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GRADIENT ESTIMATES FOR ANISOTROPIC ELLIPTIC EQUATIONS

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Abstract. We study a class of elliptic equations with different degeneracies in different directions such as

$$\left(|u_x|^{m-2}u_x\right)_x + \left(|u_y|^{q-2}u_y\right)_y = 0$$

with unequal parameters m and q, both greater than one. We show that any bounded solution of such an equation must have a bounded gradient. The ideas also apply to a much more general class of degenerate equations, which we describe in some detail.

Introduction

In this work, we consider regularity questions for elliptic equations in divergence form, modeled on the equation

$$\sum_{i=1}^{n} D_i\left(|D_iu|^{m_i-2}D_iu\right) = 0 \tag{0.1}$$

with $m_1, \dots, m_n$ constants all greater than one. The general form for our equations is

$$\operatorname{div} A(x,u,Du) + B(x,u,Du) = 0, \tag{0.2}$$

with suitable hypotheses on A and B. Such equations (and the corresponding minimization problems) have been studied by many authors (for example, see [24], [2], [8], [3], [4], [23]) and the usual regularity proved is that each component of the gradient is in a suitable L^p space with $p < \infty$ in general (and possibly depending on the particular component). Our goal here is to expand the results from [20], which show that each component of the gradient is in L^∞ if $m_i \geq 2$ for all i under the additional assumption that

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the solution is bounded. We note that this assumption is significant; there are several examples (see [23], [12], [14]) of such equations with unbounded solutions if the numbers $m_1, \dots, m_n$ are too far apart. Nonetheless, in the context of the theory of elliptic equations, such an assumption is quite natural. For example, if we are considering the solution of the Dirichlet problem for equation (0.1) with bounded Dirichlet data, then the maximum principle implies that the solution is bounded by the same bound as the data.

Our approach is a variant of the classical Moser iteration scheme [27], and our variant is based on the ideas in [20], which, in turn, uses ideas developed in [28]. In fact, the basic proof of our gradient estimate is a straightforward combination of the ideas in [20] with those in [10]; the latter work allows us to eliminate the structure conditions which require information on an upper bound for the eigenvalues of the matrix $\partial A/\partial p$ (where we use p as a dummy variable for Du). For this reason, we shall look more closely at other aspects of the gradient estimate. First, just as in [20], we base our estimate on a general Sobolev inequality, due to Michael and Simon [26]. Although the special case of this general Sobolev inequality given in [20] is adequate for our basic estimate, we provide here a modified version which allows us to consider a slightly larger class of equations and to use some more standard notation (for example, we use a^{ij} to denote the components of the matrix $\partial A/\partial p$). Moreover, we can make a clear connection between two older works on gradient estimates for general elliptic equations which do not degenerate: [28] and [25].

We begin with some preliminary notions in Section 1; although the basic ideas are all present in [20], we sharpen some of the estimates given there. Next, we introduce in Section 2 the variant of the Sobolev inequality used in [20] which will be convenient here. Our gradient estimates are presented in Section 3 and 4. We then provide an alternative estimate, which is very useful, in Section 5. Since the hypotheses for our estimates are rather complicated, we give some simpler (although less general) ones in Section 6. On the basis of these simpler conditions, Section 7 presents an approximation scheme coupled with an existence result to consider solutions without the high degree of smoothness required to prove our estimates. Simple examples appear in Section 8; these examples all assume that A depends only on p and that $B \equiv 0$. We provide examples with the full structure in Section 9, and we close in Section 10 with a comparison of the results of [28] and [25], pointing out similarities and differences, to illustrate the use of our modified Sobolev inequality.

1. PRELIMINARIES

We suppose that $b_1, \dots, b_n$ are positive, increasing, Lipschitz functions defined on $(0,\infty)$, and that there are positive constants $\delta \le \beta_0$ such that

$$\delta b_i(t) \le t b_i'(t) \le \beta_0 b_i(t) \tag{1.1}$$

for $i = 1, \dots, n$ and almost all $t > 0$. (Note that this condition is satisfied, with suitable δ and β_0, if $b_i(t) = t^{m_i}$ for some $m_i > 1$, but we shall need this generality for several reasons which will be made clear in the examples.) We define scalar functions $\bar{B}_1, \dots, \bar{B}_n$, $\bar{B}$, b, and v by

$$\bar{B}_i(t) = \int_0^t b_i(s)\,ds, \quad \bar{B} = \sum_{i=1}^n \bar{B}_i, \quad b = \sum_{i=1}^n b_i, \quad v(p) = \bar{B}^{-1}\Big(\sum_{i=1}^n \bar{B}_i(|p_i|)\Big);$$

we define the vector-valued function ν by

$$\nu_i(p) = (\operatorname{sgn} p_i)\frac{b_i(|p_i|)}{b(v(p))};$$

and we introduce three matrix-valued functions:

$$\gamma^{ij}(p) = \begin{cases} 0 & \text{if } i \ne j \\ \dfrac{\bar{B}_i(|p_i|)}{\bar{B}(v(p))} & \text{if } i = j \end{cases}, \qquad g^{ij}(p) = \begin{cases} 0 & \text{if } i \ne j \\ \dfrac{v(p) b_i'(|p_i|)}{b(v(p))} & \text{if } i = j \end{cases},$$

$$\bar{g}^{ij}(p) = g^{ij}(p) + \nu_i(p)\nu_j(p).$$

To simplify notation, we follow the Einstein summation convention in the following form: We sum over repeated indices EXCEPT those that appear as a subscript to b (or b' or $\bar{B}$).

We collect here some useful inequalities connecting v, ν, g^{ij}, and γ^{ij}. We start with two sharpened versions of [20, (4.1)]. Let $\alpha \le \delta$. Then $b_k(t)/t^\alpha$ is an increasing function of t, and therefore we have

$$\begin{aligned} v^{1+\alpha}\sum_{k=1}^n \frac{b_k(|p_k|)}{|p_k|^\alpha} &\le \max\Big\{\sum_{k=1}^n v^{1+\alpha}\frac{b_k(v)}{v^\alpha},\ \sum_{i=1}^n |p_k|^{1+\alpha}\frac{b_k(|p_k|)}{|p_k|^\alpha}\Big\} \\ &= \max\Big\{\sum_{k=1}^n v b_k(v),\ \sum_{k=1}^n |p_k| b_k(|p_k|)\Big\} \\ &\le (1+\beta_0)\max\Big\{\sum_{k=1}^n \bar{B}_k(v),\ \sum_{k=1}^n \bar{B}_k(|p_k|)\Big\} = (1+\beta_0)\bar{B}(v). \end{aligned}$$

Now we recall that

$$(1+\delta) t b(t) \le \bar{B}(t). \tag{1.2}$$

By taking $\alpha = 0$, we infer that

$$|\nu| \le \frac{1+\beta_0}{1+\delta}, \tag{1.3}$$

and taking $\alpha = \delta$, we conclude that

$$\frac{b_k(|p_k|)}{|p_k|^\delta} \le \frac{1+\beta_0}{1+\delta}\,\frac{b(v)}{v^\delta}. \tag{1.4}$$

We also have the following results

$$\nu \cdot \xi \le \Big[\frac{1+\beta_0}{\delta(1+\delta)}\Big]^{1/2}(g^{ij}\xi_i\xi_j)^{1/2}, \tag{1.5a}$$

which follows from the proof of [20, (4.2)] in conjunction with (1.2), and

$$g^{ij}\eta_{ij} \le n^{1/2}(g^{ij}\eta_{ik}g^{km}\eta_{jm})^{1/2} \tag{1.5b}$$

(which is exactly [20, (4.3)]) valid for all $p \in \mathbb{R}^n$, all vectors ξ, and all matrices η.

Next is an inequality relating γ^{ij} and g^{ij} which was only hinted at in [20]. (The inequality $|D_iu| \le G(v)$ from [20, page 518] is the basis for our inequality.) From (1.1), we have

$$\bar B_i(|p_i|) \le \frac{1}{(1+\delta)\delta}|p_i|^2 b_i'(|p_i|).$$

Moreover (compare with [20, page 518]), we have

$$\frac{\bar B_i(s)}{s^{1+\delta}} \ge \bar B_i(1)$$

if $s \ge 1$. Now we suppose that $v \ge 1$ and take $s = \bar B_i^{-1}(\bar B(v))$. First, we note that $s \ge \bar B_i^{-1}(\bar B(1))$ and that $\bar B(1) = \sum_{i=1}^n \bar B_i(1)$, so $\bar B(1) \ge \bar B_i(1)$ and therefore $\bar B_i^{-1}(\bar B(1)) \ge 1$, which means that $s \ge 1$ and hence

$$|p_i| \le \bar B_i^{-1}(\bar B(v)) \le \Big(\frac{\bar B_i(\bar B_i^{-1}(\bar B(v)))}{\bar B_i(1)}\Big)^{1/(1+\delta)} = \bar B_i(1)^{-1/(1+\delta)}\bar B(v)^{1/(1+\delta)}.$$

Therefore,

$$\bar B_i(|p_i|) \le \frac{1}{\delta}|p_i|^2 b_i'(|p_i|) \le \gamma_1 \bar B(v)^{2/(1+\delta)} b_i'(|p_i|)$$

for

$$\gamma_1 = \frac{\max\{\bar B_1(1)^{-2/(1+\delta)}, \dots, \bar B_n(1)^{-2/(1+\delta)}\}}{(1+\delta)\delta}.$$

We conclude that

$$\gamma^{ij}\xi_i\xi_j \le \gamma_1 \bar{B}(v)^{(1-\delta)/(1+\delta)}\frac{b(v)}{v}g^{ij}\xi_i\xi_j \tag{1.6}$$

for all vectors ξ.

Our concern here will be with solutions of the equation (0.2) in a domain $\Omega \subset \mathbb{R}^n$. For the most part, we shall use v to denote both the function v and its value evaluated at $p = Du$, and we use z and p as dummy variables for u and Du, respectively. We also introduce the following abbreviations:

$$a^{ij} = \frac{\partial A^i}{\partial p_j},\ \mathcal{C}^2 = a^{ij}D_{ik}uD_{jm}ug^{km}, \quad \mathcal{E} = a^{ij}D_ivD_jv.$$

(Note that these definitions are not the same as in [20].)

We assume that A is differentiable with respect to x, z, p, and that there are functions C^i_k and D^i_k such that D^i_k is differentiable with respect to x, z, p, and

$$C^i_k + D^i_k = \frac{\partial A^i}{\partial z}p_k + \frac{\partial A^i}{\partial x^k} + B\delta^i_k.$$

For simplicity, we set

$$\mathcal{D}^{ij} = \nu^k\frac{\partial D^i_k}{\partial p_j}, \quad \mathcal{F} = \Big(p_i\frac{\partial D^i_k}{\partial z} + \frac{\partial D^i_k}{\partial x^i}\Big)\nu^k.$$

2. The Sobolev inequality

In [26], Michael and Simon introduced a general Sobolev inequality; Simon [28] used it to derive gradient bounds for equations such as the minimal surface equation but also for a large class of other elliptic equations. We used a simple variation of this inequality in [20] (see Lemma 1.1 there) to study anisotropic problems. Although the form used there is adequate for our investigation, we present here a slightly different version of this inequality which allows us to examine a somewhat larger class of problems. The main innovation is that we produce an inequality involving integration with respect to standard Lebesgue measure rather than with respect to the measure μ, defined by $d\mu = v\,dx$. We shall discuss the advantages of this form in more detail in Section 10, but we point out here that it allows us to write our structure conditions in a more usual format.

Proposition 2.1. *With γ^{ij} as in Section 1, define H by $H^i = D_j(\gamma^{ij})$. Let $N = n$ if $n > 2$ and $N > 2$ be arbitrary if $n = 2$. Then*

$$\int_{B(R)} h^{2N/(N-2)}\,dx \le c(N)\Big(\int_{B(R)} h^2\,dx\Big)^{(N-n)/(N-2)} \times \Big(\int_{B(R)} \left[\gamma^{ij} D_i h D_j h + h^2|H|^2\right]\,dx\Big)^{n/(N-2)}. \tag{2.1}$$

Proof. Our first step is to verify the hypotheses of [26]. To conform with our current notation, we point out that we only need to check the hypotheses of [20, Lemma 1.1]. To this end, we set $m = 2n-1$, we define γ^{ij} for i or j in the range $n+1, \dots, 2n-1$ by $\gamma^{ij} = \delta^{ij}$, and we set $H^i = 0$ for $i \in \{n+1, \dots, 2n-1\}$. We then write $X = (x, y)$ for points in $\mathbb{R}^m$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{m-n}$ and we set $M = \{(x, y) \in \mathbb{R}^m : x \in \Omega, y = 0\}$. For an arbitrary Borel set $B \subset \mathbb{R}^m$, we define the measure $\mu(M \cap B)$ to be the Lebesgue measure of the projection of B onto M, that is, the set of all points $x \in \Omega$ such that $(x, 0) \in M \cap B$.

Then $U = \Omega \times \mathbb{R}^{m-n}$ is an open subset of $\mathbb{R}^m$ with $M \subset U$, $\mu(M \cap C)$ is finite if C is compact, and γ^{ij} and H^i are locally L^1 on M with respect to μ. Moreover,

$$\sum_{i=1}^m \gamma^{ii}(X) = n,\ \gamma^{ij}(X) = \gamma^{ji}(X), \qquad 0 \le \gamma^{ij}(X)\xi_i\xi_j \le |\xi|^2$$

for μ-almost all $X \in M$ and all $\xi \in \mathbb{R}^m$ (note that [20, (1.1a)] was mistyped). In addition,

$$\int_M \gamma^{ij} D_j h + hH^i\,d\mu = 0$$

for each $h \in C^1(U)$ with compact support and the measure $\mu(B(X, \rho))$ of a ball in M with radius ρ is equal to $\omega_n \rho^n$, so

$$\limsup_{\rho \to 0^+} \rho^{-n} \mu(B(X, \rho)) \ge \omega_n,$$

which is [20, (1.3)]. It then follows that

$$\Big(\int_M |h|^{n/(n-1)}\,d\mu\Big)^{(n-1)/n} \le C(n)\int_M |\gamma^{ij} D_i h D_j h|^{1/2} + |h||H|\,d\mu$$

for any $h \in C^1(U)$ with compact support. A straightforward use of Cauchy's inequality and Hölder's inequality (see the derivation of [28, (2.1)] for details) now gives (2.1). □

For future reference, we note that

$$H^i = -\frac{b(v)}{\bar{B}(v)^2}\bar{B}_i(|D_i u|)D_i v + \frac{b_i(|D_i u|)}{\bar{B}(v)}D_{ii}u,$$

and hence

$$v^2|H|^2 \leq C(\delta, \beta_0)[\gamma^{ij}D_i v D_j v + g^{ij}D_{ik}u\gamma^{km}D_{jm}u]. \tag{2.2}$$

3. The energy estimate

Our first step in obtaining a gradient estimate is to prove an energy inequality analogous to [28, (2.11)]. In this section and in the next three sections, we assume that u is a Lipschitz solution of (0.2) in some domain Ω and that the ball $B(R)$ is a subset of Ω. We further assume that u has second derivatives which are locally in L^2. These hypotheses can be weakened somewhat, but such a weakening will not be necessary to our applications because we use an approximation argument. To state our estimate, we assume that there are nonnegative constants $\tau_0 \geq 1$, β_1, and β_2, along with functions Λ_0, Λ_1, and Λ_2, such that

$$C^i_k\nu^k\xi_i \leq \beta_1\Lambda_0^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}, \tag{3.1a}$$

$$C^i_k g^{jk}\eta_{ij} \leq \beta_1\Lambda_0^{1/2}(a^{ij}\eta_{ik}\bar{g}^{km}\eta_{jm})^{1/2}, \tag{3.1b}$$

$$v\mathcal{D}^{ij}\eta_{ij} \leq \beta_1\Lambda_0^{1/2}(a^{ij}\eta_{ik}\bar{g}^{km}\eta_{jm})^{1/2}, \tag{3.1c}$$

$$v\mathcal{F} \leq \beta_1^2\Lambda_0 \tag{3.1d}$$

$$v|\nu^k D^i_k - \nu^i B| \leq \beta_1\Lambda_1, \tag{3.1e}$$

$$|A|\nu\cdot\xi \leq \beta_2\Lambda_2^{1/2}(a^{ij}\xi_i\xi_j)^{1/2} \tag{3.1f}$$

$$|A|g^{ij}\eta_{ij} \leq \beta_2\Lambda_2^{1/2}(a^{ij}\eta_{ik}\bar{g}^{km}\eta_{jm})^{1/2} \tag{3.1g}$$

for all $n\times n$ matrices η, all n-vectors ξ, and all $(X,z,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$ such that $z=u(X)$ and $v>\tau_0$. Note that conditions (3.1a–d) are exactly the same as [20, (2.6a–d)] (except for a slight variation in notation). Let us point out here that conditions (3.1b,c) are quite subtle. As we shall see in our examples, it is not enough to make assumptions on the size of the derivatives of A with respect to x and z.

We then have the following result.

Lemma 3.1. *Let χ be a nonnegative Lipschitz function defined on $[\tau,\infty)$ for some $\tau \geq \tau_0$ such that χ/b is increasing and set*

$$\Psi = \begin{cases} (v-\tau)\chi' + \chi - (v-\tau)\chi b'/b & \text{if } v > \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $\zeta \in C^{1,1}(\overline{B(R)})$ be nonnegative and suppose that ζ and its gradient vanish on $\partial B(R)$. If conditions (3.1) hold, then we have

$$\begin{aligned} \int_{B_\tau(R)} \Big[\Big(1-\frac{\tau}{v}\Big)\mathcal{C}^2\chi + \mathcal{E}\Psi\Big]\zeta^2\,dx &\leq 28\beta_1^2 \int_{B_\tau(R)} \Lambda_0 \Psi\zeta^2\,dx \\ &+ 8\beta_1 \int_{B_\tau(R)} \Lambda_1\chi\zeta|D\zeta|\,dx + 48\beta_2 \int_{B_\tau(R)} \Lambda_2\Psi|D\zeta|^2\,dx \\ &+ 8(2+\beta_0)\int_{B_\tau(R)} |A|v\chi[|D^2\zeta|\zeta + |D\zeta|^2]\,dx. \end{aligned} \tag{3.3}$$

Proof. We begin just as in [21, Lemma 11.10]. Let θ be a vector-valued, C^2 function which vanishes in a neighborhood of $\partial B(R)$. If we multiply the differential equation by $\operatorname{div}\theta$ and then integrate by parts, we obtain

$$\int_{B(R)} \Big[D_k A^i D_i\theta^k + BD_k\theta^k\Big]\,dx = 0.$$

An easy approximation argument shows that this identity holds for any θ which is only Lipschitz and which vanishes on $\partial B(R)$; in particular, we take $\theta = (v-\tau)_+\chi(v)\zeta^2\nu$. Then, we have

$$\begin{aligned} \int_{B(R)} D_k A^i D_i\theta^k\,dx &= \int_{B(R)} D_k A^i D_i((v-\tau)_+\chi\nu^k)\zeta^2\,dx \\ &\quad + \int_{B(R)} D_k A^i (v-\tau)_+\chi\nu^k D_i\zeta^2\,dx. \end{aligned}$$

The first integral is handled as in [20]. A direct calculation yields

$$\begin{aligned} D_k A^i D_i((v-\tau)_+\chi\nu^k) &= \Psi\Big(\mathcal{E} + \nu^k\big[C_k^i + D_k^i - B\delta_k^i\big]D_i v\Big) \\ &\quad + \Big(1-\frac{\tau}{v}\Big)_+\chi\Big(\mathcal{C}^2 + \big[C_k^i + D_k^i - B\delta_k^i\big]g^{kj}D_{ij}u\Big). \end{aligned}$$

Hence,

$$\begin{aligned} &[B\delta_k^i + D_k A^i]D_i((v-\tau)_+\chi\nu^k) \\ &= \Psi(\mathcal{E} + C_k^i\nu^k D_i v) + \Big(1-\frac{\tau}{v}\Big)_+\chi(\mathcal{C}^2 + C_k^i g^{kj}D_{ij}u) + D_k^i D_i((v-\tau)_+\chi\nu^k). \end{aligned}$$

For the second integral, we integrate by parts again (cf. the proof of [10, Lemma 2.3]) to obtain

$$\int_{B(R)} D_k A^i (v-\tau)_+ \nu^k D_i(\zeta^2)\,dx = -\int_{B(R)} A^i D_k((v-\tau)_+ \chi \nu^k) D_i(\zeta^2)\,dx + \int_{B(R)} A^i \chi (v-\tau)_+ \nu^k D_{ik}(\zeta^2)\,dx.$$

To simplify the notation, we now set

$$I = \int_{B(R)} \Big[\Big(1-\frac{\tau}{v}\Big)_+ \mathcal{C}^2\chi + \mathcal{E}\Psi\Big]\zeta^2\,dx.$$

We also perform one final integration by parts:

$$\int_{B(R)} D_k^i D_i[(v-\tau)_+\chi\nu^k]\zeta^2\,dx = -\int_{B(R)} (\mathcal{D}^{ij}D_{ij}u + \mathcal{F})[(v-\tau)_+\chi]\zeta^2\,dx - \int_{B(R)} D_k^i (v-\tau)_+\nu^k D_i(\zeta^2)\,dx.$$

Then

$$\begin{aligned} I = &-\int_{B(R)} \Psi C_k^i D_i v \nu^k \zeta^2\,dx - \int_{B(R)} \Big(1-\frac{\tau}{v}\Big)_+ \chi g^{kj} D_{ij}u\zeta^2\,dx \\ &+ \int_{B(R)} \mathcal{D}^{ij}D_{ij}u(v-\tau)_+\chi\zeta^2\,dx + \int_{B(R)} \mathcal{F}(v-\tau)_+\chi\zeta^2\,dx \\ &+ \int_{B(R)} [\nu^k D_k^i - \nu^i B](v-\tau)_+\chi D_i(\zeta^2)\,dx - \int_{B(R)} \Psi A^i\nu^k D_k v D_i(\zeta^2)\,dx \\ &- \int_{B(R)} \Big(1-\frac{\tau}{v}\Big)_+ \chi A^i g^{kj} D_{jk}u D_i(\zeta^2)\,dx - \int_{B(R)} A^i\chi(v-\tau)_+\nu^k D_{ik}(\zeta^2)\,dx. \end{aligned}$$

These terms are estimated via (3.1) and Cauchy's inequality along with the observation that $\Psi \ge \chi \ge 0$.

Specifically, (3.1a) gives

$$-\int_{B(R)} \Psi C_k^i D_i v\nu^k\zeta^2 dx \le \int_{B(R)} (\beta_1^2\Lambda_0)^{\frac12}\Psi(\mathcal{E})^{\frac12} dx \le \beta_1^2 \int_{B(R)} \Lambda_0\Psi\zeta^2 dx + \frac14 I.$$

Next, (3.1b) implies that

$$\begin{aligned} &-\int_{B(R)} \Big(1-\frac{\tau}{v}\Big)_+ \chi g^{kj} D_{ij}u\zeta^2\,dx \\ &\le \int_{B(R)} (\beta_1^2\Lambda_0)^{1/2}\chi\zeta^2\Big(\Big(1-\frac{\tau}{v}\Big)_+\mathcal{C}^2\Big)^{1/2} dx \le 2\beta_1^2\int_{B(R)} \Lambda_0\Psi\zeta^2\,dx + \frac18 I. \end{aligned}$$

From (3.1c), we see that

$$\int_{B(R)} \mathcal{D}^{ij} D_{ij} u (v-\tau)_+ \chi\zeta^2\,dx$$
$$\le \int_{B(R)} (\beta_1^2\Lambda_0)^{1/2}\chi\zeta^2\Big(\Big(1-\frac{\tau}{v}\Big)_+\mathcal{C}^2+\mathcal{E}\Big)^{1/2}\,dx \le \beta_1^2\int_{B(R)}\Lambda_0\Psi\zeta^2\,dx+\frac14 I,$$

and (3.1d) yields

$$\int_{B(R)}\mathcal{F}(v-\tau)_+\chi\zeta^2\,dx\le\beta_1^2\int_{B(R)}\Lambda_0\Psi\zeta^2\,dx.$$

Next, we infer from (3.1e) that

$$\int_{B(R)}[\nu^k D_k^i-\nu^i B](v-\tau)_+\chi D_i(\zeta^2)\,dx\le 2\beta_1\int_{B(R)}\Lambda_1\Psi\zeta|D\zeta|\,dx.$$

We then use (3.1f) to conclude that

$$\begin{aligned}-\int_{B(R)}\Psi A^i\nu^k D_k v D_i(\zeta^2)\,dx &\le \int_{B(R)}(4\beta_2^2\Lambda_2|D\zeta|^2)^{1/2}\Psi(\mathcal{E}\zeta^2)^{1/2}\,dx\\ &\le 8\beta_2^2\int_{B(R)}\Lambda_2\Psi|D\zeta|^2\,dx+\frac14 I,\end{aligned}$$

and (3.1g) gives

$$\begin{aligned}-\int_{B(R)}&\Big(1-\frac{\tau}{v}\Big)_+\chi A^i g^{kj}D_{jk}uD_i(\zeta^2)\,dx\\ &\le\int_{B(R)}(4\beta_2^2\Lambda_2|D\zeta|^2)^{1/2}\chi\Big(\Big[\Big(1-\frac{\tau}{v}\Big)_+\mathcal{C}^2+\mathcal{E}\Big]\zeta^2\Big)^{1/2}\,dx\\ &\le 4\beta_2^2\int_{B(R)}\Lambda_2\Psi|D\zeta|^2\,dx+\frac14 I.\end{aligned}$$

Finally, we use (1.3) to see that

$$\begin{aligned}-\int_{B(R)}&A^i\chi(v-\tau)_+\nu^k D_{ik}(\zeta^2)\,dx\\ &\le 2(2+\beta_0)\int_{B(R)}|A|v\Psi[|D^2\zeta|\zeta+|D\zeta|^2]\,dx.\end{aligned}$$

Adding all these inequalities and using J to denote the right-hand side of (3.3), we see that

$$I\le\frac34 I+\frac14 J,$$

which easily implies (3.3). □

Note that the constants on the right-hand side of (3.3) can be improved if we assume conditions (3.1) hold with g^{ij} in place of $\bar{g}^{ij}$ or by making different weight choices in Cauchy's inequality. On the other hand, the exact numerical values are not important for applications.

4. The gradient estimate

Our first gradient estimate reduces the pointwise estimate to an integral estimate along the lines of [28, Lemma 1]. In order to examine a large number of cases, we introduce a slightly more complicated structure than the one in [28], using [25] as a guide. We first suppose that there are functions Λ, defined on $B(R)$, and λ, defined on $(0,\infty)$, such that

$$\Lambda_0 \le \Lambda,\ \Lambda_1 \le \Lambda,\ \Lambda_2 \le \Lambda, \lambda(v) \le \Lambda \tag{4.1}$$

and that

$$|A|v \le \beta_2 \Lambda \tag{4.2}$$

for some positive constant β_2. Moreover, we assume that λ is C^1 and that

$$\lambda\Big(1 + \Big(\frac{v\lambda'}{\lambda}\Big)^2\Big)\gamma^{ij}\xi_i\xi_j \le v^2 a^{ij}\xi_i\xi_j, \tag{4.3}$$

where, here and below, we write λ for $\lambda(v)$. Further, we suppose that there are positive C^1 functions w and $\tilde{\lambda}$ satisfying the monotonicity properties:

$$w \text{ is increasing,} \tag{4.4a}$$

$$\xi^{-\beta} w(\xi) \text{ is a decreasing function of } \xi, \tag{4.4b}$$

$$w^{-\beta} b \text{ is decreasing,} \tag{4.4c}$$

$$w^{\beta}\tilde{\lambda} \text{ is increasing,} \tag{4.4d}$$

$$\xi^{-\beta}\tilde{\lambda}(\xi) \text{ is a decreasing function of } \xi \tag{4.4e}$$

for some nonnegative constant β. (Although the functions λ and Λ in [28] and [20] are assumed to be increasing, it is clear from the proof that this assumption is not needed.) We also assume that

$$(\Lambda/\lambda)^{N/2}/\Lambda \le \tilde{\lambda}(v), \tag{4.5}$$

where $N = n$ if $n > 2$ and $N > 2$ if $n = 2$. (In [28], it was assumed that equality held in (4.5) while [25] takes $\tilde{\lambda} \equiv 1$.)

We then have the following estimate in terms of our structure functions.

Lemma 4.1. *Suppose that conditions* (1.1), (3.1), *and* (4.4)–(4.5) *are satisfied. Then there is a constant* c_1 *determined only by* n, β, β_0, $\beta_1 R$, β_2, *and* δ *such that*

$$\sup_{B(R/2)} \left(1 - \frac{\tau}{v}\right)_+^N w(v) \le c_1 R^{-n} \int_{\Omega_\tau(R)} w(v)\Lambda\tilde{\lambda}(v)\,dx. \tag{4.6}$$

Proof. The proof is essentially the same as that of [20, Lemma 2.2] with Lemma 3.1 in place of [20, Lemma 2.1]. For the convenience of the reader, we sketch the proof.

With $q \ge 1 + 2\beta$ an arbitrary parameter, we set

$$\chi = \tilde{\lambda} w^q \left(1 - \frac{\tau}{v}\right)_+^{N(q-1)},$$

and we note that conditions (4.4a,c,d) imply that χ/b is increasing. In addition, (4.4e) implies that $(v-\tau)\chi' \le c(n,\beta)(1+q)\chi$. Let us now define ζ by

$$\zeta = \begin{cases} 1 & \text{if } |x| \le R/2, \\ 1 - 3\left(\frac{2|x|}{R} - 1\right)^2 + 2\left(\frac{2|x|}{R} - 1\right)^3 & \text{if } R/2 < |x| \le R, \\ 0 & \text{if } |x| > R. \end{cases}$$

We note that $|D\zeta| \le 4/R$, $|D^2\zeta| \le 12/R^2$. Therefore, Lemma 3.1 (with ζ^{Nq-N+2} in place of ζ^2) implies that

$$\int_{B_\tau(R)} \left[\left(1 - \frac{\tau}{v}\right)\mathcal{C}^2\chi + \mathcal{E}\right]\chi\zeta^{Nq-N+2}\,dx \le cq^2R^{-2}\int_{B_\tau(R)} \left(1 - \frac{\tau}{v}\right)^{N(q-1)} \tilde{\lambda}\Lambda w^q \zeta^{N(q-1)}\,dx$$

with c determined only by n, β_0, $\beta_1 R$, β_2, and δ.

With H defined as in Proposition 2.1 and h defined by

$$h^2 = \chi\lambda\left(1 - \frac{\tau}{v}\right)^2 \zeta^{Nq-N+2},$$

we find that

$$\gamma^{ij}D_ihD_jh + h^2|H|^2 \le C(\beta_0)q^2\left[\left(1 - \frac{\tau}{v}\right)\mathcal{C}^2\chi + \mathcal{E}\right]\chi\zeta^{Nq-N+2} + q^2\chi\Lambda\zeta^{Nq-N}$$

and hence (after using (2.1) and (2.2))

$$\left(\int_{B_\tau(R)} \bar{w}^{q\kappa}\,d\mu\right)^{1/\kappa} \le c(n,\beta_0,\beta_1 R,\beta_2)q^2\int_{B_\tau(R)} \bar{w}^q\,d\mu$$

for $\kappa = N/(N-2)$,

$$\bar{w} = w\zeta^N\left(1-\frac{\tau}{v}\right)^N, \quad \text{and} \quad d\mu = R^{-n}\tilde{\lambda}\Lambda\zeta^{-N}\left(1-\frac{\tau}{v}\right)^{-N} dx.$$

(Here we note that $(\lambda\tilde{\lambda})^{N/(N-2)} \geq \tilde{\lambda}\Lambda$.) A standard iteration argument due to Moser [27] (see also [21, page 119]) gives the desired result. □

Of course, this estimate is only useful if $w(v) \to \infty$ as $v \to \infty$, but this condition will be easily verified in all our examples.

In order to estimate the integral in (4.6), we need some additional structure conditions. The first one reduces our estimate to one on a much simpler integral. We assume that there are nonnegative constants β_3 and β_4 such that

$$\tilde{\lambda}\Lambda \leq \beta_3 w^{\beta_4} Du \cdot A \tag{4.7}$$

(see [28, (1.5)] or [20, (2.7)]), which (along with Lemma 4.1) reduces the pointwise estimate to an estimate of the integral $\int w^q Du\cdot A\, dX$ for $q = 1+\beta_4$. In order to estimate this integral, we assume that there are nonnegative constants β_5 and β_6, and a positive, decreasing function ε such that

$$B \leq \beta_5 Du \cdot A, \tag{4.8a}$$

$$w' A \cdot \xi \leq \beta_6^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}(Du \cdot A)^{1/2}, \tag{4.8b}$$

$$\Lambda_0 \leq \varepsilon(v) w^2 Du \cdot A. \tag{4.8c}$$

We also assume that there is a constant $\theta \in (0, 2]$ such that

$$\Lambda_1 \leq w^{2-\theta/2} Du \cdot A, \tag{4.9a}$$

$$\Lambda_2 \leq \beta_6 w^{2-\theta} Du \cdot A, \tag{4.9b}$$

$$v|A| \leq \beta_6 w^{2-\theta} Du \cdot A, \tag{4.9c}$$

$$w^{\theta/2}|A| \leq \beta_6 Du \cdot A. \tag{4.9d}$$

Lemma 4.2. *Suppose that conditions* (3.1), (4.4c), (4.8), *and* (4.9) *are satisfied. Set* $\sigma = \operatorname{osc} u$, $E = \exp(2\beta_5\sigma/\theta)$, *and* $\Sigma = \beta_6 E\sigma/R$. *Let* $q \geq 2+\beta$ *and suppose that there is a constant* $\tau_1 \geq \tau$ *such that*

$$56\beta_1^2\sigma^2\beta_6 k^2 q^2[\beta(q-2)+1]E\varepsilon(\tau_1) \leq 1. \tag{4.10}$$

Then there is a constant c_2, *determined only by* n, β_0, $\beta_1\sigma$, β_2, θ, *and* q *such that*

$$\int_{B_\tau(R/2)} w^q Du \cdot A\, dx \leq c_2 E\left[w(\tau_1) + \Sigma^{2/\theta}\right]^q \int_{B_\tau(R)} Du \cdot A\, dx. \tag{4.11}$$

Proof. We follow the proof of [28, Lemma 2] with the variations from [20, Lemma 2.3]. Define the function F by $F(s) = s\exp(\beta_5 s)$, set $\bar{u} = u - \inf_{B(R)} u$, and suppress the argument $\bar{u}$ from F and its derivatives. Also, set $f = \exp(\beta_5\bar{u})$ and $k = 2/\theta$. Now let ζ be as in Lemma 4.1 and set

$$I = \int_{B_\tau(R)} F'(w^q - w(\tau)^q)\zeta^{kq} Du\cdot A\,dx, \qquad I' = \int_{B_\tau(R)} fw^q\zeta^{kq}Du\cdot A\,dx.$$

Since

$$I = \int_{B(R)} DF\cdot (A[w^q - w(\tau)^q]_+\zeta^{kq})\,dx,$$

an integration by parts shows that $I = I_1 + I_2 + I_3$ with

$$\begin{aligned}
I_1 &= -\int_{B(R)} F \operatorname{div} A[w^q - w(\tau)^q]_+\zeta^{kq}\,dx\\
I_2 &= -\int_{B_\tau(R)} FA\cdot Dvqw^{q-1}w'\zeta^{kq}\,dx\\
I_3 &= -\int_{B_\tau(R)} FA\cdot D\zeta[w^q - w(\tau)^q]_+ q\zeta^{kq-1}\,dx.
\end{aligned}$$

The differential equation implies that

$$I_1 = \int_{B(R)} FB[w^q - w(\tau)^q]_+\zeta^{kq}\,dx,$$

so we can use (4.8a) to see that

$$I_1 \le \beta_5 \int_{B_\tau(R)} F[w^q - w(\tau)^q]\zeta^{kq}Du\cdot A\,dx.$$

A direct calculation shows that $F' = f + \beta_5 F$, so we conclude that

$$I' \le I_2 + I_3 + Ew(\tau)^q \int_{B_\tau(R)} Du\cdot A\,dx.$$

Next, we infer from (4.8b) that

$$\begin{aligned}
I_2 &\le \int_{B_\tau(R)} \left(\beta_6 q^2\sigma^2 Ew^{q-2}\mathcal{E}\right)^{1/2}\zeta^{kq}\left(fw^q Du\cdot A\right)^{1/2}\,dx\\
&\le \frac{1}{4}I' + \beta_6 q^2\sigma^2 E\int_{B_\tau(R)} \mathcal{E}w^{q-2}\zeta^{kq}\,dx.
\end{aligned}$$

To estimate this integral, we invoke Lemma 3.1 with $\chi = w^{q-2}$ after noting that χ/b is increasing by virtue of (4.4c) because $q \ge 2 + \beta$. We conclude that

$$\begin{aligned}I_2 \le \frac{1}{4}I' &+ 28\beta_1^2\sigma^2\beta_6 E k^2 q^2[\beta(q-2)+1]\int_{B_\tau(R)} \Lambda_0 w^{q-2}\zeta^{kq}\,dx \\ &+ 32\beta_1\sigma k^3 q^3 \Sigma \int_{B_\tau(R)} \Lambda_1 w^{q-1}\zeta^{kq-1}\,dx \\ &+ 768\beta_2(1+q)k^4q^4\Sigma\frac{\sigma}{R}\int_{B_\tau(R)} \Lambda_2 w^{q-2}\zeta^{kq-2}\,dx \\ &+ 224(2+\beta_0)k^4q^4\Sigma\frac{\sigma}{R}\int_{B_\tau(R)} |A| w^{q-2} v\zeta^{kq-2}\,dx.\end{aligned}$$

Using (4.8c) and (4.9) yields

$$\begin{aligned}I_2 \le{}& \frac{1}{4}I' + 28\beta_1^2\sigma^2\beta_6 E k^2 q^2[\beta(q-2)+1]\varepsilon(\tau) I' \\ &+ C\Sigma\int_{B_\tau(R)} (w\zeta^k)^{q-\theta/2} Du\cdot A\,dx + C\Sigma^2\int_{B_\tau(R)} (w\zeta^k)^{q-\theta} Du\cdot A\,dx.\end{aligned}$$

To estimate I_3, we use (4.9d) to see that

$$I_3 \le q\Sigma\int_{B_\tau(R)} (w\zeta^k)^{q-\theta/2} Du\cdot A\,dx.$$

We now replace τ by τ_1 in the preceding calculations. Taking (4.10) into account, we conclude that

$$\begin{aligned}I' \le{}& Ew(\tau_1)^q \int_{B_{\tau_1}(R)} Du\cdot A\,dx \\ &+ C\Big[\Sigma\int_{B_{\tau_1}(R)} (w\zeta^k)^{q-\theta/2} Du\cdot A\,dx + \Sigma^2\int_{B_{\tau_1}(R)} (w\zeta^k)^{q-\theta} Du\cdot A\,dx\Big],\end{aligned}$$

which we can rewrite in the form

$$\begin{aligned}\int_{B(R)} \bar w^q\,d\mu \le{}& Ew(\tau_1)^q\int_{B(R)} 1\,d\mu \\ &+ C\Sigma\int_{B(R)} \bar w^{q-\theta/2}\,d\mu + C\Sigma^2\int_{B(R)} \bar w^{q-\theta}\,d\mu\end{aligned}$$

with $\bar w = w\zeta^k$ and $d\mu = W Du\cdot A\,dx$ with W the indicator function of $B_{\tau_1}(R)$. Since $\zeta \equiv 1$ on $B(R/2)$, it follows from Young's inequality that

$$\int_{B_{\tau_1}(R/2)} w^q Du\cdot A\,dx \le CE[w(\tau_1)+\Sigma^k]^q\int_{B_{\tau_1}(R)} Du\cdot A\,dx,$$

and the proof is finished by adding this inequality to the obvious one:

$$\int_{B_{\tau_1}(R/2)\setminus B_\tau(R/2)} w^q Du\cdot A\,dx \le w(\tau_1)^q \int_{B_{\tau_1}(R)\setminus B_\tau(R)} Du\cdot A\,dx.$$

□

The integral on the right-hand side of (4.11) is estimated via [28, Lemma 3], which states that, if conditions (4.8a) and (4.9d) are satisfied and if $\tau_2 \ge \tau_0$ is taken so that

$$4\beta_6\frac{\sigma}{R} \le w(\tau_2)^{\theta/2}, \tag{4.12}$$

then

$$\int_{\Omega_{\tau_2}(R)} Du\cdot A\,dx \le 4\omega_n R^n \exp(\beta_5\sigma)\Delta(\tau_2) \tag{4.13}$$

where

$$\Delta(\tau_2) = \sup_{|p|\le\tau_2}\Big\{\sigma(B-\beta_5 p\cdot A)_+ + (p\cdot A)_- + \frac{\sigma}{\rho}|A|\Big\}. \tag{4.14}$$

5. An alternative estimate

It is sometimes convenient to have a version of Lemma 4.2 without the exponential dependence on the oscillation of u or without a restriction on $A\cdot\xi$. Here we provide a variant of that lemma, which is actually closer in spirit to the original gradient estimates for uniformly elliptic equations (see, in particular, part (ii) of the proof of [13, Theorem 15.8] or the proof of (3.14) in Chapter 4 of [15]; this particular formulation is based on [22, Lemma 4.4]). We suppose that there are constants β_6, β_7, and β_8, and a differentiable vector field $\bar A\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\tilde\lambda\Lambda \le \beta_3 w^{\beta_4} Du\cdot\bar A, \tag{5.1}$$

and

$$\bar a^{ij}\eta_{ij} \le \beta_6^{1/2}\left(a^{ij}\eta_{ik}\bar g^{km}\eta_{jm}\right)^{1/2}\left(\frac{v^2}{w^2}Du\cdot\bar A\right)^{1/2}, \tag{5.2a}$$

$$w'\bar A\cdot\xi \le \beta_6^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}(Du\cdot\bar A)^{1/2}, \tag{5.2b}$$

$$\Lambda_0 v \le \varepsilon(v) w^2 Du\cdot\bar A. \tag{5.2c}$$

We also assume that

$$\Lambda_1 \le w^{2-\theta/2} Du \cdot \bar{A}, \tag{5.3a}$$

$$\Lambda_2 \le \beta_6 w^{2-\theta} Du \cdot \bar{A}, \tag{5.3b}$$

$$v|A| \le \beta_6 w^{2-\theta} Du \cdot \bar{A}, \tag{5.3c}$$

$$w^{\theta/2} |\bar{A}| \le \beta_6 Du \cdot \bar{A}. \tag{5.3d}$$

We then have the following version of Lemma 4.2.

Lemma 5.1. *Suppose that conditions* (3.1), (4.4a– e), (5.2), *and* (5.3) *are satisfied. Set* $\sigma = \operatorname{osc} u$, *let* $q \ge 2 + \beta$, *and suppose that there is a constant* $\tau_3 \ge \tau$ *such that*

$$300\beta_1^2 \sigma^2 \beta_6 q^2 (1+\beta)^2 (\beta(q-2)+1)\varepsilon(\tau_3) \le 1. \tag{5.4}$$

Then there is a constant c_3, *determined only by* n, β_0, $\beta_1\sigma$, β_2, *and* q *such that*

$$\int_{B_\tau(R/2)} w^q Du \cdot \bar{A}\, dx \le c_3 \left[w(\tau_3) + \beta_6 \frac{\sigma}{R} \right]^q \int_{B_\tau(R)} Du \cdot A\, dx. \tag{5.5}$$

Proof. We modify the proof of Lemma 4.2 only slightly. We set

$$I = \int_{B(R)} Du \cdot \bar{A} [w^q - w(\tau)^q]_+ \zeta^{kq}\, dx,$$

$$I' = \int_{B_\tau(R)} Du \cdot \bar{A} w^q \zeta^{kq}\, dx.$$

With $\bar{u}$ and ζ as in the proof of Lemma 4.2, we integrate by parts to see that $I = I_1 + I_2 + I_3$ for

$$I_1 = -\int_{B(R)} \bar{u} \operatorname{div} \bar{A} [w^q - w(\tau)^q]_+ \zeta^{kq}\, dx,$$

$$I_2 = -\int_{B_\tau(R)} \bar{u} w' \bar{A} \cdot Dv q w^{q-1} \zeta^{kq}\, dx,$$

$$I_3 = -\int_{B(R)} \bar{u} \bar{A} \cdot D\zeta [w^q - w(\tau)^q]_+ kq \zeta^{kq-1}\, dx.$$

From (4.4b) we have that

$$[w^q - w(\tau)^q]_+ \le 2(1+\beta q) w^q \big(1 - \frac{\tau}{v}\big)_+.$$

(See [21, page 276] for details.) We now use (5.2a) and (5.3d), and we write $\operatorname{div} \bar{A} = \bar{a}^{ij} D_{ij} u$ to see that

$$I_1 \le \frac{1}{4} I' + 4(1+\beta q)^2 \beta_6 \sigma^2 \int_{B_\tau(R)} \left[\left(1 - \frac{\tau}{v}\right) \mathcal{C}^2 + \mathcal{E} \right] w^{q-2} \zeta^{kq} \, dx.$$

Also, as in Lemma 4.2, we have

$$I_2 \le \frac{1}{4} I' + \beta_6 \sigma^2 q^2 \int_{B_\tau(R)} \mathcal{E} w^{q-2} \zeta^{kq} \, dx,$$

and therefore

$$\begin{aligned} I_1 + I_2 \le & \left[\frac{1}{2} + 140 \beta_1^2 \sigma^2 \beta_6 q^2 (1+\beta)^2 (\beta(q-2)+1) \varepsilon(\tau) \right] I' \\ & + C \frac{\sigma}{R} \int_{B(R)} (w\zeta^k)^{q-\theta/2} Du \cdot \bar{A} \, dx + C \left(\frac{\sigma}{R} \right)^2 \int_{B(R)} (w\zeta^k)^{q-\theta} Du \cdot \bar{A} \, dx. \end{aligned}$$

We also conclude that

$$I_3 \le \beta_6 q \frac{\sigma}{R} \int_{B_\tau(R)} (w\zeta^k)^{q-\theta/2} Du \cdot \bar{A} \, dx,$$

and from these inequalities, we proceed exactly as in Lemma 4.2 to complete the proof. □

Note that the assumption

$$p \cdot \bar{A} \le p \cdot A \tag{5.6}$$

allows us to estimate the right-hand side of (5.5) via [28, Lemma 3] just as in Section 4.

6. General conditions implying our hypotheses

We can now verify many of our conditions under some simple, general conditions on the function A.

Lemma 6.1. *Let $\delta \le \beta_0$ be positive constants and suppose that there are Lipschitz functions $b_1, \dots, b_n$ satisfying* (1.1)*, an increasing functions F_1, a continuous function F_2 such that $tF_2(t)$ is an increasing function of t, and a constant $\tau_0 \ge 1$ such that*

$$|A(x,z,p)| \le F_1(v(p)), \tag{6.1a}$$

$$a^{ij}(x,z,p)\xi_i\xi_j \ge F_2(v) g^{ij} \xi_i \xi_j \tag{6.1b}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ *with* $v(p) \geq \tau_0$. *Then there are constants* $\beta_2(n, \beta_0, \delta)$, $\gamma(\beta_0, \delta, \bar{B}(1))$ *such that conditions* (3.1f,g) *and* (4.3) *hold with* $\Lambda_2 = F_1(v)^2/F_2(v)$ *and*

$$\lambda(v) = \gamma\Big(\int_0^v F_2(s)^{1/2}\,ds\Big)^2 b(v)^{-2/(1+\delta)} v^{2\delta/(\delta+1)}. \tag{6.2}$$

Proof. First, we have from (6.1a) and (1.5a) that

$$|A|\nu\cdot\xi \leq F_1(v)\Big[\frac{1+\beta_0}{\delta}\Big]^{1/2}(g^{ij}\xi_i\xi_j)^{1/2}.$$

Using condition (6.1b) gives (3.1f) for $\beta_2 \geq [(1+\beta_0)/\delta]^{1/2}$.

Similarly, (6.1a) and (1.5b) yield

$$|A|g^{ij}\eta_{ij} \leq n^{1/2}F_1(v)(g^{ij}\eta_{ik}g^{km}\eta_{jm})^{1/2},$$

and hence (3.1g) holds for $\beta_2 \geq n^{1/2}$.

To prove (4.3), we start by writing $\lambda = \gamma\lambda_1\lambda_2$ with

$$\lambda_1(v) = \Big(\int_0^v F_2(s)^{1/2}\,ds\Big)^2, \qquad \lambda_2(v) = b(v)^{-2/(1+\delta)}v^{2\delta/(\delta+1)}.$$

Then

$$\frac{v\lambda_1'(v)}{\lambda_1(v)} = \frac{2vF_2(v)^{1/2}}{\int_0^v F_2(s)^{1/2}\,ds},$$

and straightforward calculations from (1.1) imply that

$$\frac{2(\delta-\beta_0)}{1+\delta} \leq \frac{v\lambda_2'(v)}{\lambda_2(v)} \leq 0.$$

Moreover, $\lambda_1(v) \leq 4F_2(v)v^2$ by our monotonicity assumption on F_2, and hence

$$\Big(1+\Big(\frac{v\lambda'(v)}{\lambda(v)}\Big)^2\Big)\lambda \leq 5(1+\beta_0)^2v^2F_2(v)b(v)^{-2/(1+\delta)}v^{2\delta/(1+\delta)}.$$

From (1.1) we also infer that

$$(vb(v))^{(\delta-1)/(\delta+1)} \leq (1+\beta_0)\bar{B}(v)^{(\delta-1)/(\delta+1)},$$

so, after applying (1.6), we find that (4.3) holds with

$$\gamma = \frac{1}{5(1+\beta_0)^3\gamma_1}. \qquad \square$$

If we further restrict F_1 and F_2, then we infer most of the hypotheses for Lemma 5.1.

Lemma 6.2. *Suppose that A satisfies conditions* (6.1) *with locally Lipschitz functions F_1 and F_2 and that there is a positive constant F_0 such that*

$$0 \le \frac{sF_1'(s)}{F_1(s)} \le F_0 \tag{6.3}$$

for almost all $s \ge 0$. Suppose moreover that there is a positive constant γ_2 such that $F_1(t) \le \gamma_2 t F_2(t)$ for all $t \ge 0$. Then conditions (5.2a,b) *and* (5.3b– d) *are satisfied with $w = v$, $\theta = 2$, $\bar{A} = \alpha F_1(v)\nu$ for a positive constant $\alpha > 0$, and β_6 determined only by α, β_0, γ_2, δ, n, and F_0. In addition, if there are positive constants τ_4 and $\theta_1 < \delta/((1+\beta_0)F_0 + \beta_0)$ such that*

$$\sup |A(x, z, 0)| \le \theta_1 F(v)v/|p| \tag{6.4}$$

for $v \ge \tau_4$, then condition (5.6) *holds for α sufficiently small, determined only by β, γ_2, δ, and θ_1 and v sufficiently large.*

Proof. We begin by calculating

$$p \cdot \bar{A} = \alpha \frac{F_1(v)}{b(v)} \sum_{i=1}^{n} b_i(|p_i|)|p_i|,$$

and then (1.1) yields

$$p \cdot \bar{A} \ge \alpha \frac{F_1(v)}{b(v)} \bar{B}(v) \ge \frac{\alpha}{1+\beta_0} F_1(v)v. \tag{6.5}$$

Then, we observe that $\bar{a}^{ij} = a_1^{ij} + a_2^{ij}$ with

$$a_1^{ij} = \alpha v \Big(\frac{F_1}{b}\Big)'(v)b(v)\nu^i\nu^j, \quad a_2^{ij} = \alpha F_1(v)g^{ij}.$$

To proceed, we fix a matrix η and we define the vector ξ by $\xi_i = \eta_{ij}\nu^j$. Then

$$a_1^{ij}\eta_{ij} = \alpha v \Big(\frac{F_1}{b}\Big)'(v)b(v)\nu \cdot \xi,$$

and (6.3) and (1.1) imply that

$$\Big| v\Big(\frac{F_1}{b}\Big)'(v)\Big| \le (F_0 + \beta_0)\frac{F_1(v)}{b(v)},$$

so (1.5a) and (6.1b) imply that

$$a_1^{ij}\eta_{ij} \le \alpha(F_0+\beta_0)\Big[\frac{1+\beta_0}{\delta}\Big]^{1/2}(F_1(v))^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}.$$

A similar argument with (1.5b) in place of (1.5a) gives

$$a_2^{ij}\eta_{ij} \le \alpha n^{1/2}(F_1(v))^{1/2}[a^{ij}\eta_{ik}g^{km}\eta_{jm}]^{1/2},$$

and the combination of these two inequalities implies (5.2a) holds with β_6 determined only by α, β_0, δ, and F_0. In addition,

$$w'\bar{A}\cdot\xi = \alpha F_1(v)\nu\cdot\xi,$$

so we infer (5.2b) with $\beta_6 \ge \alpha^{1/2}(1+\beta_0)/\delta$.

Another direct calculation (see [13, Exercise 10.3]) gives

$$\begin{aligned} p\cdot A(x,z,p) - p\cdot A(x,z,0) &= \int_0^1 \frac{\partial A^i}{\partial p_j}(x,z,sp)p_ip_j\,ds \\ &\ge \int_0^1 \frac{F_1(v(sp))}{b(v(sp))}\sum_{i=1}^n b_i'(s|p_i|)p_i^2\,ds. \end{aligned}$$

Using (1.1), we find that

$$b_i'(s|p_i|) \ge \delta s^{\beta_0-1}\frac{b_i(|p_i|)}{|p_i|},$$

and a similar calculation yields

$$\frac{F_1(v(sp))}{b(v(sp))} \ge s^{(1+\beta_0)F_0}\frac{F_1(v)}{b(v)}.$$

Therefore, with $r = \beta_0 + (1+\beta_0)F_0$, we have

$$p\cdot A(x,z,p) - p\cdot A(x,z,0) \ge \delta\int_0^1 s^{r-1}\,ds\frac{F_1(v)}{b(v)}\sum_{i=1}^n b_i(|p_i|)|p_i| = \frac{\delta}{\alpha r}p\cdot\bar{A}.$$

It follows that

$$p\cdot A(x,z,p) \ge \frac{\delta}{\alpha r}p\cdot\bar{A} + p\cdot A(x,z,0).$$

Now condition (6.4) guarantees that the right-hand side of this inequality is bounded from below by a positive constant α_0 times $p\cdot\bar{A}/\alpha$, so we take $\alpha\le\alpha_0$ to complete the proof. □

7. A regularization

If the functions A and B are sufficiently nice, then there is no problem applying our estimates to weak solutions of equation (0.2) because the classical theory shows that any weak solution is actually smooth. In order to analyze our examples correctly, we need to show that A and B can be approximated, in a good way, by sufficiently nice functions. To this end, we study a regularization of A satisfying (6.1). In fact, we shall use an additional condition on b_i beyond (1.1), namely

$$\frac{tb_i'(t)}{b_i(t)} \geq 1 \quad \text{if } t < 1. \tag{7.1}$$

(Clearly, if $\delta \geq 1$, then this condition holds.)

As a first step, let us show that this stronger condition on b_i does not affect the generality of our conditions. So, suppose that $b_1, \dots, b_n$ satisfy (1.1) and set $\tilde{b}_i(t) = \min\{1, t\} b_i(t)$. Then, for $t \leq 1$, we have

$$\frac{t\tilde{b}_i'(t)}{\tilde{b}_i(t)} = \frac{tb_i'(t)}{b_i(t)} + 1 \geq 1,$$

so $\tilde{b}_i(t)$ satisfies (7.1). In addition,

$$\bar{B}_i(1) = \int_0^1 b_i(s)\, ds \leq b_i(1)$$

and

$$\bar{B}_i(1) \geq \int_{1/2}^1 sb_i(s)\, ds \geq \frac{b_i(1/2)}{2} \geq C(\beta_0) b_i(1).$$

A similar pair of inequalities holds for the function $\tilde{B}_i$, defined by

$$\tilde{B}_i(t) = \int_0^t \tilde{b}_i(s)\, ds,$$

so, if we then use $\tilde{v}$ to denote the function v using $\tilde{b}_i$ in place of b_i, it follows that, if $v(p) \geq 1$, then the ratio $\tilde{v}/v$ is bounded above and below by positive constants, and hence there is no loss of generality in replacing b_i, v, etc., by the corresponding quantities with tildes. In other words, we may assume that b_i satisfies (7.1).

Proposition 7.1. *Suppose that A satisfies* (6.1) *and that conditions* (1.1) *and* (7.1) *hold for b_i. Then there is a family of C^2 functions (A_r^i) such that $A_r^i \to A^i$ uniformly on compact subsets of $\mathbb{R}^n$ as $r \to 0$. Moreover, there is a positive constant α_1, determined only by δ, β_0, and n such that*

$$|A_r(x, z, p)| \leq F_1((1 + 2r)^{(\beta_0+1)/(\delta+1)} v(p)), \tag{7.2a}$$

$$a_r^{ij}(x, z, p)\xi_i\xi_j \geq \alpha_1 F_2(v) g^{ij}\xi_i\xi_j \tag{7.2b}$$

for $r \in (0, 1/2)$ and $v \geq \tau_0$.

Proof. (Compare with the proof of [10, Lemma 2.4].) Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a nonnegative function, with compact support in the ball $B(1)$, even with respect to each coordinate p'_i such that

$$\int_{\mathbb{R}^n} \varphi(p')\,dp' = 1,$$

let φ_1 be a nonnegative $C^\infty(\mathbb{R}^{n+1})$ function with compact support in the ball $B(1)$ such that

$$\int_{\mathbb{R}^{n+1}} \varphi(X')\,dX' = 1.$$

Using the shorthand $X = (x, z)$ and $X' = (x', z')$, we now set

$$A_r(p) = \int_{\mathbb{R}^{2n+1}} A(X + rX', p + rp')\varphi(p')\varphi_1(X')\,dp'\,dX'.$$

Next, we note that

$$\bar{B}(v(p + rp')) = \sum_{i=1}^{n} \bar{B}_i(|p_i + rp'_i|) \le \sum_{i=1}^{n} \bar{B}_i(|p_i| + r).$$

Now fix i. If $|p_i| \le 1 - r$, then

$$\bar{B}_i(|p_i| + r) \le \bar{B}_i(1),$$

while if $|p_i| > 1 - r$, then $|p_i| + r \le |p_i|/(1 - r)$. Since $r < 1/2$, we have $1/(1 - r) \le 1 + 2r$, so

$$\bar{B}_i(|p_i| + r) \le (1 + 2r)^{\beta_0+1}\bar{B}_i(|p_i|).$$

It follows that

$$\begin{aligned}\bar{B}(v(p + rp')) &\le \max\{\sum_{i=1}^{n}(1 + 2r)^{\beta_0+1}\bar{B}_i(|p_i|), \sum_{i=1}^{n}\bar{B}_i(1)\}\\ &= \max\{(1 + 2r)^{\beta_0+1}\bar{B}(v(p)), \bar{B}(1)\},\end{aligned}$$

and then, because $v(p) \ge 1$, we conclude that

$$\bar{B}(v(p + rp')) \le (1 + 2r)^{\beta_0+1}\bar{B}(v(p)) \le \bar{B}((1 + 2r)^{(\beta_0+1)/(\delta+1)}v(p)).$$

But $\bar{B}$ is strictly increasing, so

$$v(p + rp') \le (1 + 2r)^{(\beta_0+1)/(\delta+1)}v(p),$$

and therefore

$$\begin{aligned}|A_r(X,p)| &\le \int_{\mathbb{R}^{2n+1}} F_1(v(p+rp'))\varphi(p')\varphi_1(X')\,dp'\,dX' \\ &\le \int_{\mathbb{R}^{2n+1}} F_1((1+2r)^{(\beta_0+1)/(\delta+1)}v(p))\varphi(p')\varphi_1(X')\,dp'\,dX' \\ &= F_1((1+2r)^{(\beta_0+1)/(\delta+1)}v(p)),\end{aligned}$$

which is just (7.2a).

In a similar vein,

$$a_r^{ij}\xi_i\xi_j \ge \sum_{i=1}^n \xi_i^2 \int_{\mathbb{R}^n} \frac{v(p)}{b(v(p+rp'))} F_2(v(p+rp'))b_i'(|p_i+rp_i'|)\varphi(p')\,dp'.$$

Now, set

$$\Sigma = \{p' : |p'| \le 1,\ p_ip_i' \ge 0 \text{ for all } i\}.$$

Then $|p_i| \le |p_i + rp_i'|$ on Σ so $v(p+rp') \ge v(p)$ on Σ. Recalling that $v(p) \ge 1$, we see also that $b(v(p+rp')) \le Cb(v(p))$ on Σ and that $v(p) \ge Cv(p+rp')$, so

$$a_r^{ij}\xi_i\xi_j \ge C\sum_{i=1}^n \xi_i^2 \int_\Sigma \frac{v(p)}{b(v(p))} F_2(v(p))b_i'(|p_i+rp_i'|)\varphi(p')\,dp'.$$

Now we examine one part of this integral:

$$b_i'(|p_i+rp_i'|) \ge C\frac{b_i(|p_i+rp_i'|)}{|p_i+rp_i'|}.$$

If $|p_i + rp_i'| \ge 1$, then $|p_i| \le |p_i + rp_i'| \le 2|p_i|$, so

$$b_i'(|p_i+rp_i'|) \ge Cb_i'(|p_i|). \tag{7.3}$$

On the other hand, if $|p_i + rp_i'| < 1$, then $|p_i| \le 1$ as well, and we use (7.1) to conclude that

$$\frac{b_i(|p_i+rp_i'|)}{|p_i+rp_i'|} \ge \frac{b_i(|p_i|)}{|p_i|} \ge Cb_i'(|p_i|),$$

which implies (7.3) in this case as well. It follows that

$$a_r^{ij}\xi_i\xi_j \ge F_2(v(p))g^{ij}\xi_i\xi_j \int_\Sigma \varphi(p')\,dp',$$

and our symmetry assumption on φ implies that

$$\int_\Sigma \varphi(p')\,dp' \ge 2^{-n},$$

so (7.2b) is also satisfied. □

Theorem 7.2. *Suppose conditions* (6.1) *hold with* F_1 *satisfying the additional assumption*

$$\frac{tF_1'(t)}{F_1(t)} \geq \delta_1 \tag{7.4}$$

for some positive constant δ_1*. With* λ *and* Λ_2 *as in Lemma 6.1, suppose also that there are functions* Λ_0*,* Λ_1*,* w*,* $\Lambda(v)$*,* $\tilde{\lambda}$*, and a positive constant* r_0 *such that conditions* (3.1a– e), (4.4), (4.1), *and* (4.3) *hold and*

$$\Big(\frac{\Lambda((1+r_0)v)}{\lambda(v)}\Big)^{N/2}\frac{1}{\Lambda((1+r_0)v)} \leq \tilde{\lambda}(v). \tag{7.5}$$

Suppose finally that there are constants β_9 *and* β_{10} *such that*

$$\Big|\frac{\partial A^i(x,z,0)}{\partial x^i} + B(x,z,0)\Big| \leq \beta_9 \tag{7.6a}$$

for all $(x,z) \in \Omega \times \mathbb{R}$ *and*

$$\Big|\frac{\partial A^i(x,z,p)}{\partial x^i} + p_i\frac{\partial A^i(x,z,p)}{\partial z} + B(x,z,p)\Big| \leq \beta_{10}F_2(v)v \tag{7.6b}$$

for $v(p) \geq 1$*. Let* φ_2 *be a nonnegative* $C^\infty(\mathbb{R}^n)$ *function with support in* $B(1)$ *such that*

$$\int_{\mathbb{R}^n} \varphi_2(x')\,dx' = 1, \tag{7.7}$$

and, for $r > 0$ *and* $K > 0$*, define*

$$u_r(x) = \int_{\mathbb{R}^n} u(x+rx')\varphi_2(x')\,dx', \tag{7.8a}$$

$$\tilde{A}_r(x,z,p) = A_r(x,z,p) + r(1+|p|^2)^{(\delta_1-2)/2}p, \tag{7.8b}$$

$$\tilde{B}_r(x,z,p) = B_r(x,z,p) + K(u_r(x)-z). \tag{7.8c}$$

Then, for all sufficiently small r*, there is a solution* $U_r \in C^2(\overline{(B(R)})$ *of the Dirichlet problem*

$$\operatorname{div}\tilde{A}_r(x,U_r,DU_r) + \tilde{B}_r(x,U_r,DU_r) = 0 \text{ in } B(R), \tag{7.9a}$$

$$U_r = u_r \text{ on } \partial B(R). \tag{7.9b}$$

Moreover, for any $\rho \in (0,R)$*, there is a constant* $C(\rho)$*, independent of* r*, such that*

$$\sup_{B(\rho)} |DU_r| \leq C(\rho). \tag{7.10}$$

Proof. To show that there is a solution, we invoke [13, Theorem 13.8] with the operators Q_σ defined by

$$\begin{aligned} Q_\sigma U &= \operatorname{div}(\sigma A_r(x,U,DU) + r(1+|DU|^2)^{(\delta_1-2)/2}DU) \\ &\quad + \sigma B_r(x,U,DU) + K(u_r - U). \end{aligned}$$

(Note that the σ here is a parameter, and not the oscillation of u.) Hence, we only need to check that all solutions $U_{r,\sigma}$ of the Dirichlet problem

$$Q_\sigma U_{r,\sigma} = 0 \text{ in } B(R), \quad U_{r,\sigma} = \sigma u_r \text{ on } \partial B(R)$$

are uniformly bounded in $C^{0,1}(B(R))$. For simplicity of notation, we suppress the subscripts r, σ and look at U. At an interior maximum, x_0, of U, we have $D^2U \ge 0$ and $DU = 0$, so

$$\sigma\Big(\frac{\partial A_r^i(x_0,U,0)}{\partial x_i} + B_r(x_0,U,0)\Big) + K(u_r - U) \le 0.$$

It follows from (7.6a) that $U(x_0) \le \max u_r + \beta_9/K$, so $U \le \max\{\max u_r, 0\} + \beta_9/K$, and a similar argument gives a uniform lower bound for U. A uniform boundary gradient estimate follows from [13, Theorem 14.4] (by virtue of [13, (14.32)]).

The global gradient estimate will follow from the boundary gradient estimate and our interior estimate by [21, Exercise 11.2]. Hence, we only need to show that there is a uniform interior gradient estimate. To this end, we note that all our structure conditions hold for $\tilde{A}_r$ and $\tilde{B}_r$ with

$$\Lambda^{(r)}(v) = \Lambda((1+r)^{(\beta_0+1)/(\delta+1)}v)$$

in place of Λ, and suitable $\Lambda_i^{(r)}$ for $i = 0, 1, 2$, provided we use the decomposition

$$\tilde{C}^i_{r,k}(x,z,p) = C^i_{r,k} + \delta^i_k K(u_r - z).$$

For example, we have

$$\begin{aligned} C^i_{k,r}\nu^k\xi_i &\le \beta_1 \int_{\mathbb{R}^{2n+1}} \Lambda_0(X + rX', p + rp')^{1/2} \\ &\qquad \times (a^{ij}(X + rX', p + rp')\xi_i\xi_j)^{1/2}\varphi(p')\varphi_1(X')\,dp'\,dX' \\ &\le \beta_1\,(\Lambda_{0,r}(X,p))^{1/2}\,\big(a_r^{ij}(x,z,p)\xi_i\xi_j\big)^{1/2} \end{aligned}$$

with

$$\Lambda_{r,0} = \int_{\mathbb{R}^{2n+1}} \Lambda_0(X + rX', p + rp')\varphi(p')\varphi_1(X')\,dp'\,dX'.$$

In addition,

$$\delta_k^i K(u_r - z)\nu^k \xi_i = K(u_r - z)\nu \cdot \xi \le C\Big(\frac{F_1(v)^2}{F_2(v)}\Big)^{1/2} \left(a_r^{ij}\xi_i\xi_j\right)^{1/2}$$

with C determined by K, oscu, β_9, β_0, δ, $F_1(1)$, and n, so we can take $\tilde{\Lambda}_{r,0} = \Lambda_{r,0} + \Lambda_2$. In this way, we obtain a uniform gradient estimate which provides the existence result and, simultaneously, (7.10). □

Note that the approximation scheme in Theorem 7.2 does not show directly that u satisfies the corresponding gradient bound. For such a result, we also need a suitable uniqueness theorem. We shall return to this issue in later sections because the details of the structure conditions will play a critical role in the uniqueness theorem.

8. Easy examples

We now look at some examples without lower order terms, that is, in which $B \equiv 0$ and A depends only on p. Let us first note that conditions (3.1a–e) are automatically satisfied with $\Lambda_0 = \Lambda_1 \equiv 0$ and $\beta_1 = 0$. Moreover, these hypotheses imply (5.2c), (5.3a), and (5.4) with $\theta = 2$, and (6.4) holds with $\tau_4 = \theta_1 = 0$ if $A(0) = 0$, so we only need to check (6.1) and (6.3), with F_1/tF_2 bounded. In addition, we can derive an appropriate uniqueness theorem very easily.

Lemma 8.1. *Suppose conditions* (6.1) *and* (6.3) *are satisfied and that there is a positive constant* γ_2 *such that* $F_1(t) \le \gamma_2 t F_2(t)$ *for all* $t \ge 0$. *Suppose also that* A *depends only on* p *with* $A(0) = 0$ *and that* $B \equiv 0$. *Suppose finally that there is a constant* β_{11} *such that*

$$A(p) \cdot \xi \le \frac{1}{2} F_1(v(p))v(p) + \beta_{11}[1 + F_1(v(\xi))v(\xi)]. \tag{8.1}$$

Then there is a sequence $(r(m))$ *with* $\lim_{m\to\infty} r(m) = 0$ *such that the functions* $U_{r(m)}$ *from Theorem* 7.2 *converge to* u. *Hence* u *satisfies a gradient bound.*

Proof. First, we fix a function $\eta \in C^2(B(R))$ with compact support Σ. Then

$$\int_\Sigma \tilde{A}_r(DU_r) \cdot D\eta \, dx = \int_{B(R)} \tilde{A}_r(DU_r) \cdot D\eta \, dx = 0.$$

Since η has compact support, the gradients DU_r are uniformly bounded on Σ and hence, for any $\theta > 0$, there is a number r_1 such that $|A(DU_r) -$

$\tilde{A}_r(DU_r)| \le \theta$ on Σ if $r < r_1$. It follows that

$$\Big|\int_{B(R)} A(DU_r)\cdot D\eta\,dx\Big| = \Big|\int_{\Sigma} A(Du_r)\cdot D\eta\,dx\Big| \le C\theta$$

for $r < r_1$. In addition, by using the test function $u_r - U_r$ and setting $\bar{F}(s) = F_1(s)s$, we have

$$\begin{aligned}\int_{B(R)} \bar{F}(v(DU_r))\,dx &\le \int_{B(R)} DU_r\cdot\tilde{A}_r(DU_r)\,dx = \int_{B(R)} Du_r\cdot\tilde{A}_r(DU_r)\,dx \\ &\le \frac{1}{2}\int_{B(R)} \bar{F}(v(DU_r))\,dx + C\int_{B(R)} \bar{F}(v(Du_r))\,dx.\end{aligned}$$

Since

$$C\int_{B(R)} \bar{F}(v(Du_r))\,dx \le C\int_{\Omega} F(v(Du))\,dx$$

for r sufficiently small, we conclude that

$$\int_{B(R)} F_1(v(DU_r))v(DU_r)\,dx$$

is bounded uniformly in r and hence there is a sequence $(r_1(m))$ with $r_1(m) \to 0$ as $m \to \infty$ such that $U_{r_1(m)}$ converges weakly in $W^{1,1}(B(R))$ to a function U with

$$\int_{B(R)} F_1(v(DU))v(DU)\,dx$$

finite. The Kondrachev compactness theorem and our uniform interior gradient estimate allow us to extract a further subsequence $(r(m))$ such that $U_{r(m)}$ converges uniformly to U on compact subsets of $B(R)$ and the trace of $U_{r(m)}$ on $\partial B(R)$ converges to the trace of U. It follows that U is a weak solution of the boundary-value problem

$$\operatorname{div} A(DU) = 0 \text{ in } B(R), \quad U = u \text{ on } \partial B(R).$$

In addition, we have a gradient bound for U, which is uniform over any compact subset of $B(R)$. By using $U - u$ as test function in the weak form of the differential equation for u and for U, we see that $U = u$. □

We start with the example that motivated this study.

Example 1. Suppose $A^i(p) = (r_i^2 + |p_i|^2)^{(m_i-2)/2}p_i$ with constants $m_i > 1$ and $r_i \in (0,1]$. We then take $b_i = (r_i^2 + t^2)^{(m_i-2)/2}t$, so (1.1) is valid with

$\delta = \min\{2, m_1, \dots, m_n\} - 1$ and $\beta_0 = \max\{m_1, \dots, m_n\} - 1$. Then conditions (6.1) and (6.3) hold with

$$F_1(t) = \frac{1+\beta_0}{1+\delta} b(t), \quad F_2(t) = \frac{b(t)}{t},$$

and $F_0 = \beta_0$ by virtue of (1.3). Moreover, $A(0) = 0$. If we take $\Lambda(v) = vb(v)$, then all of our monotonicity conditions are easily checked (with w any positive power of v and $\tilde{\lambda} = (\Lambda/\lambda)^{N/2}/\Lambda)$, and we have

$$\left(\frac{\Lambda}{\lambda}\right)^{N/2} \le Cbv^{k+1}$$

for

$$k = \frac{(N-1-\delta)\beta_0 - (N+1+\delta)\delta}{1+\delta}.$$

If $k \ge 0$, then (4.7) holds with $w = v$ and $\beta_4 = k+1$. Hence we have a gradient estimate in this situation. On the other hand, if $k < 0$, then we can take $w = v^{-k}$ and then Lemma 4.1 implies that

$$\sup_{B(R/2)} v \le C\left(1 + \int_{B(R)} bv\,dx\right)^{-1/k}.$$

Since this integral is less than a suitable constant times

$$\int_{B(R)} Du \cdot A\,dx,$$

in this case, we obtain a gradient bound without using a maximum estimate.

Note that the assumption $r_i > 0$ allows us to apply our estimates because we can use standard existence and uniqueness results to guarantee that any weak solution is actually smooth. Of course, formally, our estimates also apply to the case $r_i = 0$, and Lemma 8.1 shows that we have a gradient estimate in this case as well because all the constants are independent of r_i.

Of course, there is nothing special about the power functions here, so we can generalize this example quite easily.

Example 2. Suppose that $b_1, \dots, b_n$ satisfy (1.1) for some positive constants $\delta \le \beta_0$, and define

$$A^i(p) = b_i(|p_i|) \operatorname{sgn} b_i.$$

Then the calculations of the previous example apply directly, and we obtain a gradient estimate in this case as well, although now we use the regularization from Proposition 7.1 (with $\tilde{b}_i$ used for conditions (6.1)).

In fact, we can prove a gradient bound under somewhat weaker conditions than those used in [2], [8], [3], [4] although we point out that the results in those papers also apply to systems.

Example 3. Let $q > 1$ be a constant, set $\delta = \min\{1, q-1\}$, and suppose that there are positive constants θ_1 and θ_2 such that

$$|A(p)| \le \theta_1 \Big(1 + \sum_{i=1}^{n-1} |p_i|^2 + |p_n|^q\Big)^{\delta/(\delta+1)},$$

$$a^{ij}(p)\xi_i\xi_j \ge \theta_2 \Big(\sum_{i=1}^{n-1} |\xi_i|^2 + |p_n|^{q-2}|\xi_n|^2\Big).$$

(In [2], [8], [3], [4], the first condition was taken to be

$$|A^i(p)| \le \theta_1 \Big(1 + \sum_{i=1}^{n-1} |p_i|^2 + |p_n|^q\Big)^{1/2}$$

for $i = 1, \dots, n-1$ and

$$|A^n(p)| \le \theta_1 \Big(1 + \sum_{i=1}^{n-1} |p_i|^2 + |p_n|^q\Big)^{1/q}.$$

It is easy to see that this condition implies ours.) We take

$$b_i(t) = \begin{cases} t^2 & \text{if } i = 1, \dots, n-1 \\ \min\{t, 1\} t^q & \text{if } i = n \end{cases}$$

and $\beta_0 = \max\{1, q-1\}$ to infer (1.1). It is clear that (6.1b) holds with $F_2(t) = Cb(t)/t$. To check that (6.1a) holds with $F_1(t) = Cb(t)$, we note that

$$\begin{aligned} |A(p)| &\le C\bar{B}(v)^{\delta/(\delta+1)} \le C(vb(v))^{\delta/(\delta+1)} \\ &= Cv^{\delta/(\delta+1)} b(v)^{\delta/(\delta+1)} \le Cb(v)^{1//(\delta+1)} b(v)^{\delta/(\delta+1)} = Cb(v). \end{aligned}$$

Hence we obtain a gradient estimate.

Next, we consider a generalization of the integral functional F_7 from [25].

Example 4. Let g and h be locally Lipschitz functions on $(0, \infty)$, set $A_1(p) = g(|p|)p/|p|$, define A_2 by

$$A_2^i(p) = \begin{cases} 0 & \text{if } i < n, \\ h(|p_n|) \operatorname{sgn} p_n & \text{if } i = n, \end{cases}$$

and let $A = A_1 + A_2$. We assume further that there are positive constants δ and Γ such that

$$\delta \le \frac{tg'(t)}{g(t)} \le \Gamma, \quad \delta \le \frac{th'(t)}{h(t)} \le \Gamma$$

for almost all $t \in (0, \infty)$. (The special case in which $\delta = 1$ and h is given by $H(t) = G(t)^\alpha$ for some $\alpha \in (1, n/(n-2))$, where

$$H(t) = \int_0^t h(s)\, ds, \quad G(t) = \int_0^t g(s)\, ds$$

is the functional F_7 from [25].) Now, we set $b_i = g + \delta^{in} h$. Then

$$g(|p|) \le C(\Gamma) \sum_{i=1}^n g(|p_i|),$$

so

$$|A(p)| \le ng(|p|) + h(|p_n|) \le (1 + nC(\Gamma))b(v),$$

and therefore (6.1a) holds with $F_1 = Cb$. In addition,

$$a^{ij}\xi_i\xi_j = \frac{g(|p|)}{|p|}\Big[|\xi|^2 + \Big(\frac{|p|g'(|p|)}{g(|p|)} - 1\Big)\frac{(p\cdot\xi)^2}{|p|^2}\Big] + h'(|p_n|)\xi_n^2,$$

so (6.1b) holds as well with $F_2(t) = Cb(t)/t$. Since $\Lambda_2 = b(v)$, we obtain a gradient estimate as in Example 1.

Our next example degenerates in one direction if $p \ne 0$. (Compare this example with Example 7 from [20].)

Example 5. Now, suppose $n = 2$ with

$$A^1(p) = |p|^2 p_1, \quad A^2(p) = p_2^3, \quad B \equiv 0.$$

Now, we take $b_1(t) = b_2(t) = t^3$. Then (6.1a) is immediate (again with $F_1 = Cb$), and a simple calculation shows that

$$a^{ij}\xi_i\xi_j = 2p_1^2\xi_1^2 + 2p_1p_2\xi_1\xi_2 + 3p_2^2\xi_2^2 \ge p_1^2\xi_1^2 + p_2^2\xi_2^2,$$

which is just (6.1b) with $F_2(t) = Cb(t)/t$.

We can also attack more complex structures.

Example 6. Here, we look at the Euler-Lagrange equation corresponding to the integral

$$\int_\Omega F\left(\sum G_i(|D_i u|)\right)\, dx$$

for suitable functions F and G, so the equation has the structure

$$A^i(p) = F'\left(\sum G_i(|D_i u|)\right) G_i'(|D_i u|)\,\operatorname{sgn} D_i u.$$

Sample functions are

$$F(t) = t^{\alpha}, \quad G_i(t) = t^{m_i}$$

with $\alpha > 0$ and $m_i > 1$ chosen so that $F(\sum G_i(p_i))$ is a strictly convex function of p. We suppose that F and $G_1, \dots, G_n$ are continuously differentiable with Lipschitz first derivatives, and that there are positive constants δ, Γ, Γ_1, and Φ with $(1+\Gamma)\Phi < \delta$ such that

$$\delta \le \frac{tG''(t)}{G'(t)} \le \Gamma, \quad -\Phi \le \frac{tF''(t)}{F'(t)} \le \Gamma_1.$$

Now, we define $\tilde{F}(p) = F(\sum_i G_i(|p_i|))$ and we note that the second derivatives $\tilde{F}^{ij} = \partial^2 \tilde{F}/(\partial p_i \partial p_j)$ satisfy the inequality

$$\begin{aligned}
\tilde{F}^{ij}\xi_i\xi_j &= F'' \sum_i \bigl(G_i'(|p_i|)\xi_i\bigr)^2 + F' \sum_i G_i''(|p_i|)\xi_i^2 \\
&\ge F'\Bigl[-\frac{\Phi}{\sum_i G_i(|p_i|)} \sum_i \bigl(G_i'(|p_i|)\xi_i\bigr)^{1/2} + \sum_i G_i''(|p_i|)\xi_i^2 \Bigr] \\
&\ge F'\Bigl[-\frac{\Phi(1+\Gamma)}{\delta} + 1 \Bigr] \sum_i G_i''(|p_i|)\xi_i^2,
\end{aligned}$$

(with F' and F'' evaluated at $\sum G_i(|p_i|)$), so our assumption relating δ and Γ to Φ guarantees that F_1 is strictly convex.

As before, we take $b_i(t) = \min\{1, t\} G_i'(t)$. Then (6.1a,b) are easily checked with $F_1 = CF'(\bar{B})b$ and $F_2(t) = CF'(\bar{B}(t))b(t)/t$, so we obtain a gradient bound.

If we modify the assumption on F somewhat to

$$\frac{t(F'b)'(t)}{(F'b)(t)} \ge \alpha$$

for some constant $\alpha > 2(\beta_0 - \delta)/(1+\delta)$ (which will be satisfied, for example, if there is a suitable positive lower bound on $tF''(t)/F'(t)$), then it is easy to see that λ is increasing, so we obtain a gradient estimate in this case provided

$$F'b(v) \le C\lambda^{N/(N-2)} \tag{8.2}$$

for v sufficiently large by taking $\tilde{\lambda} = 1$. This type of condition appears in [25], and it applies if F grows rapidly at ∞. In particular, if $F(t) = e^t$, then there is a positive constant θ (determined only by n, β_0, and δ) so that $\lambda(v) \ge Ce^v v^{-\theta}$ and $\Lambda(v) \le Ce^v v^{\theta}$. Hence (8.2) is satisfied and we obtain a gradient bound.

9. EXAMPLES WITH MORE STRUCTURE

If A depends on x (or z), then the structure conditions are more complicated. We indicate here how the situation changes in some particular cases of interest. We start with a generalization of [20, Example 4]. Throughout, $g_1, \dots, g_n$ are Lipschitz functions satisfying the inequalities

$$\delta \le \frac{tg_i'(t)}{g_i(t)} \le \Gamma \tag{9.1}$$

for positive constants $\delta \le \Gamma$, and we set $\delta_0 = \min\{\delta, 1\}$.

We start with an extension of [20, Example 4] which eliminates the assumption that $\delta \ge 1$. This example should also be contrasted with [16], which assumed that the g_i's were power functions in a small range (in our notation, the restrictions are $1 \le \delta \le \Gamma \le \min\{3, 2+\delta\}$) and that B was Lipschitz with respect to x and independent of z and p.

Example 7. We let a_i be Lipschitz functions such that $\theta_1 \le a_i \le \theta_2$, $|a_{i,x}| \le \theta_3$ for positive constants θ_1, θ_2, and θ_3. We define

$$A^i(x,p) = a_i(x) g_i(|p_i|) \operatorname{sgn} p_i.$$

Finally, we assume that $|B| = o(vg(v))$. To make our estimates quantitative, we specifically assume that there is a decreasing function ε_1 with $\varepsilon_1(t) \to 0$ as $t \to \infty$ such that

$$|B(x,z,p)| \le \varepsilon_1(v) v g(v)$$

if $v \ge 1$. We then define b_i by $b_i(t) = \min\{1, t\} g_i(t)$ so (1.1) holds with $\beta_0 = 1 + \Gamma$.

We are now ready to verify (3.1) with $D^i_k = 0$, $\Lambda_0 = \varepsilon(v) v^3 b(v)$, $\Lambda_1 = v^2 b(v)$, and $\Lambda_2 = vb(v)$ with suitable constants β_1 and β_2, and $\varepsilon = \varepsilon_1^2 + \varepsilon_2$ with $\varepsilon_2(t) = t^{-\delta_0}$.

The expression $C^i_k \nu_k \xi_i$ is easily handled. We write $C^i_k = c^i_k + d^i_k$ for $c^i_k = a_{i,k}(x) g_i(|p_i|) \operatorname{sgn} p_i$ and $d^i_k = B\delta^i_k$. Then

$$\begin{aligned} c^i_k \nu_k \xi_i &= \sum_{i,k} a_{i,k} g_i(|p_i|) \frac{b_k(|p_k|) \operatorname{sgn} p_k}{b(v)} \xi_i \le \theta_3 \sum_i |p_i| g_i'(|p_i|) \xi_i \\ &\le \theta_3 \Big(\sum_i g_i'(|p_i|) \xi_i^2 \Big)^{1/2} \Big(\sum_i g_i'(|p_i|) |p_i|^2 \Big)^{1/2} \\ &\le [\Gamma(2+\Gamma)]^{1/2} \frac{\theta_3}{\theta_1^{1/2}} \left(a^{ij} \xi_i \xi_j \right)^{1/2} (vb(v))^{1/2} \end{aligned}$$

by virtue of (1.3) and the fact that $b(v) = g(v)$ for $v \geq 1$. The term $d_k^i \nu^k \xi_i$ is estimated via (1.5a) to infer

$$d_k^i \nu^k \xi_i \leq \left(\frac{2+\Gamma}{\delta\theta_1}\right)^{1/2} (a^{ij}\xi_i\xi_j)^{1/2} (\varepsilon_1^2(v) v^3 b(v))^{1/2},$$

which implies (3.1a).

Next, we analyze the expression $C_k^i g^{jk} \eta_{ij}$: With c_k^i and d_k^i as before, we have

$$\begin{aligned} c_k^i g^{jk} \eta_{ij} &= \frac{v}{b(v)} \sum_{i,k} a_{i,k} b_k'(|p_k|) g_i(|p_i|) \eta_{ik} \\ &\leq \frac{\theta_3}{\theta_1^{1/2}} \left(a^{ij} g^{km} \eta_{ik} \eta_{jm}\right)^{1/2} \left(\frac{v}{b(v)} \sum_{i,k} b_k'(|p_k|) g_i'(|p_i|) |p_i|^2\right)^{1/2} \end{aligned}$$

because

$$\frac{v}{b(v)} \sum_{i,k} a_i b_k'(|p_k|) g_i'(|p_i|) \eta_{ik}^2 = a^{ij} g^{km} \eta_{ik} \eta_{jm}.$$

To continue, we note that

$$\begin{aligned} \frac{v}{b(v)} \sum_{i,k} b_k'(|p_k|) g_i'(|p_i|) |p_i|^2 &\leq (1+\Gamma)\Gamma \frac{v}{b(v)} \sum_k b_k'(|p_k|) G(v) \\ &\leq (1+\Gamma)^2 \Gamma v^2 \sum_k \frac{b_k(|p_k|)}{|p_k|}. \end{aligned}$$

If $|p_k| \leq 1$, then

$$\frac{b_k(|p_k|)}{|p_k|} \leq \frac{b_k(1)}{1} \leq \varepsilon_2(v) b_k(v) \leq \varepsilon_2(v) b(v).$$

On the other hand, if $|p_k| > 1$, then

$$\frac{b_k(|p_k|)}{|p_k|} \leq \frac{b_k(|p_k|)}{|p_k|^{\delta_0}} \leq (2+\beta_0)\varepsilon_2(v) b(v)$$

by (1.4). It follows that

$$c_k^i g^{jk} \eta_{ij} \leq (2+\Gamma)^2 \frac{\theta_3}{\theta_1^{1/2}} \left(a^{ij} g^{km} \eta_{ik} \eta_{jm}\right)^{1/2} \left(\varepsilon_2(v) v^2 b(v)\right)^{1/2}.$$

The estimate of $d_k^i g^{jk} \eta_{ij} = B g^{ij} \eta_{ij}$ proceeds via (1.5b):

$$d_k^i g^{jk} \eta_{ij} \leq \left(\frac{1+\delta}{\delta\theta_1}\, n\right)^{1/2} \left(a^{ij} g^{km} \eta_{ik} \eta_{jm}\right)^{1/2} \left(\varepsilon_1^2(v) v^2 b(v)\right)^{1/2}$$

so (3.1b) holds.

Since $D^i_k = 0$, conditions (3.1c) and (3.1d) are automatically satisfied, and the hypothesis on B easily gives (3.1e).

Lemma 6.1 then gives conditions (3.1f) and (3.1g) with β_1 and β_2 determined by $\varepsilon_1(1)$, δ, Γ, θ_1, θ_2, θ_3, and n.

If we take $w = v$, $\lambda = \gamma v^{2\delta/(\delta+1)} b(v)^{(\delta-1)/(\delta+1)}$ (with γ from Lemma 6.1), $\Lambda = v^3 b(v)$, and $\tilde{\lambda} = (\Lambda/\lambda)^{N/2}/\Lambda$, then the monotonicity conditions (4.4) are easily checked with β determined only by δ, Γ, and n. Conditions (4.1) and (4.2) are also easily checked, and (4.3) follows from Lemma 6.1.

A direct calculation gives (4.7) with

$$\beta_4 = \frac{(N-1-\delta)\beta_0 + N\delta}{1+\delta},$$

and conditions (5.2) and (5.3) (with $\theta = 2$) are now immediate from Lemma 6.2 and our choice of structure functions. Again, we infer a gradient estimate.

Although our estimates require very few quantitative hypotheses, we have no guarantee that our solution u is sufficiently regular to justify all our calculations. This situation is easily handled via the approximation arguments given above. The only important additional consideration is that we need to show that the boundary-value problem corresponding to $r = 0$ has a unique solution. To this end, we set up a slightly different approximation scheme. With K and ρ positive constants to be further determined, we take the constant C_ρ so that $|B(x,z,p)| \le C_\rho + \rho B(v(p))$ for all (x,z,p) and we set

$$b_0(x) = \frac{B(x,u,Du)}{C_\rho + \rho\bar{B}(v(Du))} \tag{9.2a}$$

$$\tilde{B}(x,z,p) = b_0(x)[C_\rho + \rho\bar{B}(v(p))] + K[u(x) - z], \tag{9.2b}$$

(see [30] for the choice of b_0) and note that $U = u$ is a solution of

$$\operatorname{div} A(x, DU) + \tilde{B}(x, U, DU) = 0 \text{ in } B(R), \quad U = u \text{ on } \partial B(R). \tag{9.3}$$

We then define $\tilde{A}_r$ as in Theorem 7.2, and we define $\tilde{B}_r$ as follows. With u_r from Theorem 7.2 and b_r defined by

$$b_r(x) = \int_{\mathbb{R}^n} b_0(x + rx')\varphi_2(x')\,dx',$$

we set

$$\tilde{B}_r(x,z,p) = b_r(x)[C_\rho + \rho\bar{B}(v(p))] + K[u_r(x) - z].$$

With this (slight) change in definition, the proof of that theorem shows that we obtain a unique solution U_r of (7.9) and that there is a sequence $(r(m))$

with $r(m) \to 0$ as $m \to \infty$ such that the sequence $(U_{r(m)})$ converges to a function U which is a weak solution of (9.3) such that

$$\int_{B(R)} \bar{B}(v(DU))\,dx < \infty$$

provided ρ is sufficiently small. (It suffices to recognize that the only smallness condition needed on ρ is from (5.4). By taking $K \ge C_\rho$, we obtain an oscillation estimate on U which is independent of ρ and it's easy to see that all the other constants in that condition are also independent of ρ. On the other hand, τ_2 will depend on ρ.) The final step in proving the gradient bound is to show that $u \equiv U$. For $\delta \ge 1$, this uniqueness was proved in [20, Example 4] by modifying the proof of the uniqueness theorem [15, Theorem 4.2.1] in a straightforward manner but the argument in [15] for $m < 2$, which corresponds to our assumption $\delta < 1$, relied on a special characterization of Hölder continuous functions. Since a Hölder estimate for the class of equations considered here is not currently known, we need to modify the argument in [15] more carefully.

We start by imitating the proof of [15, Lemma 4.1.3]. First we set $\bar{u} = u - U$ and $\omega = \operatorname{osc} u + \operatorname{osc} U$. Using $(U - \max U)(\bar{u}_+)^2$ as test function in the weak form of the equation for U, we find that

$$\begin{aligned}
&\theta_1 \sum_{i=1}^n \int_{B(R)} \bar{B}_i(|D_iU|)(\bar{u}_+)^2\,dx \\
&\le C_\theta \omega \int_{B(R)} (\bar{u}_+)^2\,dx + \rho\omega \sum_{i=1}^n \int_{B(R)} \bar{B}_i(|D_iU|)(\bar{u}_+)^2\,dx \\
&+ 2\theta_2 \sum_{i=1}^n \int_{B(R)} b_i(|D_iU|)|D_i\bar{u}_+||\bar{u}_+|\,dx.
\end{aligned}$$

If we use the test function $(U - \min U)(\bar{u}_-)^2$, then we obtain a similar inequality with $\bar{u}_+$ replaced by $\bar{u}_-$. By using the corresponding test functions in the weak form of the equation for u, we infer the corresponding inequalities with u in place of U. Combining all these inequalities gives

$$\sum_{i=1}^n \int_{B(R)} B_i^*(x)\bar{u}^2\,dx \le C_1(\rho) \int_{B(R)} \bar{u}^2\,dx + C_2 \sum_{i=1}^n \int_{B(R)} b_i^*(x)|D_i\bar{u}||\bar{u}|\,dx$$

for

$$B_i^*(x) = \bar{B}_i(|D_iU|) + \bar{B}_i(|D_iu|), \qquad b_i^*(x) = b_i(|D_iU|) + b_i(D_iu|)$$

provided $\rho\omega$ is sufficiently small. (It is important to note that C_2 is independent of ρ.)

Next, we use Cauchy's inequality (exactly as on page 256 of [15]) to see that

$$b_i(t)|D_i\bar{u}||\bar{u}| \le \frac{1}{2C_2}\bar{B}_i(t)|\bar{u}|^2 + C_3\frac{b_i(t)}{t}|D_i\bar{u}|^2 \tag{9.4a}$$

for any $t > 0$. Moreover Young's inequality implies that

$$\begin{aligned} b_i(t)|D_i\bar{u}||\bar{u}| &\le \frac{1}{2C_2}\bar{B}_i(t)\frac{|\bar{u}|^{(1+\delta)/\delta}}{h^{(1-\delta)/\delta}} + C_4\frac{b_i(t)}{t^\delta}|D_i\bar{u}|^{1+\delta} \\ &\le \frac{1}{2C_2}\bar{B}_i(t)|\bar{u}|^2 + C_4\frac{b_i(t)}{t^\delta}|D_i\bar{u}|^{1+\delta} \end{aligned} \tag{9.4b}$$

for $h = \omega + 1$ provided $K \ge \beta_9$, the constant from (7.6a). (Note that C_3 depends only on C_2, δ, and β_0 but C_4 depends also on ω.) Defining

$$S_i = \{x \in B(R) : |D_iU| + |D_iu| > 0,\ |D_i\bar{u}| \le \frac{1}{2}\max\{|D_iU|, |D_iu|\}\},$$
$$\Sigma_i = \{x \in B(R) : |D_iU| + |D_iu| > 0,\ |D_i\bar{u}| > \frac{1}{2}\max\{|D_iU|, |D_iu|\}\},$$

and

$$b_i^1(x) = \frac{b_i(|D_iU|)}{|D_iU|} + \frac{b_i(|D_iu|)}{|D_iu|}, \qquad b_i^\delta(x) = \frac{b_i(|D_iU|)}{|D_iU|^\delta} + \frac{b_i(|D_iu|)}{|D_iu|^\delta},$$

we now write

$$\int_{B(R)} b_i^*|D_i\bar{u}||\bar{u}|\,dx = \int_{S_i} b_i^*|D_i\bar{u}||\bar{u}|\,dx + \int_{\Sigma_i} b_i^*|D_i\bar{u}||\bar{u}|\,dx.$$

Estimating the first integral via (9.4a) and the second integral via (9.4b), we conclude that

$$C_2\int_{B(R)} b_i^*|D_i\bar{u}||\bar{u}|\,dx \le \frac{1}{2}\int_{B(R)} B_i^*|\bar{u}|^2\,dx + C_5J_i \tag{9.5}$$

for

$$J_i = \int_{S_i} b_i^1|D_i\bar{u}|^2\,dx + \int_{\Sigma_i} b_i^\delta|D_i\bar{u}|^{1+\delta}\,dx,$$

and hence

$$\sum_{i=1}^n \int_{B(R)} B_i^*\bar{u}^2\,dx \le 2C_1(\rho)\int_{B(R)} \bar{u}^2\,dx + 2C_5\sum_{i=1}^n J_i. \tag{9.6}$$

Now, we use $\bar{u}$ as test function in the weak forms of the equations for U and u along with the observation that

$$|B_i(|D_iU|) - B_i(|D_iu|)| \le [b_i(|D_iU|) + b_i(|D_iu|)]|D_i\bar{u}|$$

to conclude that

$$I \le \rho \sum_{i=1}^{n} \int_{B(R)} b_i^*(x)|D_i\bar{u}||\bar{u}|\,dx - K\int_{B(R)} \bar{u}^2\,dx \tag{9.7}$$

for

$$I = \int_{B(R)} [A(x,Du) - A(x,DU)]\cdot[Du - DU]\,dx.$$

Now we estimate I from below. We have

$$I = \sum_{i=1}^{n} \int_{B(R)} I_i(x)\,dx$$

for

$$I_i(x) = a_i(x)[g_i(|D_iu|)\operatorname{sgn} D_iu - g_i(|D_iU|)\operatorname{sgn} D_iU][D_iu - D_iU],$$

and

$$\begin{aligned} g_i(|D_iu|)\operatorname{sgn} D_iu &- g_i(|D_iU|)\operatorname{sgn} D_iU \\ &= \int_0^1 g_i'(|sD_iu + (1-s)D_iU|)\,ds[D_iu - D_iU]. \end{aligned}$$

We next observe that, if $|D_iU| \ge |D_iu|$ and $|D_iU| > 0$, then $\frac{1}{2}|D_iU| \le |sD_iu + (1-s)D_iU| \le |D_iU|$ for $s \in (0,1/4)$ and hence

$$\begin{aligned} \int_0^1 g_i'(|sD_iu + (1-s)D_iU|)\,ds &\ge \int_0^{1/4} g_i'(|sD_iu + (1-s)D_iU|)\,ds \\ &\ge C\frac{g_i(|D_iU|)}{|D_iU|}. \end{aligned}$$

Similarly, if $|D_iU| \le |D_iu|$, then

$$\begin{aligned} \int_0^1 g_i'(|sD_iu + (1-s)D_iU|)\,ds &\ge \int_{3/4}^1 g_i'(|sD_iu + (1-s)D_iU|)\,ds \\ &\ge C\frac{g_i(|D_iu|)}{|D_iu|}. \end{aligned}$$

Let us now set $P_i = \max\{|D_iU|, |D_iu|\}$ to conclude that

$$[g_i(|D_iu|)\operatorname{sgn} D_iu - g_i(|D_iU|)\operatorname{sgn} D_iU][D_iu - D_iU] \ge C_6\frac{g_i(P_i)}{P_i}|D_i\bar{u}|^2.$$

On S_i, we have $\frac{1}{2}P_i \le |D_iU| \le P_i$ and $\frac{1}{2}P_i \le |D_iu| \le P_i$, so

$$\int_{S_i} I_i(x)\,dx \ge C_7 \int_{S_i} b_i^1(x)|D_i\bar{u}|^2\,dx.$$

On Σ_i, we have

$$\frac{|D_i\bar{u}|}{P_i} \ge \frac{|D_i\bar{u}|^\delta}{P_i^\delta}$$

and we always have

$$\frac{g_i(P_i)}{P_i^\delta} \ge \max\left\{\frac{b_i(|D_iu|)}{|D_iu|^\delta}, \frac{b_i(|D_iU|)}{|D_iU|^\delta}\right\},$$

so

$$\int_{\Sigma_i} I_i\,dx \ge C_8 \int_{\Sigma_i} b_i^\delta(x)|D_i\bar{u}|^{1+\delta}\,dx.$$

Therefore, $I \ge C_9 J$ for $J = \sum_{i=1}^n J_i$ and $C_9 = \min\{C_7, C_8\}$. Combining this inequality with (9.5), (9.6), and (9.7) then shows that

$$C_9 J \le \rho\Big(\frac{C_1(\rho)}{2C_2}\int_{B(R)} \bar{u}^2\,dx + C_5\frac{1+C_2}{C_2}J\Big) - K\int_{B(R)} \bar{u}^2\,dx.$$

If we also have ρ so small that

$$C_5\rho\frac{1+C_2}{C_2} \le \frac{1}{2},$$

we take $K \ge \rho C_1(\rho)/(2C_2)$ to infer that $J = 0$ and hence $\bar{u} = 0$.

Of course, there are other choices for the regularization of such problems and we examine here a simple alternative that arose in [11]. Suppose now that each g_i is a power function and that each a_i is a constant. Thus, we look at

$$A^i(p) = |p_i|^{m_i-1}\operatorname{sgn} p_i, \quad B(x,z,p) = f(x)$$

with $f \in L^\infty$ and each $m_i \ge 2$. We take the regularization (see [11])

$$A_r^i(p) = r(1+|p|^2)^{(\delta-2)/2}p + A^i(p).$$

with $\delta = \min\{m_1,\dots,m_n\}$. For $g_i(t) = t^{m_i}$, it is easy to check that conditions (6.1) hold. Moreover, (3.1a– e) are obvious with $\Lambda_0 = \lambda_1 = b(v)$, so we obtain a uniform gradient estimate in this case as well. We can also allow powers less than 2 in our structure. This situation is a special case of our general structure above, but it can also be handled if we use the regularization

$$A_r^i(p) = r(1+|p|^2)^{(\delta-1)/2}p + (r+|p_i|^2)^{(m_i-1)/2}p_i.$$

We leave the verification of our hypotheses in this case to the reader.

We close with an example that extends the so-called $p(x)$-Laplacian equation (see [9] or [1]) to our anisotropic setting.

Example 8. Now let $\alpha_1, \dots, \alpha_n$ be positive Lipschitz functions of x, let $r_1, \dots, r_n$ be constants in $(0,1]$, and set

$$A^i(p) = (r_i^2 + |p_i|^2)^{(\alpha_i(x)-1)/2} p_i.$$

We also take $B(x,z,p) \equiv 0$ for simplicity. To see how this structure fits into our general scheme, we set $\delta_i = \inf_{B(R)} \alpha_i$, $\omega_i = \mathrm{osc}_{B(R)} \alpha_i$, $\delta = \min\{1, \delta_1, \dots, \delta_n\}$, and $\beta_0 = 1 + \max\{\delta_1 + \omega_1, \dots, \delta_n + \omega_n\}$. Then (1.1) and (7.1) are clearly satisfied with $b_i(t) = \min\{1, t\} t^{\delta_i}$.

It is clear that

$$a^{ij}\xi_i\xi_j \geq \delta \sum_{i=1}^n (r_i^2 + |p_i|^2)^{(\alpha_i(x)-1)/2} \xi_i^2, \tag{9.8}$$

so conditions (6.1) are satisfied with $F_1 = Cb^{1+\omega}$ and $F_2(t) = cb(t)/t$ for

$$\omega = \max\Big\{\frac{\omega_1}{\delta_1}, \dots, \frac{\omega_n}{\delta_n}\Big\}.$$

In addition, by noting that $t^s \log(t)$ is bounded for $t \in (0,1)$ and $t^{-s}\log(t)$ goes to zero as $t \to \infty$ for any positive number s, we easily see that (3.1a) holds with $\Lambda_0 = vb(v)^{1+\omega}$ and β_1 determined only by δ, β_0, and ω. Similarly (3.1b) holds in this case upon taking (9.8) into account. In this way, we obtain the estimate

$$\sup v(Du) \leq C\Big(1 + \int_{B(R)} v^k Du \cdot \bar{A}\, dx\Big)$$

for $\bar{A}^i = b_i(|p_i|) \operatorname{sgn} p_i$ and some positive constant k.

To estimate this integral via our results, we need to restrict R. First, we note that we can make $\omega_i \leq 1$ if R is sufficiently small. In addition, we have $b(v) \leq b(1)v^\Gamma$ for $v \geq 1$ and $\Gamma = 2 + \max\{\delta_1, \dots, \delta_n\}$, so by taking R still smaller, if necessary, we may assume that $\omega\Gamma \leq 1/2$. Then we choose $w = v$, $\theta = 1$, and $\varepsilon(t) = 1/t$ to infer (5.2) and (5.3). Thus, we can estimate

$$\int_{B(R)} v^k Du \cdot \bar{A}\, dx$$

in terms of

$$\int_{B(R)} Du \cdot A\, dx,$$

and hence we have a gradient bound again. Note that the estimate is independent of the constants $r_1, \dots, r_n$, so we also have a bound for $r_1 = \cdots = r_n = 0$.

This last example suggests that it should be possible to analyze still more complicated structures. For example, equations with A^i given by $A^i(x,p) = \partial F(x,p)/\partial p_i$ with F a generalized vector-valued N-function (see [29]) should be amenable to the techniques of this paper; however, it is not clear (at present) exactly how to handle the x-dependence in this generality.

10. Discussion of the Sobolev inequality

As we have already pointed out, our Sobolev inequality is a variant of the one used in [20], which makes it closer in some ways to the classical Sobolev inequality than to the Michael-Simon inequality. In this section, we consider the corresponding inequality for isotropic equations. In particular, we consider the connections between the gradient estimates in [28] and those in [25].

First, we point out some fundamental differences between the two papers: [28] is concerned with smooth solutions of the equation in question while [25] studies weak solutions directly. Hence, [28] has only a brief remark about weak solutions (on pages 845 and 846) while a large fraction of [25] is devoted to technical matters concerning weak solutions and the existence of second derivatives for such solutions. On the other hand, the structure conditions in [28] emphasize the behavior of the coefficients A and B as $|p| \to \infty$ (certainly an important factor in trying to estimate Du) while the structure conditions [25] are required to hold for all values of p. Finally (although this list is not meant to be exhaustive in detailing the differences between the papers), the list of examples in [25] is much more extensive than that in [28]. In particular, [28] focuses on a gradient bound that applies to both mean curvature type equations and uniformly elliptic equations (except for a cursory remark in footnote (1), page 854) while [25] focuses on a broad spectrum of equations.

We now show how to adapt our Sobolev inequality to these isotropic equations. To this end, we redefine the quantities g^{ij}, γ^{ij}, v, ν, and H as follows. (Except for H, these are the standard definitions as in [28].) First, we set $v = (1 + |Du|^2)^{1/2}$ and $\nu = Du/v$. Then we set $g^{ij} = \gamma^{ij} = \delta^{ij} - \nu^i \nu^j$ and

$$H^i = D_i(g^{ij}) = -\frac{1}{v}\left(g^{ik} D_{ik} u \nu^j + g^{jk} D_{ik} u \nu^i\right).$$

Note that, in place of (2.2), we have

$$v|H|^2 \le C(n)[g^{ij} D_i v D_j v + g^{ij} D_{ik} u g^{km} D_{jm} u].$$

and the Sobolev inequality holds in the form (2.1) using this notation.

With this form of the Sobolev inequality, we can use the argument in [28] to prove a gradient bound which is closer in spirit to the bound in [25]. Specifically, we suppose that A and B satisfy the structure conditions (3.1a–d) for some nonnegative constant β_1 and some function Λ_0. (Of course, the decomposition using the functions C_k^i and D_k^i is not present in [28], but it was first introduced in [17] and is certainly present in [20].) We also suppose that there is a positive function $\bar{\mu}$ such that

$$a^{ij}\psi_i\xi_j \le (\bar{\mu}|\psi|^2)^{1/2}(a^{ij}\xi_i\xi_j)^{1/2}$$

for any vectors ψ and ξ. Moreover, we assume that there are C^1 functions w, Λ, λ, and $\tilde{\lambda}$ satisfying (4.3), (4.4a,b,d,e), and

$$\bar{\mu}v^2 + \Lambda_0 + \lambda \le \Lambda.$$

Then there is a constant C, determined only by N, β, and $\beta_1 R$ such that (4.6) holds. This estimate is almost [28, Lemma 1] but with some important changes (and some other minor technical changes, such as replacing w^2 there by w here, which we will not discuss further). First, the factor v in the integral is eliminating and the functions Λ and λ here are, typically, the products of the corresponding functions in [28] with v, so the integral on the right-hand side of (4.6) is smaller than the corresponding integral in [28].

Let us look at this estimate in one particular case: when $A = \nu$, which makes our differential equation one of prescribed mean curvature. Gradient estimates for such equations are well known so our purpose here is to see that the estimate can be reproduced in the current framework. Following the example on page 843 of [28], we suppose that there is a positive constant γ_1 such that

$$|B(x,z,p)| \le \gamma_1, \quad \nu \cdot \frac{\partial B}{\partial x} + \frac{|p|^2}{v}\frac{\partial B}{\partial z} \le \gamma_1, \quad v^2\left|\frac{\partial B}{\partial p}\right| \le \gamma_1.$$

Then the hypotheses outlined here are satisfied with $\bar{\mu} = \frac{1}{v}$, $\lambda = \frac{v}{2}$, $\Lambda_0 = v$, $\Lambda = 2v$, $\tilde{\lambda} = 1$, $w(v) = \log v$, $\beta = 1$, $\beta_1 = \gamma_1$ provided $\tau_0 = 2$. It follows that

$$\sup_{B(R/2)} v \le c_1 \exp\left(c_2 R^{-n} \int_{B(R)} (\log v) v\, dx\right)$$

with constants c_1 and c_2 determined only by n and $\gamma_1 R$. (To compare this result directly with those in [28], we should take $w = (\log v)^{1/2}$ there.)

Further estimation of the integral here follows exactly along the lines in [28], so we omit it here. Suffice it to say that the usual form of the gradient estimate in this situation can be obtained by using the refined arguments in [28, Section 3].

Estimates for other equations can be obtained by this method as well. For simplicity, we suppose that A depends only on p and that $B \equiv 0$. We suppose also that there are constants $q \leq r$ such that

$$a^{ij}\xi_i\xi_j \geq v^q|\xi|^2, \quad |a^{ij}| \leq Cv^r$$

and that $a^{ij} = a^{ji}$. (We shall see presently how to deal with A when $[a^{ij}]$ is nonsymmetric.) Now our conditions are satisfied with $\bar{\mu} = v^r$, $\lambda = v^{q+2}$, $\Lambda_0 = 0$, $\Lambda = Cv^{r+2}$, $\tilde{\lambda} = (\Lambda/\lambda)^{N/2}/\Lambda$, $w(v) = v^k$ for any positive constant k and some constants β (determined by k, q, and r) and β_1. Hence, we obtain a bound in the form

$$\sup v \leq C\Big(1 + R^{-n}\int_{B(R)} v^{k+(r-q)N/2}\,dx\Big)^{1/k}.$$

Noting that $Du \cdot A \geq Cv^{q+2}$, we can estimate the right-hand side of this inequality, for suitably small k under the additional restriction $(r-q)N/2 < q+2$ which is the same as

$$r < \frac{N+2}{2}q + N.$$

In fact, by using [25, Theorem 4.1], it is possible to estimate the integral of v^{r+2} if $r < (qN+4)/(N-2)$ and that this inequality between r and q implies that $(r-q)N/2 < r+2$ just as in [24]. Had we used the gradient estimate in [28] directly (because of the differences in notation, the structure functions in [28] are $\bar{\mu} = v^{r+1}$, $\lambda = v^{q+1}$, $\Lambda = v^{r+1}$, and the equivalents of Λ_0 and $\tilde{\lambda}$ are the same as above) we would obtain a gradient bound in the form

$$\sup v \leq C\Big(1 + R^{-n}\int_{B(R)} v^{k+1+(r-q)N/2}\,dx\Big)^{1/k}.$$

Hence this estimate leads to a gradient bound under the stronger assumption $r < (qN+2)/(N-2)$.

Our structure conditions also include those in [25] although our estimates apply only to smooth solutions of the equation (or to arbitrary solutions that can be approximated in a suitable fashion by smooth solutions) while the estimates of [25] apply directly to weak solutions. On the other hand, we only need to make assumptions about the behavior of the coefficients in the equation as $|p| \to \infty$.

For our purposes, we can write the structure conditions in [25] in the following form: First, there exist two increasing functions g_1 and g_2 and a constant γ_2 such that $g_1 \le g_2$ and

$$g_2(\sqrt{n}t)(1+t^2) \le \gamma_2\Big(\int_0^t (g_1(s))^{1/2}\,ds\Big)^{2N/(N-2)} \tag{10.1}$$

for $t \ge 1$. Second, there are positive constants m_1 and m_2 such that

$$a^{ij}\xi_i\xi_j \ge m_1 g_1(|p|), \quad |a^{ij}| \le m_2 g_2(|p|),$$
$$|a^{ij} - a^{ji}| \le m_2\left(g_1(|p|)g_2(|p|)\right)^{1/2}.$$

Next, we can write $B = C + D$ for functions C and D with D differentiable and

$$\Big|\frac{\partial A^i}{\partial x}\Big| + (1+|p|)\Big|\frac{\partial A^i}{\partial z}\Big| + |C| \le m_2(1+|p|)\left(g_1(|p|)g_2(|p|)\right)^{1/2},$$
$$\Big|\frac{\partial D}{\partial x}\Big| + (1+|p|)\Big|\frac{\partial D}{\partial z}\Big| \le m_2 g_2(|p|),$$
$$\Big|\frac{\partial A^i}{\partial x}\Big| \le m_2\left(g_1(|p|)g_2(|p|)\right)^{1/2}.$$

A straightforward calculation (including [25, Lemma 3.5]) shows that our previous hypotheses are satisfied with

$$\bar\mu = m_2 g_2(|p|), \quad \Lambda_0 = m_2 g_2(|p|)v^2, \quad \Lambda = 4m_2 g_2(|p|)v^2,$$
$$\lambda = \theta_1\Big(\int_0^t (g_1(s))^{1/2}\,ds\Big)^2, \quad \tilde\lambda = 1, \quad w = v^k$$

with k any positive constant, θ_1 sufficiently small (determined by m_1, m_2, and γ_2) and suitable β and β_0. This gives us a bound of the form

$$\sup v \le C\Big(1 + R^{-n}\int_{B(R)} v^k g_2(|Du|)(1+|Du|^2)\,dx\Big)^{1/k},$$

which is [25, Theorem 2.3]. Note that, by considering smooth solutions, we can improve condition (10.1) slightly: the factor $\sqrt{n}$ on the left-hand side can be replaced by 1 in this case. Because of the difference quotient arguments in [25], it does not seem possible to rewrite the structure conditions there in terms of C_k^i and D_k^i; but, of course, the test functions arguments in that paper, when applied to smooth solutions, do allow the use of these structure functions (and the replacement of $\sqrt{n}$ by 1 in our structure conditions).

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