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SINGULAR SOLUTIONS TO A QUASILINEAR ODE

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Abstract. In this paper, we prove the existence of infinitely many radial solutions having a singular behaviour at the origin for a superlinear problem of the form $-\Delta_p u = |u|^{\delta-1}u$ in $B(0,1)\setminus\{0\}\subset\mathbb{R}^N$, $u=0$ for $|x|=1$, where $N>p>1$ and $\delta>p-1$. Solutions are characterized by their nodal properties. The case $\delta+1<\frac{Np}{N-p}$ is treated. The study of the singularity is based on some energy considerations and takes into account the classification of the behaviour of the possible solutions available in the literature. By following a shooting approach, we are able to deduce the main multiplicity result from some estimates on the rotation numbers associated to the solutions.

1. Introduction

In this paper we are concerned with the existence of radial solutions with prescribed nodal properties and singular behavior at the origin for the problem

$$-\Delta_p u = |u|^{\delta-1}u, \quad x\in B^* = B(0,1)\setminus\{0\}\subset\mathbb{R}^N, \qquad u=0 \quad \text{for} \quad |x|=1,$$

where as usual $\Delta_p u := \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, with $N>p>1$, in the superlinear case. That is, solutions to

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}|u|^{\delta-1}u, \quad r\in(0,1), \\ \lim_{r\to 0^+} u(r) = \pm\infty, \quad u(1)=0, \end{gathered} \tag{1.1}$$

where $N>p>1$ and $\delta>p-1$.

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The case of *regular* solutions (that is, solutions which are bounded at 0) was done in [15], in a much more general framework. In order to state a simplified version of this result, for ϕ an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, and f a continuous function on $\mathbb{R}$, let us define the numbers

$$p_\infty := \liminf_{x\to+\infty} \frac{x\phi(x)}{\Phi(x)}, \quad \delta_\theta + 1 := \limsup_{x\to+\infty} \frac{xf(x)}{F(\theta x)},\ \theta \in (0,1), \tag{1.2}$$

and set $\delta^- := \limsup_{x\to-\infty} \frac{xf(x)}{F(x)} - 1$, where

$$\Phi(x) = \int_0^x \phi(t)dt, \quad \text{and} \quad F(x) = \int_0^x f(t)dt.$$

The result proved in [15] is the following.

Theorem *Let ϕ be an odd increasing homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, let f be a continuous function on $\mathbb{R}$ satisfying $xf(x) > 0$, for $x \neq 0$ such that*

$$\lim_{|x|\to\infty} \frac{f(x)}{\phi(x)} = +\infty, \tag{1.3}$$

and assume that ϕ, ϕ^{-1}, and f satisfy a Δ_2 condition at infinity. If $p_\infty < N$ and there exists $\theta \in (0,1)$ such that

$$\delta_\theta + 1 < \frac{Np_\infty}{N - p_\infty} \quad \text{and} \quad \delta^- + 1 < \frac{Np_\infty}{N-p_\infty}, \tag{1.4}$$

then there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, problem

$$(P) \qquad -(r^{N-1}\phi(u'))' = r^{N-1}f(u),\ r \in (0,1), \quad u(1) = 0$$

has at least one (*regular*) *solution which has exactly n zeros in* $(0,1)$.

This result was proved by using a continuation theorem in the framework of the Leray-Schauder degree, which is a variant of that in [3], and was related to that of Castro and Kurepa in [4] and [5], where ϕ is the identity, and to that of De Thélin and El Hachimi in [22], where ϕ is the one-dimensional p-Laplacian. In fact our result was strongly motivated by theirs.

In the present case of (1.1) we have $p_\infty = p$, $\delta_\theta + 1 = \frac{\delta+1}{\theta^{\delta+1}}$, and $\delta^- = \delta$, condition (1.3) reads $\delta + 1 > p$ (f superlinear), and condition (1.4) reads $\delta < \frac{N(p-1)+p}{N-p}$ (f subcritical). Our main result is the following.

Theorem 1.1. *Let $0 < p-1 < \delta$, $N > p$ and $\delta < \frac{N(p-1)+p}{N-p}$. Then, for every $n \in \mathbb{N}$, the problem*

$$-(r^{N-1}|u'|^{p-2}u')' = r^{N-1}|u|^{\delta-1}u,\ r \in (0,1), \quad u(1) = 0$$

has at least two singular solutions u_n, v_n which have exactly n zeros in $(0,1)$ and which satisfy, respectively,

$$\lim_{r\to 0^+} u_n(r) = +\infty \quad \textit{and} \quad \lim_{r\to 0^+} v_n(r) = -\infty.$$

In order to prove our result, we will use shooting techniques. More precisely, for $\varepsilon,\ \beta > 0$ we will consider the initial-value problem

$$\begin{aligned} &(t^{N-1}|w'|^{p-2}w')' + t^{N-1}|w|^{\delta-1}w = 0, \\ &w(\varepsilon) = 0, \quad |w'(\varepsilon)| = \varepsilon^{-\frac{\delta+1}{\delta+1-p}}\beta, \end{aligned} \tag{1.5}$$

and prove that for any $n \in \mathbb{N}$, there exists $\varepsilon \in (0,1)$ and $\beta > 0$ such that $\lim_{t\to 0} w_{\varepsilon,\beta}(t) = \pm\infty$, $w_{\varepsilon,\beta}$ does not change sign in $(0,\varepsilon)$, $w_{\varepsilon,\beta}(1) = 0$ and it has exactly $n-1$ zeros in $(\varepsilon, 1)$. To achieve this goal, we will make strong use of the behavior of the singular solutions to the equation in (1.1) (see [2], [16]) and the *rotation number* associated to $w_{\varepsilon,\beta}$. For this rotation number we refer the reader to the works of Castro and Lazer (see [6]), Castro and Kurepa (see [4, 5]), De Thélin and El Hachimi (see [22]), Del Pino, Manásevich and Murúa ([10]), Rebelo ([21]), Margheri, Rebelo and Zanolin (see [19], where a relation between rotation number and Maslov index is established), and Dalbono and Zanolin (see [7]).

Very recently we found out that Dolbeault, Esteban and Ramaswamy (see [11]), studied the problem of radially symmetric singular solutions for

$$\Delta u + \lambda f(u) = 0 \quad \text{in} \quad B^*$$

for the case when f is asymptotic to some power $|u|^{\delta-1}u$ at infinity. In this paper, the authors establish existence and classify the singular behavior at the origin of the possibly sign-changing solutions for δ subcritical or critical; i.e., $\delta \le \frac{N+2}{N-2}$. In the subcritical case their result is the same as in [2] (which is done for the p-Laplacian operator but for the homogeneous case $f(u) = |u|^{\delta-1}u$), but for a more general nonlinearity f. In the critical case $\delta = \frac{N+2}{N-2}$, they are able to construct singular solutions with a prescribed number of zeros, whenever a bounded solution with the same prescribed number of zeros exists. By means of an order-preserving property, they first analyze the behavior of some particular solutions to the equation near the origin. Secondly, while studying the behavior of these particular solutions near 1, they introduce a suitable parametrization of the set of bounded solutions, which, combined with some topological and energy arguments, will lead to their multiplicity result.

We organize this paper as follows. In Section 2, we prove the key lemmas that will be used to prove our results concerning the singular behavior of the solutions at the origin, then in Section 3 we will prove the crucial lemmas concerning the nodal properties of the solutions. Finally, in Section 4, we prove Theorem 1.1.

2. Key lemmas concerning singular behavior of the solutions at the origin

The homogeneity of the equation involved in our problem plays a fundamental role in our analysis. Indeed, by setting $w(t) = \varepsilon^{-p/(\delta+1-p)}u(r)$, $t = \varepsilon r$, we see that if u satisfies the equation in (1.1), then so does w, and u is singular at 0 if and only if w is. This suggests to us the idea of constructing singular solutions to (1.1) with a prescribed number of zeros in $(0,1]$ by constructing solutions to (1.5) which are singular at 0, do not change sign in $(0,\varepsilon)$, and have the same prescribed number of zeros in $[\varepsilon,1]$. We begin this section by proving a crucial result concerning the singular behavior and positivity of solutions to an initial-value problem.

Lemma 2.1. *Let $0 < p-1 < \delta$, $N > p$, and $\delta < \frac{N(p-1)+p}{N-p}$. Then, there exists $\beta_0 > 0$ such that for any $0 < |\beta| \le \beta_0$, it holds that any solution to the initial-value problem*

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}|u|^{\delta-1}u, \quad r \in (0,1) \\ u(1) = 0, \quad u'(1) = -\beta, \end{gathered} \tag{2.1}$$

is singular and does not change sign in $(0,1)$.

Proof. We will prove our result first for $p-1 < \delta < \frac{N(p-1)}{N-p}$. Since this proof is contained in a much more general one given in [14], we only sketch it and, for simplicity, we will only treat the case of $\beta > 0$. Set

$$\beta_0 = \Big(\frac{N(p-1)}{N-p} - \delta\Big)^{\frac{1}{\delta+1-p}}\Big(\frac{N-p}{p-1}\Big)^{\frac{\delta+1}{\delta+1-p}},$$

and let $0 < \beta < \beta_0$. It follows from the general theory of ordinary differential equations (see for example [17]) that the initial-value problem (2.1) has a solution u defined at some left neighborhood $(r_1,1)$ of 1 which is positive in this interval. Also, in $(r_1,1)$, u satisfies the integral equation

$$u(r) = \int_r^1 s^{\frac{1-N}{p-1}}\phi_{p'}\Big(\beta^{p-1} - \int_s^1 \tau^{N-1}|u|^{\delta-1}u(\tau)\,d\tau\Big)ds, \tag{2.2}$$

where $\phi_{p'}(x) = |x|^{(2-p)/(p-1)}x$. The positiveness of u implies that the inequality

$$u(r) \le h_\beta(r) = \beta(\frac{p-1}{N-p})r^{\frac{p-N}{p-1}}$$

holds for all $r \in (r_1, 1]$ and this holds while the solution u remains positive. On the other hand we can see that condition $\beta < \beta_0$ implies that

$$\int_0^1 \tau^{N-1} h_\beta^\delta(\tau)d\tau < \beta^{p-1},$$

and thus u has to be positive in its left maximal interval of definition (contained in $[0, 1]$). All this together implies that u is defined in $(0, 1]$ and is positive there.

It only remains to prove that u is singular. Let us set

$$\Theta := \beta^{p-1} - \int_0^1 \tau^{N-1} h_\beta^\delta(\tau)d\tau,$$

then from $\beta < \beta_0$ we have $\Theta > 0$. Now from (2.2), and using the generalized L'Hôpital rule, we find that

$$\liminf_{r\to 0^+} \frac{u(r)}{h_\beta(r)} \ge \liminf_{r\to 0^+} \frac{\Theta^{1/(p-1)} r^{\frac{1-N}{p-1}}}{\beta r^{\frac{1-N}{p-1}}} = \frac{\Theta^{1/(p-1)}}{\beta} > 0,$$

implying that u is singular. Assume next that

$$\frac{N(p-1)}{N-p} \le \delta < \frac{N(p-1)+p}{N-p}.$$

We will prove first that there exists $\beta_1 > 0$ such that, for all $|\beta| \le \beta_1$, solutions to (2.1) are singular, that is, $\lim_{r\to 0^+} u(r) = \infty$. Assume by contradiction that there is a sequence $\beta_n \to 0$ such that the solutions u_{β_n} to (2.1) with $\beta = \beta_n$ are not singular. Theorem 3.2 and Theorem 6.2 in [2] guarantee that such solutions u_{β_n} are bounded. Then, by using the equation in (2.1) we obtain that

$$\lim_{r\to 0^+} r^{N-1}|u'_{\beta_n}(r)|^{p-2}u'_{\beta_n}(r) = 0,$$

implying

$$r^{N-1}|u'_{\beta_n}(r)|^{p-1} \le \int_0^r s^{N-1}|u_{\beta_n}(s)|^\delta ds \le C_n\, r^N$$

for some positive constant C_n, and, consequently, $u'_{\beta_n}(0) = 0$ for all $n \in \mathbb{N}$. It follows then from [15] that there exists a positive constant M such that

$$|u_{\beta_n}(r)| \le M \quad \text{for all} \quad r \in [0, 1] \quad \text{and all } n. \tag{2.3}$$

Then, since u_{β_n} satisfies

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u_{\beta_n}|^{\delta+1-p}|u|^{p-2}u = 0, \quad u'(0) = 0,$$

we find that the possible zeros of u_{β_n} are bounded away from 0 (see [9, Lemma 4.1]), in fact they are greater than the first zero ρ_0 of a nontrivial solution to

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}M^{\delta+1-p}|u|^{p-2}u = 0, \quad u'(0) = 0. \tag{2.4}$$

Our aim consists now in proving that there exists $n_0 \in \mathbb{N}$ such that u_{β_n} does not change sign on $[\rho_0, 1)$ for every $n \geq n_0$. Let us suppose, by contradiction, that there exists a subsequence of u_{β_n} (still called u_{β_n}) such that for every $n \in \mathbb{N}$ the function u_{β_n} has at least one zero in $[\rho_0, 1)$. Let us define $r_n := \min\{r \in [\rho_0, 1) : u_{\beta n}(r) = 0\}$.

Also, let us consider the Pohozaev-type energy

$$E(u_{\beta_n})(r) = r^N\Big(\frac{|u'_{\beta_n}|^p}{p'} + \frac{|u_{\beta_n}|^{\delta+1}}{\delta+1}\Big) + \frac{N}{\delta+1}r^{N-1}u_{\beta_n}|u'_{\beta_n}|^{p-2}u'_{\beta_n}, \tag{2.5}$$

and its derivative

$$E(u_{\beta_n})'(r) = \Big(\frac{N}{\delta+1} - \frac{N-p}{p}\Big)r^{N-1}|u'_{\beta_n}|^p. \tag{2.6}$$

Since $\delta + 1 < \frac{Np}{N-p}$, we see that E is strictly increasing on $(0, 1)$. Hence, it is easy to prove that

$$r_n^N|u_{\beta_n}(r)|^{\delta+1} \leq \frac{\delta+1}{p'}|\beta_n|^p \quad \text{for all } r \in [r_n, 1]. \tag{2.7}$$

Let us analyze now the behaviour of the function u_{β_n} in the interval $[\rho_0, r_n]$, when $\rho_0 < r_n$. Since $u'_{\beta_n}(0) = 0$ and $u_{\beta_n}(r) \neq 0$ for all $r \in [0, \rho_0)$, one can deduce that $u_{\beta_n}(r)\, u'_{\beta_n}(r) < 0$ for every $r \in [\rho_0, r_n)$. From equation (2.1) we obtain that the function $r \mapsto r^{N-1}|u'_{\beta_n}(r)|^{p-1}$ is strictly increasing in $[\rho_0, r_n]$. In particular, recalling the monotonicity of the energy, we get

$$r^{N-1}|u'_{\beta_n}(r)|^{p-1} < r_n^{N-1}|u'_{\beta_n}(r_n)|^{p-1} < r_n^{\frac{N-p}{p}}|\beta_n|^{p-1}$$

for every $r \in [\rho_0, r_n]$. Furthermore, from (2.3) and the monotonicity of the energy, it follows that

$$r^N\frac{|u_{\beta_n}(r)|^{\delta+1}}{\delta+1} < \frac{|\beta_n|^p}{p'} + \frac{N}{\delta+1}r^{N-1}|u'_{\beta_n}(r)|^{p-1}|u_{\beta_n}(r)|$$

$$\leq \frac{|\beta_n|^p}{p'} + \frac{N\,M\,r_n^{\frac{N-p}{p}}}{\delta+1}|\beta_n|^{p-1}$$

for each $r \in [\rho_0, r_n]$. Taking into account (2.7), we can conclude that

$$\rho_0^N |u_{\beta_n}(r)|^{\delta+1} \;\leq\; \frac{\delta+1}{p'}|\beta_n|^p + N\,M\,|\beta_n|^{p-1} \;:=\; f(\beta_n) \quad \text{ for all } r \in [\rho_0, 1].$$

It is important to note that $\lim\limits_{n\to+\infty} f(\beta_n) = 0$. Hence, by comparison with the nontrivial solutions to

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}\Big(\frac{f(\beta_n)}{\rho_0^N}\Big)^{\frac{\delta+1-p}{\delta+1}}|u|^{p-2}u = 0, \quad u'(0) = 0 \tag{2.8}$$

(see [18, Theorem 1.1]), we conclude that for n large enough u_{β_n} does not change sign on $(0,1)$, a contradiction. Thus, we infer that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$u_{\beta_n} \text{ does not change sign on } (0,1).$$

Using now the uniqueness of such a solution proved by Nabana and de Thelin (see [20]), we have a contradiction and thus we conclude that there is $\beta_1 > 0$ such that for all $|\beta| \leq \beta_1$ the solution u_β must be singular.

Finally, we will prove that there exists $\beta_2 \in (0, \beta_1]$ such that for all $|\beta| \leq \beta_2$ all the (singular) solutions u_β do not change sign in $(0,1)$. Indeed, let us first treat the case $\delta = \frac{N(p-1)}{N-p}$. We will make the following change of variable:

$$s = r^{\frac{p-N}{p-1}}, \quad y(s) = u(r). \tag{2.9}$$

Then the problem (2.1) is equivalent to the following

$$\begin{aligned} &\Big(\tfrac{N-p}{p-1}\Big)^p \Big(|y_s|^{p-2}\,y_s\Big)_s + s^{\frac{(N-1)p}{p-N}}|y|^{\delta-1}y \;=\; 0, \qquad s \in (1,+\infty) \\ &\qquad y(1) = 0, \qquad y_s(1) = \beta\;\tfrac{p-1}{N-p}, \end{aligned} \tag{2.10}$$

where f_s denotes $\frac{\mathrm{d}f}{\mathrm{d}s}$ for any given function $s \mapsto f(s)$. In particular, if we consider the Pohozaev-type energy defined in (2.5) as a function of the new variable s, we note that it is strictly decreasing in $[1,+\infty)$, since in such an interval

$$E_s(r(s)) \;=\; E'(r(s))\;r_s \;=\; -\frac{p-1}{N-p}\;s^{\frac{N-1}{p-N}}\,E'(r(s)) \;\leq\; 0. \tag{2.11}$$

Moreover, the energy in the variable s is expressed by

$$E(s) \;=\; s\frac{1}{p'}\;|y_s|^p\left(\frac{N-p}{p-1}\right)^p + s^{\frac{(p-1)N}{p-N}}\,\frac{|y|^{\delta+1}}{\delta+1} - \frac{N}{\delta+1}\left(\frac{N-p}{p-1}\right)^{p-1} y\,|y_s|^{p-2}\,y_s.$$

If we define $\gamma(s) = |y_s|^{p-2}\,y_s$ and we assume, without loss of generality, $\beta > 0$, the system we deal with can be written in the following form

$$\begin{aligned} y_s &= \;|\gamma|^{\frac{1}{p-1}}\,\mathrm{sign}\,\gamma, \quad s\in(1,+\infty)\\ \gamma_s &= \;-s^{\frac{(N-1)p}{p-N}}\,\left(\tfrac{p-1}{N-p}\right)^p\,|y|^{\frac{(p-1)p}{N-p}}\,|y|^{p-1}\mathrm{sign}\,y, \quad s\in(1,+\infty)\\ & y(1)=0, \quad \gamma(1)=\beta^{p-1}\,\left(\tfrac{p-1}{N-p}\right)^{p-1}>0. \end{aligned} \tag{2.12}$$

Suppose, by contradiction, that there is a sequence $\beta_n \in (0,\beta_1)$ with $\lim_{n\to+\infty}\beta_n = 0$ such that the solutions y_{β_n} to (2.10) with $\beta=\beta_n$ have a zero in $(1,+\infty)$. More precisely, there exists $s_n \in (1,+\infty)$ such that $y_{\beta_n}(s_n) = y_{\beta_n}(1) = 0$, $\left(y_{\beta_n}\right)_s(s_n) < 0$ and y_{β_n} is strictly positive in $(1,s_n)$.

Our aim consists in proving that there is $n_0\in\mathbb{N}$ such that for every $n\ge n_0$

$$s^{\frac{(N-1)p}{p-N}}\,\left(\frac{p-1}{N-p}\right)^p |y_{\beta_n}|^{\frac{(p-1)p}{N-p}} < \left(\frac{p}{p-1}\right)^{-p} s^{-p} \qquad s\in[1,s_n]. \tag{2.13}$$

Let us denote by $\widetilde{s}_n$ the maximum of y_{β_n} in the interval $(1,s_n)$. From (2.11) we know that $E_n(s)\le\frac{\beta_n^p}{p'}$ for every $s\ge1$. In particular, we obtain that

$$|y_{\beta_n}(s)|^{\delta+1} \le \frac{\delta+1}{p'}\;\beta_n^p\;s^{\frac{(p-1)N}{N-p}}, \qquad s\in[\widetilde{s}_n,s_n],$$

which, for our choice of δ, leads to

$$s^{\frac{(N-1)p}{p-N}}\left(\frac{p-1}{N-p}\right)^p|y_{\beta_n}|^{\frac{(p-1)p}{N-p}} \;\le\; C(p,N)\beta_n^{\frac{p(p-1)}{N-1}}\,s^{\frac{1-pN}{N-1}} \le C(p,N)\beta_n^{\frac{p(p-1)}{N-1}}\,s^{-p},$$

where $C(N,p)$ is a suitable positive constant and $s\in[\widetilde{s}_n,s_n]$. In particular, there exists $n_1 = n_1(p,N)$ such that, for any $n\ge n_1$, (2.13) holds in the interval $[\widetilde{s}_n,s_n]$. Thus, it remains to verify the validity of (2.13) in the interval $[1,\widetilde{s}_n]$. We point out that for n sufficiently large

$$y_{\beta_n}(s)\;\le\;\beta_n^{\frac{N-p}{N}}\;s, \qquad s\in[1,\widetilde{s}_n]. \tag{2.14}$$

Indeed, it is sufficient to choose n large enough to guarantee that

$$(y_{\beta_n})_s(1) < \beta_n^{\frac{N-p}{N}}$$

and to observe that the function $s \mapsto ((y_{\beta_n})_s)^{p-1}(s)$ and, consequently, $s \mapsto (y_{\beta_n})_s(s)$ is decreasing on $[1, \widetilde{s}_n]$. From (2.14), we finally infer that

$$s^{\frac{(N-1)p}{p-N}} \Big(\frac{p-1}{N-p}\Big)^p |y_{\beta_n}|^{\frac{(p-1)p}{N-p}} \le \Big(\frac{p-1}{N-p}\Big)^p \beta_n^{\frac{p(p-1)}{N}} s^{-p}, \quad s \in [1, \widetilde{s}_n].$$

This implies the existence of $n_0 \ge n_1$ such that for any $n \ge n_0$ the inequality (2.13) holds in all of the interval $[1, s_n]$.

Acting along the same lines of [18, proof of Theorem 2.1], we compare (2.12), in which β is chosen equal to β_n for a fixed natural number $n \ge n_0$, with

$$\begin{aligned} y_s = &\ |\gamma|^{\frac{1}{p-1}} \operatorname{sign} \gamma, \qquad s \in [1, s_n] \\ \gamma_s = &\ -\left(\tfrac{p}{p-1}\right)^{-p} s^{-p}\, |y|^{p-1} \operatorname{sign} y, \qquad s \in [1, s_n]. \end{aligned} \tag{2.15}$$

It is easy to see that system (2.15) admits the positive solution $y(s) = s^{\frac{1}{p'}}$ and $\gamma(s) = (\frac{p}{p-1})^{-(p-1)}\, s^{-\frac{1}{p'}}$ defined in $[\varepsilon, +\infty)$, for any $\varepsilon > 0$. Since the comparison theorem and the corresponding note in [18] guarantee that every solution of (2.15) has at least one zero on the interval $(1, s_n)$, the contradiction follows.

Let us treat now the case $\frac{N(p-1)}{N-p} < \delta < \frac{N(p-1)+p}{N-p}$. By writing

$$r = e^{-t}, \quad v(t) := r^{\frac{p}{\delta+1-p}} u(r), \tag{2.16}$$

we find that (v, z), where $z(t) = |v_t + \theta v|^{p-2}(v_t + \theta v)$, $\theta = \frac{p}{\delta+1-p}$, satisfies

$$\begin{aligned} v_t &= -\theta v + |z|^{p'-2} z, \quad t \in (0, +\infty) \\ z_t &= -|v|^{\delta-1} v + Bz, \quad t \in (0, +\infty) \\ v(0) &= 0, \quad z(0) = \beta\, |\beta|^{p-2}, \end{aligned} \tag{2.17}$$

with $B = N - \theta\delta > 0$. The change of variables (2.16) is classical; it was first introduced by Fowler and it is generally used in order to classify the solutions to problem (1.1) (cf., for instance, [1], [2], [16]). For simplicity, we will sketch the proof for the case $\beta > 0$. In order to prove that all the singular solutions u_β do not change sign for β "sufficiently small," we use a result proved in [2]. More precisely, if one considers a singular solution u of (2.1), it is guaranteed in [2] that the solution (v, z) of the corresponding problem (2.17) is such that both the functions v and z admit a constant, nonzero limit at $+\infty$.

Moreover, Nabana and de Thelin (see [20]), ensured the existence of a $\widetilde{\beta} > 0$ for which $u_{\widetilde{\beta}}$ is positive and regular. This implies that also $v_{\widetilde{\beta}}$ and $z_{\widetilde{\beta}}$

are positive and regular and, consequently (see again [2]),

$$\lim_{t\to+\infty} v_{\widetilde{\beta}}(t) = \lim_{t\to+\infty} z_{\widetilde{\beta}}(t) = 0,$$

according to the notation for which $(v_{\widetilde{\beta}}, z_{\widetilde{\beta}})$ represents a solution of (2.17) with $\beta = \widetilde{\beta}$. Taking into account that solutions to system (2.17) with different initial data cannot intersect, the result follows by a careful phase-plane analysis of the system. □

Remark 2.1. The positiveness of the singular solutions to (2.1) for $|\beta| \le \beta_0$ in the case $p-1 < \delta < \frac{N(p-1)}{N-p}$ can also be proved by using the change of variables (2.9) and by acting exactly as in the case $\delta = \frac{N(p-1)}{N-p}$.

As we mentioned before, the change of variable $w(t) = \varepsilon^{-p/(\delta+1-p)} u(r)$, $t = \varepsilon r$, allows us to state an equivalent version of Lemma 2.1. More precisely,

Lemma 2.2. *Let $0 < p-1 < \delta$, $N > p$ and $\delta < \frac{N(p-1)+p}{N-p}$. Then, there exists $\beta_0 > 0$ such that, for any $0 < |\beta| \le \beta_0$ and for every $\varepsilon > 0$, it holds that any solution to the initial-value problem*

$$\begin{aligned} &-(t^{N-1}|w'|^{p-2}w')' = t^{N-1}|w|^{\delta-1}w, \quad t \in (0,\varepsilon) \\ &w(\varepsilon) = 0, \quad w'(\varepsilon) = -\varepsilon^{-\frac{\delta+1}{\delta+1-p}}\,\beta, \end{aligned}$$

is singular and does not change sign in $(0,\varepsilon)$.

3. Key results concerning the nodal properties of the solutions

In order to prove Theorem 1.1, it remains to study the multiplicity of solutions defined in the interval $(\varepsilon, 1)$ for problems of the form (1.5). For this reason, in what follows we will establish some properties of the solution to

$$\begin{aligned} &(r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u|^{\delta-1}u = 0, \quad r \in (\varepsilon,1) \\ &u(\varepsilon) = 0, \quad |u'(\varepsilon)| = \varepsilon^{-\frac{\delta+1}{\delta+1-p}}\,\bar{\beta} > 0, \end{aligned} \tag{3.1}$$

with $\bar{\beta}$ fixed such that $0 < \bar{\beta} < \beta_0$ (β_0 as in Lemma 2.1), which will be useful when computing the associated rotation number.

Lemma 3.1. *Let $N > p$, with $1 < p < \delta+1 < \frac{Np}{N-p}$. Then, given $M > 0$, there is $\varepsilon_{M,\bar{\beta}} \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_{M,\bar{\beta}}]$, the solution to problem (3.1) satisfies $|u(r)| + |u'(r)| > M \quad \forall\, r \in [\varepsilon, 1]$.*

Proof. Let u be any solution of

$$-\left(r^{N-1}\,|u'|^{p-2}u'\right)' = r^{N-1}\,|u|^{\delta-1}\,u \qquad r\in(0,1).$$

We consider the energy (2.5) for the solution u, namely

$$E(u)(r) = r^N\Big(\frac{|u'|^p}{p'} + \frac{|u|^{\delta+1}}{\delta+1}\Big) + \frac{N}{\delta+1} r^{N-1}u|u'|^{p-2}u', \tag{3.2}$$

which is monotone nondecreasing. Hence, for any $\varepsilon\in(0,1)$, it holds that $E(r)\ge E(\varepsilon)$ for all $r\in[\varepsilon,1]$. Hence, if u satisfies the initial conditions $u(\varepsilon)=0,\quad |u'(\varepsilon)| = \varepsilon^{-\frac{\delta+1}{\delta+1-p}}\bar\beta$, we have that there is a positive constant C depending only on N and δ such that

$$C\Big(\frac{|u'(r)|^p}{p'} + \frac{|u(r)|^{\delta+1}}{\delta+1} + \frac{|u(r)|^p}{p}\Big) \ge \varepsilon^{-\alpha}\,\frac{\bar\beta^p}{p'},$$

where

$$\alpha = \frac{Np-(\delta+1)(N-p)}{\delta+1-p} > 0. \tag{3.3}$$

Thus the result follows by choosing $\varepsilon_{M,\bar\beta}>0$ small enough. □

The following two lemmas will provide estimates from below and from above, respectively, on the number of zeros in $(\varepsilon,1]$ to the solutions of (3.1), provided that suitable conditions on ε and β hold.

Lemma 3.2. *Let β_0 be as in Lemma 2.1, let $\bar\beta\in(0,\beta_0\,]$, let $\varepsilon_0\in(0,1)$, and let $m\in\mathbb{N}$, $m\ge\frac{1-\varepsilon_0}{4\pi_p}\frac{2(N-1)}{p\varepsilon_0}$. There exists $\varepsilon_{m,0}=\varepsilon(m,\varepsilon_0,\bar\beta)\in(0,\varepsilon_0]$ such that, for any $\varepsilon\in(0,\varepsilon_{m,0}\,]$, the solution u to the initial-value problem*

$$(r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u|^{\delta-1}u = 0,\quad u(\varepsilon)=0,\quad |u'(\varepsilon)| = \varepsilon^{-\frac{\delta+1}{\delta+1-p}}\bar\beta. \tag{3.4}$$

has at least m zeros in $(\varepsilon_0,1)\subset(\varepsilon,1)$.

Proof. Let us take

$$m\ge\frac{1-\varepsilon_0}{4\pi_p}\;\frac{2(N-1)}{p\,\varepsilon_0} \tag{3.5}$$

and set

$$L = \Big(\frac{4\pi_p\,m}{1-\varepsilon_0}\Big)^p \ge L_0 := \Big(\frac{2(N-1)}{p\,\varepsilon_0}\Big)^p.$$

Moreover, let k be a fixed positive constant depending only on p and δ satisfying

$$|x|^{\delta+1}\ge L|x|^p - k\,L^{\frac{\delta+1}{\delta+1-p}}, \qquad \forall\,x\in\mathbb{R}. \tag{3.6}$$

Let u be the solution of the initial-value problem (3.4) with ε chosen small enough, say $0 < \varepsilon \le \varepsilon_M$, where $M = M(L, k, \delta, p)$ and $\varepsilon_{M,\bar\beta}$ is such that

$$4kL^{\frac{\delta+1}{\delta+1-p}} \, < \, (p-1)|u'(r)|^p \, + \, L\,|u(r)|^p \qquad \forall r \in [\varepsilon, 1].$$

The existence of such a $\varepsilon_{M,\bar\beta}$ is guaranteed by the previous Lemma 3.1. It is not restrictive to suppose $\varepsilon_{M,\bar\beta} \le \varepsilon_0$. Recalling the dependence of $\varepsilon_{M,\bar\beta}$ on L, we will denote such a constant by $\varepsilon_{m,0}$.

In what follows, we will make strong use of the so-called $\sin_p$ and $\cos_p$ functions introduced by Del Pino, Elgueta and Manásevich (see [8]), as the solutions of

$$(\varphi_p(w_\vartheta))_\vartheta \, + \, \lambda\varphi_p(w) \, = \, 0 \qquad w(0) = 0, \qquad w_\vartheta(0) \, = 1$$

(see also [10]), and we will use the coordinate system (see [12], [13], where it was first introduced)

$$u = \frac{\pi_p}{L^{\frac{1}{p}}}\,\rho\, w(\vartheta), \quad \varphi_p(u') = -\,\rho^{p-1}\,\varphi_p\left(w_\vartheta(\vartheta)\right), \tag{3.7}$$

with $\rho = \rho(r) \in \mathbb{R}^+_0$, $\vartheta = \vartheta(r)$ and where $w(\vartheta) := \pi_p^{-1}\,\sin_p(\pi_p\,\vartheta)$ is a solution to

$$(\varphi_p(w_\vartheta))_\vartheta + \pi_p^p\,\varphi_p(w) \, = \, 0, \quad w(0) = 0, \qquad w_\vartheta(0) \, = 1. \tag{3.8}$$

For this change of variables, we also quote the work by Zhang (see [23, 24]), where the only difference is the power of ρ. We recall that $\varphi_p(x) = |x|^{p-2}\,x$ and that π_p can be explicited by the following integral expression (see for example [8])

$$\pi_p \, = \, 2\,(p-1)^{\frac{1}{p}} \int_0^1 \frac{\mathrm{d}t}{(1-z^p)^{\frac{1}{p}}}.$$

Furthermore, by definition (cf. [8]), $\sin_p : \mathbb{R} \to \mathbb{R}$ is a $2\pi_p-$periodic function and $\sin_p(\mu) = 0$ if and only if $\mu = n\,\pi_p$ with $n \in \mathbb{Z}$. In particular, from (3.7), we obtain that

$$u(r) \, = \, 0 \qquad \Longleftrightarrow \qquad \vartheta(r) \, \in \, \mathbb{Z}. \tag{3.9}$$

Let us consider now $\varepsilon \in (0, \varepsilon_{m,0}]$. The problem in (3.4) is equivalent to the following

$$\begin{cases} u' = \varphi_{p'}(z) \\ z' = -\dfrac{N-1}{r}\,z \, - \, |u|^{\delta-1}\,u \\ u(\varepsilon) = 0, \qquad |z(\varepsilon)| = \varepsilon^{-\frac{(\delta+1)(p-1)}{\delta+1-p}}\,\bar\beta^{\,p-1}, \end{cases}$$

where $z = \varphi_p(u')$. Taking into account (3.7) and (3.8), it is possible to deduce that for every $r \in [\varepsilon, 1]$

$$-\vartheta' = \frac{L^{\frac{1}{p}}}{\pi_p} \; \frac{(p-1)\,|u'|^p + \frac{N-1}{s}\, u\; \varphi_p(u') + |u|^{\delta+1}}{(p-1)\,|u'|^p + L\,|u|^p}. \tag{3.10}$$

By Young's inequality, it follows that for every $r \in [\varepsilon_0, 1]$

$$2\,\frac{N-1}{r}\,|u|\,|u'|^{p-1} = p\Big(\,|u'|^{p-1}\;\frac{2\,(N-1)}{p\,r}\,|u|\Big) \le p\frac{|u'|^p}{p'} + L_0\,|u|^p.$$

In particular, since $L \ge L_0$, we obtain

$$2\,\frac{N-1}{r}\,|u|\,|u'|^{p-1} \le (p-1)|u'|^p + L\,|u|^p\,. \tag{3.11}$$

Thus, according to (3.10), we get

$$-\vartheta' \ge \frac{L^{\frac{1}{p}}}{2\,\pi_p}\frac{(p-1)\,|u'|^p - L\,|u|^p + 2\,|u|^{\delta+1}}{(p-1)\,|u'|^p + L\,|u|^p} \qquad \forall r \in [\varepsilon_0, 1].$$

Moreover, from the choice of k in (3.6), we obtain that for every $r \in [\varepsilon_0, 1]$

$$-\vartheta' \ge \frac{L^{\frac{1}{p}}}{2\,\pi_p}\Big[\frac{(p-1)\,|u'|^p + L\,|u|^p}{(p-1)\,|u'|^p + L\,|u|^p} - \frac{2k\,L^{\frac{\delta+1}{\delta+1-p}}}{(p-1)\,|u'|^p + L\,|u|^p}\Big],$$

that is,

$$-\vartheta' \ge \frac{L^{\frac{1}{p}}}{2\,\pi_p}\Big[1 - \frac{2k\,L^{\frac{\delta+1}{\delta+1-p}}}{(p-1)\,|u'|^p + L\,|u|^p}\Big].$$

Thus, by the choice of $\varepsilon_{m,0}$ and by the definition of L, we have

$$-\vartheta'(r) > \frac{1}{4}\,\frac{L^{\frac{1}{p}}}{\pi_p} = \frac{m}{1-\varepsilon_0} \qquad \forall\, r \in [\varepsilon_0, 1].$$

Hence, we infer that

$$\vartheta(1) - \vartheta(\varepsilon_0) < -m\,, \tag{3.12}$$

which means that at least m integers belong to the interval $(\vartheta(1), \vartheta(\varepsilon_0))$. By the continuity of $\vartheta(\cdot)$ and according to (3.9), the thesis follows. □

Remark 3.1. Lemma 3.1 ensures that Lemma 3.2 still holds for every solution to the initial-value problem (3.4) written with β instead of $\bar{\beta}$, whenever $\beta \in [\bar{\beta}, \beta_0]$. In such a case, the constant $\varepsilon_{m,0}$ will coincide with $\varepsilon(m, \varepsilon_0, \bar{\beta})$ introduced in Lemma 3.2 and it will not depend on β.

Remark 3.2. We stress the fact that the thesis of Lemma 3.2 could be equivalently written in the form

$$\vartheta(1) - \vartheta(\varepsilon) < -m, \tag{3.13}$$

where ε is as in (3.4) and, in particular, $\vartheta(\varepsilon) \in \mathbb{Z}$. This is a consequence of the property of the angular function ϑ which ensures that $\vartheta'(\overline{r}) < 0$ for every $\overline{r} \in [\varepsilon, 1]$ such that $\vartheta(\overline{r}) = n \in \mathbb{Z}$. In particular, if there exists such a constant $\overline{r} \in [\varepsilon, 1]$, then $\vartheta(r) > n$ for every $r \in [\varepsilon, \overline{r})$, while $\vartheta(s) < n$ for every $s \in (\overline{r}, 1]$.

Let us choose $\varepsilon_* = \varepsilon(\beta_0) \in (0,1)$ satisfying

$$\pi_p^{\frac{\delta+1}{\delta+1-p}} \left(\frac{p'}{\delta+1}\right)^{\frac{1}{p}} \varepsilon_*^{\frac{(N+p-2)(\delta+1)}{(p-1)(\delta+1-p)}} \leq \beta_0, \tag{3.14}$$

with β_0 as in Lemma 2.1.

Lemma 3.3. *Let β_0 be as in Lemma 2.1, let $\tilde{\varepsilon} \in (0, \varepsilon_*]$. Then, there exists $\tilde{\beta} = \tilde{\beta}(\tilde{\varepsilon}) \in (0, \beta_0]$ such that for any $\beta \in (0, \tilde{\beta}\,]$ the solution u to the initial-value problem*

$$\begin{aligned} &(r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u|^{\delta-1}u = 0, \\ &u(\tilde{\varepsilon}) = 0, \quad |u'(\tilde{\varepsilon})| = \tilde{\varepsilon}^{-\frac{\delta+1}{\delta+1-p}}\,\beta \end{aligned} \tag{3.15}$$

has no zeros in $(\tilde{\varepsilon}, 1]$.

Proof. Let us take a solution $u = u_{(\tilde{\varepsilon},\beta)}$ to the initial-value problem (3.15). Let us define

$$\tilde{\beta} = \tilde{\beta}(\tilde{\varepsilon}) := \pi_p^{\frac{\delta+1}{\delta+1-p}} \left(\frac{p'}{\delta+1}\right)^{\frac{1}{p}} \tilde{\varepsilon}^{\,\frac{(N+p-2)(\delta+1)}{(p-1)(\delta+1-p)}}.$$

Since $\tilde{\varepsilon} \leq \varepsilon_*$, from (3.14) it follows that $\tilde{\beta} \in (0, \beta_0]$.

Consider now $\beta \in (0, \tilde{\beta}\,]$. Problem (3.15) can be equivalently written in the form

$$\begin{cases} u'(r) = \varphi_{p'}\left(\frac{y(r)}{r^{N-1}}\right) \\ y'(r) = -\,r^{N-1}\,|u(r)|^{\delta-1}\,u(r) \\ u(\tilde{\varepsilon}) = 0 \qquad |y(\tilde{\varepsilon})| = \tilde{\varepsilon}^{\,\frac{(\delta+1)(N-p)-p(N-1)}{\delta+1-p}}\,\beta^{p-1}. \end{cases} \tag{3.16}$$

We introduce the coordinate system (see [12])

$$\begin{aligned} u &= \frac{\pi_p}{\mu^{\frac{1}{p}}}\,\rho\, w(\theta) \\ y &= -\,\rho^{p-1}\,\varphi_p\left(w_\theta(\theta)\right), \end{aligned} \tag{3.17}$$

with $\rho = \rho(r) \in \mathbb{R}_0^+$, $\theta = \theta(r)$ and where $w(\theta) := \pi_p^{-1} \sin_p(\pi_p \theta)$ solves (3.8), written with θ instead of ϑ. It follows that for every $r \in [\tilde{\varepsilon}, 1]$

$$-\theta' = \frac{\mu^{\frac{1}{p}}}{\pi_p} \frac{(p-1)\, r^{-\frac{N-1}{p-1}} |y|^{p'} + r^{N-1} |u|^{\delta+1}}{(p-1)\, |y|^{p'} + \mu\, |u|^p}. \tag{3.18}$$

We remark that θ is well defined for every $r \in [\tilde{\varepsilon}, 1]$, whenever we consider a solution u to the initial-value problem (3.15). Indeed, since the energy (3.2) is monotone nondecreasing, arguing as in the proof of Lemma 3.1, from the initial condition in (3.15) we get the existence of a positive constant C such that

$$C\Big(\frac{|u'(r)|^p}{p'} + \frac{|u(r)|^{\delta+1}}{\delta+1} + \frac{|u(r)|^p}{p}\Big) \geq \tilde{\varepsilon}^{-\alpha} \frac{\beta^p}{p'} > 0,$$

with α defined in (3.3). This implies that u and u' (and, consequently, u and y) cannot contemporaneously vanish. In such a way, the angle θ is well defined.

Moreover, for every $r \in [\tilde{\varepsilon}, 1]$ we obtain

$$-\theta' = \frac{\mu^{\frac{1}{p}}}{\pi_p\, r^{\frac{N-1}{p-1}}} \frac{(p-1)|y|^{p'} + r^{\frac{(N-1)p}{p-1}} |u|^{\delta+1}}{(p-1)|y|^{p'} + \mu|u|^p} \leq \frac{\mu^{\frac{1}{p}}}{\pi_p \tilde{\varepsilon}^{\frac{N-1}{p-1}}} \frac{(p-1)|y|^{p'} + |u|^{\delta+1}}{(p-1)|y|^{p'} + \mu|u|^p}.$$

In order to improve the estimate above, we consider the energy

$$\widetilde{E}(u)(r) = \frac{|u'|^p}{p'} + \frac{|u|^{\delta+1}}{\delta+1}. \tag{3.19}$$

Since its derivative

$$\widetilde{E}(u)'(r) = -\frac{(N-1)}{r} |u'|^p \leq 0 \qquad \forall\, r \in (0,1], \tag{3.20}$$

we conclude that

$$\widetilde{E}(u)(r) \leq \widetilde{E}(u_{(\tilde{\varepsilon},\beta)})(\tilde{\varepsilon}) = \frac{1}{p'} \tilde{\varepsilon}^{-\frac{(\delta+1)p}{\delta+1-p}} \beta^p \qquad \forall\, r \in [\tilde{\varepsilon}, 1].$$

In order to simplify the notation, we set $\widetilde{E}(\tilde{\varepsilon}, \beta) := \widetilde{E}(u_{(\tilde{\varepsilon},\beta)})(\tilde{\varepsilon})$. Moreover, by definition of the energy, we obtain that for every $r \in [\tilde{\varepsilon}, 1]$

$$|u(r)| \leq (\delta+1)^{\frac{1}{\delta+1}} (\widetilde{E}(u)(r))^{\frac{1}{\delta+1}} \leq (\delta+1)^{\frac{1}{\delta+1}} (\widetilde{E}(\tilde{\varepsilon}, \beta))^{\frac{1}{\delta+1}},$$

which implies that for every $r \in [\tilde{\varepsilon}, 1]$

$$|u(r)|^{\delta+1} = |u(r)|^{\delta+1-p} |u(r)|^p \leq (\delta+1)^{\frac{\delta+1-p}{\delta+1}} (\widetilde{E}(\tilde{\varepsilon}, \beta))^{\frac{\delta+1-p}{\delta+1}} |u(r)|^p.$$

Now, we are able to improve the previous upper estimate on $-\theta'$. More precisely, we get

$$-\theta' \le \frac{\mu^{\frac{1}{p}}}{\pi_p\,\tilde{\varepsilon}^{\frac{N-1}{p-1}}}\,\frac{(p-1)\,|y|^{p'} + (\delta+1)^{\frac{\delta+1-p}{\delta+1}}\,(\widetilde{E}(\tilde{\varepsilon},\beta))^{\frac{\delta+1-p}{\delta+1}}\,|u|^p}{(p-1)\,|y|^{p'} + \mu\,|u|^p},$$

for every $r \in [\tilde{\varepsilon}, 1]$. Choosing

$$\mu = (\delta+1)^{\frac{\delta+1-p}{\delta+1}}\,(\widetilde{E}(\tilde{\varepsilon},\beta))^{\frac{\delta+1-p}{\delta+1}},$$

we deduce that

$$-\theta' \le \frac{1}{\pi_p}\,(\delta+1)^{\frac{\delta+1-p}{p(\delta+1)}}\,\frac{(\widetilde{E}(\tilde{\varepsilon},\beta))^{\frac{\delta+1-p}{p(\delta+1)}}}{\tilde{\varepsilon}^{\frac{N-1}{p-1}}} = \frac{1}{\pi_p}\left(\frac{\delta+1}{p'}\right)^{\frac{\delta+1-p}{p(\delta+1)}}\beta^{\frac{\delta+1-p}{\delta+1}}\,\tilde{\varepsilon}^{-\frac{N+p-2}{p-1}}.$$

From the choice of β, we finally infer that

$$-\theta'(r) \le 1 \qquad \forall\, r \in [\tilde{\varepsilon}, 1],$$

which implies that

$$\theta(1) - \theta(r) \ge -1 + r > -1 \quad \forall\, r \in [\tilde{\varepsilon}, 1).$$

Taking into account that the function θ is decreasing and that $\tilde{\varepsilon}$ is a zero of u, according to (3.9), written with θ instead of ϑ, we can conclude that $\theta(r) \notin \mathbb{Z}$ for each $r \in (\tilde{\varepsilon}, 1]$ and, consequently, the thesis follows. □

Remark 3.3. We observe that Lemma 3.3 still holds if we consider solutions u to the initial-value problem (3.15) written with ε instead of $\tilde{\varepsilon}$, whenever $\varepsilon \in [\tilde{\varepsilon}, \varepsilon_*]$. In such a case, the constant $\tilde{\beta}$ will coincide with $\tilde{\beta}(\tilde{\varepsilon})$ introduced in Lemma 3.3 and it will not depend on ε.

Remark 3.4. We point out that even if the angular function ϑ defined by (3.10) is different from the function θ introduced in (3.18), both of them present the same characterization in terms of number of zeros in $(\tilde{\varepsilon}, 1]$ of the solution to a Cauchy problem of the form (3.15). In particular, the thesis of Lemma 3.3 could be also read as follows

$$\vartheta(1) - \vartheta(\tilde{\varepsilon}) > -1, \tag{3.21}$$

where $\tilde{\varepsilon}$ is as in (3.15).

4. MAIN RESULT

We are now ready to prove our main result, which will follow directly from the next theorem and Lemma 2.2.

Theorem 4.1. *Let $0 < p-1 < \delta$, $N > p$ and $\delta < \frac{N(p-1)+p}{N-p}$ and let β_0 be as in Lemma 2.1. For every $n \in \mathbb{N}$, there exist three constants $\varepsilon_n = \varepsilon(n, \beta_0)$, $\beta_n^{\pm} = \beta(n, \beta_0, \varepsilon_n)$ with $\beta_n^+ > 0 > \beta_n^-$, $|\beta_n^{\pm}| \le \beta_0$ and at least two solutions u_n and v_n of the Dirichlet problem*

$$\begin{gathered} -(r^{N-1}|u'|^{p-2}u')' = r^{N-1}|u|^{\delta-1}u, \quad r \in (\varepsilon_n, 1) \\ u(\varepsilon_n) = 0 = u(1), \end{gathered} \tag{4.1}$$

having exactly n zeros in $[\varepsilon_n, 1)$ and satisfying

$$u_n'(\varepsilon_n) = -\varepsilon_n^{-\frac{\delta+1}{\delta+1-p}}\,\beta_n^+\,, \qquad v_n'(\varepsilon_n) = -\varepsilon_n^{-\frac{\delta+1}{\delta+1-p}}\,\beta_n^-\,. \tag{4.2}$$

Proof. Let β_0 be as in Lemma 2.1 and ε_* be as in (3.14) and $n \in \mathbb{N}$. Let us apply Lemma 3.2 with $\bar{\beta} := \beta_0$, $m := n$ and $\varepsilon_0 \in (0,1)$ satisfying the inequality (3.5).

Thus, according to Remark 3.2, there exists $\varepsilon_{n,0} \in (0, \varepsilon_0]$ such that, choosing $\varepsilon_n := \min(\varepsilon_{n,0}, \varepsilon_*) \in (0, \varepsilon_{n,0}\,]$, the angular function $\vartheta_{\varepsilon_n,\beta_0}$ associated by means of the relation (3.7) to the solution $u_{\varepsilon_n,\beta_0}$ of the initial-value problem

$$\begin{gathered} (r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u|^{\delta-1}u = 0, \quad r \in (\varepsilon_n, 1) \\ u(\varepsilon_n) = 0, \quad |u'(\varepsilon_n)| = \varepsilon_n^{-\frac{\delta+1}{\delta+1-p}}\beta_0 \end{gathered}$$

satisfies

$$\vartheta_{\varepsilon_n,\beta_0}(1) - \vartheta_{\varepsilon_n,\beta_0}(\varepsilon_n) < -n.$$

Moreover, if we set $\tilde{\varepsilon} = \varepsilon_n \in (0, \varepsilon_*]$ in Lemma 3.3, according to Remark 3.4, we infer that there exists $\tilde{\beta} = \tilde{\beta}(\varepsilon_n) \in (0, \beta_0]$ such that the angular function $\vartheta_{\varepsilon_n,\tilde{\beta}}$ associated by means of the relation (3.7) to the solution $u_{\varepsilon_n,\tilde{\beta}}$ of the initial-value problem

$$\begin{gathered} (r^{N-1}|u'|^{p-2}u')' + r^{N-1}|u|^{\delta-1}u = 0, \quad r \in (\varepsilon_n, 1) \\ u(\varepsilon_n) = 0, \quad |u'(\varepsilon_n)| = \varepsilon_n^{-\frac{\delta+1}{\delta+1-p}}\,\tilde{\beta} \end{gathered}$$

satisfies

$$\vartheta_{\varepsilon_n,\tilde{\beta}}(1) - \vartheta_{\varepsilon_n,\tilde{\beta}}(\varepsilon_n) > -1.$$

Now, we are in a position to apply a shooting method. From the continuous dependence of a solution $u = u_{\varepsilon,\beta}$ of (1.5) on the initial data and by the definition of $\vartheta = \vartheta_{\varepsilon,\beta}$ associated to $u_{\varepsilon,\beta}$ by (3.7), we can deduce the continuity of the function $\vartheta_{\varepsilon,\beta}$ with respect to the variable β, whenever $\varepsilon > 0$ is fixed. Hence, from the intermediate value theorem, one can conclude that there exist $\beta_n^+ \in (\tilde{\beta}(\varepsilon_n), \beta_0)$ and $\beta_n^- \in (-\beta_0, -\tilde{\beta}(\varepsilon_n))$ such that

$$\vartheta_{\varepsilon_n,\beta_n^\pm}(1) - \vartheta_{\varepsilon_n,\beta_n^\pm}(\varepsilon_n) = -n.$$

In particular, we have found two solutions u_n and v_n to (4.1), having exactly n zeros in $[\varepsilon_n, 1)$ and satisfying the initial conditions (4.2), respectively. This completes the proof. □

As we mentioned at the beginning of this section, Theorem 1.1 follows by combining Lemma 2.2 with Theorem 4.1.

Remark 4.1. The shooting argument could be applied also with respect to the variable ε. We have decided to use the shooting method with respect to the other variable, because it allows us to achieve, for every natural number n, the existence of an uncountable set of solutions to our main problem (1.1), having exactly n zeros in $(0, 1)$ (as it is said in [1]). Indeed, Theorem 4.1 has been proved by fixing $\varepsilon_n := \min(\varepsilon_{n,0}, \varepsilon_*)$, but it can equivalently be proved by replacing ε_n with any positive constant $\widetilde{\varepsilon}_n < \varepsilon_n$. This means that we can find solutions to problem (1.1), with the required nodal properties and having ε as a first zero on the interval $[0, 1]$, for any $\varepsilon \in (0, \varepsilon_n)$.

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