),$$

where $a = \big(meas(S^2)\big)^{-1}$, $B_\varepsilon = B(0, \delta\varepsilon M_{1,\varepsilon}^2)$, ν is the outward normal to ∂B_ε, and G_{B_ε} is the Green's function of Δ under Dirichlet boundary conditions in B_ε. Using (3.3) and (3.4) yields

$$\begin{aligned} \mid w_\varepsilon(y) \mid &\leq c \int_{B_\varepsilon} \frac{dx}{\mid y-x \mid \big(1+ \mid x \mid\big)^4} + \frac{c}{m_\varepsilon \delta\varepsilon M_{1,\varepsilon}^2} \\ &\leq c\Big(\big(1+ \mid y \mid\big)^{-2} + \big(m_\varepsilon \delta\varepsilon M_{1,\varepsilon}^2\big)^{-1}\Big). \end{aligned} \tag{3.5}$$

It follows that w_ε is bounded and by standard elliptic estimates w_ε converges, up to some subsequence, in the C^2_{loc}-norm to a function w satisfying

$$\begin{cases} \Delta w + 5\delta^4_{(0,\alpha_0)}(y) w(y) = 0 \quad \text{in} \quad \mathbb{R}^3 \\ \mid w(y) \mid \leq c\big(1+ \mid y \mid\big)^{-2}. \end{cases} \tag{3.6}$$

By Lemma $A1$ of [5] (see also Lemma 2.4 of [10]), every solution of (3.6) can be written as

$$w(y) = \sum_{j=1}^{3} a_j \frac{\partial \delta_{(0,\alpha_0)}}{\partial y_j} + a_0\big(y \cdot \nabla \delta_{(0,\alpha_0)}(y) + \frac{1}{2}\delta_{(0,\alpha_0)}(y)\big)$$

for some constants $a_j \in \mathbb{R}$, $j = 0, ..., 3$. Since $w(0) = \frac{\partial w}{\partial y_j}(0) = 0$, we obtain that $a_j = 0$ for $0 \leq j \leq 3$; namely, $w \equiv 0$. Since $w_\varepsilon(y_\varepsilon) = 1$, it follows that $\mid y_\varepsilon \mid \to +\infty$ as $\varepsilon \to 0$. Applying (3.5) at $y = y_\varepsilon$ gives

$$1 = \mid w_\varepsilon(y_\varepsilon) \mid \leq c\Big(\big(1+ \mid y_\varepsilon \mid\big)^{-2} + \big(m_\varepsilon \delta\varepsilon M_{1,\varepsilon}^2\big)^{-1}\Big). \tag{3.7}$$

Since the right-hand side of (3.7) goes to zero, as $\varepsilon \to 0$, we derive a contradiction. Thus $m_\varepsilon \varepsilon \delta M_{1,\varepsilon}^2$ must be bounded and the proof of our lemma follows. □

Lemma 3.4. *Let δ be the positive constant stated in Lemma* 3.3*. Then we have*

$$\int_{B(x_{1,\varepsilon}, \delta\varepsilon)} u_\varepsilon^6 = S_3 + o(1) \quad as \quad \varepsilon \to 0,$$

where $S_3 = S^{3/2}$ and S is the Sobolev constant defined in (2.6)*.*

Proof. We have

$$\int_{B(x_{1,\varepsilon},\delta\varepsilon)} u_\varepsilon^6 = \int_{B(0,\delta\varepsilon M_{1,\varepsilon}^2)} v_\varepsilon^6$$
$$= \int_{B(0,\delta\varepsilon M_{1,\varepsilon}^2)} \delta_{(0,\alpha_0)}^6 + O\Big(\int_{B(0,\delta\varepsilon M_{1,\varepsilon}^2)} \delta_{(0,\alpha_0)}^5 |v_\varepsilon - \delta_{(0,\alpha_0)}| + |v_\varepsilon - \delta_{(0,\alpha_0)}|^6\Big)$$
$$= \int_{B(0,\delta\varepsilon M_{1,\varepsilon}^2)} \delta_{(0,\alpha_0)}^6 + O\Big(\mid v_\varepsilon - \delta_{(0,\alpha_0)} \mid_{L^6(B(0,\delta\varepsilon M_{1,\varepsilon}^2))}\Big).$$

Using Lemma 3.3 and the fact that $\varepsilon M_{1,\varepsilon}^2 \to +\infty$ as $\varepsilon \to 0$, we easily derive our lemma. □

Now, we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. We distinguish two cases:

Case 1. $\int_{A_\varepsilon} \mid u_\varepsilon - P_\varepsilon \delta_{(x_{1,\varepsilon},\lambda_{1,\varepsilon})} \mid^6 \to 0$ as $\varepsilon \to 0$, where $\lambda_{1,\varepsilon} = \alpha_0 M_{1,\varepsilon}^2$. In this case we are done; the number of blow up points in the theorem is reduced to 1; that is, $p = 1$.

Case 2. $\int_{A_\varepsilon} \mid u_\varepsilon - P_\varepsilon \delta_{(x_{1,\varepsilon},\lambda_{1,\varepsilon})} \mid^6 \not\to 0$ as $\varepsilon \to 0$. We are going to study this case. First, let us prove that

$$\int_{A_\varepsilon \setminus B(x_{1,\varepsilon},\delta\varepsilon)} u_\varepsilon^6 \not\to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.8}$$

Observe that

$$\int_{A_\varepsilon \setminus B(x_{1,\varepsilon},\delta\varepsilon)} P_\varepsilon \delta_{(x_{1,\varepsilon},\lambda_{1,\varepsilon})}^6 \le \int_{A_\varepsilon \setminus B(x_{1,\varepsilon},\delta\varepsilon)} \delta_{(x_{1,\varepsilon},\lambda_{1,\varepsilon})}^6 \tag{3.9}$$
$$= \int_{\tilde{A}_\varepsilon \setminus B(0,\delta\varepsilon M_{1,\varepsilon}^2)} \delta_{(0,\alpha_0)}^6 \to 0 \text{ as } \varepsilon \to 0,$$

where we have used the fact that $\varepsilon M_{1,\varepsilon}^2 \to \infty$ and $\delta_{(0,\alpha_0)} \in L^6(\mathbb{R}^3)$.

By Lemma 3.3 and the fact that $\varepsilon M_{1,\varepsilon}^2 \to \infty$, it is easy to derive

$$\int_{B_\varepsilon} |u_\varepsilon - P\delta_{(x_{1,\varepsilon},\lambda_{1,\varepsilon})}|^6 \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.10}$$

Clearly, (3.9) and (3.10) imply (3.8). Now, we set

$$u_\varepsilon(x_{2,\varepsilon}) = \max_{A_\varepsilon \setminus B(x_{1,\varepsilon},\delta\varepsilon)} u_\varepsilon := M_{2,\varepsilon}.$$

It is clear that $|x_{1,\varepsilon} - x_{2,\varepsilon}| \ge \delta\varepsilon$.

By (3.8), there exists $c > 0$ such that

$$c \leq \int_{A_\varepsilon \setminus B(x_{1,\varepsilon},\delta\varepsilon)} u_\varepsilon^6 \leq M_{2,\varepsilon}^4 \int_{A_\varepsilon} u_\varepsilon^2(x)dx.$$

But, we have

$$\int_{A_\varepsilon} u_\varepsilon^2(x)dx = \varepsilon^3 \int_{D_\varepsilon} \tilde{u}_\varepsilon^2(X)dX \leq \frac{\varepsilon^3}{c_\varepsilon} \int_{D_\varepsilon} \mid \nabla \tilde{u}_\varepsilon(X) \mid^2 dX$$
$$= \frac{\varepsilon^2}{c_\varepsilon} \int_{A_\varepsilon} \mid \nabla u_\varepsilon(x) \mid^2 dx \leq \frac{C\varepsilon^2}{c_\varepsilon},$$

where $\tilde{u}_\varepsilon(X) = u_\varepsilon(\varepsilon X)$, $D_\varepsilon = \varphi(A_\varepsilon)$, with $\varphi : x \mapsto \varphi(x) = \varepsilon^{-1}x$, and $c_\varepsilon > 0$. By Lin [14], we have $c_\varepsilon \to c > 0$ as $\varepsilon \to 0$. We derive that $\varepsilon M_{2,\varepsilon}^2 \not\to 0$ as $\varepsilon \to 0$ and therefore as in Lemma 2.3 of [6], we have that $M_{2,\varepsilon}^2 d(x_{2,\varepsilon}, \partial A_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. This implies that $M_{2,\varepsilon}^2 \mid x_{1,\varepsilon} - x_{2,\varepsilon} \mid \to +\infty$ as $\varepsilon \to 0$. Now, for $y \in E_\varepsilon := M_{2,\varepsilon}^2 \big(A_\varepsilon - x_{2,\varepsilon}\big)$, we introduce the following function

$$U_\varepsilon(y) = M_{2,\varepsilon}^{-1} u_\varepsilon\big(x_{2,\varepsilon} + M_{2,\varepsilon}^{-2} y\big).$$

It is easy to see that U_ε is bounded by 1 in $B(0, (1/2)M_{2,\varepsilon}^2 \mid x_{2,\varepsilon} - x_{1,\varepsilon} \mid)$. Therefore, $U_\varepsilon \to \delta_{(0,\alpha_0)}$ in $C_{loc}^2(\mathbb{R}^3)$ as $\varepsilon \to 0$. Thus, we have obtained in Case 2 a second blow up point. It is clear that we can iterate such a process. But, since the energy of u_ε is bounded such a process stops after finitely many steps, and the proof of our theorem is thereby completed. $\square$

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. To this aim, we first study the location of blow up points that we found in Section 3. To this goal, we need a rather delicate analysis and careful estimates. First, we start with the general setting. Let, for $p \in \mathbb{N}^*$ and $\eta > 0$ given,

$$V_\varepsilon(p,\eta) = \Big\{ u \in \Sigma^+(A_\varepsilon) \ : \exists\, y_1, ..., y_p \in A_\varepsilon, \exists\, \lambda_1, ..., \lambda_p > \frac{1}{\eta} \text{ with}$$
$$\Big|\Big| u - C(p) \sum_{i=1}^{p} P_\varepsilon \delta_{(y_i,\lambda_i)} \Big|\Big| < \eta,\ \lambda_i\, d(y_i, \partial A_\varepsilon) > \frac{1}{\eta} \,\forall\, i, \quad \varepsilon_{ij} < \eta \,\forall i \neq j \Big\},$$

where $\Sigma^+(A_\varepsilon) = \{u \in H_0^1(A_\varepsilon) : u > 0,\ ||u|| = 1\}$ and $\varepsilon_{ij} = (\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j|y_i - y_j|^2)^{-1/2}$.

If a function u belongs to $V_\varepsilon(p,\eta)$, then, for $\eta > 0$ small enough, the minimization problem

$$\min_{\alpha_i,\lambda_i>0,\ y_i\in A_\varepsilon} \Big|\Big|u - \sum_{i=1}^{p} \alpha_i P_\varepsilon \delta_{(y_i,\lambda_i)}\Big|\Big| \tag{4.1}$$

has a unique solution, up to permutation (see Lemma A.2 in [3]).

Therefore, for $\varepsilon > 0$ sufficiently small, u_ε (solution of (P_ε)) can be uniquely written as

$$\widetilde{u}_\varepsilon := \frac{u_\varepsilon}{||u_\varepsilon||} = \sum_{i=1}^{p} \alpha_{i,\varepsilon} P_\varepsilon \delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})} + v_\varepsilon, \tag{4.2}$$

where v_ε satisfies the following conditions:

$$(V_0)\ \langle v_\varepsilon, P_\varepsilon\delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}\rangle = \langle v_\varepsilon, \frac{\partial P_\varepsilon \delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}}{\partial \lambda_{i,\varepsilon}}\rangle = \langle v_\varepsilon, \frac{\partial P_\varepsilon \delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}}{\partial (x_{i,\varepsilon})_k}\rangle = 0 \quad \forall\, i,$$

where $(x_{i,\varepsilon})_k$ is the kth component of $x_{i,\varepsilon}$, $k \in \{1,2,3\}$, and $\alpha_{i,\varepsilon}$ satisfies:

$$J(u_\varepsilon)^3 \alpha_{j,\varepsilon}^4 = 1 + o(1)\ \forall j.$$

To simplify the notation, we write $\alpha_i, x_i, \lambda_i, \delta_i$, $P\delta_i$, and θ_i instead of $\alpha_{i,\varepsilon}$, $x_{i,\varepsilon}, \lambda_{i,\varepsilon}, \delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}$, $P_\varepsilon\delta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}$, and $\theta_{(x_{i,\varepsilon},\lambda_{i,\varepsilon})}$ respectively, and we also write u_ε instead of $\widetilde{u}_\varepsilon$.

As a consequence of Theorem 3.1, it is easy to obtain the following result:

Corollary 4.1. *For each i, we denote by $B_i := B(x_i, \alpha d_i/4)$. For $i \neq j$, we have*

$$(a)\quad \varepsilon_{ij} \le \frac{c}{(\lambda_i d_i \lambda_j d_j)^{1/2}}, \quad (b)\quad \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{1}{2}\varepsilon_{ij}(1+o(1)), \quad (c)\quad B_i \cap B_j = \emptyset.$$

Proof. The proof is immediate since $|x_i - x_j| \ge \alpha\varepsilon$ for each $i \neq j$ and $d_i \le \varepsilon$ for each i. □

Now, let us recall the estimate of the v_ε-part of u_ε.

Proposition 4.2. [6] *Let v_ε be defined by (4.2). Then, we have the following estimate*

$$||v_\varepsilon|| \le c \sum_i \frac{1}{\lambda_i d_i} + c \sum_{i\neq j} \varepsilon_{ij} \big(Log \varepsilon_{ij}^{-1}\big)^{1/3}.$$

In the next propositions, we give useful expansions of the gradient of J which allows us to characterize the concentration points given by Theorem 3.1.

Regarding the estimate of $||v_\varepsilon||^2$, it is negligible with respect to the principle part of Proposition 3.2 of [6]; however it is of the same order as the principle part of Proposition 3.3 of [6]. Following an idea introduced by O. Rey [18] and using the fact that the balls B_i are disjoints, we are able to improve the terms which contain v_ε and therefore we can obtain the analogue of Proposition 3.3 of [6].

Proposition 4.3. *For each i, we have the following expansion*

$$\langle \nabla J(u_\varepsilon), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = 2J(u_\varepsilon)c_1\Big(-\frac{\alpha_i}{2}\frac{H_\varepsilon(x_i,x_i)}{\lambda_i}(1+o(1)) \\ -\sum_{j\neq i}\alpha_j\big(\lambda_i\frac{\partial \varepsilon_{ij}}{\partial \lambda_i}+\frac{1}{2}\frac{H_\varepsilon(x_i,x_j)}{(\lambda_i\lambda_j)^{1/2}}\big)(1+o(1))+R\Big),$$

where c_1 is a positive constant and

$$R = O\big(\sum_1^p (\lambda_k d_k)^{-2} + \sum_{k\neq r}\varepsilon_{kr}^2\big(Log\varepsilon_{kr}^{-1}\big)^{2/3}\big).$$

Proof. It follows from Lemma 5.1, Proposition 4.2 and the fact that v_ε satisfies (V_0). □

Proposition 4.4. *For each i, we have the following expansion*

$$\langle \nabla J(u_\varepsilon), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial x_i}\rangle = J(u_\varepsilon)c_1\Big(-2\sum_{j\neq i}\alpha_j\big(\frac{1}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial x_i} - \frac{1}{\lambda_i(\lambda_i\lambda_j)^{1/2}}\frac{\partial H_\varepsilon(x_i,x_j)}{\partial x_i}\big) \\ +\frac{\alpha_i}{\lambda_i^2}\frac{\partial H_\varepsilon(x_i,x_i)}{\partial x_i} + o\Big(\sum_1^p \frac{1}{\big(\lambda_k d_k\big)^2}\Big)\Big).$$

Proof. It follows from Lemmas 5.2, 5.3, 5.6, 5.7, 5.8, Proposition 4.2, and the fact that v_ε satisfies (V_0). The negligible terms which appear in those estimates can be written as $o((\lambda_1 d_1)^{-2})$ since $|x_i - x_j| \geq \alpha\varepsilon$ for each $i \neq j$ and $d_k \leq \varepsilon$ for each k. □

Now, we order all the $\lambda_i d_i$'s : $\lambda_1 d_1 \leq \lambda_2 d_2 \leq ... \leq \lambda_p d_p$.

First, we introduce the set of indices i such that $\lambda_i d_i$ and $\lambda_1 d_1$ are of the same order. Let C_1 be a large positive constant and define

$$I = \{1\} \cup \{i : \lambda_k d_k \leq C_1 \lambda_{k-1} d_{k-1} \text{ for each } k \leq i\} := \{1, 2, ..., l\}. \tag{4.3}$$

Secondly, we define a subset of I such that the distance between the points is at most comparable to their distances to the boundary. Let C_0 be a large positive constant; we define

$$B = \{i \in I : \exists k_1, ..., k_m \in I : k_1 = i, ..., k_m = 1\,;\, |x_{k_j} - x_{k_{j+1}}|\} \\ \leq C_0 \min(d_{k_j}, d_{k_{j+1}}). \tag{4.4}$$

Lemma 4.5. *Let B be defined by* (4.4)*. Then,* $\{1\} \subsetneq B$*.*

Proof. First, we remark that Proposition 4.3 implies immediately that $p \geq 2$. To prove our lemma, we argue by contradiction. We assume that $B = \{1\}$.

Using Proposition 4.3, and the fact that $H_\varepsilon(x_i, x_i) \sim c/d_i$ (see [1]), we derive

$$0 = \langle \nabla J(u_\varepsilon), \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \rangle \leq -\frac{c}{(\lambda_1 d_1)} + O(\sum_{k \neq 1} \varepsilon_{1k}). \tag{4.5}$$

Two cases may occur. If $k > l$ where l is defined by (4.3), then by Corollary 4.1, we have

$$\varepsilon_{1k} \leq \frac{c}{(\lambda_k d_k \lambda_1 d_1)^{1/2}} \leq \frac{1}{C_1^{1/2}} \frac{1}{((\lambda_l d_l)(\lambda_1 d_1))^{1/2}} = o\big(\frac{1}{\lambda_1 d_1}\big)$$

for C_1 large enough. In the other case, we have $|x_1 - x_k| \geq C_0$ min (d_1, d_k); then

$$\varepsilon_{1k} \leq \big(\frac{1}{\lambda_1 \lambda_k |x_1 - x_k|^2}\big)^{1/2} \leq \frac{2}{C_0^{1/2}} \frac{1}{\big((\lambda_1 d_1)(\lambda_k d_k)\big)^{1/2}} = o\big(\frac{1}{\lambda_1 d_1}\big)$$

for C_0 large enough. Thus, (4.5) yields a contradiction and the result follows. □

Next, our goal is to prove the following crucial result:

Proposition 4.6. *Let $x_{1,\varepsilon}$,..., $x_{p,\varepsilon}$ be the points given by Theorem* 3.1*. Then, we have $p \geq 2$ and there exist $k \in \{2, .., p\}$, i_1,..., $i_k \in \{1, ..., p\}$ such that*

$$d\rho_\varepsilon(x_{i_1,\varepsilon}, ..., x_{i_k,\varepsilon}) \to 0 \quad \text{and} \quad d^2 \nabla \rho_\varepsilon(x_{i_1,\varepsilon}, ..., x_{i_k,\varepsilon}) \to 0, \quad \text{as } \varepsilon \to 0,$$

where $d = \min_{1 \leq r \leq k} d(x_{i_r,\varepsilon}, \partial A_\varepsilon)$. In addition, we have for all $m, r \in \{1, ..., k\}$ $|x_{i_m,\varepsilon} - x_{i_r,\varepsilon}| \leq C_0' d$, where C_0' is a positive constant independent of ε.

Proof. Let $k =$ card B; that is, $B = \{i_1, ..., i_k\}$. By Lemma 4.5, we have $k \geq 2$.

Let $M_B = (m_{ij})_{i,j \in B}$ be the matrix defined by (2.3) and let $\rho_B = \rho_\varepsilon(x_{i_1,\varepsilon}, \ldots, x_{i_k,\varepsilon})$ be the smallest eigenvalue associated to M_B. We denote by e the eigenvector associated to ρ_B whose norm is 1. We know that all components of e are strictly positive (see [4]). Let $\eta > 0$ be such that, for any γ belonging to a neighborhood $C(e,\eta) \subset \{y \in (\mathbb{R}_+^*)^k : ||y|^{-1}y - e| < \eta\}$, we have

$$^T\gamma M_B\gamma - \rho_B|\gamma|^2 \leq \frac{c_2}{d}|\gamma|^2 \text{ and } {}^T\gamma\frac{\partial M_B}{\partial x_i}\gamma = \big(\frac{\partial \rho_B}{\partial x_i} + o(\frac{1}{d^2})\big)|\gamma|^2, \qquad (4.6)$$

and for $\gamma \in (\mathbb{R}_+^*)^k \setminus C(e,\eta)$, we have

$$^T\gamma M_B\gamma - \rho_B|\gamma|^2 \geq c_3|\gamma|^2 d^{-1}. \qquad (4.7)$$

First, we study the vector Λ defined by $\Lambda = \big(\lambda_{i_1}^{-1/2}, ..., \lambda_{i_k}^{-1/2}\big)$.

Claim 1. We have $\Lambda \in C(e,\eta)$.

Proof of Claim 1. We argue by contradiction. Assume that $\Lambda \in (\mathbb{R}_+^*)^k \setminus C(e,\eta)$. Let

$$\Lambda(t) = |\Lambda|\frac{(1-t)\Lambda + t|\Lambda|e}{|(1-t)\Lambda + t|\Lambda|e|} := \frac{y(t)}{|y(t)|}.$$

From Proposition 4.3, we derive

$$\langle \nabla J(u_\varepsilon), Z\rangle_{|t=0} = -c\frac{d}{dt}\big({}^T\Lambda(t)M_B\Lambda(t)\big) + O\big(\sum_{i\in B, j\notin B}\varepsilon_{ij}\big) + o\big(\frac{1}{\lambda_1 d_1}\big),$$

where Z is the vector field defined on the variables λ along the flow line defined by $\Lambda(t)$. Observe that

$$\begin{aligned}\frac{d}{dt}\big({}^T\Lambda(t)M_B\Lambda(t)\big) &= \frac{d}{dt}\big(\frac{{}^T\Lambda(t)M_B\Lambda(t)}{|\Lambda(t)|^2}|\Lambda(0)|^2\big)\\ &= |\Lambda(0)|^2\frac{d}{dt}\big(\rho_B + \frac{(1-t)^2}{|y(t)|^2}({}^T\Lambda(0)M_B\Lambda(0) - \rho_B|\Lambda(0)|^2)\big)\\ &= |\Lambda(0)|^2\big(\frac{2(1-t)}{|y(t)|^4}({}^T\Lambda(0)M_B\Lambda(0) - \rho_B|\Lambda(0)|^2)\\ &\quad \times(-(1-t)|\Lambda(0)| < e,\Lambda(0) > -t|\Lambda|^2)\big).\end{aligned}$$

Thus,

$$\begin{aligned}\langle \nabla J(u_\varepsilon), Z\rangle_{|t=0} = &-\frac{2c}{|\Lambda|^2}({}^T\Lambda M_B\Lambda - \rho_B|\Lambda|^2)(-|\Lambda| < e,\Lambda(0) >)\\ &+ o\big(\frac{1}{\lambda_1 d_1}\big) + O\Big(\sum_{i\in B, j\notin B}\varepsilon_{ij}\Big).\end{aligned}$$

Since $|e| = 1$, then there exists m such that $e_{i_m} \geq \frac{1}{k}$. Thus,

$$< e,\Lambda(0) >= \sum_j e_{i_j}\Lambda_{i_j} \geq \frac{1}{k}\Lambda_{i_m}.$$

Using (4.7), we obtain

$$\begin{aligned}\langle \nabla J(u_\varepsilon), Z\rangle_{|t=0} &\geq \frac{cc_3}{d}|\Lambda|\Lambda_{i_m} + o\big(\frac{1}{\lambda_1 d_1}\big) + O\Big(\sum_{i\in B, j\notin B}\varepsilon_{ij}\Big)\\ &\geq \frac{c}{\big(\lambda_1 d_1 \lambda_{i_m} d_{i_m}\big)^{1/2}} + o\big(\frac{1}{\lambda_1 d_1}\big) + O\Big(\sum_{i\in B,\, j\notin B}\varepsilon_{ij}\Big).\end{aligned}$$

As in the proof of Lemma 4.5, we have

$$\varepsilon_{ij} = o\big(\frac{1}{(\lambda_1 d_1 \lambda_{i_m} d_{i_m})^{1/2}}\big) \quad \forall\, i \in B, \forall\, j \notin B. \tag{4.8}$$

Thus,

$$0 \geq \frac{c}{(\lambda_1 d_1 \lambda_{i_m} d_{i_m})^{1/2}} + o\big(\frac{1}{\lambda_1 d_1}\big) \geq \frac{1}{\lambda_1 d_1}\big(\frac{c}{C_1^{k/2}} + o(1)\big) > 0.$$

This yields a contradiction and our claim follows.

Now, we will prove that

$$d\rho_B \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0. \tag{4.9}$$

Using Proposition 4.3 and (4.8), we have

$$\begin{aligned} 0 = \sum_{i\in B}\langle \nabla J(u_\varepsilon), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle &= \sum_{i\in B}\Big[\frac{H_\varepsilon(x_i,x_i)}{\lambda_i}(1+o(1))\\ &- \sum_{j\neq i, j\in B}(\varepsilon_{ij} - \frac{H_\varepsilon(x_i,x_j)}{(\lambda_i\lambda_j)^{\frac{1}{2}}})(1+o(1)) + O(\sum_{j\notin B}\varepsilon_{ij}) + R\Big]\\ &=^T \Lambda M_B \Lambda + o\big(\frac{1}{\lambda_1 d_1}\big). \end{aligned} \tag{4.10}$$

We assume, arguing by contradiction, that $d\rho_B \not\longrightarrow 0$, when $\varepsilon \longrightarrow 0$. Therefore, there exists $C_4 > 0$ such that $d|\rho_B| \geq C_4$. Now, we distinguish two cases:

First case: $\rho_B > 0$. In this case, we derive from (4.10)

$$0 \geq \rho_B|\Lambda|^2 + o\big(\frac{1}{\lambda_1 d_1}\big) \geq C_4\frac{|\Lambda|^2}{d} + o\big(\frac{1}{\lambda_1 d_1}\big) > 0.$$

This yields a contradiction.

Second case: $\rho_B < 0$. In this case, using Claim 1, we derive from (4.6) and (4.10),

$$0 \leq \rho_B|\Lambda|^2 + \frac{c_2|\Lambda|^2}{d} + o\big(\frac{1}{\lambda_1 d_1}\big) \quad \leq \quad \frac{|\Lambda|^2}{d}\big(\rho_B\, d + c_2\big) + o\big(\frac{1}{\lambda_1 d_1}\big)$$

$$\leq \quad \frac{|\Lambda|^2}{d}(-C_4+c_2)+o\big(\frac{1}{\lambda_1 d_1}\big).$$

If we choose $c_2 \leq \frac{1}{2}C_4$, we obtain a contradiction. Thus, (4.9) follows.

In order to complete the proof of Proposition 4.6, it remains to prove that:

$$d^2\nabla\rho_B \to 0, \qquad \text{as } \varepsilon \to 0. \tag{4.11}$$

We assume, arguing by contradiction, that $d^2\nabla\rho_B \not\longrightarrow 0$ when $\varepsilon \longrightarrow 0$.

For $i \in B$, using Proposition 4.4, we derive

$$0 =^T \Lambda\frac{\partial M_B}{\partial x_i}\Lambda + O\big(\sum_{j\notin B}\frac{\partial \varepsilon_{ij}}{\partial x_i} - \frac{1}{(\lambda_i\lambda_j)^{\frac{1}{2}}}\frac{\partial H_\varepsilon}{\partial x_i}(x_i,x_j)\big) + o\big(\frac{1}{d_i}\frac{1}{(\lambda_1 d_1)}\big).$$

Observe that $|\partial H_\varepsilon/\partial x_i(x_i,x_j)| \leq c(d_i|x_i-x_j|)^{-1}$. Thus, as in the proof of Lemma 4.5, we prove that, for $i \in B$ and $j \notin B$,

$$\big(|\frac{\partial \varepsilon_{ij}}{\partial x_i}| + \frac{1}{(\lambda_i\lambda_j)^{1/2}}\big)|\frac{\partial H_\varepsilon}{\partial x_i}(x_i,x_j)| = o\big(\frac{1}{d(\lambda_1 d_1)}\big).$$

Therefore, by (4.6), we have

$$0 =^T \Lambda\frac{\partial M_B}{\partial x_i}\Lambda + o\big(\frac{1}{d(\lambda_1 d_1)}\big) = \big(\frac{\partial \rho_B}{\partial x_i}d^2 + o(1)\big)\frac{|\Lambda|^2}{d^2} + o\big(\frac{1}{d(\lambda_1 d_1)}\big),\ \forall\, i \in B.$$

Thus,

$$0 \geq \big(|\nabla\rho_B|d^2 + o(1)\big)\frac{|\Lambda|^2}{d^2} + o\big(\frac{1}{d(\lambda_1 d_1)}\big) \geq C_6\frac{|\Lambda|^2}{d^2} + o\big(\frac{1}{d(\lambda_1 d_1)}\big) > 0.$$

This yields a contradiction. Hence (4.11) follows. The proof of Proposition 4.6 is thereby completed. □

Proof of Theorem 1.1. Arguing by contradiction, we assume that (P_ε) has a solution whose energy is bounded. Using Theorem 1.5 of [6] and Proposition 4.6, we deduce Theorem 1.1. □

5. Appendix

In this section, we collect some estimates needed to prove Propositions 4.3 and 4.4. Here we will denote by $u_\varepsilon := \sum_{j=1}^{p}\alpha_j P\delta_{(x_j,\lambda_j)} + v_\varepsilon$ the function defined in Theorem 3.1. Thus, we have $|x_i - x_j| \geq \alpha\varepsilon$ for each $i \neq j$ and $\lambda_i d_i \to \infty$ as $\varepsilon \to 0$ for each i. In the sequel, we denote by $\varphi_{i,k} = \lambda_i^{-1}\partial P\delta_i/\partial(x_i)_k$ where $(x_i)_k$ is the kth component of x_i, $k \in \{1,2,3\}$.

Recall that B_i denotes $B(x_i, \alpha d_i/4)$ and we have, for each $i \neq j$, $B_i \cap B_j = \emptyset$.

Furthermore, all the estimates needed in this Appendix will be done in the balls B_i and in $A_\varepsilon \backslash B_i$. The integral over the ball B_i is computed through rescaling, while for the integral on the complement, using the fact that $\lambda_i d_i \to \infty$ as $\varepsilon \to 0$ we have the following estimate

$$\int_{A_\varepsilon \backslash B_i} \delta^6_{(a_i,\lambda_i)} \leq \int_{\mathbb{R}^3 \backslash B_i} \delta^6_{(a_i,\lambda_i)} = \int_{\mathbb{R}^3 \backslash B(0,\alpha\lambda_i d_i/4)} \delta^6_{(0,1)} \leq \frac{c}{(\lambda_i d_i)^3}, \qquad (5.1)$$

where c is a positive constant independent of ε.

Another fact which we use in this Appendix is the estimate of the function $P\delta_{(a,\lambda)}$. In fact, using the maximum principle, $P\delta_{(a,\lambda)}$ can be written as

$$P\delta_{(a,\lambda)}(x) = c_0 \frac{H_\varepsilon(a,x)}{\lambda^{1/2}} + f_{(a,\lambda)}(x)$$

where $f_{(a,\lambda)}$ satisfies

$$|f_{(a,\lambda)}|_{L^\infty} \leq \frac{c}{\lambda^{5/2} d^3}; \quad \Big|\frac{\partial f_{(a,\lambda)}}{\partial \lambda}\Big|_{L^\infty} \leq \frac{c}{\lambda^{7/2} d^3}, \quad \Big|\frac{\partial f_{(a,\lambda)}}{\partial a}\Big|_{L^\infty} \leq \frac{c}{\lambda^{5/2} d^4},$$

where c is a constant independent of ε.

For the integrals in the ball B_i, in the final step, we compute the integral in $\mathbb{R}^3$ and another integral outside of the balls as in (5.1). Hence, all the constants and the O which we will obtain are independent of ε.

Lemma 5.1. *For $i \neq j$, we have the following estimates*

1) $$\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = \frac{c_1}{2} \frac{H_\varepsilon(a_i,a_i)}{\lambda_i} + O\big(\frac{1}{(\lambda_i d_i)^2}\big)$$

2) $$\langle P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = c_1\big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{1}{2}\frac{H_\varepsilon(a_i,a_j)}{(\lambda_i\lambda_j)^{1/2}}\big) + O\Big(\varepsilon_{ij}^2\big(Log\varepsilon_{ij}^{-1}\big)^{2/3} + \sum_{k=i,j}\frac{1}{(\lambda_k d_k)^2}\Big),$$

3) $$\int_{A_\varepsilon} P\delta_i^5 \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = 2\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + O\big(\frac{1}{(\lambda_i d_i)^2}\big),$$

4) $$\int_{A_\varepsilon} P\delta_j^5 \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \langle P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + O\big(\varepsilon_{ij}^2\big(Log\varepsilon_{ij}^{-1}\big)^{2/3} + \frac{1}{(\lambda_j d_j)^2}\big),$$

5) $$5\int_{A_\varepsilon} P\delta_j\big(P\delta_i^4 \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\big) = \langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + O\big(\varepsilon_{ij}^2\big(Log\varepsilon_{ij}^{-1}\big)^{2/3} + \frac{1}{(\lambda_i d_i)^2}\big),$$

6) $$\int_{\mathbb{R}^3} \delta_i^3 \delta_j^3 = O\big(\varepsilon_{ij}^3 Log\varepsilon_{ij}^{-1}\big),$$

where $c_1 = c_0 \int_{\mathbb{R}^3} \delta_{(0,1)}^5$ *where* c_0 *is defined in* (2.4) *and* O *is independent of* ε.

Proof. For the proof, we argue as we explained in the introduction of this section to take care of the dependence with respect to ε, then the remaining estimates are very similar to those performed in [2], [17], and [18]. □

Lemma 5.2. *For* $i \in \{1, ..., p\}$ *and* $j \neq i$, *we have the following estimates*

$$
\begin{aligned}
&1) \quad \langle P\delta_i, \varphi_{i,k}\rangle = -\frac{c_1}{2\lambda_i^2}\frac{\partial H_\varepsilon}{\partial (x_i)_k}(x_i, x_i) + O\big(\frac{1}{(\lambda_i d_i)^3}\big),\\
&2) \quad \int_{A_\varepsilon} P\delta_i^5 \varphi_{i,k} = 2\langle P\delta_i, \varphi_{i,k}\rangle + O\big(\frac{Log(\lambda_i d_i)}{(\lambda_i d_i)^3}\big),\\
&3) \quad \langle P\delta_j, \varphi_{i,k}\rangle = \frac{-c_1}{\lambda_i^{3/2}\lambda_j^{1/2}}\frac{\partial H_\varepsilon}{\partial (x_i)_k}(x_j, x_i) + \frac{c_1}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial (x_i)_k}\\
&\qquad\qquad + O\big(\frac{1}{(\lambda_1 d_1)^3} + \lambda_j |x_i - x_j| \varepsilon_{ij}^4\big),\\
&4) \quad \int_{A_\varepsilon} P\delta_j^5 \varphi_{i,k} = \langle P\delta_j, \varphi_{i,k}\rangle + O\big(\frac{1}{(\lambda_1 d_1)^{5/2}}\big),\\
&5) \quad \int_{A_\varepsilon} 5P\delta_j P\delta_i^4 \varphi_{i,k} = \langle P\delta_j, \varphi_{i,k}\rangle + O\big(\frac{1}{(\lambda_1 d_1)^{5/2}}\big).
\end{aligned}
$$

Proof. Claims 1, 2, and 3 are proved in [2] and [17]. We will prove Claim 4. We have

$$
\begin{aligned}
\int_{A_\varepsilon} P\delta_j^5 \varphi_{i,k} &= \int_{A_\varepsilon} (\delta_j^5 + O(\delta_j^4 \theta_j))\varphi_{i,k}\\
&= \langle P\delta_j, \varphi_{i,k}\rangle + O\big(\int_{B_j} \delta_j^4 \theta_j |\varphi_{i,k}| + \int_{A_\varepsilon \setminus B_j} \delta_j^5 \delta_i\big).
\end{aligned}
$$

For the second integral, using Holder's inequality, we obtain

$$\int_{\mathbb{R}^3 \setminus B_j} \delta_j^5 \delta_i = O\big(\frac{1}{(\lambda_j d_j)^{5/2}}\big). \tag{5.2}$$

By Corollary 4.1, we have $B_i \cap B_j = \emptyset$ and therefore, for any $x \in B_j$, we get

$$\sup_{B_j}\Big|\frac{1}{\lambda_i}\frac{\partial \delta_i}{\partial (x_i)_k}\Big| \leq C \sup_{B_j}\big(\frac{1}{\lambda_i^{3/2}|x - x_i|^2}\big) = O\big(\frac{1}{\lambda_i^{3/2}\max^2(d_i, d_j)}\big), \tag{5.3}$$

$$\sup_{B_j}\Big|\frac{1}{\lambda_i}\frac{\partial \theta_i}{\partial (x_i)_k}\Big| \leq \frac{C}{\lambda_i d_i}\sup_{B_j}\theta_i = O\big(\frac{1}{\lambda_i^{3/2} d_i \max(d_i, d_j)}\big). \tag{5.4}$$

Thus, we obtain

$$\int_{B_j} \delta_j^4 \theta_j |\varphi_{i,k}| \le \frac{c}{\lambda_i^{3/2} \lambda_j^{3/2} d_i d_j \max(d_i, d_j)} \le \frac{c}{(\lambda_1 d_1)^3}. \tag{5.5}$$

Combining (5.5) and (5.2), the claim follows.

It remains to prove Claim 5. We have

$$\begin{aligned} 5 \int_{A_\varepsilon} P\delta_j P\delta_i^4 \varphi_{i,k} &= 5 \int_{A_\varepsilon} \big(\delta_i^4 - 4\delta_i^3\theta_i + O\big(\delta_i^2\theta_i^2\big)\big) P\delta_j \big(\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} - \frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial (x_i)_k}\big) \\ &= \langle P\delta_j, \varphi_{i,k} \rangle + O\big(\int_{B_i} \delta_j \delta_i^4 \Big|\frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial (x_i)_k}\Big|\big) - 20 \int_{B_i} P\delta_j \delta_i^3 \theta_i \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \\ &\quad + O\big(\int_{B_i} \delta_i^3 \theta_i^2 \delta_j + \int_{\mathbb{R}^3 \setminus B_i} \delta_i^5 \delta_j\big). \end{aligned}$$

Observe that

$$\sup_{B_i} |D\theta_i| \le \frac{C}{d_i} \sup_{B_i} \theta_i \le \frac{C}{\lambda_i^{\frac{1}{2}} d_i^2} \quad ; \quad \sup_{B_i} \delta_j \le \frac{c}{\lambda_j^{\frac{1}{2}} \max(d_i, d_j)}, \tag{5.6}$$

$$\sup_{B_i} |DP\delta_j| \le \sup_{B_i} |D\delta_j| + \sup_{B_i} |D\theta_j| \le \frac{C}{\lambda_j^{\frac{1}{2}} \max^2(d_i, d_j)} + \frac{C}{\lambda_j^{\frac{1}{2}} d_i \max(d_i, d_j)}. \tag{5.7}$$

Thus, we derive

$$\int_{B_i} \delta_i^3 \theta_i^2 \delta_j \le |\delta_j \theta_i^2|_{L^\infty} \int_{B_i} \delta_i^3 \le \frac{c Log(\lambda_i d_i)}{(\lambda_j d_j)^{1/2} (\lambda_i d_i)^{5/2}}, \tag{5.8}$$

$$\int_{B_i} \delta_j \delta_i^4 \Big|\frac{1}{\lambda_i} \frac{\partial \theta_i}{\partial (x_i)_k}\Big| \le \frac{c}{(\lambda_j d_j)^{1/2} (\lambda_i d_i)^{5/2}}, \tag{5.9}$$

$$\begin{aligned} \int_{B_i} P\delta_j \delta_i^3 \theta_i \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} &= O\big(\sup_{B_i} |D\big(\theta_i P\delta_j\big)| \int_{B_i} \delta_i^4 |x - x_i|\big) \\ &= O\big(\frac{Log(\lambda_i d_i)}{(\lambda_i d_i)^{\frac{5}{2}} (\lambda_j d_j)^{\frac{1}{2}}}\big). \end{aligned} \tag{5.10}$$

Using (5.2), (5.8), (5.9), and (5.10), the lemma follows. □

Lemma 5.3. *For each i, we have*

$$\int_{A_\varepsilon} \Big(\sum_{j=1}^p \alpha_j P\delta_j\Big)^5 \varphi_{i,k} = 2 \sum_{j=1}^p \alpha_j \langle P\delta_j, \varphi_{i,k} \rangle + O\big(\frac{1}{(\lambda_1 d_1)^{9/4}}\big).$$

Proof. Notice that

$$\Big(\sum_{j=1}^{p}\alpha_j P\delta_j\Big)^5=\sum_{j=1}^{p}\Big(\alpha_j P\delta_j\Big)^5+5(\alpha_i P\delta_i)^4\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big)\tag{5.11}$$

$$+10(\alpha_i P\delta_i)^3\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big)^2+O\Big(\sum_{j\neq i}\delta_i^2\delta_j^3+\sum_{j\notin\{i,r\}}\delta_j^4\delta_r\Big).$$

Since $B_j\cap B_i=\emptyset$ and $B_j\cap B_r=\emptyset$, using (5.3) and (5.4), we derive

$$\int_{B_j}\delta_j^4\delta_r|\varphi_{i,k}|\leq\frac{c}{\lambda_r^{1/2}\max(d_r,d_j)\lambda_i^{3/2}d_i\max(d_i,d_j)}\int_{B_j}\delta_j^4\leq\frac{c}{(\lambda_1 d_1)^3},\tag{5.12}$$

$$\int_{A_\varepsilon\setminus B_j}\delta_j^4\delta_r^2\leq\int_{A_\varepsilon\setminus B_j}\delta_j^6+\int_{A_\varepsilon\setminus B_j}\delta_j^3\delta_r^3\leq\frac{c}{(\lambda_1 d_1)^{5/2}}.\tag{5.13}$$

Now we will estimate the third term. Using (5.6) and (5.7), we obtain

$$\int_{B_i}P\delta_i^3\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big)^2\varphi_{i,k}\tag{5.14}$$

$$=\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big)^2(x_i)\int_{B_i}\big(\delta_i^3+O\big(\delta_i^2\theta_i\big)\big)\frac{1}{\lambda_i}\Big(\frac{\partial\delta_i}{\partial(x_i)_k}-\frac{\partial\theta_i}{\partial(x_i)_k}\Big)$$

$$+O\Big(\sup_{B_i}\Big|D\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big)^2\Big|\int_{B_i}\delta_i^4|x-x_i|\Big)=O\big(\sum_{j\neq i}\frac{Log(\lambda_i d_i)}{\lambda_j\max^2(d_i,d_j)}\frac{1}{\lambda_i^2 d_i}\big).$$

Combining (5.11),...,(5.14), and Lemma 5.2, the result follows. □

To improve the estimates of the integrals involving v_ε, we use an original idea due to Rey [18]; namely we write

$$v_\varepsilon=\sum_{i=1}^{p}v_i^\varepsilon+w,\tag{5.15}$$

where v_i^ε denotes the projection of v_ε onto $H_0^1(B_i)$; that is

$$\Delta v_i^\varepsilon=\Delta v_\varepsilon\quad\text{in}\quad B_i;\quad v_i^\varepsilon=0\quad\text{on}\quad\partial B_i,\tag{5.16}$$

where $B_i=B(x_i,\alpha d_i/4)$ is defined in Corollary 4.1. v_i^ε can be assumed to be defined in all A_ε since it can be continued by 0 in $A_\varepsilon\setminus B_i$. We have

$$v_\varepsilon=v_i^\varepsilon+w\quad\text{in}\quad B_i,\quad\text{with}\quad\Delta w=0\text{ in }B_i.\tag{5.17}$$

We split v_i^ε into an even part $v_i^{\varepsilon,e}$ and an odd part $v_i^{\varepsilon,o}$ with respect to $(x-x_i)_k$, thus we have

$$v_\varepsilon = v_i^{\varepsilon,e} + v_i^{\varepsilon,o} + w \quad \text{in} \quad B_i \quad \text{with} \quad \Delta w = 0 \text{ in } B_i. \tag{5.18}$$

Lemma 5.4. *We have*

$$\int_{B_i} \delta_i^3 v_\varepsilon^2 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} = O\big(||v_i^{\varepsilon,o}||||v_\varepsilon|| + \frac{||v_\varepsilon||^2}{(\lambda_i d_i)^{1/2}}\big).$$

Proof. Using (5.18) and the fact that the even part of v_ε^2 has no contribution to the integrals, we obtain

$$\int_{B_i} \delta_i^3 v_\varepsilon^2 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} = \int_{B_i} \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \big(2v_\varepsilon - w\big) w + O\big(||v_i^{\varepsilon,o}||||v_i^{\varepsilon,e}||\big). \tag{5.19}$$

Let ψ be the solution of

$$\Delta \psi = \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \big(2v_\varepsilon - w\big) \quad \text{in} \quad B_i; \ \psi = 0 \quad \text{on} \quad \partial B_i.$$

Thus, we have

$$\int_{B_i} \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \big(2v_\varepsilon - w\big) w = \int_{B_i} \Delta \psi . w = \int_{\partial B_i} \frac{\partial \psi}{\partial \nu} w. \tag{5.20}$$

Let G_i be the Green's function for the Laplacian on B_i; that is,

$$G_i(x,y) = \frac{1}{|x-y|} - \frac{\alpha \delta d_i}{4|x||y - \frac{(\alpha\delta d_i)^2 x}{16|x|^2}|}, \quad (x,y) \in B_i^2.$$

Therefore, ψ is given by

$$\psi(y) = -\int_{B_i} G_i(x,y) \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \big(2v_\varepsilon - w\big) dx, \quad y \in B_i \tag{5.21}$$

and its normal derivative by

$$\frac{\partial \psi}{\partial \nu}(y) = -\int_{B_i} \frac{\partial G_i}{\partial \nu}(x,y) \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} \big(2v_\varepsilon - w\big) dx, \quad y \in \partial B_i. \tag{5.22}$$

Notice that

$$\text{for } x \in B_i \setminus B(y, \alpha d_i/8), \quad \text{we have} \quad \frac{\partial G_i}{\partial \nu}(x,y) = O\big(\frac{1}{d_i^2}\big), \tag{5.23}$$

$$\text{for } x \in B_i \cap B(y, \alpha d_i/8), \quad \text{we have} \quad \frac{\partial G_i}{\partial \nu}(x,y) = O\big(\frac{1}{|x-y|^2}\big), \tag{5.24}$$

$$\text{for } x \in B_i \cap B(y, \alpha d_i/8), \quad \text{we have} \quad \delta_i^3 \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (x_i)_k} = O\big(\frac{1}{\lambda_i^2 d_i^4}\big). \tag{5.25}$$

Therefore,

$$\Big|\frac{\partial\psi}{\partial\nu}(y)\Big| \leq C\int_{B_i\cap\big(|x-y|\geq\alpha d_i/8\big)}|2v_\varepsilon - w|\frac{\delta_i^4}{d_i^2}dx \tag{5.26}$$
$$+ C\int_{B_i\cap\big(|x-y|\leq\alpha d_i/8\big)}\frac{|2v_\varepsilon - w|}{\lambda_i^2 d_i^4|x-y|^2}dx \leq \frac{C}{\lambda_i^{1/2}d_i^2}||v_\varepsilon||, \quad \forall y\in\partial B_i.$$

Using (5.26), (5.20) becomes

$$\int_{B_i}\delta_i^3\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k}\big(2v_\varepsilon - w\big)w = O\big(\frac{||v_\varepsilon||}{\lambda_i^{1/2}d_i^2}\int_{\partial B_i}|w|\big). \tag{5.27}$$

To estimate the right-hand side of (5.27), we introduce the following functions

$$\bar w(X) = (\alpha d_i/4)^{1/2}w(x_i + \alpha d_i X/4); \quad \bar v_\varepsilon(X) = (\alpha d_i/4)^{1/2}v_\varepsilon(x_i+\alpha d_i X/4).$$

$\bar w$ satisfies

$$\Delta\bar w = 0 \quad \text{in} \quad B := B(0,1); \quad \bar w = \bar v_\varepsilon \quad \text{on} \quad \partial B. \tag{5.28}$$

We deduce that

$$\int_{\partial B}|\bar w| \leq C\big(\int_B|\nabla\bar v_\varepsilon|^2\big)^{1/2} = C\big(\int_{B_i}|\nabla v_\varepsilon|^2\big)^{1/2}. \tag{5.29}$$

But, we have

$$\int_{\partial B}|\bar w| = \int_{\partial B}(\alpha d_i/4)^{1/2}|w(x_i+\alpha d_i X/4)| = \frac{1}{(\alpha d_i/4)^{3/2}}\int_{\partial B_i}|w|. \tag{5.30}$$

Thus,

$$\int_{\partial B_i}|w| \leq c d_i^{3/2}\big(\int_{B_i}|\nabla v_\varepsilon|^2\big)^{1/2}. \tag{5.31}$$

Using (5.19), (5.27) and (5.31), the lemma follows. □

Lemma 5.5. *For ε small, we have*

$$\int_{A_\varepsilon}\delta_i^3 v_\varepsilon\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k} = O\big(\frac{||v_i^{\varepsilon,o}||}{\lambda_i^{1/2}} + \frac{||v_\varepsilon||}{\lambda_i d_i^{1/2}}\big).$$

Proof. Lemma 5.5 can be proved in the same way as Lemma 5.4. So we omit its proof. □

Lemma 5.6. *For ε small and $i\neq j$, we have*

$$\int_{A_\varepsilon}\Big(\sum_{j=1}^p\alpha_j P\delta_j\Big)^4 v_\varepsilon\varphi_{i,k} = O\Big(||v_i^{\varepsilon,o}||\frac{1}{\lambda_1 d_1} + ||v_\varepsilon||\frac{1}{(\lambda_1 d_1)^{3/2}}\Big).$$

Proof. We notice that

$$\Big(\sum_1^p \alpha_j P\delta_j\Big)^4 = \big(\alpha_i P\delta_i\big)^4 + 4(\alpha_i P\delta_i)^3\Big(\sum_{j\neq i}\alpha_j P\delta_j\Big) + O\Big(\delta_i^2\sum_{j\neq i}\delta_j^2 + \sum_{j\neq i}\delta_j^4\Big). \tag{5.32}$$

For the last term in (5.32), we have, using (5.2) and (5.3),

$$\begin{aligned}\int_{A_\varepsilon}\delta_j^4 v_\varepsilon\varphi_{i,k} &= \int_{B_j}\delta_j^4 v_\varepsilon\varphi_{i,k} + \int_{\mathbb{R}^3\setminus B_j}\delta_j^4 v_\varepsilon\varphi_{i,k} \\ &= O\big(\frac{||v_\varepsilon||}{\lambda_i^{3/2}d_i\max(d_i,d_j)\lambda_j^{1/2}} + \frac{||v_\varepsilon||}{(\lambda_j d_j)^2}\big).\end{aligned} \tag{5.33}$$

For the third term in (5.32), we use Holder's inequality and we obtain

$$\int_{A_\varepsilon}\delta_i^2\delta_j^2|v_\varepsilon||\varphi_{i,k}| \le \int_{A_\varepsilon}\delta_i^3\delta_j^2|v_\varepsilon| \le c||v_\varepsilon||\varepsilon_{ij}^2\big(Log\varepsilon_{ij}^{-1}\big)^{\frac{2}{3}} \le c\frac{||v_\varepsilon||}{(\lambda_1 d_1)^{\frac{3}{2}}}. \tag{5.34}$$

Regarding the first term in (5.32), we write

$$\begin{aligned}\int_{A_\varepsilon}P\delta_i^4 v_\varepsilon\varphi_{i,k} &= \int_{A_\varepsilon}\big(\delta_i^4 - 4\delta_i^3\theta_i + O\big(\delta_i^2\theta_i^2\big)\big)\frac{v_\varepsilon}{\lambda_i}\big(\frac{\partial\delta_i}{\partial(x_i)_k} - \frac{\partial\theta_i}{\partial(x_i)_k}\big) \\ &= -4\theta_i(x_i)\int_{B_i}\delta_i^3 v_\varepsilon\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k} + O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^2}\big).\end{aligned}$$

Using Lemma 5.5, we derive that

$$\int_{A_\varepsilon}P\delta_i^4 v_\varepsilon\varphi_{i,k} = O\big(\frac{||v_i^{\varepsilon,o}||}{\lambda_i d_i} + \frac{||v_\varepsilon||}{(\lambda_i d_i)^{3/2}}\big). \tag{5.35}$$

Finally, we deal with the second term in (5.32)

$$\begin{aligned}\int_{B_i}P\delta_i^3 P\delta_j v_\varepsilon\varphi_{i,k} &= P\delta_j(x_i)\int_{B_i}\big(\delta_i^3 + O\big(\delta_i^2\theta_i\big)\big)\frac{v_\varepsilon}{\lambda_i}\big(\frac{\partial\delta_i}{\partial(x_i)_k} - \frac{\partial\theta_i}{\partial(x_i)_k}\big) \\ &\quad + O\big(\sup_{B_i}|DP\delta_j|\int_{B_i}\delta_i^4|v_\varepsilon||x-x_i|\big).\end{aligned} \tag{5.36}$$

Observe that, by (5.6), we have

$$P\delta_j(x_i)\int_{B_i}\delta_i^3\Big(\theta_i + \frac{1}{\lambda_i}|\frac{\partial\theta_i}{\partial(x_i)_k}|\Big)|v_\varepsilon| = O\big(\frac{||v_\varepsilon||}{\lambda_j^{1/2}d_i\max(d_i,d_j)\lambda_i^{3/2}}\big). \tag{5.37}$$

Using (5.7), we derive that

$$\sup_{B_i}|DP\delta_j|\int_{B_i}\delta_i^4|v_\varepsilon||x-x_i| = O\big(\frac{||v_\varepsilon||}{\lambda_j^{1/2}d_i\max(d_i,d_j)\lambda_i^{3/2}}\big). \tag{5.38}$$

By Lemma 5.5, (5.37), and (5.38), (5.36) become

$$\int_{B_i} P\delta_i^3 P\delta_j v_\varepsilon \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(x_i)_k} = O\Big(\frac{||v_i^{\varepsilon,o}||}{\big(\lambda_i d_i \lambda_j d_j\big)^{1/2}} + \sum_{r\in\{i,j\}} \frac{||v_\varepsilon||}{(\lambda_r d_r)^{3/2}}\Big). \tag{5.39}$$

For the integral on $\mathbb{R}^3 \setminus B_i$, we use Holder's inequality and obtain

$$\int_{\mathbb{R}^3\setminus B_i} P\delta_i^3 P\delta_j |v_\varepsilon| \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(x_i)_k} \leq \int_{\mathbb{R}^3\setminus B_i} \delta_i^4 \delta_j |v_\varepsilon| = O\Big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^2}\Big). \tag{5.40}$$

Using (5.33), (5.34), (5.35), (5.39), and (5.40), the lemma follows. □

Lemma 5.7. *For $i \neq j$ we have*

$$\int_{A_\varepsilon} \Big(\sum_{j=1}^{p} \alpha_j P\delta_j\Big)^3 v_\varepsilon^2 \varphi_{i,k} = O\big(||v_i^{\varepsilon,o}||||v_\varepsilon|| + \frac{||v_\varepsilon||^2}{(\lambda_1 d_1)^{1/2}}\big).$$

Proof. We have

$$\big(\sum_{1}^{p} \alpha_j P\delta_j\big)^3 = \alpha_i^3\delta_i^3 + O\big(\delta_i^2\theta_i\big) + O\big(\sum_{j\neq i}\big(\delta_i^2\delta_j + \delta_j^3\big)\big).$$

We now observe that

$$\int_{A_\varepsilon} \big(\delta_i^3\theta_i + \sum_{j\neq i}\big(\delta_i^3\delta_j + \delta_j^3\delta_i\big)\big)|v_\varepsilon|^2 = O\big(||v_\varepsilon||^2\big(\frac{1}{\lambda_i d_i} + \sum_{j\neq i}\varepsilon_{ij}\big(Log\varepsilon_{ij}^{-1}\big)^{1/3}\big)\big), \tag{5.41}$$

$$\int_{\mathbb{R}^3\setminus B_i} \delta_i^4 |v_\varepsilon|^2 = O\big(\frac{||v_\varepsilon||^2}{(\lambda_i d_i)^2}\big), \tag{5.42}$$

$$\int_{B_i} \delta_i^3 v_\varepsilon^2 \big(\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k} - \frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(x_i)_k}\big) = O\big(||v_i^{\varepsilon,o}||||v_\varepsilon|| + \frac{||v_\varepsilon||^2}{(\lambda_i d_i)^{1/2}}\big), \tag{5.43}$$

where we have used Lemma 5.4 in the last equality. Clearly, (5.41), (5.42), and (5.43) imply our lemma. □

Lemma 5.8. *For ε small, we have*

$$||v_i^{\varepsilon,o}|| = O\big(\frac{1}{\big(\lambda_1 d_1\big)^{9/8}}\big).$$

Proof. We write

$$v_i^{\varepsilon,o} = \tilde{v}_i^o + aP\delta_i + b\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i} + \sum_{r=1}^{3} C_r \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial(x_i)_r} \tag{5.44}$$

with

$$\langle \tilde{v}_i^o, P\delta_i\rangle = \langle \tilde{v}_i^o, \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = \langle \tilde{v}_i^o, \frac{\partial P\delta_i}{\partial (x_i)_r}\rangle = 0 \text{ for each } r = 1, 2, 3.$$

Taking the scalar product in $H_0^1(A_\varepsilon)$ of (5.44) with $P\delta_i$, $\lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$, $\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}$, $1 \le r \le 3$, provides us with the following invertible linear system in a, b, C_r (with $1 \le r \le 3$)

$$(S) \begin{cases} \langle P\delta_i, v_i^{\varepsilon,o}\rangle = a(C' + o(1)) + b\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + \sum_{r=1}^3 C_r \langle P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle \\ \langle \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, v_i^{\varepsilon,o}\rangle = a\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + b(C'' + o(1)) \\ \qquad\qquad + \sum_{r=1}^3 C_r \langle \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle \\ \langle \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}, v_i^{\varepsilon,o}\rangle = a\langle P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}\rangle + b\langle \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}\rangle \\ \qquad\qquad + \sum_{r=1}^3 C_r \langle \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle. \end{cases}$$

Observe that

$$\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = O\big(\frac{1}{\lambda_i d_i}\big); \quad \langle \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle = O\big(\frac{1}{(\lambda_i d_i)^2}\big);$$

$$\langle P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle = O\big(\frac{1}{(\lambda_i d_i)^2}\big);$$

$$\langle \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_r}\rangle = (C''' + o(1))\delta_{jr} + O\big(\frac{1}{(\lambda_i d_i)^2}\big),$$

where δ_{jr} denotes the Kronecker symbol.

Now, because of evenness of δ_i and oddness of $v_i^{\varepsilon,o}$ with respect to $(x-x_i)_k$ we obtain

$$\langle P\delta_i, v_i^{\varepsilon,o}\rangle = \int_{A_\varepsilon} \nabla P\delta_i . \nabla v_i^{\varepsilon,o} = \int_{B_i} \nabla P\delta_i . \nabla v_i^{\varepsilon,o} = \int_{B_i} \delta_i^5 v_i^{\varepsilon,o} = 0. \tag{5.45}$$

In the same way we have

$$\langle \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, v_i^{\varepsilon,o}\rangle = \langle \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_j}, v_i^{\varepsilon,o}\rangle = 0 \quad \text{ for each } j \ne k.$$

We also have

$$\langle \varphi_{i,k}, v_i^{\varepsilon,o}\rangle = \int_{B_i} \nabla \varphi_{i,k} . \nabla\big(v_\varepsilon - v_i^{\varepsilon,e} - w\big) = -\int_{A_\varepsilon \setminus B_i} \nabla \varphi_{i,k} . \nabla v_\varepsilon - \int_{B_i} \nabla \varphi_{i,k} . \nabla w \tag{5.46}$$

since v_ε satisfies (V_0), $v_i^{\varepsilon,e}$ is even with respect to $(x-x_i)_k$, and $v_i^{\varepsilon,e}=0$ on ∂B_i. On one hand

$$\Big|\int_{A_\varepsilon\setminus B_i}\nabla\varphi_{i,k}.\nabla v_\varepsilon\Big|\le C||v_\varepsilon||\big(\int_{A_\varepsilon\setminus B_i}|\nabla\varphi_{i,k}|^2\big)^{\frac{1}{2}}\le\frac{C||v_\varepsilon||}{(\lambda_i d_i)^{\frac{3}{2}}}. \tag{5.47}$$

On the other hand, let ψ_2 be such that

$$\Delta\psi_2=\Delta\varphi_{i,k} \text{ in } B_i;\quad \psi_2=0 \text{ on } \partial B_i.$$

Writing

$$\psi_2=\varphi_{i,k}+\theta,\quad \text{with } \Delta\theta=0 \text{ in } B_i, \tag{5.48}$$

we obtain

$$\int_{B_i}\nabla\big(\varphi_{i,k}\big).\nabla w=\int_{B_i}\nabla\psi_2.\nabla w-\int_{B_i}\nabla\theta.\nabla w=-\int_{\partial B_i}\frac{\partial\theta}{\partial\nu}w. \tag{5.49}$$

Using an integral representation for ψ_2, as in (5.22), we obtain for $y\in\partial B_i$

$$\frac{\partial\psi_2}{\partial\nu}(y)=\int_{B_i}\frac{\partial G_i}{\partial\nu}(x,y)\big(5\delta_i^4\varphi_{i,k}\big)dx. \tag{5.50}$$

In $B_i\setminus B(x_i,\alpha d_i/8)$, we argue as in (5.26), using (5.23) and (5.24), to obtain

$$\int_{B_i\setminus B(x_i,\alpha d_i/8)}\frac{\partial G_i}{\partial\nu}(x,y)\big(5\delta_i^4\varphi_{i,k}\big)dx=O\big(\frac{1}{\lambda_i^{5/2}d_i^4}\big).$$

Furthermore, since

$$\Big|\nabla\frac{\partial G_i}{\partial\nu}(x,y)\Big|=O\big(\frac{1}{d_i^3}\big)\quad\text{for}\quad (x,y)\in B(x_i,\alpha d_i/8)\times\partial B_i,$$

we obtain

$$\begin{aligned}\int_{B(x_i,\alpha d_i/8)}\frac{\partial G_i}{\partial\nu}(x,y)\big(5\delta_i^4\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k}\big)dx&\le\frac{c}{d_i^3}\int_{B(x_i,\alpha d_i/8)}\delta_i^5|x-x_i|\\&=O\big(\frac{1}{\lambda_i^{3/2}d_i^3}\big),\end{aligned}$$

where we have used the evenness of δ_i and the oddness of its derivative. Thus

$$\frac{\partial\psi_2}{\partial\nu}(y)=O\big(\frac{1}{\lambda_i^{3/2}d_i^3}\big) \tag{5.51}$$

so that on ∂B_i

$$\frac{\partial\theta}{\partial\nu}=\frac{\partial\psi_2}{\partial\nu}-\frac{\partial}{\partial\nu}\big(\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial(x_i)_k}\big)+\frac{\partial}{\partial\nu}\big(\frac{1}{\lambda_i}\frac{\partial\theta_i}{\partial(x_i)_k}\big)=O\big(\frac{1}{\lambda_i^{3/2}d_i^3}\big). \tag{5.52}$$

It follows from (5.46), (5.47), (5.49), (5.52), and (5.31) that

$$\langle \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial (x_i)_k}, v_i^{\varepsilon,o}\rangle = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{3/2}}\big). \tag{5.53}$$

Inverting the linear system (S), we deduce from the above estimates

$$a = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{\frac{7}{2}}}\big),\ b = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{\frac{7}{2}}}\big),\ C_k = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{\frac{3}{2}}}\big),\ C_r = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{\frac{7}{2}}}\big), \tag{5.54}$$

$r \neq k$. This implies through (5.44)

$$||v_i^{\varepsilon,o} - \tilde{v}_i^o|| = O\big(\frac{||v_\varepsilon||}{(\lambda_i d_i)^{3/2}}\big),\ \ ||v_i^{\varepsilon,o}||^2 = ||\tilde{v}_i^o||^2 + O\big(\frac{||v_\varepsilon||^2}{(\lambda_i d_i)^3}\big). \tag{5.55}$$

We turn now to the last step, which consists in estimating $||\tilde{v}_i^o||$. Since $\nabla J_\varepsilon(u_\varepsilon) = 0$, we obtain

$$\begin{aligned} 0 &= \langle \sum_{r=1}^p \alpha_r P\delta_r + v_\varepsilon, v_i^{\varepsilon,o}\rangle - J_\varepsilon(u_\varepsilon)^3 \int_{A_\varepsilon} \big(\sum_{r=1}^p \alpha_r P\delta_r + v_\varepsilon\big)^5 v_i^{\varepsilon,o} \qquad (5.56)\\ &= \sum_{r=1}^p \alpha_r \int_{B_i} \delta_r^5 v_i^{\varepsilon,o} + \int_{B_i} \nabla v_\varepsilon . \nabla v_i^{\varepsilon,o} - J_\varepsilon(u_\varepsilon)^3 \int_{B_i} \big(\sum_{r=1}^p \alpha_r P\delta_r + v_\varepsilon\big)^5 v_i^{\varepsilon,o}. \end{aligned}$$

Concerning the first integral, it is equal to 0 if $r = i$ because of the oddness of $v_i^{\varepsilon,o}$ and the evenness of δ_i. For $r \neq i$, using Holder's inequality, we obtain

$$\int_{B_i} \delta_r^5 v_i^{\varepsilon,o} = O\big(\frac{||v_i^{\varepsilon,o}||}{(\lambda_r d_r)^{5/2}}\big). \tag{5.57}$$

Let us consider the second integral. Using (5.18), we obtain

$$\int_{B_i} \nabla v_\varepsilon . \nabla v_i^{\varepsilon,o} = \int_{B_i} \nabla\big(v_i^{\varepsilon,o} + v_i^{\varepsilon,e} + w\big). \nabla v_i^{\varepsilon,o} = \int_{B_i} |\nabla v_i^{\varepsilon,o}|^2. \tag{5.58}$$

For the last integral, we write

$$\begin{aligned} \Big(\sum_{r=1}^p \alpha_r P\delta_r + v_\varepsilon\Big)^5 &= (\alpha_i P\delta_i)^5 + 5(\alpha_i P\delta_i)^4 \Big(\sum_{r\neq i} \alpha_r P\delta_r + v_\varepsilon\Big)\\ &\quad + O\Big(\delta_i^3 \Big(\sum_{r\neq i} \delta_r^2 + v_\varepsilon^2\Big) + \sum_{r\neq i} \delta_r^5 + |v_\varepsilon|^5\Big), \qquad (5.59) \end{aligned}$$

and we have to estimate the contribution of each term. We notice that

$$\int_{B_i}\Big(\delta_i^3\Big(\sum_{r\neq i}\delta_r^2+v_\varepsilon^2\Big)+\sum_{r\neq i}\delta_r^5+|v_\varepsilon|^5\Big)|v_i^{\varepsilon,o}|\leq C||v_i^{\varepsilon,o}||\big(\sum\frac{1}{(\lambda_j d_j)^2}+||v_\varepsilon||^2\big). \tag{5.60}$$

Using (5.6), (5.7), and the oddness of $\delta_i^4 v_i^{\varepsilon,o}$, we obtain for $r\neq i$

$$\begin{aligned}\int_{B_i}P\delta_i^4 P\delta_r v_i^{\varepsilon,o}&=\int_{B_i}\big(\delta_i^4+O\big(\delta_i^3\theta_i\big)\big)\big(P\delta_r(x_i)+O\big(\frac{|x-x_i|}{\lambda_r^{1/2}d_i\max(d_i,d_r)}\big)\big)v_i^{\varepsilon,o}\\&=O\big(\int_{B_i}\delta_i^3\theta_i\delta_r|v_i^{\varepsilon,o}|+\int_{B_i}\delta_i^4\frac{|x-x_i||v_i^{\varepsilon,o}|}{\lambda_r^{1/2}d_i\max(d_i,d_r)}\big)\\&=O\big(||v_i^{\varepsilon,o}||\big(\frac{1}{(\lambda_i d_i)^2}+\frac{1}{(\lambda_r d_r)^2}\big)\big).\end{aligned}\tag{5.61}$$

Now, we write

$$\begin{aligned}\int_{B_i}P\delta_i^4 v_\varepsilon v_i^{\varepsilon,o}&=\int_{B_i}P\delta_i^4(v_i^{\varepsilon,o}+v_i^{\varepsilon,e}+w)v_i^{\varepsilon,o}\\&=\int_{B_i}P\delta_i^4(v_i^{\varepsilon,o}+v_i^{\varepsilon,e})v_i^{\varepsilon,o}+\int_{B_i}P\delta_i^4 w v_i^{\varepsilon,o}.\end{aligned}$$

For the first integral in the right side, we have

$$\begin{aligned}\int_{B_i}P\delta_i^4(v_i^{\varepsilon,o}+v_i^{\varepsilon,e})v_i^{\varepsilon,o}&=\int_{B_i}\big(\delta_i^4+O\big(\delta_i^3\theta_i\big)\big)(v_i^{\varepsilon,o}+v_i^{\varepsilon,e})v_i^{\varepsilon,o}\\&=\int_{B_i}\delta_i^4(v_i^{\varepsilon,o})^2+O\big(\frac{||v_i^{\varepsilon,o}||||v_\varepsilon||}{\lambda_i d_i}\big).\end{aligned}\tag{5.62}$$

To deal with the term $\int_{B_i}P\delta_i^4 w v_i^{\varepsilon,o}$, we introduce the following function

$$\Delta\psi_3=P\delta_i^4 v_i^{\varepsilon,o}\ \text{in}\ B_i;\quad \psi_3=0\ \text{on}\ \partial B_i.$$

As in (5.26), we obtain

$$\frac{\partial\psi_3}{\partial\nu}(y)=O\big(\frac{||v_i^{\varepsilon,o}||}{\lambda_i^{1/2}d_i^2}\big)\quad\text{for}\quad y\in\partial B_i.$$

Using (5.31), we find

$$\int_{B_i}P\delta_i^4 w v_i^{\varepsilon,o}=\int_{B_i}\Delta\psi_3 w=\int_{\partial B_i}\frac{\partial\psi_3}{\partial\nu}w=O\big(\frac{||v_i^{\varepsilon,o}||||v_\varepsilon||}{(\lambda_i d_i)^{1/2}}\big).\tag{5.63}$$

Lastly, we write

$$\int_{B_i} P\delta_i^5 v_i^{\varepsilon,o} = \int_{B_i} \big(\delta_i^5 - 5\delta_i^4\theta_i + O\big(\delta_i^3\theta_i^2\big)\big)v_i^{\varepsilon,o} \tag{5.64}$$

$$= O\big(\sup_{B_i}|D\theta_i|\int_{B_i}\delta_i^4|x-x_i||v_i^{\varepsilon,o}|\big) + O\big(\int_{B_i}\delta_i^3\theta_i^2|v_i^{\varepsilon,o}|\big) = O\big(\frac{||v_i^{\varepsilon,o}||}{(\lambda_i d_i)^2}\big).$$

Using (5.57), ..., (5.64), and the estimate of $||v_\varepsilon||$, (5.56) becomes

$$0 = \int_{B_i}|\nabla v_i^{\varepsilon,o}|^2 - 5J_\varepsilon(u_\varepsilon)^3\alpha_i^4\int_{B_i}\delta_i^4(v_i^{\varepsilon,o})^2 + O\big(\sum\frac{||v_i^{\varepsilon,o}||}{(\lambda_r d_r)^{11/8}}\big). \tag{5.65}$$

Since $J_\varepsilon(u_\varepsilon)^3\alpha_i^4 = 1 + o(1)$ and the quadratic form

$$v \mapsto \int_{A_\varepsilon}|\nabla v|^2 - 5\int_{A_\varepsilon}\delta_i^4 v^2$$

is positive definite on the subset $\big[\mathrm{Span}\,\big(P\delta_i, \frac{\partial P\delta_i}{\partial\lambda_i}, \frac{\partial P\delta_i}{\partial(x_i)_j} : 1 \le j \le 3\big)\big]^{\perp}_{H_0^1(A_\varepsilon)}$, we obtain

$$\int_{A_\varepsilon}|\nabla\tilde{v}_i^o|^2 - 5\int_{A_\varepsilon}\delta_i^4(\tilde{v}_i^o)^2 = O\big(\sum\frac{1}{(\lambda_j d_j)^{9/4}}\big), \tag{5.66}$$

where we have used (5.55), (5.65), and Proposition 4.2 and therefore our lemma follows. □

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