Advances in Differential Equations Volume 10, Number 7 (2005), 721–746

GLOBAL BIFURCATION BRANCHES FOR RADIALLY SYMMETRIC SCHRÖDINGER EQUATIONS

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Abstract. We prove a new result on bifurcating branches of bound states for the nonlinear radially symmetric Schrödinger equation

$$-\Delta u = w(|x|)|u|^{\sigma}u - \lambda^2 u \quad \text{on } \mathbb{R}^N.$$

We show that, under suitable assumptions on w and σ, there exist infinitely many continua of nontrivial bound states u_λ which emanate from the trivial solution branch at $\lambda = 0$. These continua reach arbitrarily large values of λ, and they are distinguished by the number of nodal domains of the corresponding solutions u_λ.

1. Introduction

We consider the superlinear Schrödinger equation

$$-\Delta u = w(x)|u|^{\sigma}u - \lambda^2 u, \quad x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0 \tag{1.1}$$

for $N \geq 2$ and $\sigma > 0$, where $w \in L^\infty(\mathbb{R}^N)$ is a positive function and λ is a parameter. This equation attracted extensive attention during the last 30 years. It arises for instance in nonlinear optics and plasma physics (see e.g. [26] and [27, Chapter 1]). Most of the results obtained in the literature are of one of the following two types.

(I) Existence and multiplicity results for solutions for fixed λ.

(II) Bifurcation results establishing properties of solution sets $\{u_\lambda\}$ for parameter values close to $\lambda = 0$, the lowest point of the (essential) spectrum of $-\Delta$ on $\mathbb{R}^N$.

For type (I) results we refer the reader to [29, 16] and the references therein. The results of the present paper are of type (II). Let S denote the set of all pairs (λ, u) such that u is a nontrivial solution of (1.1). We introduce the following definitions.

Accepted for publication: January 2005.
AMS Subject Classifications: 35Q55, 34A12.

We call $\lambda = 0$ an L^p*-bifurcation point* for (1.1) if there is a sequence $(\lambda_k, u_k)_k \subset S$ such that $(\lambda_k, u_k) \to (0,0)$ in $\mathbb{R} \times L^p(\mathbb{R}^N)$.

We call it an L^p*-branching point* if there is a connected subset $\mathcal{C} \subset S$ such that $(0,0)$ belongs to the closure of $\mathcal{C}$ in $\mathbb{R} \times L^p(\mathbb{R}^N)$.

These definitions are motivated by [1, 25]. Due to extensive research by many authors within the last two decades, the role of $\lambda = 0$ as a possible bifurcation point for (1.1) is by now well understood. Necessary and sufficient conditions on w, σ, and p for L^p-bifurcation were established e.g. in [2, 24, 21, 31, 6]. The most general results on bifurcating sequences were obtained by variational methods. For an excellent survey and more references, we refer the reader to [22].

Much less understood is the role of $\lambda = 0$ as a possible branching point of (1.1). In fact, no general approach is available for the study of continuous bifurcation branches emanating from the essential spectrum of a linear operator. In particular, purely variational arguments do not provide continua of solutions (λ, u). A general approach would complement the degree-theoretic mechanism developed by Rabinowitz [20] to investigate the global branching properties of the (pure point) spectrum of a compact operator. This mechanism applies for equation (1.1) on a *bounded* domain Ω and yields the existence of a global solution branch emanating from the first (Dirichlet or Neumann) eigenvalue of $-\Delta$, see [20]. Moreover, multiple branches can be found in the case where Ω and w are radially symmetric. Degree theory has also been successfully applied in the study of branching properties of isolated eigenvalues of Schrödinger operators $-\Delta + V$ with nonvanishing potential V; see e.g. [19, 4].

In the autonomous case w equal to a constant, the parameter λ in (1.1) can be rescaled to 1, therefore one solution already gives rise to a global branch. In particular, for $N = 1$ and w equal to a constant, (1.1) has precisely one branch of solutions which can be calculated explicitly, see [15]. One of the first results for the nonautonomous case is due to Toland [28]. He studied (1.1) with radially symmetric and nonincreasing w by a bounded domain approximation, and he proved the existence of a global branch of positive solutions. A similar result was obtained by Stuart [23] via a rescaling of (1.1) to

$$-\Delta u = w(\frac{x}{\lambda})|u|^\sigma u - u. \tag{1.2}$$

If w has a limit $w_\infty := \lim_{|x|\to\infty} w(x)$, then positive solutions of the autonomous limit equation

$$-\Delta u = w_\infty |u|^\sigma u - u, \tag{1.3}$$

have to be considered as starting points for solution branches of (1.2). In [23], for $N = 1$ and an even function w, a branch of positive radial solutions is found via the implicit function theorem starting from the unique even positive solution of (1.3). Local branches of positive solutions have also been constructed in the case where w is not radially symmetric, see [17, 25] for early results and [2, 3, 5, 13] for recent progress. The proofs use suitable generalizations of the Poincaré-Melnikow method, where the version developed by Ambrosetti and Badiale [2] leads to the most general results.

In the present paper we complement the results on branches of positive solutions by constructing branches of solutions with arbitrarily many sign changes. Moreover, we give sufficient conditions ensuring that these branches are global; i.e., they reach arbitrarily large values of λ. We make the following general assumptions:

(A_1) $w \in C^1(\mathbb{R}^N)$ is radially symmetric, and

$$0 < \inf_{\mathbb{R}^N} w \leq \sup_{\mathbb{R}^N} w < \infty.$$

(A_2) There is $R_0 > 0$ such that $r \mapsto w(r)$ is nonincreasing on $[R_0, \infty)$.

(A_3) $0 < \sigma < \frac{4}{N-2}$ for $N \geq 3$ and $\sigma > 0$ for $N = 2$.

Here and in the following, we freely identify a radial function $\mathbb{R}^N \to \mathbb{R}$ with the induced function $[0, \infty) \to \mathbb{R}$ of the radial variable $r = |x|$. For some of our results, we also need the following (stronger) condition:

(A_4) Either

(i) $N = 2$, or

(ii) $N \geq 3$ and $0 < \sigma \leq \frac{2}{N-2}$, or

(iii) $N \geq 3$, $0 < \sigma < \frac{4}{N-2}$, and $r \mapsto w(r)$ is nondecreasing.

We consider the Hilbert space $\mathcal{H} := \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}$ and the sets $S_n = \{(\lambda, u) \in (0, \infty) \times \mathcal{H} : u \text{ is a nontrivial solution of (1.1) having precisely } n \text{ zeros}\}$ for $n \in \mathbb{N} \cup \{0\}$.

Here the number of zeros refers to u as a function of $r = |x|$. By standard elliptic estimates, $S_n \subset W^{1,p}(\mathbb{R}^N)$ for $1 \leq p \leq \infty$. Our main results are the following.

Theorem 1.1. *Suppose that (A_1)–(A_3) hold, and let $n \in \mathbb{N} \cup \{0\}$. Then there is a subset $\mathcal{C}_n \subset S_n$ with the following properties.*

(i) *$\mathcal{C}_n$ is connected in $(0, \infty) \times W^{1,p}(\mathbb{R}^N)$ for $1 \leq p \leq \infty$.*

(ii) *$(0, \infty) \subset \{\lambda : (\lambda, u) \in \mathcal{C}_n\}$.*

(iii) *If $\sigma < \frac{2p}{N}$ and (A_4) holds, then $(0, 0)$ belongs to the closure of $\mathcal{C}_n$ in $[0, \infty) \times W^{1,p}(\mathbb{R}^N)$.*

In the case that (A_4) is not satisfied, we can still show the existence of solution continua emanating from $(\lambda, u) = (0, 0)$, but we have to give up assertion (ii) of the preceding theorem. We have the following:

Theorem 1.2. *Suppose that (A_1)–(A_3) hold, and let $n \in \mathbb{N} \cup \{0\}$. Then there is a subset $\mathcal{C}_n \subset S_n$ with the following properties :*

(i) *$\mathcal{C}_n$ is connected in $(0, \infty) \times W^{1,p}(\mathbb{R}^N)$ for $1 \le p \le \infty$.*

(ii) *If $\sigma < \frac{2p}{N}$, then $(0, 0)$ belongs to the closure of $\mathcal{C}_n$ in $[0, \infty) \times W^{1,p}(\mathbb{R}^N)$.*

Moreover, either $(0, \infty) \subset \{\lambda : (\lambda, u) \in \mathcal{C}_n\}$, or $\mathcal{C}_n$ contains a sequence $(\lambda_k, u_k)_k$ such that $\lambda_k \to 0$ as $k \to \infty$, and u_k converges pointwise to a nontrivial solution of the equation

$$-\Delta u = w(x)|u|^\sigma u. \tag{1.4}$$

Some remarks are in order. First, the restriction $N \ge 2$ on the dimension is not technical; the results cannot be true for $N = 1$. Indeed, an elementary phase plane analysis shows that the equation $-u''(x) = |u|^\sigma u - u$, $x \in \mathbb{R}$ has no sign-changing solutions which tend to zero at infinity. Second, the bound for σ in the above theorems is sharp. By a result of Stuart [24], $\lambda = 0$ is not an L^p-bifurcation point if $\sigma = \frac{2p}{N}$. We also note that in [28, 23] a global branch of positive solutions is constructed under the assumption that $r \mapsto w(r)$ is nonincreasing on $[0, \infty)$, whereas we only require (A_2). On the other hand, neither (A_2) nor (A_4) is needed for the mere existence of solutions of (1.1) with precisely n zeros. In fact, the existence of such solutions for fixed λ has been established in a much more general setting; see [7] and the references therein. This paper seems to be the first to study continua of sign-changing solutions for (1.1). A major difficulty, even for constructing local branches, is the lack of a classification of sign-changing solutions to (1.3). It is not known whether (1.3) has exactly one radial solution with precisely n zeros, or whether it has a nondegenerate sign-changing solution.

In this paper, we treat (1.2) by a parameter-dependent shooting method for the singular initial-value problem

$$\begin{aligned} -u'' - \frac{N-1}{r}u' &= w(\frac{r}{\lambda})|u|^\sigma u - u, \\ u(0) = a, \qquad & u'(0) = 0, \end{aligned} \tag{1.5}$$

Here the prime stands for $\frac{\partial}{\partial r}$. This approach is motivated by Jones and Küpper [12, 11], Grillakis [9], as well as McLeod, Troy and Weissler [18]. For autonomous and parameter-independent variants of (1.1), these authors used shooting arguments to prove the existence of solutions with a prescribed number of zeros. In order to study continuous bifurcation branches for the nonautonomous case, we generalize the techniques from [18] and [9] and implement new ideas in order to overcome the difficulties caused by the parameter dependence and the r-dependence of w. The shooting method reduces the original problem to a two-dimensional one, where the remaining free variables are λ and the initial value a. By a careful analysis we characterize the parameter pairs (λ, a) for which the corresponding solution $u(r)$ has precisely n positive zeros and decays exponentially as $r \to \infty$, thus representing a bound state of (1.1). Here assumption (A_2) is used in a crucial way. This assumption ensures that the 'energy'

$$E(r) = \frac{1}{2}(u'^2(r) - u^2(r)) + \frac{1}{\sigma+2} w(\frac{r}{\lambda})|u(r)|^{\sigma+2}$$

assigned to a solution u of (1.5) is nonincreasing for r large. Moreover, we will show that (A_4) implies that every radial solution of (1.4) has infinitely many zeros. Combining these two facts with a blow up argument, we establish useful a priori bounds for solutions with a fixed number of zeros. Finally, we complete the proofs of Theorems 1.1 and 1.2 by a purely topological argument in the plane.

The paper is organized as follows. In Section 2, we introduce the rescaled problem and reformulate our main results; see Theorems 2.1 and 2.2. The rescaled problem will be treated in Section 3 by the shooting method for (1.5). In Section 4, we complete the proof of our main theorems. In Section 5, we prove that, under assumption (A_4), every nontrivial radial solution of (1.4) is oscillating.

We conclude by mentioning some related work for a different type of equations. Heinz [10] studied the *sublinear* problem

$$-\Delta + w(x)|u|^{\sigma} u = \lambda^2 u, \quad x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0, \tag{1.6}$$

and he constructed global continuous branches of radial solutions with a prescribed number of zeros in the case where $w(x) = e^{f(|x|)}$ with a positive and nondecreasing function f. For a similar one-dimensional problem, a continuous branch of positive solutions was already exhibited by Küpper [14]. Positive solution branches have also been constructed by Giacomoni [8] for more general sublinear equations on $\mathbb{R}^N$ via bounded domain approximations.

Acknowledgments. The paper was written while the author was visiting the Department of Mathematics at the University of Minnesota, to which he is deeply grateful for its hospitality. His research and the visit was supported by DFG Grant WE 2821/2-1 (Germany). He also wishes to thank Professor Hans-Peter Heinz for helpful and inspiring discussions.

2. PRELIMINARIES

We fix some notation. For $1 \le p \le \infty$ we denote by $|\cdot|_p$ the usual norm of $L^p(\mathbb{R}^N)$. We consider the Banach spaces

$$\mathcal{E}_s = \Big\{u \in C^1(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N} e^{s|x|}|u(x)| < \infty,\ \sup_{x\in\mathbb{R}^N} e^{s|x|}|\nabla u(x)| < \infty\Big\}, \quad s > 0,$$

endowed with norms

$$\|u\|_s := \max\Big\{\sup_{x\in\mathbb{R}^N} e^{s|x|}|u(x)|,\ \sup_{x\in\mathbb{R}^N} e^{s|x|}|\nabla u(x)|\Big\}.$$

Note that we have continuous embeddings

$$\mathcal{E}_t \hookrightarrow \mathcal{E}_s \hookrightarrow W^{1,p}(\mathbb{R}^N) \qquad \text{for} \quad t > s > 0,\ 1 \le p \le \infty. \tag{2.1}$$

Throughout the paper, we assume that (A_1)–(A_3) hold. In particular, the limit

$$w_\infty := \lim_{|x|\to\infty} w(x)$$

exists and is positive. We define the functions $w_\lambda \in C^1(\mathbb{R}^N)$ by

$$w_\lambda(x) = \begin{cases} w(\frac{x}{\lambda}), & \lambda > 0, \\ w_\infty, & \lambda = 0. \end{cases} \tag{2.2}$$

If u is a solution of the problem

$$-\Delta u = w_\lambda(x)|u|^\sigma u - u, \qquad u \in \mathcal{E}_s \quad \text{for} \quad s < 1, \tag{2.3}$$

then $v(x) = \lambda^{\frac{2}{\sigma}} u(\lambda x)$ is a solution of (1.1). This holds for every $\lambda > 0$. Moreover,

$$|v|_p = \lambda^{\frac{2}{\sigma}-\frac{N}{p}}|u|_p \qquad \text{and} \qquad |\nabla v|_p = \lambda^{\frac{2}{\sigma}+1-\frac{N}{p}}|\nabla u|_p. \tag{2.4}$$

We have the following results for (2.3).

Theorem 2.1. *Let $0 < s < 1$, $n \in \mathbb{N} \cup \{0\}$. Then there exists a connected subset $\Gamma \subset (0,\infty) \times \mathcal{E}_s$ with the following properties.*

(i) *If $(\lambda, u) \in \Gamma$, then u is a nontrivial solution of (2.3) having precisely n zeros.*

(ii) $(0,\infty) \subset \{\lambda : (\lambda, u) \in \Gamma\}$.

(iii) *If* (A_4) *holds, then the closure of* Γ *in* $[0,\infty)\times\mathcal{E}_s$ *intersects the set* $\{(0,u) : u\in\mathcal{E}_s\setminus\{0\}\}$.

Theorem 2.2. *Let* $0<s<1$, $n\in\mathbb{N}\cup\{0\}$. *Then there exists a connected subset* $\Sigma\subset[0,\infty)\times\mathcal{E}_s$ *with the following properties.*

(i) *If* $(\lambda,u)\in\Sigma$, *then* u *is a nontrivial solution of (2.3) having precisely* n *zeros.*

(ii) *The closure of* Σ *in* $[0,\infty)\times\mathcal{E}_s$ *intersects the set* $\{(0,u) : u\in\mathcal{E}_s\setminus\{0\}\}$.

Moreover, either $(0,\infty)\subset\{\lambda : (\lambda,u)\in\Sigma\}$, *or* Σ *contains a sequence* $(\lambda_k,u_k)_k$ *such that* $\lambda_k\to 0$ *and*

$$0<\lim_{k\to\infty}\lambda_k^{\frac{2}{\sigma}}u_k(0)\le\lim_{k\to\infty}\lambda_k^{\frac{2}{\sigma}}|u_k|_\infty<\infty.$$

Using these theorems, we now may complete the proofs of our main results from Section 1.

Proof of Theorem 1.1 (completed). Fix $n\in\mathbb{N}\cup\{0\}$, and put $\mathcal{C}_n=\big\{\big(\lambda,\lambda^{\frac{2}{\sigma}}u(\lambda\,\cdot)\big) : (\lambda,u)\in\Gamma\big\}$. From Theorem 2.1 and (2.1) it follows that $\mathcal{C}_n$ is a subset of S_n which is connected in $(0,\infty)\times W^{1,p}(\mathbb{R}^N)$ for $1\le p\le\infty$. Moreover, $(0,\infty)\subset\{\lambda : (\lambda,u)\in\mathcal{C}_n\}$. Next suppose that (A_4) holds. Then Theorem 2.1(iii) and (2.4) imply that $(0,0)$ belongs to the closure of $\mathcal{C}_n$ in $[0,\infty)\times W^{1,p}(\mathbb{R}^N)$ provided that $\sigma<\frac{2p}{N}$. □

Proof of Theorem 1.2 (completed). Fix $n\in\mathbb{N}\cup\{0\}$, and put $\mathcal{C}_n=\big\{\big(\lambda,\lambda^{\frac{2}{\sigma}}u(\lambda\,\cdot)\big) : (\lambda,u)\in\Sigma\big\}$. From Theorem 2.2 and (2.1) it follows that $\mathcal{C}_n$ is a subset of S_n which is connected in $(0,\infty)\times W^{1,p}(\mathbb{R}^N)$ for $1\le p\le\infty$. Moreover, Theorem 2.2(ii) and (2.4) imply that $(0,0)$ belongs to the closure of $\mathcal{C}_n$ in $[0,\infty)\times W^{1,p}(\mathbb{R}^N)$ provided that $\sigma<\frac{2p}{N}$. Finally, either $(0,\infty)\subset\{\lambda : (\lambda,u)\in\mathcal{C}_n\}$, or $\mathcal{C}_n$ contains a sequence $(\lambda_k,u_k)_k$ such that $\lambda_k\to 0$ as $k\to\infty$ and

$$0<\lim_{k\to\infty}u_k(0)\le\lim_{k\to\infty}|u_k|_\infty<\infty.$$

In the second case, elliptic estimates show that, passing to a subsequence, we have $u_k\to u$ in $W^{2,s}_{loc}(\mathbb{R}^N)$ for all $s<\infty$ and $u_k\to u$ in $C^1_{loc}(\mathbb{R}^N)$, where u is a solution of (1.4). □

The rest of the paper is devoted to the proofs of Theorems 2.1 and 2.2. For convenience we set $q:=\sigma+2$. Moreover, for $r=|x|\ge 0$, we write $w_\lambda(r)$ instead of $w_\lambda(x)$.

3. The shooting method

We consider the singular initial-value problem

$$\begin{aligned} &-u'' - \frac{N-1}{r}u' = w_\lambda(r)|u|^\sigma u - u, \\ &u(0) = a, \qquad u'(0) = 0, \end{aligned} \tag{3.1}$$

where w_λ is defined in (2.2). We recall that $w'_\lambda(r) \le 0$ for $r = |x| \ge \lambda R_0$ as a consequence of (A_2). We will show in Lemma 3.1 below that (3.1) has a unique solution $u = u(\lambda, a;\,\cdot\,)$ for every $\lambda \ge 0$, $a \in \mathbb{R}$. By symmetry we may restrict our attention to initial values $a \ge 0$. It is useful to consider the quantity $E(\lambda, a;\,\cdot\,)$, the energy of $u = u(\lambda, a;\,\cdot\,)$, given by

$$E(\lambda, a; r) = \frac{1}{2}[(u'(r))^2 - u^2(r)] + \frac{1}{q}w_\lambda(r)|u(r)|^q, \qquad r \ge 0.$$

Since $q > 2$ and $\inf w > 0$ by (A_1), an easy calculation shows that there are constants $\kappa_1, \kappa_2 > 0$ (independent of λ, a) such that

$$|u(r)|^q \le \kappa_1 E(\lambda, a; r) + \kappa_2. \tag{3.2}$$

Lemma 3.1. (*i*) *For every $\lambda \ge 0$ and every $a \in \mathbb{R}$ the initial-value problem* (3.1) *has a unique solution $u = u(\lambda, a;\cdot)$. It is defined on the whole interval $[0,\infty)$ and it is bounded.*

(*ii*) *If $\lambda_k \ge 0$, $a_k \in \mathbb{R}$, $k \in \mathbb{N}$ are such that $\lambda_k \to \lambda_0$, $a_k \to a_0$, then*

$$u(\lambda_k, a_k;\cdot) \to u(\lambda_0, a_0;\cdot) \qquad \text{in} \quad C^1_{loc}([0,\infty)).$$

(*iii*) *The function $r \mapsto E(\lambda, a; r)$ is decreasing on the interval $[\lambda R_0, \infty)$.*

(*iv*) *There are constants $C_1, C_2 > 0$ (independent of λ, a) such that*

$$|u(\lambda, a; r)|^q \le C_1|u(\lambda, a; 0)|^q + C_2 \qquad \text{for} \quad r \ge 0.$$

Proof. Step 1: Local existence and continuous dependence of λ, a. We only consider the case $N \ge 3$; the proof for $N = 2$ is similar. We put

$$f(\lambda, r, t) = w_\lambda(r)|t|^\sigma t - t \qquad \text{for} \quad \lambda \ge 0,\ r \ge 0,\ t \in \mathbb{R},$$

and we note that u is a solution of (3.1) if and only if it satisfies the integral equation

$$u(r) = a - \frac{1}{N-2}\int_0^r \Big(1 - \Big(\frac{s}{r}\Big)^{N-2}\Big) s f(\lambda, s, u(s))\, ds, \qquad r \ge 0. \tag{3.3}$$

Obviously $(r,t) \to f(\lambda, r, t)$ is locally bounded and locally Lipschitz continuous in t. More precisely, for $c > 0$ there are constants $M_i = M_i(c) > 0$, $i =$

$1,2$ such that

$$|f(\lambda,r,t)| \le M_1 \qquad \text{for } \lambda \ge 0,\ ,r \ge 0,\ |t| \le c, \tag{3.4}$$

$$|f(\lambda,r,t) - f(\lambda,r,s)| \le M_2|t-s| \quad \text{for } \lambda \ge 0,\ ,r \ge 0,\ t,s \in [-c,c]. \tag{3.5}$$

We fix $a_0 \in \mathbb{R}$, and we let $M_1, M_2 > 0$ satisfy (3.4), (3.5) for $c := |a_0| + 2$. Moreover we fix ε such that

$$0 < \varepsilon < \min_{i=1,2} \Big(\frac{2(N-2)}{M_i}\Big)^{\frac{1}{2}}.$$

We put $K := \{u \in C([0,\varepsilon]) : |u - a_0|_\infty \le 2\}$. For $a \in \mathbb{R}$, $\lambda \ge 0$ we define integral operators $T_{\lambda,a} : K \to C([0,\varepsilon])$ by

$$(T_{\lambda,a}\, u)(r) = a - \frac{1}{N-2}\int_0^r \Big(1 - \Big(\frac{s}{r}\Big)^{N-2}\Big) s f(\lambda, s, u(s))\, ds$$

for $u \in K$, $r \in [0,\varepsilon]$. We note that for $|a - a_0| \le 1$, $u \in K$ and $r \in [0,\varepsilon]$ we have

$$\begin{aligned}
|(T_{\lambda,a}u)(r) - a_0| &\le |a - a_0| + \frac{1}{N-2}\int_0^r |\Big(1 - \Big(\frac{s}{r}\Big)^{N-2}\Big) s f(\lambda, s, u(s))|\, ds \\
&\le 1 + \frac{M_1}{N-2}\int_0^r \Big(1 - \Big(\frac{s}{r}\Big)^{N-2}\Big) s\, ds \\
&\le 1 + \frac{M_1}{N-2}\int_0^r s\, ds \le 1 + \frac{M_1\varepsilon^2}{2(N-2)} < 2,
\end{aligned}$$

hence,

$$T_{\lambda,a}(K) \subset K \qquad \text{for} \qquad |a - a_0| \le 1, \lambda \ge 0.$$

Moreover, we have for $u, v \in K$ and $r \in [0,\varepsilon]$

$$\begin{aligned}
&|(T_{\lambda,a}u)(r) - (T_{\lambda,a}v)(r)| \\
&\le \frac{1}{N-2}\int_0^r \Big(1 - \Big(\frac{s}{r}\Big)^{N-2}\Big) s |f(\lambda, s, u(s)) - f(\lambda, s, v(s))|\, ds \\
&\le \frac{M_2}{N-2}\int_0^r s|u(s) - v(s)|\, ds \le \frac{M_2\varepsilon^2}{2(N-2)}|u - v|_\infty,
\end{aligned}$$

where $\frac{M_2\varepsilon^2}{2(N-2)} < 1$ by our choice of ε. As a consequence, the map $T_{\lambda,a} : K \to K$ is a contraction for $\lambda \ge 0$, $|a - a_0| \le 1$, and hence it has a unique fixed point $u = u(\lambda, a\,;\cdot) \in K$. Now let $a_k \in [a_0 - 1, a_0 + 1]$ and $\lambda_k \ge 0$, $k \in \mathbb{N}$ be such that $a_k \to a_0$, $\lambda_k \to \lambda_0$. Then it is easy to see that the corresponding fixed points $u_k = u(\lambda_k, a_k\,;\cdot)$ form an equicontinuous sequence in K. Hence a standard argument based on the Arzela-Ascoli theorem shows that $u_k \to u_0 = u(\lambda_0, a_0; \cdot)$ in $C([0,\varepsilon])$. Moreover, using that

$$u_k'(r) = -\int_0^r \frac{s^{N-1}}{r^{N-1}} f(\lambda_k, s, u_k(s))\, ds \quad \text{and}$$
$$u_0'(r) = -\int_0^r \frac{s^{N-1}}{r^{N-1}} f(\lambda_k, s, u_0(s))\, ds,$$

we find that $u_k' \to u_0'$ in $C([0,\varepsilon])$.

Step 2: Global existence and boundedness We only consider $\lambda > 0$, $a \in \mathbb{R}$; the proof for $\lambda = 0$ is similar and in fact easier. We denote by R_{max} the maximal existence time of the solution $u = u(\lambda, a; \cdot)$. Then for $r \in [0, R_{max})$ we have

$$\begin{aligned}\frac{\partial}{\partial r} E(\lambda, a; r) &= u'(r)[u''(r) - u(r) + w_\lambda(r)|u(r)|^\sigma u(r)] + \frac{1}{q}\frac{\partial w_\lambda}{\partial r}(r)|u(r)|^q \\ &= -\frac{N-1}{r}(u'(r))^2 + \frac{1}{q\lambda} w'(\frac{r}{\lambda})|u(r)|^q \le \frac{1}{q\lambda} w'(\frac{r}{\lambda})|u(r)|^q.\end{aligned}$$

Moreover, by (3.2) and since $w'(s) \le 0$ for $s \ge R_0$, we have

$$\frac{1}{q\lambda} w'(\frac{r}{\lambda})|u(r)|^q \le \frac{1}{q\lambda}\Big(\max_{0\le s\le R_0} w'(s)\Big)|u(r)|^q \le \frac{c_1}{\lambda} E(\lambda, a; r) + \frac{c_2}{\lambda}.$$

Here and in the following, $c_1, c_2, c_3, \ldots$ are positive constants independent of λ, a. We deduce that

$$\frac{\partial}{\partial r}\Big(E(\lambda, a; r) e^{-\frac{c_1}{\lambda} r}\Big) \;\le\; \frac{c_2}{\lambda} e^{-\frac{c_1}{\lambda} r} \;\le\; \frac{c_2}{\lambda}$$

and therefore

$$E(\lambda, u; r) \le [E(\lambda, a; 0) + \frac{c_2}{\lambda} r] e^{\frac{c_1}{\lambda} r} \qquad \text{for} \quad r \le R_{\max}.$$

In particular, we have

$$E(\lambda, a; r) \;\le\; c_3 E(\lambda, a; 0) + c_4 \qquad \text{for} \quad r \le \min\{R_{\max}, \lambda R_0\}.$$

Combining this with the fact that

$$\frac{\partial}{\partial r} E(\lambda, a; r) = -\frac{N-1}{r}(u'(r))^2 + \frac{1}{q\lambda} w'(\frac{r}{\lambda})|u(r)|^q \le 0 \qquad \text{for} \quad r \ge \lambda R_0 \tag{3.6}$$

by (A_2), we conclude

$$E(\lambda, a; r) \;\le\; c_3 E(\lambda, a; 0) + c_4 \qquad \text{for} \quad r \le R_{\max}. \tag{3.7}$$

Now (3.2) and the definition of E imply that u and u' remain bounded on $[0, R_{\max})$, hence $R_{\max} = \infty$. Thus we have shown assertions (i) and (ii), whereas (iii) follows from (3.6).

To see (iv), we note that

$$E(\lambda, a; 0) \le \frac{w(0)}{q} |u(\lambda, a; 0)|^q.$$

Combining this with (3.2) and (3.7), we obtain (iv) with $C_1 := \frac{\kappa_1 c_3 w(0)}{q}$ and $C_2 := \kappa_1 c_4 + \kappa_2$. □

The next lemma is concerned with uniform bounds for the solution $u(\lambda, a, \cdot)$ depending on its number of zeros. Here we use the oscillation result proved in Section 5 below.

Lemma 3.2. *Let $n \in \mathbb{N} \cup \{0\}$.*

(i) *There exist $A = A(n) > 0$ and $B = B(n) > 0$ such that the solution $u(\lambda, a; \cdot)$ has more than n zeros provided that*

$$\lambda = 0, \qquad a \ge A \tag{3.8}$$

or

$$\lambda > 0, \qquad A \le a \le B\lambda^{-\frac{2}{\sigma}}. \tag{3.9}$$

(ii) *There is $C = C(n) > 0$ such that for $\lambda > 0$, $a \ge \max\{C, C\lambda^{-\frac{2}{\sigma}}\}$, the solution $u(\lambda, a; \cdot)$ has more than n zeros.*

(iii) *Suppose that (A_4) holds. Then there exists $D = D(n) > 0$ such that for $\lambda \ge 0$, $a \ge D$ the solution $u(\lambda, a; \cdot)$ has more than n zeros.*

Proof. We prove all three parts by contradiction. For this we consider $a_k, \lambda_k > 0,\ k \in \mathbb{N}$ such that

$$a_k \to \infty, \quad \text{and} \quad u_k := u(\lambda_k, a_k, \cdot) \text{ has at most } n \text{ zeros for every } k.$$

By Lemma 3.1(iv) we have

$$|u_k(r)|^q \le C_1 a_k^q + C_2 \qquad \text{for} \qquad r \ge 0. \tag{3.10}$$

We define $v_k \in C^1(\mathbb{R}^N)$ by $v_k(x) = a_k^{-1} u_k(a_k^{-\frac{\sigma}{2}} x)$. Thus $v_k(0) = 1$, and v_k is a solution of the equation

$$-\Delta v_k = w_k(x)|v_k|^\sigma v_k - a_k^{-\sigma} v_k, \tag{3.11}$$

with $w_k(x) = w(\frac{x}{\mu_k})$, $r \ge 0$ and $\mu_k := \lambda_k a_k^{\frac{\sigma}{2}}$ for $k \in \mathbb{N}$. Moreover, (3.10) implies that

$$|v_k(r)|^q \le C_1 + C_2 a_k^{-q} \qquad \text{for} \qquad r \ge 0,$$

hence the functions $v_k(\cdot)$, $k \in \mathbb{N}$ are uniformly bounded on $[0, \infty)$. Using this, (3.11), and elliptic estimates, we may pass to a subsequence such that $v_k \to v$ in $W^{2,s}_{loc}(\mathbb{R}^N)$ for all $s < \infty$, and hence $v_k \to v$ in $C^1_{loc}(\mathbb{R}^N)$. Obviously the function v has at most n zeros. We now distinguish three cases.

Case 1: $\mu_k \to 0$ **as** $k \to \infty$. In this case, $w_k(x) - w_\infty \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for every $s < \infty$, and hence v satisfies

$$-\Delta v = w_\infty |v|^\sigma v, \qquad v(0) = 1. \tag{3.12}$$

We may apply Theorem 5.2 to the constant function $w \equiv w_\infty$ which, together with $\sigma < \frac{4}{N-2}$, satisfies (A_4)(iii). However, Theorem 5.2 implies that every radial solution of (3.12) has infinitely many zeros, and this is a contradiction. Hence Case 1 does not occur. Consequently, there exist $A, B > 0$ such that, if (3.9) holds, then $u(\lambda, a; \cdot)$ has more than n zeros. A similar argument shows that, for A sufficiently large, (3.8) implies that $u(\lambda, a; \cdot)$ has more than n zeros. Thus assertion (i) is proved.

Case 2: $\mu_k \to \infty$ **as** $k \to \infty$. In this case, $w_k(r) \to w(0)$ uniformly on compact intervals, and hence v satisfies

$$-\Delta v = w(0)|v|^\sigma v, \qquad v(0) = 1. \tag{3.13}$$

Again we come to a contradiction, since every radial solution of (3.13) has infinitely many zeros by Theorem 5.2 (applied to the constant function $w \equiv w(0)$). Hence Case 2 cannot occur, and assertion (ii) follows.

Case 3: $\mu_k \to \mu > 0$ **as** $k \to \infty$. In this case the function v satisfies

$$-\Delta v = w(\frac{x}{\mu})|v|^\sigma v, \qquad v(0) = 1. \tag{3.14}$$

This however cannot be true if (A_4) holds for w. Indeed, then also $w(\frac{\cdot}{\mu})$ satisfies (A_4), and every radial solution of (3.14) would have infinitely many zeros by Theorem 5.2. From this contradiction we deduce assertion (iii). $\square$

Lemma 3.3. *Let $\lambda \geq 0$, $a > 0$, and put $u := u(\lambda, a; \cdot)$.*

- (i) *If $\xi > 0$ satisfies $u'(\xi) = 0$ and $u(\xi)u''(\xi) \leq 0$, then $|u(\xi)| \geq \left(\frac{1}{w_\lambda(\xi)}\right)^{\frac{1}{\sigma}}$. In particular this holds for any local maximum point ξ of $|u|$.*
- (ii) *If $E(\lambda, a; y) \geq 0$ for some $y > \lambda R_0$, then u neither has a positive local minimum nor a negative local maximum in the interval $(\lambda R_0, y)$. In particular this holds if $y > \lambda R_0$ is a zero of u.*
- (iii) *If $\lim_{r \to \infty} u(r) = 0$, then u neither has a positive local minimum nor a negative local maximum in the interval $(\lambda R_0, \infty)$.*

(iv) *If* $y_2 > y_1 \geq \lambda R_0$ *are consecutive zeros of* u*, then*

$$\max_{y_1<r<y_2} |u(r)| \geq \Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}}.$$

(v) *If* u *has a largest zero* $\bar{y} \geq \lambda R_0$*, then* $\sup_{r>\bar{y}} |u(r)| \geq \big(\frac{1}{|w|_\infty}\big)^{\frac{1}{\sigma}}$.

Proof. (i) Let $\xi \geq \lambda R_0$. If ξ satisfies $u(\xi) > 0$, $u'(\xi) = 0$, and $u''(\xi) \leq 0$, then

$$w_\lambda(\xi)u(\xi)^{\sigma+1} - u(\xi) = -u''(\xi) \geq 0$$

by (3.1), hence $u(\xi) \geq \big(\frac{1}{w_\lambda(\xi)}\big)^{\frac{1}{\sigma}}$. Similarly, if $u(\xi) < 0$, $u'(\xi) = 0$ and $u''(\xi) \geq 0$, then $u(\xi) \leq -\big(\frac{1}{w_\lambda(\xi)}\big)^{\frac{1}{\sigma}}$.

(ii) Suppose by contradiction that $\xi \in (\lambda R_0, y)$ satisfies $u(\xi) > 0$, $u'(\xi) = 0$, and $u''(\xi) \geq 0$. Then (3.1) yields

$$w_\lambda(\xi)u(\xi)^{\sigma+1} - u(\xi) = -u''(\xi) \leq 0$$

and hence

$$E(\lambda, a; \xi) = \frac{1}{q}u^q(\xi) - \frac{1}{2}u^2(\xi) \leq \Big(\frac{1}{q} - \frac{1}{2}\Big)u^2(\xi) < 0 \leq E(\lambda, a; y),$$

contrary to Lemma 3.1(iii).

(iii) follows from (ii), since $\lim_{r\to\infty} u(r) = 0$ implies $E(\lambda, a; y) \geq 0$ for $y \geq \lambda R_0$ by Lemma 3.1(iii).

(iv) follows directly from (i).

(v) We suppose that u is positive on $(\bar{y}, \infty)$. If u' has a zero in $(\bar{y}, \infty)$, then we are done. Indeed, in this case the smallest zero of u' in $(\bar{y}, \infty)$ satisfies the assumptions of (i). Hence we may assume that $u'(r) > 0$ on $(\bar{y}, \infty)$. Since u is bounded, there exists

$$u_\infty := \lim_{r\to\infty} u(r).$$

Moreover, $E(\lambda, a; \cdot)$ is bounded from below on $(\lambda R_0, \infty)$ and decreasing by Lemma 3.1(iii), hence

$$e_\infty := \lim_{r\to\infty} u(r)$$

exists as well. Consequently, the limit

$$\lim_{r\to\infty} (u'(r))^2 = 2\Big(e_\infty - w_\infty \frac{|u_\infty|^q}{q} - \frac{u_\infty^2}{2}\Big)$$

exists, and this limit must be zero. Using (3.1) we find

$$0 = \lim_{r\to\infty} u''(r) = w_\infty u_\infty^{\sigma+1} - u_\infty.$$

Hence $u_\infty = \left(\frac{1}{w_\infty}\right)^{\frac{1}{\sigma}} \geq \left(\frac{1}{|w|_\infty}\right)^{\frac{1}{\sigma}}$, which shows the assertion. □

Lemma 3.4. *Let $\lambda \geq 0$, $a > 0$ and put $u := u(\lambda, a; \cdot)$. Then the following are equivalent*:

(i) *$u \in \mathcal{E}_s$ for any $0 < s < 1$.*
(ii) *$u \in \mathcal{E}_s$ for some $s > 0$.*
(iii) *u has finitely many zeros, and $E(\lambda, a; r) \geq 0$ for $r \geq \lambda R_0$.*

Proof. Trivially, (i) implies (ii). If (ii) is satisfied, then u has finitely many zeros by Lemma 3.3(iv). Moreover, $E(\lambda, a; r) \to 0$ as $r \to \infty$, and thus Lemma 3.1(iii) implies that $E(\lambda, a; r) \geq 0$ for $r \geq \lambda R_0$.

Now suppose that (iii) holds. By Lemma 3.3(ii), there is $R > 0$ such that $u|_{[R,\infty)}$ is monotone and of one sign. Hence both limits

$$u_\infty = \lim_{r\to\infty} u(r), \qquad e_\infty = \lim_{r\to\infty} E(\lambda, a; r)$$

exist. Moreover, $e_\infty \geq 0$ by assumption.

Now, as in the proof of Lemma 3.3(v), we have

$$0 = \frac{1}{2} \lim_{r\to\infty} (u'(r))^2 = e_\infty - w_\infty \frac{|u_\infty|^q}{q} - \frac{u_\infty^2}{2},$$

and $0 = \lim_{r\to\infty} u''(r) = w_\infty u_\infty^{\sigma+1} - u_\infty$. Hence $u_\infty = 0$ or $w_\infty u_\infty^\sigma = 1$. Since

$$u_\infty^2 \Big(\frac{w_\infty |u_\infty|^\sigma}{q} - \frac{1}{2}\Big) = e_\infty \geq 0,$$

we conclude $u_\infty = 0$. Consequently,

$$V(r) := w_\lambda(r)|u(r)|^\sigma \to 0 \qquad \text{as} \quad r \to \infty.$$

Since u is a bounded solution of the equation $\Delta u + [V(r) - 1]u = 0$, [24, Proposition 4.4] implies that $\lim_{r\to\infty} u(r)e^{sr} = 0$ for $s < 1$. Then standard elliptic estimates show that $\lim_{r\to\infty} u'(r)e^{sr} = 0$. Hence $u \in \mathcal{E}_s$ for $s < 1$. □

Next we study how the number and position of zeros of $u(\lambda, a; \cdot)$ depend locally on the parameter λ and initial value a. This is the most difficult part.

Lemma 3.5. *There is a neighborhood $\mathcal{N}$ of $\{(\lambda, 0) : \lambda \geq 0\}$ in $[0,\infty)^2$ such that for $(\lambda, a) \in \mathcal{N}$, $a > 0$ the function $u(\lambda, a; \cdot)$ remains positive on $[0,\infty)$.*

Proof. We suppose by contradiction that there are $\lambda_k \geq 0$, $a_k > 0$, $k \in \mathbb{N}$ such that

$$\lambda_k \to \lambda \geq 0, \quad a_k \to 0 \qquad \text{as} \quad k \to \infty,$$

and such that every $u_k := u(\lambda_k, a_k; \cdot)$ has a zero in $(0,\infty)$. We may assume that

$$u_k(0) = a_k < \Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}} \qquad \text{for every } k.$$

Hence

$$u_k'(r) = \frac{1}{r^{N-1}} \int_0^r s^{N-1}\Big(1 - u_k(s)^\sigma w_{\lambda_k}(s)\Big) u_k(s)\, ds \;>\; 0$$

for every k and $r > 0$ close to zero. We put

$$\xi_k := \min\{r > 0 : u_k'(r) = 0\} \in (0,\infty), \qquad k \in \mathbb{N},$$

and we note that u_k is strictly increasing on $(0, \xi_k)$. By Lemma 3.3(i) we have

$$u_k(\xi_k) \geq \Big(\frac{1}{w_{\lambda_k}(\xi_k)}\Big)^{\frac{1}{\sigma}} \geq \Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}}.$$

We pick numbers c_1, c_2 such that

$$0 < c_1 < c_2 < \Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}},$$

and we fix $k_0 \in \mathbb{N}$ such that $u_k(0) = a_k < c_1$ for $k \geq k_0$. Then, since u_k is strictly increasing on $(0, \xi_k)$, there exist unique points $0 < s_k < r_k < \xi_k$ such that $u(s_k) = c_1$ and $u(r_k) = c_2$ for $k \geq k_0$. We note that $s_k, r_k \to \infty$ as $k \to \infty$, since $u_k \to 0$ in C^1_{loc} by Lemma 3.1(ii). Hence there is $k_1 \geq k_0$ such that

$$s_k > \lambda_k R_0 \qquad \text{for } k \geq k_1. \tag{3.15}$$

We now put $E_k := E(\lambda_k, a_k; \cdot)$. By our choice of c_2 we have

$$E_k(r) - \frac{1}{2} {u_k'}^2(r) \leq \Big(\frac{1}{q}|w|_\infty u_k^\sigma - \frac{1}{2}\Big) u_k^2 \leq -c_3 \quad \text{for} \quad r \in (\lambda_k R_0, r_k),\ k \geq k_1, \tag{3.16}$$

where $c_3 := \big(\frac{1}{2} - \frac{(c_2)^\sigma |w|_\infty}{q}\big) > 0$. We claim that

$$E_k(r_k) < 0 \qquad \text{for } k \text{ sufficiently large.} \tag{3.17}$$

Indeed, suppose by contradiction that, passing to a subsequence, we have $E_k(r_k) \geq 0$ for every k. Then E_k is nonnegative on $[\lambda_k R_0, r_k]$ by monotonicity. Hence $u_k'(r) \geq (2c_3)^{\frac{1}{2}}$ on $[s_k, r_k]$ by (3.16), which yields $r_k - s_k \leq (c_2 - c_1)(2c_3)^{-\frac{1}{2}}$. Using this, (3.16), and the estimate

$$E_k'(r) \leq -\frac{N-1}{r} {u_k'}^2(r) \qquad \text{for} \quad r \geq \lambda_k R_0 \tag{3.18}$$

(cf. (3.6)), we obtain

$$\begin{aligned} s_k E_k(s_k) &\geq s_k E_k(s_k) - r_k E_k(r_k) = -\int_{s_k}^{r_k} [E_k(r) + rE_k'(r)]\,dr \\ &\geq \frac{2N-3}{2}\int_{s_k}^{r_k} (u_k'(r))^2\,dr \geq \frac{2N-3}{2(r_k - s_k)}\Big(\int_{s_k}^{r_k} u_k'(r)\,dr\Big)^2 \\ &= \frac{2N-3}{2(r_k - s_k)}(u_k(r_k) - u_k(s_k))^2 = \frac{2N-3}{2(r_k - s_k)}(c_2 - c_1)^2 \\ &\geq (2N-3)(2c_3)^{\frac{1}{2}}(c_2 - c_1) =: c_4 > 0 \end{aligned} \tag{3.19}$$

On the other hand, we have

$$\begin{aligned} c_1 &\geq \int_{\lambda_k R_0}^{s_k} u_k'(r) \geq (s_k - \lambda_k R_0) \min_{\lambda_k R_0 \leq r \leq s_k} u_k'(r) \\ &\geq 2(s_k - \lambda_k R_0) \min_{\lambda_k R_0 \leq r \leq s_k} \sqrt{E_k(r)} \geq 2(s_k - \lambda_k R_0)\sqrt{E_k(s_k)}. \end{aligned} \tag{3.20}$$

Here we used (3.16) again. Combining (3.19) and (3.20) we find

$$s_k \geq \frac{4c_4}{c_1^2}(s_k - \lambda_k R_0)^2,$$

contrary to the fact that $s_k \to \infty$ as $k \to \infty$. Thus (3.17) is proved. However, (3.17) contradicts our assumption that u_k has a zero in (r_k, ∞) for every k. The proof is finished. □

Lemma 3.6. *Let $\lambda \geq 0, a > 0$ be such that $u(\lambda, a, \cdot) \in \mathcal{E}_s$ for some $0 < s < 1$. Then there exists $R = R(\lambda, a)$ and a neighborhood N of (λ, a) in $[0, \infty)^2$ with the following property:*

If $(\tilde{\lambda}, \tilde{a}) \in N$, then the function $\tilde{u} = u(\tilde{\lambda}, \tilde{a}, \cdot)$ has at most one zero in $[R, \infty)$. Moreover, if $\tilde{u} \in \mathcal{E}_s$, then $\tilde{u}$ has no zero in $[R, \infty)$

Proof. We put $u := u(\lambda, a, \cdot) \in \mathcal{E}_s$. By Lemma 3.4, u has finitely many zeros $0 < y_1 < ... < y_n < \infty$. Let $R := y_n + 1$. Without loss we may assume that u is negative on the interval (y_n, ∞). Hence we may fix $\tilde{R} > \max\{y_n, \lambda R_0\}$ such that

$$-\Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}} < u(\tilde{R}) < 0 \qquad \text{and} \qquad u'(\tilde{R}) > 0. \tag{3.21}$$

Now we let $a_k > 0$, $\lambda_k \geq 0$, $k \in \mathbb{N}$ be such that $a_k \to a$, $\lambda_k \to \lambda$, and we show the following:

$$\begin{aligned}&\text{For } k \text{ sufficiently large, } u_k := u(\lambda_k, a_k; \cdot) \text{ has at most one zero in } (R,\infty).\\ &\text{Moreover, if (for } k \text{ large) } u_k \text{ has one zero in } (R,\infty)\text{, then } u_k \notin \mathcal{E}_s.\end{aligned} \tag{3.22}$$

Clearly u_k has at least n zeros for k large, and

$$y_{j,k} \to y_j \qquad \text{as} \quad k \to \infty \qquad (j = 1, ..., n), \tag{3.23}$$

where $y_{1,k} < ... < y_{n,k}$ denote the first n zeros of u_k. We now suppose that, passing to a subsequence, u_k has at least one more zero $y_{n+1,k} > y_{n,k}$ for every k. Since $\lambda_k \to \lambda$, we have

$$\tilde{R} > \lambda_k R_0 \qquad \text{for } k \text{ large}. \tag{3.24}$$

Moreover, since

$$y_{n+1,k} \to \infty \qquad \text{as} \quad k \to \infty,$$

we also have

$$y_{n,k} < \tilde{R} < y_{n+1,k} \qquad \text{for } k \text{ large}. \tag{3.25}$$

Finally, we have

$$-\Big(\frac{1}{|w|_\infty}\Big)^{\frac{1}{\sigma}} < u_k(\tilde{R}) < 0 \qquad \text{and} \qquad u_k'(\tilde{R}) > 0 \qquad \text{for } k \text{ large}. \tag{3.26}$$

We now proceed similarly as in the proof of Lemma 3.5. We pick numbers c_1, c_2 such that $0 < c_1 < c_2 < \big(\frac{1}{|w|_\infty}\big)^{\frac{1}{\sigma}}$. As a consequence of Lemma 3.3, there exist unique points $r_k > s_k > y_{n+1,k}$ such that u_k is increasing on $(\tilde{R}, r_k)$ and

$$u(s_k) = c_1, \qquad u(r_k) = c_2, \qquad k \in \mathbb{N}. \tag{3.27}$$

Note that $s_k, r_k \to \infty$, since $u_k \to u$ in C^1_{loc}. We put $E_k := E(\lambda_k, a_k; \cdot)$, and we find that

$$E_k(r) - \frac{1}{2}{u_k'}^2(r) \leq \Big(\frac{1}{q}|w|_\infty u_k^\sigma - \frac{1}{2}\Big)u_k^2 \leq -c_3 \ \text{ for } \ r \in (\tilde{R}, r_k), \ k \geq k_1, \tag{3.28}$$

where $c_3 > 0$ is a constant. We claim that

$$E_k(r_k) < 0 \qquad \text{for } k \text{ large enough}. \tag{3.29}$$

Indeed, suppose by contradiction that, passing to a subsequence, we have $E_k(r_k) \geq 0$ for every k. Then E_k is positive on $(\tilde{R}, \infty)$ by (3.24) and the monotonicity of E_k. Hence $u_k'(r) \geq (2c_3)^{\frac{1}{2}}$ on $[s_k, r_k]$ by (3.16), which yields $r_k - s_k \leq (c_2 - c_1)(2c_3)^{-\frac{1}{2}}$. Using this, (3.18), and (3.28), we obtain

$$s_k E_k(s_k) \geq c_4 > 0 \tag{3.30}$$

by precisely the same argument as in the proof of Lemma 3.5. From (3.28) we also deduce

$$\begin{aligned} c_1 + 1 &\geq \int_{\tilde{R}}^{s_k} u_k'(r) \geq (s_k - \tilde{R}) \min_{\tilde{R} \leq r \leq s_k} u_k'(r) \\ &\geq 2(s_k - \tilde{R}) \min_{\tilde{R} \leq r \leq s_k} \sqrt{E_k(r)} \geq 2(s_k - \tilde{R})\sqrt{E_k(s_k)}. \end{aligned} \tag{3.31}$$

Combining (3.19) and (3.31) we find

$$s_k \geq \frac{4c_4}{(c_1+1)^2}(s_k - \tilde{R})^2$$

which contradicts the fact that $s_k \to \infty$ as $k \to \infty$. Thus (3.29) is proved. Consequently, u_k has no zero in (r_k, ∞) for k large, and $u_k \notin \mathcal{E}_s$ by Lemma 3.4. This proves (3.22), as required. □

We close this section with two important corollaries of Lemma 3.6.

Corollary 3.7. *Let $\lambda \geq 0$, $a > 0$, and $0 < s < 1$ be such that $u(\lambda, a, \cdot\,) \in \mathcal{E}_s$ has n zeros. Then for $(\tilde{\lambda}, \tilde{a})$ close to (λ, a) the function $u(\tilde{\lambda}, \tilde{a}, \cdot\,)$ has at most $n+1$ zeros.*

Proof. This follows easily from Lemma 3.1(ii) and Lemma 3.6. □

Corollary 3.8. *Let $\lambda \geq 0, a > 0$ and $0 < s < 1$ be such that $u = u(\lambda, a, \cdot\,) \in \mathcal{E}_s$. Moreover, let $a_k > 0$, $\lambda_k \geq 0$, $k \in \mathbb{N}$ be such that $a_k \to a$, $\lambda_k \to \lambda$ and*

$$u_k := u(\lambda_k, a_k; \cdot\,) \in \mathcal{E}_s \qquad \text{for all} \quad k \in \mathbb{N}.$$

Then $\|u_k - u\|_s \to 0$.

Proof. From the assumption and Lemma 3.6 we infer that there is $R > 0$ such that u_k has no zeros in $[R, \infty)$ for every k. Without loss we may assume that $R > \lambda_k R_0$ for every k, and that u_k is positive on $[R, \infty)$. We pick $\tilde{R} > R$ such that $u'(\tilde{R}) < 0$ and $[u(\tilde{R})]^\sigma < \frac{1-s}{2|w|_\infty}$. Then also $u_k'(\tilde{R}) < 0$ and $[u_k(\tilde{R})]^\sigma < \frac{1-s}{2|w|_\infty}$ for k sufficiently large. Hence Lemma 3.3(iii) implies that u_k is positive and decreasing on $[\tilde{R}, \infty)$ for k large enough. Consequently, the function $f_k = 1 - w_{\lambda_k} u_k^\sigma$ satisfies

$$f_k(r) \geq \frac{1+s}{2} \qquad \text{for } r \geq \tilde{R} \text{ and } k \text{ sufficiently large.}$$

Since $s' := \frac{1+s}{2} < \sqrt{\frac{1+s}{2}}$ and u_k is a solution of the equation

$$-\Delta u_k + f_k(r) u_k = 0,$$

we can proceed as in the proof of [24, Proposition 4.4] to deduce *uniform* decay estimates of the form

$$|u_k(r)| \leq c_1 e^{-s' r}, \qquad |u_k'(r)| \leq c_2 e^{-s' r}$$

for $r \geq \tilde{R}$ and k sufficiently large. Here c_1, c_2 are positive constants. Since $s < s'$, we find that

$$e^{s r} \max\{|u_k(r)|, |u_k'(r)|\} \to 0 \qquad \text{as} \quad r \to \infty$$

uniformly in k. Combining this with the fact that $u_k \to u$ in C^1_{loc}, we conclude that $\|u_k - u\|_s \to 0$. □

4. Proof of the main Theorems

In this section we complete the proofs of Theorems 2.1 and 2.2. We fix $n \in \mathbb{N} \cup \{0\}$, and we let $A, B, C, D > 0$ be the constants (depending on n) given by Lemma 3.2. We consider the set

$$\mathcal{B} := \{(\lambda, a) \in [0, \infty) \times (0, \infty) : u(\lambda, a; \cdot) \text{ has at least } n+1 \text{ zeros}\}. \tag{4.1}$$

From Lemma 3.1(ii) we deduce that $\mathcal{B}$ is open in $[0, \infty)^2$. Moreover we have

$$\overline{\mathcal{B}} \cap \{(\lambda, 0) : \lambda \geq 0\} = \varnothing \tag{4.2}$$

by Lemma 3.5. The following observation is crucial.

Proposition 4.1. *If $(\lambda, a) \in \partial\mathcal{B}$, then $u(\lambda, a; \cdot) \in \mathcal{E}_s$ for $0 < s < 1$, and u has precisely n zeros.*

Proof. Let $(\lambda, a) \in \partial\mathcal{B}$. Since $\mathcal{B}$ is open, $u(\lambda, a; \cdot)$ has at most n zeros. Moreover,

$$E(\lambda, a, r) \geq 0 \qquad \text{for} \quad r \geq \lambda R_0. \tag{4.3}$$

Indeed, if $(\lambda_k, a_k)_k \subset \mathcal{B}$ is a sequence with $\lambda_k \to \lambda$, $a_k \to a$ and x_k is the $(n+1)$-st zero of $u(\lambda_k, a_k \cdot)$, then $x_k \to \infty$ as $k \to \infty$. Since $E(\lambda_k, a_k, r) \geq 0$ for $r \in [\lambda_k R_0, x_k)$, (4.3) follows.

Now Lemma 3.4 yields that $u(\lambda, a; \cdot) \in \mathcal{E}_s$ for $s < 1$. Moreover, Corollary 3.7 implies that $u(\lambda, a; \cdot)$ has at least n zeros, as required. □

Lemma 4.2. *There exist unbounded connected subsets Γ_*, Σ_* of $\partial\mathcal{B} \setminus \{(0, a) : a \geq 0\}$ such that*

(i) $(0, \infty) \subset \{\lambda : (\lambda, a) \in \Gamma_*\}$.

(ii) *Either the closure of* Γ_* *in* $[0,\infty)^2$ *intersects the set* $\{(0,a) : 0 < a < A\}$*, or* Γ_* *contains a sequence* $(\lambda_k, a_k)_k$ *such that* $\lambda_k \to 0$ *and* $0 < \lim \lambda_k^{\frac{2}{\sigma}} a_k < \infty$.

(iii) *The closure of* Σ_* *in* $[0,\infty)^2$ *intersects the set* $\{(0,a) : 0 < a < A\}$.

(iv) *Either* $(0,\infty) \subset \{\lambda : (\lambda, a) \in \Sigma_*\}$*, or* Σ_* *contains a sequence* $(\lambda_k, a_k)_k$ *such that* $\lambda_k \to 0$ *and* $0 < \lim \lambda_k^{\frac{2}{\sigma}} a_k < \infty$.

Remark 4.3. If (A_4) holds, then the set $\{a : (\lambda, a) \in \partial B\}$ is bounded by Lemma 3.2(iii). Consequently, the first alternative holds in parts (ii) and (iv) of Lemma 4.2.

Proof. We define a continuous map $\Phi : [0,\infty)^2 \to \mathbb{R}$ by

$$\Phi(\lambda, a) = \begin{cases} \operatorname{dist}((\lambda, a), \partial\mathcal{B}) & (\lambda, a) \in \overline{\mathcal{B}}, \\ -\operatorname{dist}((\lambda, a), \partial\mathcal{B}) & (\lambda, a) \in [0,\infty)^2 \setminus \overline{\mathcal{B}}. \end{cases}$$

Moreover, we define $g : (0,\infty) \times [0,1] \to \mathbb{R}^2$ by

$$g(\lambda, s) = (\lambda, s \max\{C, C\lambda^{-\frac{2}{\sigma}}\}).$$

For every $\lambda \in (0,\infty)$ we have $(\Phi \circ g)(\lambda, 0) < 0$ by Lemma 3.5 and $(\Phi \circ g)(\lambda, 1) > 0$ by Lemma 3.2(ii), hence

$$\deg\Big((\Phi \circ g)(\lambda, \cdot), [0,1], 0\Big) = 1.$$

Now a standard argument based on the Leray-Schauder continuation principle (see e.g. [30, Theorem 14.C]) implies that there exists a connected set $\Gamma_0 \subset (0,\infty) \times (0,1)$ such that

$$g(\lambda, s) \in \Phi^{-1}(0) = \partial\mathcal{B} \qquad \text{for all } (\lambda, s) \in \Gamma_0,$$
$$(0,\infty) \subset \{\lambda : (\lambda, s) \in \Gamma_0\}.$$

Hence $\Gamma_* := g(\Gamma_0)$ has property (i). Moreover, (ii) follows immediately from Lemma 3.2(i).

For the construction of Σ_* we consider $h : (0,\infty) \times [0,1] \to \mathbb{R}^2$ defined by

$$h(t, s) = (1 - s) \cdot (t, 0) + s \cdot (0, A + t)$$

For every $t \in (0,\infty)$ we have $(\Phi \circ h)(t, 0) < 0$ by Lemma 3.5 and $(\Phi \circ h)(t, 1) > 0$ by Lemma 3.2(i), hence

$$\deg\Big((\Phi \circ h)(t, \cdot), [0,1], 0\Big) = 1.$$

Again a standard argument based on the Leray-Schauder continuation principle yields the existence of a connected set $\Sigma_0 \subset (0,\infty)\times(0,1)$ such that

$$h(t,s) \in \Phi^{-1}(0) = \partial\mathcal{B} \qquad \text{for all } (t,s)\in\Sigma_0,$$
$$(0,\infty) \subset \{t : (t,s)\in\Sigma_0\}.$$

We put $\Sigma_* = h(\Sigma_0)$. Then Σ_* has property (iii) by construction, and (iv) is a direct consequence of Lemma 3.2. □

Proof of Theorem 2.1 (completed). We put

$$\Gamma := \Big\{\big(\lambda, u(\lambda,a;\cdot)\big) : (\lambda,a)\in\Gamma_*\Big\} \subset (0,\infty)\times\mathcal{E}_s,$$

where Γ_* is given by Lemma 4.2. Then Γ is connected by Corollary 3.8. The remaining assertions now follow from Proposition 4.1, Lemma 4.2, and Remark 4.3. □

Proof of Theorem 2.2 (completed). We put

$$\Sigma := \Big\{\big(\lambda, u(\lambda,a;\cdot)\big) : (\lambda,a)\in\Sigma_*\Big\} \subset (0,\infty)\times\mathcal{E}_s,$$

where Σ_* is given by Lemma 4.2. Then Σ is connected by Corollary 3.8. The remaining assertions now follow Proposition 4.1 and Lemma 4.2. Note here that, if Σ contains a sequence (λ_k,u_k) such that $\lambda_k\to 0$ and

$$0 < \lim_{k\to\infty} \lambda_k^{\frac{2}{\sigma}} u_k(0) < \infty,$$

then $\lambda_k^{\frac{2}{\sigma}}|u_k|_\infty$ remains bounded as $k\to\infty$ by Lemma 3.1(iv). □

5. Appendix: An oscillation result

In this section we study radial solutions of the equation

$$-\Delta u = w(|x|)|u|^\sigma u, \tag{5.1}$$

and throughout this section we assume that $N\geq 2$, $\sigma>0$, and that w is a positive function. As before, we let $q=\sigma+2$. Some of the following arguments are inspired by Toland [28, Section 3] and by Grillakis [9, Appendix]. Note however that in [28] only positive solutions are considered, and in [9] it is assumed that w is constant. We start with a simple observation.

Lemma 5.1. *Suppose that $w\geq 0$. If u is a radial pointwise solution of* (5.1) *with finitely many zeros, then*

$$\liminf_{r\to\infty} |u(r)|r^{N-2} > 0 \qquad \textit{if} \quad N\geq 3, \tag{5.2}$$

and

$$\liminf_{r\to\infty} |u(r)|\ln(r) > 0 \qquad \textit{if} \quad N=2. \tag{5.3}$$

Proof. Without loss we assume that u is positive on $[r_0,\infty)$ for some $r_0>0$. We denote by ρ the fundamental solution of $-\Delta$, and we choose $c_0>0$ such that

$$u(r_0)-c_0\rho(r_0)\ge 0.$$

Since

$$-\Delta(u-c_0\rho)=w(|x|)|u|^\sigma u\ge 0 \qquad \text{on} \quad \mathbb{R}^N\setminus B_{r_0}(0), \tag{5.4}$$

the function $u-c_0\rho$ has no interior local minima in $\mathbb{R}^N\setminus B_{r_0}(0)$ unless it is constant. Noting moreover that $\liminf_{r\to\infty}(u-c_0\rho)(r)\ge 0$, we deduce that

$$u(r)\ge c_0\rho(r) \qquad \text{for} \quad r\in(r_0,\infty).$$

This gives (5.2) and (5.3). □

Theorem 5.2. *Suppose that* (A_1), (A_2), *and* (A_4) *hold. Then every radial solution of* (5.1) *has infinitely many zeros.*

Remark 5.3. An example given by Toland [28, page 348] shows that the bound on σ in (A_4) is sharp for this result.

Proof of Theorem 5.2. As before we let $w_\infty=\lim_{r\to\infty} w(r)>0$. We assume by contradiction that there is a nontrivial radial solution $u=u(r)$ of (5.1) having only finitely many zeros. We denote by $r_0>0$ the last zero of u. We may assume that u is positive on (r_0,∞). Note that $u=u(r)$ satisfies

$$-u''(r)-\frac{N-1}{r}u'(r)=w(r)|u|^\sigma u, \qquad r>0, \tag{5.5}$$

and from this we deduce that u neither has a local minimum nor a degenerate critical point in (r_0,∞). Hence u' changes sign at most once in (r_0,∞), and thus $u_\infty=\lim\limits_{r\to\infty}u(r)\in[0,\infty]$ exists. Moreover, by (A_2) we have $w'(r)\le 0$ on $[R_0,\infty)$. From this and (5.5) it follows that the function

$$r\to E(r):=\frac{1}{2}u'(r)^2+w(r)\frac{|u|^q}{q}$$

is decreasing on $[R_0,\infty)$. Hence u remains bounded, and consequently $u_\infty<\infty$. Let $e_\infty:=\lim\limits_{r\to\infty}E(r)$. Then the limit

$$\lim_{r\to\infty}(u'(r))^2=2\Big(e_\infty-w_\infty\frac{|u_\infty|^q}{q}\Big)$$

exists, and this limit must be zero. Now (5.5) implies

$$\lim_{r\to\infty} u''(r) = -w_\infty u_\infty^{\sigma+1},$$

and this forces $u_\infty = 0$. Hence there exists precisely one local maximum point $\hat{r}$ of u in the interval (r_0, ∞). In what follows, we let $c_0, c_1, ...,$ denote various positive constants. For $r > \hat{r}$ we have

$$\begin{aligned} u'(r) &= -\frac{1}{r^{N-1}} \int_{\hat{r}}^{r} w(s)u(s)^{\sigma+1} s^{N-1}\,ds \\ &\le -c_0 \frac{1}{r^{N-1}} u(r)^{\sigma+1} \int_{\hat{r}}^{r} s^{N-1}\,ds \qquad (5.6)\\ &= -c_1 u(r)^{\sigma+1}\Big(r - \frac{\hat{r}^N}{r^{N-1}}\Big). \qquad (5.7) \end{aligned}$$

Hence

$$\frac{\partial}{\partial r} u^{-\sigma}(r) = -\sigma u^{-(\sigma+1)}(r)u'(r) \ge c_2\Big(r - \frac{\hat{r}^N}{r^{N-1}}\Big) \qquad \text{for} \quad r > \hat{r}.$$

Consequently, we get for $N \ge 3$

$$\begin{aligned} u^{-\sigma}(r) &= u^{-\sigma}(\hat{r}) + \int_{\hat{r}}^{r} \frac{\partial}{\partial s} u^{-(\sigma)}(s)\,ds \\ &\ge c_3 + c_2 \int_{\hat{r}}^{r} \Big(s - \frac{\hat{r}^N}{s^{N-1}}\Big)\,ds = c_4 + c_2\Big(\frac{r^2}{2} + \frac{\hat{r}^N}{(N-2)r^{N-2}}\Big), \end{aligned}$$

while for $N = 2$ we obtain

$$u^{-\sigma}(r) \ge c_4 + c_2(\frac{r^2}{2} - \hat{r}^2 \log r).$$

In both cases we deduce

$$u(r) = O(r^{-\frac{2}{\sigma}}) \qquad \text{as} \quad r \to \infty. \qquad (5.8)$$

In case $N = 2$ this contradicts (5.3). Moreover, if $N \ge 3$ and $\sigma < \frac{2}{N-2}$, then (5.8) contradicts (5.2). Next we assume that $N \ge 3$ and $\sigma = \frac{2}{N-2}$. Then

$$0 < \liminf_{r\to\infty} u(r)r^{N-2} \le \limsup_{r\to\infty} u(r)r^{N-2} < \infty \qquad (5.9)$$

by (5.2) and (5.8). Using this, we obtain for $r > \hat{r}$

$$\begin{aligned} u'(r) &= -\frac{1}{r^{N-1}} \int_{\hat{r}}^{r} w(s)u(s)^{\sigma+1} s^{N-1}\,ds \\ &\le -\frac{c_5}{r^{N-1}} \int_{\hat{r}}^{r} s^{N-1-(N-2)(\sigma+1)}\,ds = -\frac{c_5}{r^{N-1}}(\ln(r) - \ln(\hat{r})), \end{aligned}$$

and hence

$$\begin{aligned} u(r)r^{N-2} &= -r^{N-2}\int_r^\infty u'(r)\,dr \geq c_5 r^{N-2}\int_r^\infty \frac{1}{s^{N-1}}(\ln(s)-\ln(\hat r))\,ds \\ &\geq c_5 r^{N-2}(\ln(r)-\ln(\hat r))\int_r^\infty \frac{1}{s^{N-1}}\,ds \\ &\geq c_5(N-2)(\ln(r)-\ln(\hat r)) \;\to\; \infty \qquad \text{as} \quad r\to\infty, \end{aligned}$$

which contradicts (5.9).

Finally, we assume (A_4)(iii). Thus, $N \geq 3$, $0<\sigma<\frac{4}{N-2}$, and the function $r \mapsto w(r)$ is nondecreasing on $[0,\infty)$. Hence (5.8) implies

$$|u(r)|^q = O(r^{-N-\varepsilon}) \qquad \text{as} \quad r\to\infty, \tag{5.10}$$

where $\varepsilon = \frac{4}{\sigma}-(N-2)>0$. Consequently,

$$\int_{\mathbb{R}^N} |u|^q = |S_N|\int_0^\infty r^{N-1}|u(r)|^q\,dr < \infty.$$

Here $|S_N|$ denotes the measure of the unit sphere in $\mathbb{R}^N$. Multiplying the equation (5.1) by u and integrating over $B_r = B_r(0)$, $r>\hat r$ we get

$$\int_{B_r}|\nabla u|^2\,dx - \int_{B_r} w(r)|u|^q\,dx = |S_N|r^{N-1}u'(r)u(r) < 0,$$

and therefore

$$\int_0^\infty r^{N-1}(u'(r))^2\,dr = \int_{\mathbb{R}^N}|\nabla u|^2\,dx \leq \int_{\mathbb{R}^N} w(r)|u|^q\,dx < \infty. \tag{5.11}$$

We now put

$$\alpha := \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx - \frac{N}{q}\int_{\mathbb{R}^N} w(r)|u|^q\,dx - \frac{1}{q}\int_{\mathbb{R}^N} w'(r)r|u|^q\,dx$$

Multiplying the equation (5.1) by $x\cdot\nabla u$ and integrating over B_r, $r>\max\{\hat r, r_1\}$ we get

$$\begin{aligned} \frac{N-2}{2}\int_{B_r}|\nabla u|^2\,dx - \frac{N}{q}\int_{B_r} w(r)|u|^q\,dx - \frac{1}{q}\int_{B_r} w'(r)r|u|^q\,dx \\ = -\,|S_N|\left\{r^N\left(\frac{(u'(r))^2}{2}+\frac{1}{q}|u(r)|^q\right)\right\} \end{aligned}$$

and hence

$$r^N(u'(r))^2 \to -\frac{2\alpha}{|S_N|} \qquad \text{as} \quad r\to\infty$$

by (5.10). Now (5.11) forces $\alpha = 0$. Thus,

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &= \frac{N}{q}\int_{\mathbb{R}^N} w(r)|u|^q\,dx + \frac{1}{q}\int_{\mathbb{R}^N} w'(r)r|u|^q\,dx \\ &\geq \frac{N}{q}\int_{\mathbb{R}^N} w(r)|u|^q\,dx, \end{aligned} \tag{5.12}$$

since $w' \geq 0$ by assumption. Combining this with (5.11) we get

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \frac{N}{q}\int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

and hence $q \geq \frac{2N}{N-2}$, contrary to our assumption on $\sigma = q - 2$. □

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