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EXACT CONTROLLABILITY OF FINITE ENERGY STATES FOR AN ACOUSTIC WAVE/PLATE INTERACTION UNDER THE INFLUENCE OF BOUNDARY AND LOCALIZED CONTROLS

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Abstract. In this work, we derive a result of exact controllability for a structural acoustic partial differential equation (PDE) model, comprised of a three-dimensional interior acoustic wave equation coupled to a two-dimensional Kirchoff plate equation, with the coupling being accomplished across a boundary interface. For this PDE system, we show that by means of boundary controls, the interior wave and Kirchoff plate initial data can be steered to an arbitrary *finite energy* state. In this work, key use is made of recent, microlocally-derived, $L^2 \times H^{-1}$ "recovery" estimates for wave equations with Dirichlet boundary data. Moreover, the coupling of the disparate acoustic wave/Kirchoff plate dynamics is reconciled by means of sharp regularity estimates which are valid for hyperbolic equations of second order.

1. Introduction and statement of main results

In this work, we will consider a coupled partial differential equation (PDE) structure of hyperbolic type with an interface. This model, which is defined on a three-dimensional bounded domain Ω, is a *hybrid* PDE, thematically like that which appears in [35]: in particular, the model (in (1.1) below) comprises a wave equation coupled to a two-dimensional plate equation of hyperbolic type, with the coupling taking place on an interface which coincides with a portion of the boundary of Ω. This type of model arises in the context of structural acoustic interactions–see e.g., [40, 20, 34, 14, 11] and

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references therein. The aim of this paper is to establish optimal reachability properties with respect to *finite energy* states, by the sole means of actuation on the boundary. In fact, we shall prove that, in some finite time T, any state of finite energy can be driven (or steered) to another specified finite energy state, by the utilization of an appropriate forcing term on $\partial\Omega$. More precisely, we shall use *Neumann boundary controls* applied to the (interior) wave equation, in tandem with *localized* controls applied to a small portion of the interface. In this way, the control function we invoke acts upon the structural acoustic system as "inobstrusively" as possible, with this function being supported only on $\partial\Omega$. In fact, it is known that due to the hybrid connection of wave and plate, exact controllability of finite energy states by means of controls (even full internal controls) affecting the wave only is not possible [34]. Thus, placing localized controls on an arbitrary small area near the boundary of the interface is the best one can hope for.

Our interest in studying this particular problem is motivated not only by the physical relevance and technological implications associated with control-theoretic results for the model (1.1), but also, and primarily, by mathematical aspects related to the propagation of an underlying hyperbolicity and associated behavior of traces of (wave) solutions. Ultimately, this trace behavior will have a critical bearing on the answer to our "boundary reachability" question. In fact, the recently developed trace regularity theory, for both Lopatinski and non-Lopatinski types of hyperbolic boundary-value problems [27, 29, 30, 32, 43, 45, 38] is the main vehicle and driving force behind the results presented below.

It is the presence of the coupling on the interface (both from the wave to the plate and from the plate to the wave), and the fact that we are insisting on physically relevant *finite energy states*–associated with classical weak solutions–which will necessitate the use of the "sharp" trace theory which is now available for solutions to second-order hyperbolic equations. These sharp trace regularity results, which for solutions to wave equations cannot be obtained from any known interior regularity and the classical Sobolev trace theorem, will allow the generation of appropriate "inverse-type" estimates which are intrinsically related to controllability-observability inequalities. In sum, our paper is geared to showing how these hyperbolic trace results can be used not only successfully, but indeed critically, for the purpose of proving exact reachability properties of coupled structures. In order to illustrate this idea, we shall focus below on the prototype model (1.1), which governs a given acoustic-structure interaction. However, our analysis can be also applied to other interactive structures, such as electromagnetic or fluid structure interactions.

1.1. **Statement of the controllability problem.** Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set, with sufficiently smooth (two-dimensional) boundary Γ. Moreover, we assume that $\Gamma = \Gamma_1 \cup \Gamma_0$, where Γ_0, an interface, is a flat surface. For this geometry, we have the following controlled partial differential equation (PDE) model:

$$\begin{cases} z_{tt} = \Delta z \quad \text{on } (0,T)\times\Omega \\ \dfrac{\partial z}{\partial \nu} = u_1 \quad \text{on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t \quad \text{on } (0,T)\times\Gamma_0 \\ v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + z_t|_{\Gamma_0} = a(x)u_0 \quad \text{on } (0,T)\times\Gamma_0 \\ v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T)\times\partial\Gamma_0 \\ [z(0), z_t(0), v(0), v_t(0)] = [\vec{z}_0, \vec{v}_0]\,. \end{cases} \tag{1.1}$$

Here, the functions u_i are the "controls" of the system, invoked here so as to influence the behavior of the dynamics in a way we shall describe presently. Also, $a(x)$ is a C^2 coefficient which is locally distributed on Γ_0, with moreover $\partial\Gamma_0 \subset \operatorname{supp}(a)$. In addition, we specify that $0 \le a(x) \le 1$, with $a(x)$ being identically 1 on a connected portion of $\overline{\Gamma_0}$, with this portion containing $\partial\Gamma_0$. (One might think of a as a "smoothing out" of a characteristic function, with respect to this connected portion of $\overline{\Gamma_0}$.) Moreover, the "moment of rotational inertia parameter" γ which appears in the plate component of (1.1) is strictly positive, in which case the PDE, when $u_1 = 0$ and $u_0 = 0$, manifests *hyperbolic* dynamics.

Our primary goal in this paper is to study controllability properties of the controlled model (1.1) in the relevant *finite energy space* of initial data. Our emphasis here that we steer arbitrary data of finite energy, is not only due to physical considerations, but also to the implications of such controllability properties on other control-theoretic questions related to structural acoustic flow. For example, exact controllability for finite energy data is directly applicable to optimal control theory, inasmuch as the validation of such a controllability property means that the so-called "finite cost condition", or sufficient condition for optimization problem, is satisfied. With this goal in mind, our space of well posedness $\mathbf{H}$ is here given to be

$$\mathbf{H} = H_1 \times H_0, \tag{1.2}$$

where

$$H_1 \equiv H^1(\Omega)\times L^2(\Omega) \text{ and } H_0 \equiv \left[H^2(\Gamma_0)\cap H_0^1(\Gamma_0)\right]\times H_0^1(\Gamma_0). \tag{1.3}$$

It is a well-known fact that the spaces H_1 (respectively, H_0) are classical finite energy spaces associated with weak solutions to wave (respectively, Kirchoff plate) equations. Given then these "natural spaces," our objective in this paper is to find controls u_1, u_0 which will steer, exactly, the given (finite energy) initial data to an arbitrary target state, at time $T > 0$. In this work, we are invoking the classical definition of exact controllability:

Definition 1. *We say that the PDE* (1.1) *is exactly controllable at time* $T > 0$ *if, for given initial data* $[\vec{z}_0, \vec{v}_0] \in \mathbf{H}$ *and arbitrary terminal state* $[\vec{z}_T, \vec{v}_T] \in \mathbf{H}$*, there are functions* $u_1 \in L^2(\Sigma_1)$ *and* $u_0 \in L^2(0, T; H^{-1}(\Gamma_0))$ *such that the corresponding solution of* (1.1) *satisfies* $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$.

Remark 2. Our choice here of the control space $L^2(\Sigma_1)$ (respectively, $L^2(0, T; H^{-1}(\Gamma_0)))$ is optimal, inasmuch as it is optimal with respect to the control of the uncoupled wave (respectively, Kirchhoff plate) equation (see [31, 24], and also [21], wherein sharp controllability estimates are obtained, via Carleman's estimates, for each of these two models).

Remark 3. Since our goal is to obtain exact controllability of *all finite energy states*, the imposition of control on Γ_1 is unavoidable [34]. We could proceed to specify geometrical conditions for which control on Γ_1 is limited to just a portion thereof, by invoking, say, the precise statements in [28] and [32]; but this would add additional technicalities detracting from the principle aim of the paper: namely, to validate the exact controllability property given in Definition 1, with the *locally distributed* control coefficient $a(x)$ in place.

1.2. **About the model, physical motivation and techniques used.** The PDE model is a representative of the class of so-called *hybrid structures* (see the pioneering paper on the subject [35]). More specifically, the PDE system (1.1) is an example of a *structural acoustic interaction*; namely, it comprises an acoustic wave equation (in the variable z) on the interior of Ω, coupled to a structural plate equation (in the variable v) on the flat boundary portion Γ_0.

Although the PDE modelling of structural acoustic interactions has been an ongoing enterprise for decades–see e.g., [40],[14]–the last fifteen years in particular have seen a flourishing in the literature of those studies concerned with structural acoustic PDE's. The principal reason for this increased interest is the recent development of engineering technologies which have allowed the implementation of practicable schemes for the active control of external noise, as it enters a given structural acoustic system. Recent contributions to the literature deal with the extensive study of such topics as optimal control (with associated Riccati equations), stabilization, controllability, regularity, numerical computation ([11],[5],[6],[3], [34],[20],[37],[16],[8],[9],[1],[26],[25],[15],[18]).

In all these papers, a wave equation is typically invoked in the modelling, so as to describe an interior acoustic field; on the other hand, either wave, plate, beam, or shell equations, subject to various degrees of damping (be it viscous, structural, or thermal), have been utilized so as to model the structural component of the coupled PDE system. But amongst all these distinct structural acoustic PDE examples, there is one unifying complication which makes the mathematical analysis of structural acoustic interactions a worthy and challenging field of endeavor: namely, the mode of coupling between the acoustic wave and structural PDE dynamics is by means of *unbounded trace terms* (with respect to the topology of the underlying state space) on the boundary interface. It is the need to reconcile the presence of these acoustic wave trace terms–for which the classical Sobolev trace theorem cannot be applied–which has essentially dictated the analysis underpinning many of the known control-theoretic results obtained for this model (see e.g., [5] and [25], along with references therein).

In the present work, we will be concerned with the *exact controllability problem* associated with the PDE (1.1), as stated explicitly in Definition 1. It should be noted that the derivation of *exact* controllability properties constitutes a particularly challenging problem for coupled structures. This circumstance is due to the fact that the solution of such a problem tends to depend upon the existence of an inverse-type of estimate–generally referred to as an *observability estimate*–which allows for a reconstruction of the total energy from overdetermined boundary data.

In particular, the methodologies invoked to derive such observability estimates are very sensitive to the presence of the unbounded coupling traces which occur at the structural acoustic interface. The coupling within the structure leads to the presence of boundary traces in the estimates, traces which will not allow the direct application of (by now) standard "recovery estimates" for uncoupled plate and wave equations. These terms not only can not be absorbed by the quantities available to observation, but also they are not bounded with respect to the state space topology. Because of this complication, very few results are available in the literature on the topic of exactly controlling structural acoustic dynamics. In fact, the results which are available at present deal mostly with special geometries–e.g., rectangular, two-dimensional domains–and with a topological setting which is geometry and operator (eigenfunction expansions) dependent and nonintrinsic to the model; see [37, 1] and [34], with the latter paper providing very interesting results of a negative nature.

On the other hand, our present work deals with the domain Ω being of a general geometrical configuration. More importantly, the controllability concept we are studying is the classical one which is explicitly stated in Definition 1. This definition underscores our prime concern to work within the *finite energy state space* $\mathbf{H}$.

As an example of a "finite energy" reachability result in structural acoustics,a property of boundary controllability is stated in [10] for a canonical wave-wave interaction; i.e., a wave equation on Γ_0 is invoked in [10], in place of a Kirchoff plate–with this property being valid for *arbitrary* initial data in the relevant finite energy space. In particular, in [10], precise geometrical configurations are given which will ensure exact controllability for the both acoustic and structural wave states, in the case that control is implemented *only* on the active portion of the boundary (which, again, is Γ_0 in the present work). (This specification of the geometry in [10] is no violation of the now classic work [13], which gives sufficient conditions for the boundary control of waves on a bounded open set.)However, as seen in [10], the price to pay for controlling the given wave-wave interaction on Γ_0 only, is that the controls must be very "rough" (distributional) with respect to time. If square integrability in time is imposed on the class of controls, then the results in [10] provide the finite energy controllability but with two controls acting on the entire boundary Γ and Γ_1.

Since square integrability in time of control functions is demanded by many physical applications, the goal in this paper is to work not only within a class of square integrable in time controls, but also with controls of minimal support on the interface Γ_0. To accomplish this we develop an approach which is based on "sharp trace theory" and allows us to deal with the generic unboundedness of these boundary traces. This approach partly involves an exploiting of recent results in both the "hidden regularity" of Kirchoff plates, and "sharp trace regularity" for solutions of wave equations under the Neuman boundary conditions (i.e., the "non-Lopatinski" case) [27, 29, 30, 32, 43, 45, 38]. In order to take advantage of "sharp" regularity results, observability estimates for Kirhchoff plates on the *negative* scale of the energy are necessary to be developed -see Remark 14. This is in contrast with a "classical," uncoupled controllability problem formulated for plate equation -see [24] and references therein.

Here are a few words about the techniques employed. As seen in the formulation of the problem in the present paper, we assume that the coefficient $a(x)$ in (1.1) has its localized support situated about the boundary. Thus, after a decoupling of the system, to be done as in [9] by the usage of microlocally-derived trace regularity results, we will ultimately be confronted with the problem of controlling the Kirchoff plate by means of locally distributed control, with respect to the *relevant space of finite energy* (in addition to dealing with the aforementioned wave equation under Neumann control). This controllability of the plate component–the proof of which will be driven by the need to handle *a priori* not-necessarily-defined boundary traces (on the finite energy space) of the wave velocity–is the heart of the matter in this work.

In order to obtain the continuous observability inequality which is equivalent to the exact control of (1.1) (see (1.15) below), and in particular to handle the contribution from the Kirchoff component, we will invoke an inverse-type of estimate which was derived in [4], and which is valid in "negative norms" for wave equations under Dirichlet control and forcing terms of a particular form. In particular, this recovery estimate reconstructs the initial energy of the Kirchoff plate energy from measurements of boundary terms. Subsequently, in order to handle these boundary terms, it will become necessary to invoke the "hidden regularity" results in [27] on a *negative scale* of Sobolev spaces. To conclude our introductory remarks, we cite the main technical ingredients of the present work, which are also of independent mathematical interest. They are: [(i)] Reconstruction/observability estimates in negative ("dual") norms, with applications to the boundary controllability of the Kirchoff plate component in (1.1), which is driven by a source resulting from the coupling with the boundary trace of the interior wave velocity. [(ii)] Sharp trace regularity theory, corresponding to the wave equation with Neumann (non-Lopatinski) boundary conditions. [(iii)] Hidden regularity of normal derivatives on the boundary *in negative norms* for Kirchoff plates, under Dirichlet boundary data.

1.3. **Statement of the main result.** We assert that to achieve the exact controllability result which is posted in Definition 1, it is enough to consider the exact controllability question for the following PDE system, again within the class of controls $L^2(\Sigma_1) \times L^2(0,T;H^{-1}(\Gamma_0))$:

$$\begin{aligned}
&z_{tt} = \Delta z \quad \text{on } (0,T)\times\Omega \\
&\begin{cases} \dfrac{\partial z}{\partial \nu} + z = u_1 \quad \text{on } (0,T)\times\Gamma_1 \\ \dfrac{\partial z}{\partial \nu} = v_t \quad \text{on } (0,T)\times\Gamma_0 \end{cases} \\
&v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + z_t|_{\Gamma_0} = a(x)u_0 \quad \text{on } (0,T)\times\Gamma_0 \\
&v|_{\partial\Gamma_0} = \Delta v|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T)\times\partial\Gamma_0 \\
&[z(0), z_t(0), v(0), v_t(0)] = [\vec{z}_0, \vec{v}_0]\,.
\end{aligned} \tag{1.4}$$

By way of justifying our claim, we start by deriving the following regularity result for solutions of (1.4), corresponding to given initial and boundary data:

Lemma 4. *For given data* $\{[\vec{z}_0, \vec{v}_0], u_1, u_0\} \in \mathbf{H} \times L^2(\Sigma_1) \times L^2(0,T; H^{-1}(\Gamma_0))$, *the solution* $[\vec{z}, \vec{v}]$ *of* (1.4) *satisfies the following estimate:*

$$\begin{aligned} & \|z, z_t\|_{C([0,T]; H^{\frac{3}{5}-\epsilon}(\Omega) \times H^{-\frac{2}{5}-\epsilon}(\Omega))} + \|[v, v_t]\|_{C([0,T]; H_0)} \\ \leq \; & C_T \Big(\|[\vec{z}_0, \vec{v}_0], u_0, u_1\|_{\mathbf{H} \times L^2(\Sigma_1) \times L^2(0,T; H^{-1}(\Gamma_0))} \Big). \end{aligned} \tag{1.5}$$

Proof. Let $A_D : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$ (respectively, $P_\gamma : D(A_D) \subset L^2(\Gamma_0) \to L^2(\Gamma_0)$) be the positive-definite, self-adjoint operator defined by

$$\begin{aligned} A_D g &= -\Delta g, \;\; D(A_D) = H^2(\Gamma_0) \cap H_0^1(\Gamma_0); \\ P_\gamma g &= (I + \gamma A_D), \; D(P_\gamma) = D(A_D). \end{aligned} \tag{1.6}$$

From the now classic results in [17], one has the following characterizations:

$$D(A_D) \approx H^2(\Gamma_0) \cap H_0^1(\Gamma_0); \; D(\sqrt{P_\gamma}) \approx H_0^1(\Gamma_0).$$

These Sobolev space identifications of $D(A_D)$ and $D(\sqrt{P_\gamma})$ allow the H_0-norm in (1.3) to be given by

$$\left\| \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_{H_0}^2 = \|v_0\|_{D(A_D)}^2 + \|v_1\|_{D(\sqrt{P_\gamma})}^2 .$$

Now, if we let $A_0 : D(A_0) : H_0 \to H_0$ be defined by

$$A_0 = \begin{bmatrix} 0 & I \\ -P_\gamma^{-1} A_D^2 & 0 \end{bmatrix}, \text{ with } D(A_0) = D(A_D^{\frac{3}{2}}) \times D(A_D),$$

then it is readily seen that A_0 generates a C_0-group $\{e^{A_0 t}\}_{t \in \mathbb{R}}$ on H_0. Accordingly, for $0 \leq s \leq T$, the plate component of the solution of (1.4) may be written as

$$\begin{aligned} \begin{bmatrix} v(s) \\ v_t(s) \end{bmatrix} &= e^{A_0 s} \vec{v}_0 - \int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1} \left(z_t|_{\Gamma_0} - a u_0 \right) \end{bmatrix} d\tau \\ &= e^{A_0 s} \vec{v}_0 + A_0 \int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1} \, z|_{\Gamma_0} \end{bmatrix} d\tau \\ &+ \int_0^s e^{A_0(s-\tau)} \begin{bmatrix} 0 \\ P_\gamma^{-1}(a u_0) \end{bmatrix} d\tau - \begin{bmatrix} 0 \\ P_\gamma^{-1} \, z|_{\Gamma_0} \end{bmatrix} \Bigg|_0^s . \end{aligned} \tag{1.7}$$

Estimating this will yield now, for all $0 \leq s \leq T$,

$$\|[v(s), v_t(s)]\|_{H_0} \tag{1.8}$$

$$\leq C_T \left(\left\| z|_{\Gamma_0} \right\|_{L^\infty(0,s;L^2(\Gamma_0))} + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))} + \|[\vec{z}_0, \vec{v}_0]\|_{H_1 \times H_0} \right).$$

In order to obtain the "forward" estimate (1.5), one needs to handle the boundary traces $z|_{\Gamma_0}$ defined on the interface. We note that the classical regularity results in [33] for solutions to wave equations under Neumann boundary data, which is L^2 in time and space, yield that $z \in C([0,T]; H^{1/2}(\Omega))$. This degree of interior regularity will not allow usage of the Sobolev trace theorem, so as to infer a legitimate notion of z on the boundary. In this connection, we recall the following boundary data regularity result in [43] (see also [29]). We note that Theorem 5 below gives a much stronger result than that needed at this particular moment. Indeed, this theorem lifts–with respect to the classical results [33]–the interior regularity by almost $\frac{1}{10}$ of a derivative (see also [45] for related regularity results), while what is needed now for the estimate (1.8) is merely an ϵ improvement of differentiability in order to give an "almost everywhere" meaning to the trace $z|_\Gamma$.

Theorem 5. (See Theorem 3.1 of [43].) *Let ϕ satisfy the following wave equation on $(0,s) \times \Omega$, where Ω is a smooth bounded domain:*

$$\begin{aligned}
&\phi_{tt} = \Delta\phi \quad \text{on } (0,s) \times \Omega \\
&\frac{\partial \phi}{\partial \nu} + \alpha(x)\phi = g \ \text{ on } (0,s) \times \Gamma \quad (\text{where the } L^\infty \text{coefficient } \alpha(x) \geq 0 \text{ on } \Gamma) \\
&[\phi(0), \phi_t(0)] = \vec{z}_0.
\end{aligned}$$

Then we have, continuously, $\{\vec{z}_0, g\} \in H_1 \times L^2((0,s) \times \Gamma) \Rightarrow [\phi, \phi_t] \in C([0,s]; H^{\frac{3}{5}-\epsilon}(\Omega) \times H^{-\frac{2}{5}-\epsilon}(\Omega))$.

Appealing to this result, with

$$g(t,x) = \begin{cases} u_1(t,x), & x \in \Gamma_1 \\ v_t(t,x), & x \in \Gamma_0 \end{cases}$$

(so that $g \in L^2((0,s) \times \Gamma)$), and subsequently applying the classical Sobolev trace theorem, the estimate (1.8) becomes, for all $0 \leq s \leq T$,

$$\begin{aligned}
&\|[v(s), v_t(s)]\|_{H_0}^2 \\
&\leq C_T \Big(\int_0^s \|v_t\|_{L^2(\Gamma_0)}^2 \, dt + \|u_1\|_{L^2(\Sigma_1)}^2 + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))}^2 + \|[\vec{z}_0, \vec{v}_0]\|_{\mathbf{H}}^2 \Big).
\end{aligned}$$

Gronwall's inequality now gives, for $0 \leq t \leq T$,

$$\|[v(t), v_t(t)]\|_{H_0}^2 \leq C_T \left(\|u_1\|_{L^2(\Sigma_1)}^2 + \|u_0\|_{L^2(0,T;H^{-1}(\Gamma_0))}^2 + \|[\vec{z}_0, \vec{v}_0]\|_{\mathbf{H}}^2 \right). \tag{1.9}$$

In turn, combining this estimate with Theorem 5 gives

$$\begin{aligned}&\|[z, z_t]\|_{C([0,T];H^{\frac{3}{5}-\epsilon}(\Omega)\times H^{-\frac{2}{5}-\epsilon}(\Omega))}\\ &\leq C_T\left(\|u_1\|^2_{L^2(\Sigma_1)} + \|u_0\|^2_{L^2(0,T;H^{-1}(\Gamma_0))} + \|[\vec{z}_0, \vec{v}_0]\|^2_{\mathbf{H}}\right). \qquad (1.10)\end{aligned}$$

The estimates (1.9) and (1.10) complete the proof of Lemma 4. □

With Lemma 5 in hand, suppose now that the controls $[u_1, u_0] \in L^2(\Sigma_1)\times L^2(0, T; H^{-1}(\Gamma_0))$ are such that the corresponding solution of (1.4) satisfies $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$, for given target data $[\vec{z}_T, \vec{v}_T] \in \mathbf{H}$. Then from Lemma 4 and the Sobolev embedding theorem, we have (conservatively) $z|_\Gamma \in L^2(\Sigma)$. Subsequently, if we set $[u_1^*, u_0^*] = [u_1 - z|_{\Gamma_1}, u_0]$, then $[\vec{z}, \vec{v}]$ satisfies the PDE (1.1), with controls $[u_1^*, u_0^*]$ in place, and as we said, satisfy the reachability property $[\vec{z}(T), \vec{v}(T)] = [\vec{z}_T, \vec{v}_T]$.

The reason we will consider the problem (1.4), instead of (1.1), is that on the afore-defined space $\mathbf{H}$, one will avoid the acoustic wave "steady states" which are inherent in the original problem (1.1), inasmuch as pure Neumann boundary conditions are being considered for the z variable. In fact, if $\mathbf{H}$ is topologized through the inner product

$$\begin{aligned}([\vec{z}, \vec{v}], [\vec{\varsigma}, \vec{v}])_{\mathbf{H}} = \int_\Omega \nabla z_0 \cdot \nabla \varsigma_0 d\Omega + (z_0|_{\Gamma_1}, \varsigma_0|_{\Gamma_1})_{L^2(\Gamma_1)} + (z_1, \varsigma_1)_{L^2(\Omega)}\\ +(A_D v_0, A_D \upsilon_0)_{L^2(\Gamma_0)} + (P_\gamma^{\frac{1}{2}} v_0, P_\gamma^{\frac{1}{2}} \upsilon_0)_{L^2(\Gamma_0)}, \qquad (1.11)\end{aligned}$$

then an invocation of the Lumer-Phillips theorem will yield that the structural acoustic flow, described in (1.4), is associated with the generator of a C_0-group on $\mathbf{H}$. Consequently, this semigroup generation will yield the wellposedness,

$$\{[\vec{z}_0, \vec{v}_0] \in \mathbf{H},\ u_1 = 0,\ u_0 = 0\} \Longrightarrow [\vec{z}, \vec{v}] \in C([0, T]; \mathbf{H}). \qquad (1.12)$$

On the other hand, when boundary data is present in the model (1.4), Lemma 4 states that the spatial regularity of the wave component is below the level of finite energy. This circumstance is not unexpected, since it is essentially "inherited" from uncoupled wave equations subject to $L^2(\Sigma)$ boundary data (see [29]).

But as we said, our principle intent here is to investigate reachability properties of the structural acoustic model (1.1) for *arbitrary* initial data $[\vec{z}_0, \vec{v}_0]$ in $\mathbf{H}$, by means of controls $[u_1, u_0]$ in $L^2(\Sigma_1) \times L^2(0, T; H^{-1}(\Gamma_0))$. In turn, by the reasoning which we gave above, it suffices to study the associated exact controllability problem for the (Robin) system (1.4). In this connection, our main result in this paper is as follows:

Theorem 6. *For terminal time $T > 0$ large enough, the PDE* (1.4) *is exactly controllable in the sense of Definition* 1.

Remark 7. The result of Theorem 6 could be extended to the case when the boundary Γ_0 is not necessarily assumed to be "flat". However this would necessitate our considering shells rather than plates to model the dynamics on the boundary portion Γ_0. In consequence, one would be compelled to use the tools of Riemanian geometry, along with appropriate differential geometry-based multipliers [49, 22, 36]. This procedure would make the exposition of the paper much more technical, a circumstance which would detract from the main objective. For this reason we prefer to work within the Euclidean (rather than Riemannian) metric.

The proof of Theorem 6 hinges on ascertaining the surjectivity of the *control to terminal state* map, which is associated with the exact controllability of (1.4). We could proceed to flesh out this abstract map by generating the necessary (and cumbersome) operator theoretic quantities; but we desist from doing so, and instead direct the reader, if he or she is unfamiliar with the classic functional analytical argument relating exact controllability to its "dual problem," to [41] and [28]. (See also [7] which deals with exact controllability relative to a different coupled PDE system). The relevant observability inequality, which is dual to the exact controllability of (1.4), is as follows: Consider the homogenous PDE,

$$\begin{array}{l} \phi_{tt} = \Delta\phi \text{ on } (0,T)\times\Omega \\ \left\{ \begin{array}{l} \dfrac{\partial\phi}{\partial\nu} + \phi = 0 \text{ on } (0,T)\times\Gamma_1 \\ \dfrac{\partial\phi}{\partial\nu} = \omega_t \text{ on } (0,T)\times\Gamma_0 \end{array} \right. \\ \omega_{tt} - \gamma\Delta\omega_{tt} + \Delta^2\omega + \phi_t|_{\Gamma_0} = 0 \text{ on } (0,T)\times\Gamma_0 \\ \phi|_{\Gamma_0} = \Delta\phi|_{\Gamma_0} = 0 \text{ on } (0,T)\times\Gamma_0 \\ [\phi(T),\phi_t(T),\omega(T),\omega_t(T)] = [\phi_0,\phi_1,\omega_0,\omega_1] \in \mathbf{H}. \end{array} \tag{1.13}$$

With respect to this structural acoustic system, we make the denotation $\mathcal{E}(t) \equiv \mathcal{E}_\phi(t) + \mathcal{E}_\omega(t)$, where the respective acoustic wave and Kirchoff plate energies are given by

$$2\mathcal{E}_\phi(t) = \int_\Omega \left[|\nabla\phi(t)|^2 + \phi_t^2(t) \right] d\Omega + \int_{\Gamma_1} \left(\phi(t)|_{\Gamma_1} \right)^2 d\Gamma_1;$$

$$2\mathcal{E}_\omega(t) = \|A_D\omega(t)\|^2_{L^2(\Gamma_0)} + \left\| P_\gamma^{\frac{1}{2}}\omega_t(t) \right\|^2_{L^2(\Gamma_0)}.$$

Note that one can verify directly that, as expected, the energy of the system (1.13) is *conserved*; i.e.,

$$\mathcal{E}(t) \equiv \mathcal{E}(s), \text{ for all } 0 \leq s, t \leq T. \tag{1.14}$$

With the backwards problem (1.13) in mind, the PDE (1.4) will be exactly controllable, within the class of controls $L^2(\Sigma_1) \times L^2(0,T; H^{-1}(\Gamma_0))$, if the solution variables $[\phi, \phi_t, \omega, \omega_t]$ of (1.13) satisfy the following inequality, for $T > 0$ large enough:

$$\mathcal{E}(T) \leq C_T \Big(\int_0^T \int_{\Gamma_1} \phi_t^2 d\Gamma_1 dt + \int_0^T \left\| P_\gamma^{\frac{1}{2}} (a\omega_t) \right\|^2_{L^2(\Gamma_0)} \Big) dt. \tag{1.15}$$

The rest of the paper is accordingly geared towards the derivation of the inequality (1.15). In the course of our work, we will propagate many so-called "lower-order terms". By lower-order terms, denoted throughout as l.o.t. $(\vec{\phi}, \vec{\omega})$, we mean

$$\text{l.o.t.}(\vec{\phi}, \vec{\omega}) = \mathcal{O}\Big(\left\| [\vec{\phi}, \vec{\omega}] \right\|_{C([0,T]; H^{1-\epsilon}(\Omega) \times H^{-\epsilon}(\Omega) \times H^{2-\epsilon}(\Gamma_0) \times H^{1-\epsilon}(\Gamma_0))} \Big), \tag{1.16}$$

where again, $[\vec{\phi}, \vec{\omega}]$ is the solution component of (1.13).

2. Reconstruction of the energy for the wave component of (1.13).

2.1. **An energy relation for the wave component of (1.13).** As we said above, the proof of Theorem 6 is tantamount to deriving the inequality (1.15), to which end we will devote our energies. To start, we will derive a relation for the interior acoustic variables $[\phi, \phi_t]$, which we will find useful throughout. To wit, multiplying the interior wave component of (1.13) by ϕ_t, and integrating with respect to time and space, we have for all $0 \leq t \leq T$,

$$\mathcal{E}_\phi(T) = \mathcal{E}_\phi(t) + \int_t^T \langle \omega_t, \phi_t \rangle_{H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)} dt. \tag{2.1}$$

The fact that the duality pairing on the right-hand side may be taken in the $H^{\frac{1}{2}}(\Gamma_0) \times H^{-\frac{1}{2}}(\Gamma_0)$ topology is a consequence of the result in [38], a work which addresses the regularity of second-order hyperbolic equations, under the action of prescribed Neumann data. Here again, we remind the reader that finite energy solutions to wave equations in the non-Lopatinski case generally do not produce $H^1(\Sigma)$ boundary traces (in contrast to the *Dirichlet* case, wherein the "hidden regularity' provides for the corresponding wave traces $[z|_\Gamma, z_t|_\Gamma]$ being well defined, in the L^2 sense; see [27]). For the Neumann case we have instead,

Lemma 8. (See Theorem 3, page 443 of [38].) *Let* $[\varphi, \varphi_t]$ *solve the following boundary-value problem on* $(0,T)\times\Omega$:

$$\begin{aligned}
&\varphi_{tt} - \Delta\varphi = 0 \quad on\ (0,T)\times\Omega,\\
&\frac{\partial\varphi}{\partial\nu} + \alpha(x)\phi = g \in L^2(0,T;H^{\frac{1}{2}}(\Gamma)) \quad on\ (0,T)\times\Gamma\\
&\qquad (where\ the\ L^\infty coefficient\ \alpha(x)\geq 0\ on\ \Gamma)\\
&[\varphi(0),\varphi_t(0)] = [\phi_0,\varphi_1] \in H^1(\Omega)\times L^2(\Omega).
\end{aligned}$$

Then we have the estimate

$$\int_0^T [\|\varphi_t\|^2_{H^{-\frac{1}{2}}(\Gamma)}]\,dt \leq C_T\Big\{\,\|[\phi_0,\varphi_1]\|^2_{H^1(\Omega)\times L^2(\Omega)} + \int_0^T \|g\|^2_{H^{\frac{1}{2}}(\Gamma)}\,dt\Big\}.^1$$

2.2. **The main estimate for the wave component of (1.13).** Throughout, we will invoke the denotations $Q_\delta \equiv (\delta, T-\delta)\times\Omega$, $\Sigma_\delta = (\delta, T-\delta)\times\Gamma$ and $\Sigma_{i,\delta} = (\delta, T-\delta)\times\Gamma_i$. Let $h(x)$ be a $\left[C^2(\overline{\Omega})\right]^n$ vector field which is to be specified below. An invocation of the classical wave multiplier $h\cdot\nabla\phi$ (e.g., see [39], [42], [24], [48]) to the wave component in (1.13) gives

$$\begin{aligned}
&\int_{Q_\delta} H(x)\nabla\phi\cdot\nabla\phi dQ_\delta = \int_{\Sigma_{0,\delta}} \omega_t\,(h\cdot\nabla\phi)\,d\Sigma_{0,\delta} - \int_{\Sigma_{1,\delta}} \phi\,(h\cdot\nabla\phi)\,d\Sigma_{1,\delta}\\
&-\frac{1}{2}\int_{\Sigma_\delta} \phi_t^2 h\cdot\nu d\Sigma_\delta - \frac{1}{2}\int_{\Sigma_\delta} |\nabla\phi|^2\,h\cdot\nu d\Sigma_\delta\\
&+\frac{1}{2}\int_{Q_\delta}\left(|\nabla\phi|^2 - \phi_t^2\right) div(h) dQ_\delta - \Big[(\phi_t, h\cdot\nabla\phi)_{L^2(\Omega)}\Big]_\delta^{T-\delta}. \qquad (2.2)
\end{aligned}$$

Here, the matrix $H(x)$ is the Jacobian matrix associated with the vector field $\vec{h}$.

We next apply the quantity $\phi(x)div(\tilde{h})$ to the wave component in (1.13), where $\tilde{h}(x)$ is any vector field in $\left[C^2(\overline{\Omega})\right]^3$ (to be eventually specified). Integrating in time and space, and subsequently using Green's theorem and the identity

$$\nabla\phi\cdot\nabla(\phi\,div(\tilde{h})) = \phi\nabla(div(\tilde{h}))\cdot\nabla\phi + |\nabla\phi|^2\,div(\tilde{h}),$$

we obtain

$$\int_{Q_\delta}\left(\phi_t^2 - |\nabla\phi|^2\right) div(\tilde{h}) dQ_\delta = \Big[\left\langle\phi_t, \phi div(\tilde{h})\right\rangle_{H^{-\epsilon}(\Omega)\times H^\epsilon(\Omega)}\Big]_\delta^{T-\delta} \qquad (2.3)$$

[1]This estimate is now known not to be sharp with respect to the boundary datum g (see [43],[29],[45],[2]). But since this estimate will suffice for the present paper, we beg off from citing the pluperfect boundary trace regularity.

$$+\int_{Q_\delta}\phi\nabla\left(div(\tilde{h})\right)\nabla\phi dQ_\delta-\int_{\Sigma_{0,\delta}}\omega_t\phi div(\tilde{h})d\Sigma_{0,\delta}+\int_{\Sigma_{1,\delta}}\phi^2 div(\tilde{h})d\Sigma_{1,\delta}.$$

Upon estimating this relation, by means of Sobolev trace theory and the fact that $2ab\leq a^2+b^2$, we obtain

$$\begin{aligned}&\Big|\int_{Q_\delta}\left(\phi_t^2-|\nabla\phi|^2\right)div(\tilde{h})dQ_\delta\Big|\\ \leq\ & C_\epsilon\int_{\Sigma_{0,\delta}}\omega_t^2 d\Sigma_{0,\delta}+\epsilon\int_{Q_\delta}|\nabla\phi|^2\,dQ_\delta+\mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).\end{aligned}\tag{2.4}$$

Setting now $h(x)=\tilde{h}(x)\equiv x-x_0$, where, say, $x_0\in\mathbb{R}^3$ is any point of Γ_0–and so necessarily $(x-x_0)\cdot\nu(x)=0$ on Γ_0–we have upon combining (2.2) and (2.4), the estimate

$$\begin{aligned}&\int_{Q_\delta}|\nabla\phi|^2\,dQ\\ &\leq C_\epsilon\Big(\Big|\int_{\Sigma_{0,\delta}}\omega_t\big(h\cdot\nabla\phi\big)d\Sigma_0\Big|+\int_{\Sigma_{1,\delta}}\Big|\frac{\partial\phi}{\partial\tau}\Big|^2 d\Sigma_1+\int_{\Sigma_{1,\delta}}\phi_t^2 d\Sigma_1\Big)\\ &+C_\epsilon\int_{\Sigma_{0,\delta}}|\omega_t|^2\,d\Sigma_0+C_{h,\epsilon}\mathcal{E}_\phi(T)+C_{h,\epsilon}\int_0^T\Big|\langle\omega_t,\phi_t\rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\Big|\,dt\\ &+\mathsf{l.o.t.}(\vec{\phi},\vec{\omega})\end{aligned}\tag{2.5}$$

(in deriving this estimate, we have also implicitly used the relation (2.1)). Looking at the estimate in (2.5) it is manifest that the problematic terms therein are the tangential derivatives $\frac{\partial\phi}{\partial\tau}$ on the boundary Γ_0 (and also Γ_1), inasmuch as these terms are not bounded on the finite energy space H_1. In order to cope with these tangential derivatives we shall use two different sharp regularity results (on Γ_0 and Γ_1, respectively) which will allow the absorption of these unbounded traces.

To handle the first term on the right-hand side: Since $h(x)=x-x_0$, with x_0 being on Γ_0, then $h(x)$ is a scalar multiple of some (unit) vector $\tau(x)$, say, where τ is a vector tangent to Γ_0. In short, we have

$$\int_{\Sigma_{0,\delta}}\omega_t\,(h\cdot\nabla\phi)\,d\Sigma_0=C\int_{\Sigma_{0,\delta}}\omega_t\frac{\partial\phi}{\partial\tau}d\Sigma_0.\tag{2.6}$$

With the term $\frac{\partial\phi}{\partial\tau}$ in mind, we recall the following "sharp" regularity result for boundary traces of solutions to wave equations:

Lemma 9. (See [43], page 113, Theorem 3.1, Theorem 3.3(a); see also [45]). *Set the parameter* η *as*

$$\eta = \begin{cases} \frac{1}{4}, & \text{if } \Omega \text{ is a parallelepiped;} \\ \frac{1}{3}, & \text{if } \Omega \text{ is a smooth, bounded domain.} \end{cases}$$

With Γ_0 *being a flat portion of the boundary of* Ω*, then if* w *solves the wave equation*

$$\begin{cases} w_{tt} = \Delta w \quad \text{on } (0,T)\times\Omega \\ \begin{cases} \dfrac{\partial w}{\partial \nu} = 0, \text{ on } (0,T)\times\Gamma_1 \\ \dfrac{\partial w}{\partial \nu} = g \in L^2(0,T;H^\eta(\Gamma_0)) \quad \text{on } (0,T)\times\Gamma_0 \end{cases} \\ [w(0), w_t(0)] = [w_0, w_1] \in H^1(\Omega)\times L^2(\Omega), \end{cases}$$

one has the estimate

$$\|w\|_{L^2(0,T;H^{1-\eta}(\Gamma_0))} \le C_T \left(\|g\|_{L^2(0,T;H^\eta(\Gamma_0))} + \|[w_0,w_1]\|_{H^1(\Omega)\times L^2(\Omega)} \right).$$ [2]

Applying this estimate with $g = w_t|_\Gamma$ to (2.6) results then in

$$\int_0^T \left\| \frac{\partial \phi}{\partial \tau} \right\|^2_{H^{-\eta}(\Gamma_0)} dt \le C_T \Big(\mathcal{E}_\phi(T) + \int_0^T \|w_t\|^2_{H^\eta(\Gamma_0)}\, dt \Big);$$

consequently,

$$\begin{aligned} \int_{\Sigma_{0,\delta}} \omega_t\, (h\cdot\nabla\phi)\, d\Sigma_0 &= C \int_\delta^{T-\delta} \left\langle \omega_t, \frac{\partial \phi}{\partial \tau} \right\rangle_{H^\eta(\Gamma_0)\times H^{-\eta}(\Gamma_0)} dt \\ &\le C_T\, \|\omega_t\|_{L^2(0,T;H^\eta(\Gamma_0))} \left(\sqrt{\mathcal{E}_\phi(T)} + \|\omega_t\|_{L^2(0,T;H^\eta(\Gamma_0))} \right) \\ &\le \mathcal{E}_\phi(T) + \mathsf{l.o.t}(\vec{\phi}, \vec{\omega}). \end{aligned} \tag{2.7}$$

To deal with the second term on the right-hand side of (2.5), involving the tangential derivative of ϕ on Γ_1, the nature of the requisite estimate will be very different; it will necessitate an additional control (which is available to us) of wave velocity traces on Γ_1. More specifically we recall the following, microlocally derived, inequality:

Lemma 10. (See Lemma 7.21 of [32]). *Let* ϕ *be a solution of the wave equation on* $(0,T)\times\Omega$*, or more generally, any second-order hyperbolic equation*

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[2] As explicitly stated in Corollary 3.4(b), Theorem 3.3(a), we have $\{g, [w_0,w_1] = 0\} \in H^\eta(\Sigma) \implies w|_\Gamma \in H^{1-\eta}(\Sigma)$. However, in the details of the proof, it is evident that we have, continuously, $g \in L^2(0,T;H^\eta(\Gamma_0)) \implies w \in L^2(0,T;H^{1-\eta}(\Gamma_0))$.

with smooth space-dependent coefficients. Then, for all $\delta > 0$, we have the estimate

$$\int_{\delta}^{T-\delta}\int_{\Gamma_*}\Big(\frac{\partial\phi}{\partial\tau}\Big)^2 dtd\Gamma_* \\ \leq C_{T,\delta}\Big(\int_0^T\int_{\Gamma_*}\phi_t^2 dtd\Gamma_* + \int_0^T\int_{\Gamma}\Big(\frac{\partial\phi}{\partial\nu}\Big)^2 dtd\Gamma\Big) + \mathsf{l.o.t}(\vec{\phi},\vec{\omega}), \tag{2.8}$$

where Γ_* is any smooth connected segment of boundary Γ .

Applying the estimates in (2.7) and (2.8)–with $\Gamma_* = \Gamma_1$ therein–to the right-hand side of (2.5), and accounting for the boundary conditions, gives then

$$\int_{Q_\delta}|\nabla\phi|^2\, dQ \tag{2.9}$$
$$\leq C\Big(\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathcal{E}_\phi(T) + \int_0^T\Big|\langle\omega_t,\phi_t\rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\Big|dt\Big) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}),$$

where the constant C above does not depend on the terminal time $T > 0$.

Subsequently, we can combine this estimate with that in (2.3)–where, therein, $\tilde{h}$ is any vector field satisfying $div(\tilde{h}) = 1$–so as to have

$$\int_{\delta}^{T-\delta}\mathcal{E}_\phi(t)dt \\ \leq C\Big(\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathcal{E}_\phi(T) + \int_0^T\Big|\langle\omega_t,\phi_t\rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\Big|dt\Big) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}),$$

where, again, the constant C above does not depend on the time T. Using again the acoustic wave relation (2.1), we have then, for T large enough,

$$(T - C - 2\delta)\,\mathcal{E}_\phi(T) \\ \leq C_T\Big(\int_0^T\Big|\langle\omega_t,\phi_t\rangle_{H^{\frac{1}{2}}(\Gamma_0)\times H^{-\frac{1}{2}}(\Gamma_0)}\Big|dt + \int_{\Sigma_1}\phi_t^2 d\Sigma_1\Big) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).$$

Applying Lemma 8, we get, finally,

$$(T - C - 2\delta)\,\mathcal{E}_\phi(T) \\ \leq C_T\Big(\int_0^T\|\omega_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)}\,dt\Big)^{\frac{1}{2}}\Big\{\mathcal{E}_\phi(T) + \Big(\int_0^T\|\omega_t\|^2_{H^{\frac{1}{2}}(\Gamma_0)}\,dt\Big)\Big\}^{\frac{1}{2}} \\ + C_T\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}) \leq \delta\mathcal{E}_\phi(T) + C_T\int_{\Sigma_1}\phi_t^2 d\Sigma_1 + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).$$

By these means, we have thus derived the following:

Theorem 11. *For terminal time $T > 0$ large enough, the solution of the interior acoustic wave component in the coupled system* (1.13) *satisfies the following estimate:*

$$\mathcal{E}_\phi(T) \leq C_T \int_{\Sigma_1} \phi_t^2 d\Sigma_1 + l.o.t.(\vec{\phi}, \vec{\omega}). \tag{2.10}$$

3. Reconstruction of the energy for the plate component

In a fashion wholly analogous to that which led to the relation (2.1), we have, for all $0 \leq s \leq t \leq T$,

$$\mathcal{E}_\omega(t) = \mathcal{E}_\omega(s) - \int_s^t \langle \phi_t, \omega_t \rangle_{H^{-\frac{1}{2}}(\Gamma_0)\times H^{\frac{1}{2}}(\Gamma_0)} dt, \tag{3.1}$$

where, again, the topology of the duality pairing is validated by Lemma 8. In this section, we shall prove the following:

Theorem 12. *For terminal time $T > 0$, the plate component of* (1.13) *satisfies the following estimate for all $0 \leq t \leq T$:*

$$\mathcal{E}_\omega(t) \leq C_T \Big(\int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_0^T \left\| P_\gamma^{\frac{1}{2}}(a\omega_t) \right\|^2_{L^2(\Gamma_0)} dt \Big) + \text{l.o.t.}(\vec{\phi}, \vec{\omega}). \tag{3.2}$$

Proof of Theorem 12. To start, we set $\psi \equiv \Delta\omega$, where again ω is the plate variable in (1.13). This gives the boundary-value problem,

$$\begin{array}{rcl} \gamma\psi_{tt} - \Delta\psi &=& \phi_t|_{\Gamma_0} + \omega_{tt} \ \text{ on } (0,T)\times\Gamma_0 \\ \psi|_{\partial\Gamma_0} &=& 0 \ \text{ on } (0,T)\times\partial\Gamma_0 \\ [\psi(T), \psi_t(T)] &=& [\Delta\omega_0, \Delta\omega_1]. \end{array} \tag{3.3}$$

In terms of this problem, we will be reconstructing the energy for the variables ψ and ψ_t.

Step 1. Our change of variable to ψ suggests that the correct reconstruction of the energy for the wave equation (3.3) will not be undertaken at the level of the usual "energy space" $H^1(\Omega)\times L_2(\Omega)$, but rather the "negative energy" space $L_2(\Omega)\times H^{-1}(\Omega)$ (see[24, 23]). It is known that the derivation of observability estimates in "negative" energy spaces is a subtle technical problem in general, typically involving the invocation of a pseudodifferential–rather than differential–calculus). In our particular case, the relevant estimate can be obtained from a more general "negative" estimate which was derived in [4]. In fact, for the boundary-value problem in (3.3), we recall the following "negative energy recovery estimate" for the wave operator [4]. This estimate will be critical for observability estimates of the plate model under consideration.

Lemma 13. (See Theorem 4.2 of [4]). *Suppose that φ satisfies the following wave equation on $(0,T)\times\Gamma_0$, where Γ_0 is a (smooth) bounded and open set:*

$$\gamma\varphi_{tt}-\Delta\varphi-F(\phi)=\frac{d}{dt}f_1+f_2 \quad \text{on } (0,T)\times\Gamma_0.$$

Here, F is a first-order linear differential operator in time and space, with $L^\infty((0,T)\times\Gamma_0)$ coefficients. Moreover, the forcing terms $f_1\in L^2(0,T;L^2(\Gamma_0))\cap C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))$, and $f_2\in L^2(0,T;H^{-1}(\Gamma_0))$. Then for $T>2\gamma\, diam(\Omega)$, we have the estimate for all $t\in[0,T]$ and arbitrary $\epsilon>0$,

$$\begin{aligned}&\|\varphi(t)\|^2_{L^2(\Gamma_0)}+\|\varphi_t(t)\|^2_{H^{-1}(\Gamma_0)}\\ &\le C_{T,\epsilon}\Big\{\|\varphi\|^2_{L^2((0,T)\times\partial\Gamma_0)}+\Big\|\frac{\partial\varphi}{\partial n}\Big\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))}+\|\varphi\|^2_{H^{-1}((0,T)\times\Gamma_0)}\\ &+\int_0^T\Big[\|f_1\|^2_{L^2(\Gamma_0)}+\|f_2\|^2_{H^{-1}(\Gamma_0)}\Big]dt+\|f_1\|^2_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0)}\Big\}.\end{aligned} \tag{3.4}$$

(Here, $n(x)$ is the unit normal vector which is exterior to $\partial\Gamma_0$.)

Remark 14. We note that "standard" (see [23, 24] and references therein) observability estimates for the wave equation provide reconstructions of $H^1(\Gamma_0)\times L_2(\Gamma_0)$ norms of the solution from $L_2(\partial\Gamma_0\times(0,T))$ measurements of the normal derivatives on the boundary.

In addition, these earlier estimates apply to a wave equation with potential function which is represented by a zero-order differential operator. Thus, the novelty of the estimate in (3.4) is that (i) the reconstruction of the energy is on the negative scale of the energy $L_2(\Gamma_0)\times H^{-1}(\Gamma_0)$ in terms of negative norms $H^{-1}(\partial\Gamma_0\times(0,T))$ of normal derivatives, and (ii) the potential function is allowed to be a first-order differential operator. It is known that "multipliers method" [23] can not provide observability estimates for the wave equation with the potential function of the first order, unless the latter is assumed "sufficiently small".

Likewise, the "multipliers method" alone can not provide negative norm estimates for the wave equation (with or without the potential). In fact, first application and derivation of "negative norm" estimates for the wave operator were given in [32] in the context of uniform stabilization of the wave equation with Dirichlet boundary conditions. This was achieved by introducing a change of variables that involves a pseudodifferential operator of the order -1 defined in a cotangent bundle (time and space) [47]. Pseudodifferential calculus along with micro-local analysis lead to the desired estimate. Later, this method combined with Carleman's estimates was substantially extended in [46] to more general models of wave equations that also include first-order potentials. The proof of the estimate (3.4) given in [4] draws on these aforementioned techniques but it is more direct in addressing the equation of interest with a specific form of the forcing terms.

We proceed now to apply this estimate (3.4) to the wave equation in (3.3), with

$$f_1 \equiv \omega_t + \phi|_{\Gamma_0}\,;\; F = f_2 = 0.$$

Doing so, we obtain then, for all $0 \le t \le T$,

$$\begin{aligned}
&\|\psi(t)\|^2_{L^2(\Gamma_0)} + \|\psi_t(t)\|^2_{H^{-1}(\Gamma_0)} \\
\le\;& C_T\Big\{\Big\|\frac{\partial\psi}{\partial n}\Big\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))} + \|\psi\|^2_{H^{-1}((0,T)\times\Gamma_0)} \\
&+ \int_0^T \big\|\omega_t + \phi|_{\Gamma_0}\big\|^2_{L^2(\Gamma_0)}\,dt + \big\|\omega_t + \phi|_{\Gamma_0}\big\|^2_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))}\Big\} \\
\le\;& C_T\left\|\frac{\partial\psi}{\partial n}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0))} + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega})
\end{aligned} \tag{3.5}$$

(in writing down the last estimate, we are using the fact that

$$\big\|\phi|_{\Gamma_0}\big\|_{C([0,T];H^{-\frac{1}{2}+\epsilon}(\Gamma_0))}$$

and other terms are bounded by lower-order terms, by virtue of the Sobolev trace theorem).

Step 2. Apparently, we must estimate the term $\|\frac{\partial\psi}{\partial n}\|_{H^{-1}((0,T)\times\partial\Gamma_0)}$, inasmuch as this term is not bounded via any known interior regularity result, and subsequent application of trace theory. The key to our estimation argument will be the usage of an underlying "hidden regularity", on an appropriate negative energy space. To see this, we let $\tilde{a}(x)$ be a smooth cutoff function which is *identically one* on $\partial\Gamma_0$, and moreover satisfies

$$\operatorname{supp}(\tilde{a}) \subset \{x \in \operatorname{supp}(a) : a(x) = 1\} \tag{3.6}$$

(again, $a(x)$ is the locally-distributed coefficient which appears in (1.1) and (1.4)).

Subsequently, we multiply the boundary-value problem in (3.3) by the coefficient $\tilde{a}(x)$, and make the change of variable

$$\tilde{\psi} \equiv \tilde{a}(x)\psi. \tag{3.7}$$

As defined, we then have that $[\tilde{\psi}, \tilde{\psi}_t]$ solves the following problem:

$$\gamma\tilde{\psi}_{tt} - \Delta\tilde{\psi} = -\left(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi\right) + \tilde{a}\omega_{tt} + \tilde{a}\;\phi_t|_{\Gamma_0} \quad \text{on } (0,T)\times\Gamma_0$$

$$\widetilde{\psi}\Big|_{\partial\Gamma_0} = 0 \quad \text{on } (0,T)\times\partial\Gamma_0 \tag{3.8}$$
$$[\tilde{\psi}(T), t(\tilde{\psi})_t(T)] = [\tilde{a}(x)\Delta\omega_0, \tilde{a}(x)\Delta\omega_1].$$

Concerning this boundary-value problem, we recall the following "hidden regularity" result for the normal derivative of hyperbolic solutions, with initial data defined on $L^2(\Gamma_0)\times H^{-1}(\Gamma_0)$.

Lemma 15. (see [27], Theorem 2.3 and Remark 2.8, therein.) *Suppose that φ satisfies the following wave equation on $(0,T)\times\Gamma_0$, where Γ_0 is a (smooth) bounded and open set:*

$$\begin{array}{l} \gamma\varphi_{tt} - \Delta\varphi = \frac{d}{dt}f_1 + f_2 \quad on \ (0,T)\times\Gamma_0, \\ \varphi|_{\Gamma_0} = g \quad on \ (0,T)\times\partial\Gamma_0 \\ [\varphi(0),\varphi_t(0)] = [\phi_0,\varphi_1] \end{array}$$

where forcing terms $f_1 \in L^2(0,T;L^2(\Gamma_0))$, $f_2\in L^2(0,T;H^{-1}(\Gamma_0))$, boundary data $g\in L^2(0,T;L^2(\partial\Gamma_0))$ and initial data $[\phi_0,\varphi_1]\in L^2(\Gamma_0)\times H^{-1}(\Gamma_0)$. Then, continuously, we have

$$[\varphi,\varphi_t]\in C([0,T];L^2(\Gamma_0)\times H^{-1}(\Gamma_0)); \ \frac{\partial\varphi}{\partial\nu}\in H^{-1}((0,T)\times\partial\Gamma_0).$$

Applying the Lemma 15 to the boundary-value problem (3.8), with

$$f_1 \equiv \tilde{a}\left(\omega_t + \phi|_{\Gamma_0}\right); \quad f_2 \equiv -\left(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi\right),$$

we obtain the estimate, for all $0\le t\le T$,

$$\begin{aligned} &\left\|\left[\tilde{\psi}(t),\tilde{\psi}_t(t)\right]\right\|_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} + \left\|\frac{\partial\tilde{\psi}}{\partial\nu}\right\|^2_{H^{-1}((0,T)\times\partial\Gamma_0)} \\ \le\ & C_T\Big(\int_0^T \|\psi\Delta\tilde{a}+2\nabla\tilde{a}\cdot\nabla\psi\|^2_{H^{-1}(\Gamma_0)}\,dt + \int_0^T \left\|\tilde{a}\left(\omega_t+\phi|_{\Gamma_0}\right)\right\|^2_{L^2(\Gamma_0)}dt \\ & + \|[\tilde{a}(x)\Delta\omega_0,\tilde{a}(x)\Delta\omega_1]\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}\Big) \\ \le\ & C_T\Big(\int_0^T \|\psi\Delta\tilde{a}+2\nabla\tilde{a}\cdot\nabla\psi\|^2_{H^{-1}(\Gamma_0)}dt \\ & + \|[\tilde{a}(x)\Delta\omega_0,\tilde{a}(x)\Delta\omega_1]\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}\Big) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}). \end{aligned} \tag{3.9}$$

To refine the right-hand side of this inequality, we note (after recalling $\psi = \Delta\omega$), that

$$\int_0^T \|\psi\Delta\tilde{a}\|^2_{H^{-1}(\Gamma_0)}dt \le C\int_0^T \|\psi\Delta\tilde{a}\|^2_{L^2(\Gamma_0)}dt \le C_{\tilde{a}}\int_0^T\int_{\text{supp}(\tilde{a})}(\Delta\omega)^2 d\Gamma_0 dt. \tag{3.10}$$

Moreover, for any $\varphi \in H_0^1(\Gamma_0)$, we have

$$\langle \nabla \tilde{a} \cdot \nabla \psi, \varphi \rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} = -\int_{\Gamma_0} \psi \left(\nabla \tilde{a} \cdot \nabla \varphi + \varphi \Delta \tilde{a} \right) d\Gamma_0;$$

whence we obtain the norm estimate

$$\| \nabla \tilde{a} \cdot \nabla \psi \|_{H^{-1}(\Gamma_0)} \leq C_{\tilde{a}} \int_{\mathrm{supp}(\tilde{a})} (\Delta \omega)^2 \, d\Gamma_0. \tag{3.11}$$

Applying (3.10) and (3.11) to the right-hand side of (3.9) gives now, for all $0 \leq t \leq T$,

$$\begin{aligned} &\left\| \left[\tilde{\psi}(t), \tilde{\psi}_t(t) \right] \right\|_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)} + \left\| \frac{\partial \tilde{\psi}}{\partial \nu} \right\|^2_{H^{-1}((0,T) \times \partial \Gamma_0)} \\ &\leq C_{T,\tilde{a}} \Big(\int_0^T \int_{\mathrm{supp}(\tilde{a})} (\Delta \omega)^2 \, d\Gamma_0 dt + \| [\tilde{a}(x) \Delta \omega_0, \tilde{a}(x) \Delta \omega_1] \|^2_{L^2(\Gamma_0) \times H^{-1}(\Gamma_0)} \Big) \\ &+ \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \end{aligned} \tag{3.12}$$

Step 3. We proceed now to estimate the first term on the right-hand side of (3.12).

Proposition 16. *The plate variable of the system* (1.13) *satisfies the relation*

$$\begin{aligned} \int_0^T \| a \Delta \omega \|^2_{L^2(\Gamma_0)} \, dt \leq C \Big(&\int_0^T \left\| P_\gamma^{\frac{1}{2}} (a \omega_t) \right\|^2_{L^2(\Gamma_0)} dt \\ &+ \left| \left[\left\langle P_\gamma^{\frac{1}{2}} (a \omega_t), P_\gamma^{\frac{1}{2}} (a \omega) \right\rangle_{H^{-1}(\Gamma_0) \times H_0^1(\Gamma_0)} \right]_{t=0}^{t=T} \right| \Big) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \end{aligned}$$

Proof. The proof relies on a rather classical "partition of energy" type of argument. Applying the coefficient $a(x)$ to the plate component in (1.13), we obtain the system

$$\begin{aligned} &(a\omega)_{tt} - \gamma \Delta (a\omega)_{tt} + \Delta^2 (a\omega) + a \left. \phi_t \right|_{\Gamma_0} \\ &\qquad = \left[\Delta^2, a \right] \omega - \gamma \left(\omega_{tt} \Delta a + 2 \nabla a \cdot \nabla \omega_{tt} \right) \quad \text{on } \Gamma_0 \\ &\left. a\omega \right|_{\Gamma_0} = 0; \quad \left. \Delta (a\omega) \right|_{\Gamma_0} = \left. 2 \nabla a \cdot \nabla \omega \right|_{\partial \Gamma_0} \quad \text{on } \partial \Gamma_0 \\ &(a\omega(T), a\omega_t(T)) = (a\omega_0, a\omega_1) \end{aligned} \tag{3.13}$$

(above, as usual, $[P_1, P_2]$ denotes the action of commutation with respect to two pseudodifferential operators P_1,P_2 in $OPS^{s_1}(\mathbb{R}^2)$ and $OPS^{s_1}(\mathbb{R}^2)$, respectively). Thereto, we multiply both sides of the PDE by $a\omega$ and subsequently integrate in time and space. Integrating by parts, we eventually obtain the following relation:

$$\int_0^T \|\Delta\,(a\omega)\|^2_{L^2(\Gamma_0)}\,dt = \int_0^T \Big[\Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}$$
$$+\Big(\left. 2\nabla a\cdot\nabla\omega\right|_{\partial\Gamma_0}, \frac{\partial(a\omega)}{\partial n}\Big)_{L^2(\partial\Gamma_0)} + \big\langle\left[\Delta^2, a\right]\omega, a\omega\big\rangle_{[H^2(\Gamma_0)]'\times H^2(\Gamma_0)}\Big]dt$$
$$-\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} - \big(a\,\phi|_{\Gamma_0}, a\omega\big)_{L^2(\Gamma_0)}\Big]_{t=0}^{t=T}$$
$$+\int_0^T(\gamma(\Delta a)\omega_t + a\,\phi|_{\Gamma_0}, a\omega_t)_{L^2(\Gamma_0)}dt + 2\gamma\int_0^T\langle\nabla a\cdot\nabla\omega_t, a\omega_t\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}dt$$
$$-\Big[\gamma\left((\Delta a)\omega_t, a\omega\right)_{L^2(\Gamma_0)} + 2\gamma\left\langle\nabla a\cdot\nabla\omega_t, a\omega\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}.$$

Estimating the right-hand side of this expression, by making use of the known Sobolev regularity of the commutator–see e.g., [47]; in particular, $[\Delta^2, a]\in\mathcal{L}(H^1(\Gamma_0), [H^2(\Gamma_0)]')$–the Sobolev embedding theorem and Leibniz' formula, we have

$$\int_0^T \|a\Delta\omega\|^2_{L^2(\Gamma_0)}\,dt \le \epsilon\int_0^T \|a\omega\|^2_{H^2(\Gamma_0)}\,dt + C\int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}dt$$
$$+\left|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\right|$$
$$+\int_0^T\Big\|\left. 2\nabla a\cdot\nabla\omega\right|_{\partial\Gamma_0} + \frac{\partial(a\omega)}{\partial n}\Big\|^2_{L^2(\partial\Gamma_0)}dt + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega})$$
$$\le \epsilon\int_0^T \|a\Delta\omega\|^2_{L^2(\Gamma_0)}\,dt + \left|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\right|$$
$$+C\int_0^T\Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}dt + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}), \tag{3.14}$$

where in the last step we have used the fact that the map

$$\omega\to \left. 2\nabla a\cdot\nabla\omega\right|_{\partial\Gamma_0} + \frac{\partial(a\omega)}{\partial n}\in\mathcal{L}(H^{\frac{3}{2}+\epsilon}(\Gamma_0), H^{\epsilon}(\partial\Gamma_0)).$$

This concludes the proof of Proposition 16. □

Applying the last proposition to the estimate (3.12) gives now, for all $0\le t\le T$,

$$\Big\|\Big[\tilde{\psi}(t), \tilde{\psi}_t(t)\Big]\Big\|_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} + \Big\|\frac{\partial\tilde{\psi}}{\partial\nu}\Big\|^2_{H^{-1}((0,T)\times\partial\Gamma_0)}$$

$$\leq C_T \left\|[\tilde{a}(x)\Delta\omega_0, \tilde{a}(x)\Delta\omega_1]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} + C_T\Big(\int_0^T \left\|P_\gamma^{\frac{1}{2}}(a\omega_t)\right\|^2_{L^2(\Gamma_0)} dt$$

$$+ \left|\left[\left\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\right]_{t=0}^{t=T}\right|\Big) + \mathsf{l.o.t.}(\vec{\phi}, \vec{\omega}). \tag{3.15}$$

Step 4. To conclude the proof of Theorem 12, we must deal with the terminal and initial data on the right-hand side of (3.15). (In general, the reconstruction of initial and/or terminal data is a critical step in the derivation of any given observability estimate.) There is a technical complication in our case, owing to the fact that reconstruction of the energy for ψ takes place at the "dual" level, $L_2 \times H^{-1}$. To cope with this issue we shall apply an energy method, with an operator theoretic, i.e., nonlocal, multiplier. To this end, we multiply the wave equation in (3.8) by $A_D^{-1}\tilde{\psi}_t$, and integrate in time and space (where again, $\tilde{\psi} = \tilde{a}\Delta\omega$). This gives for all $0 \leq s \leq T$ (after an integration by parts),

$$\frac{1}{2}\left\|[\tilde{a}\Delta\omega_0, \gamma\tilde{a}\Delta\omega_1]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} = \frac{1}{2}\left\|\left[\tilde{\psi}(s), \gamma\tilde{\psi}_t(s)\right]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}$$

$$+ \int_s^T \left\langle \tilde{a}\omega_{tt} + \tilde{a}\ \phi_t|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau$$

$$- \int_s^T \left\langle \psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi, A_D^{-1}\tilde{\psi}_t\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau. \tag{3.16}$$

Now concerning the right-hand side of this expression, we have, after using the wave equation in (3.8),

$$\int_s^T \left\langle \tilde{a}\omega_{tt} + \tilde{a}\ \phi_t|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau$$

$$= -\int_s^T \left\langle \tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_{tt}\right\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau$$

$$+ \left[\left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t\right)_{L^2(\Gamma_0)}\right]_s^T$$

$$= -\frac{1}{\gamma}\int_s^T \Big(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, \frac{\partial}{\partial t}A_D^{-1}(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0})$$

$$- \tilde{\psi} - A_D^{-1}(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi)\Big)_{L^2(\Gamma_0)} d\tau + \left[\left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, A_D^{-1}\tilde{\psi}_t\right)_{L^2(\Gamma_0)}\right]_s^T$$

$$= -\frac{1}{2\gamma}\left[\left\|A_D^{-\frac{1}{2}}\left(\tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}\right)\right\|^2_{L(\Gamma_0)}\right]_s^T \tag{3.17}$$

$$+\int_s^T \left\langle \tilde{a}\omega_t + \tilde{a}\ \phi|_{\Gamma_0}, \tilde{a}\Delta\omega\right\rangle_{H^{\epsilon}(\Gamma_0)\times H^{-\epsilon}(\Gamma_0)} + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}) = \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}),$$

where, in this last step, we are using the fact that $\Delta \in \mathcal{L}(H^{2-\epsilon}(\Gamma_0), H^{-\epsilon}(\Gamma_0))$ (as well as the Sobolev trace theorem, applied to the term $\phi|_{\Gamma_0}$).

Moreover, for the third term on the right-hand side of (3.16),

$$\begin{aligned}
&\Big|\int_s^T \Big\langle \psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi, A_D^{-1}\tilde{\psi}_t\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)} d\tau\Big| \\
&= \Big|\int_s^T \left\langle A_D^{-1}\left(\psi\Delta\tilde{a} + 2\nabla\tilde{a}\cdot\nabla\psi\right), \Delta\left(\tilde{a}\omega_t\right) - \omega_t\Delta\tilde{a} - 2\nabla\tilde{a}\cdot\nabla\omega_t\right\rangle_{H_0^1(\Gamma_0)\times H^{-1}(\Gamma_0)} d\tau\Big| \\
&\leq C_{\tilde{a}}\Big(\int_0^T \Big(\int_{\mathrm{supp}(\tilde{a})} (\Delta\omega)^2 d\Gamma_0\Big)dt + \int_0^T \Big(\int_{\mathrm{supp}(\tilde{a})} \Big[|\omega_t|^2 + \gamma\,|\nabla\omega_t|^2\Big] d\Gamma_0\Big)dt\Big) \\
&+ \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}) \\
&\leq C\Big(\int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)} dt + \Big|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\Big|\Big) \\
&+ \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}),
\end{aligned} \tag{3.18}$$

after using the estimate (3.11) and Proposition 16. Applying (3.17) and (3.18) to the right-hand side of (3.16) gives then the estimate, for all $0 \leq s \leq T$,

$$\begin{aligned}
&\frac{1}{2}\left\|[\tilde{a}\Delta\omega_0, \gamma\tilde{a}\Delta\omega_1]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} \leq \frac{1}{2}\Big\|\Big[\tilde{\psi}(s), \gamma\tilde{\psi}_t(s)\Big]\Big\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} \\
&+ C_T\Big(\int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)} dt + \Big|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\Big|\Big) \\
&+ \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).
\end{aligned} \tag{3.19}$$

Integrating both sides of this relation from 0 to T we subsequently have

$$\begin{aligned}
&\left\|[\tilde{a}\Delta\omega_0, \gamma\tilde{a}\Delta\omega_1]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} \leq C_T\int_0^T \Big\|\Big[\tilde{\psi}(s), \gamma\tilde{\psi}_t(s)\Big]\Big\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)} ds \\
&+ C_T\Big(\int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)} dt + \Big|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\Big|\Big) \\
&+ \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}).
\end{aligned} \tag{3.20}$$

Estimating the right-hand side of (3.20) by another invocation of Proposition 16 gives now

$$\left\|[\tilde{a}\Delta\omega_0, \gamma\tilde{a}\Delta\omega_1]\right\|^2_{L^2(\Gamma_0)\times H^{-1}(\Gamma_0)}$$

$$\leq C_T\Big(\int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}dt + \Big|\Big[\Big\langle P_\gamma^{\frac{1}{2}}(a\omega_t), P_\gamma^{\frac{1}{2}}(a\omega)\Big\rangle_{H^{-1}(\Gamma_0)\times H_0^1(\Gamma_0)}\Big]_{t=0}^{t=T}\Big|\Big)$$
$$+ \text{l.o.t.}(\vec{\phi},\vec{\omega}). \tag{3.21}$$

Now combining (3.5), (3.15), and (3.21), with the fact that $\tilde{a} = 1$ on $\partial\Gamma_0$, we have for all $0 \leq t \leq T$,

$$\begin{aligned}
&\|A_D\omega(t)\|^2_{L^2(\Gamma_0)} + \Big\|P_\gamma^{\frac{1}{2}}\omega_t(t)\Big\|^2_{L^2(\Gamma_0)} \\
&\leq \Big(1 + \Big\|P_\gamma^{\frac{1}{2}}A_D^{-1}\Big\|^2_{\mathcal{L}(H^{-1}(\Gamma_0),L^2(\Gamma_0))}\Big)\Big\|\psi(t)\Big\|^2_{L^2(\Gamma_0)} + \|\psi_t(t)\|^2_{H^{-1}(\Gamma_0)} \qquad (3.22)\\
&\leq C_T\Big\|\frac{\partial\tilde{\psi}}{\partial n}\Big\|_{H^{-1}((0,T)\times\partial\Gamma_0)} + \text{l.o.t.}(\vec{\phi},\vec{\omega}) \\
&\leq \tilde{C}_T\Big(\Big|\Big[\Big(a\omega_t, P_\gamma(a\omega)\Big)_{L^2(\Gamma_0)}\Big]_0^T\Big| + \int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}dt\Big) + \text{l.o.t.}(\vec{\phi},\vec{\omega}).
\end{aligned}$$

To deal with the first term on the right-hand side of (3.22): Using the inequality $ab \leq \delta\frac{a^2}{2} + \frac{b^2}{2\delta}$, we have

$$\begin{aligned}
&\Big|\Big[(a\omega_t, P_\gamma(a\omega))_{L^2(\Gamma_0)}\Big]_0^T\Big| \\
&\leq C_a\Big(\|A_D\omega(T)\|_{L^2(\Gamma_0)}\|\omega_t(T)\|_{L^2(\Gamma_0)} + \|A_D\omega(0)\|_{L^2(\Gamma_0)}\|\omega_t(0)\|_{L^2(\Gamma_0)}\Big) \\
&+ \text{l.o.t.}(\vec{\phi},\vec{\omega}) \\
&\leq \frac{\epsilon}{\tilde{C}_T}\Big(\|A_D\omega(T)\|^2_{L^2(\Gamma_0)} + \|A_D\omega(0)\|^2_{L^2(\Gamma_0)}\Big) + \text{l.o.t.}(\vec{\phi},\vec{\omega}), \qquad (3.23)
\end{aligned}$$

where $\tilde{C}_T$ is the same constant as that in (3.22). Inserting this estimate into the right-hand side of (3.22), followed by two applications of the relation (3.1), gives finally, for fixed t,

$$\begin{aligned}
\mathcal{E}_\omega(t) \leq \epsilon\mathcal{E}_\omega(t) + C_T\Big(\int_0^T \Big|\langle\phi_t,\omega_t\rangle_{H^{-\frac{1}{2}}(\Gamma_0)\times H^{\frac{1}{2}}(\Gamma_0)}\Big|dt \\
+ \int_0^T \Big\|P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)}dt\Big) + \text{l.o.t.}(\vec{\phi},\vec{\omega}).
\end{aligned}$$

A final usage of Lemma 8 and Theorem 11 completes the proof of Theorem 12. □

Theorems 11 and 12 give now the (polluted) estimate, for $T > 0$ large enough,

$$\mathcal{E}(T) \leq C_T\Big(\int_{\Sigma_1} \phi_t^2 d\Sigma_1 + \int_0^T \Big\| P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)} dt\Big) + \mathsf{l.o.t.}(\vec{\phi},\vec{\omega}), \quad (3.24)$$

which is the continuous observability inequality (1.15), modulo lower-order terms.

4. Completion of the proof of Theorem 6

The argument to eliminate the lower-order terms in the estimate (3.24) is by now standard; it is an argument by contradiction which makes use of the classic Holmgren's uniqueness theorem (see, e.g., [19]). But, for the sake of completeness, we sketch out the argument here.

Lemma 17. *For T large enough, there exists a constant $C_T > 0$ such that the solution of* (1.13) *satisfies the following inequality:*

$$\begin{aligned}&\Big\|\Big[\vec{\phi},\vec{\omega}\Big]\Big\|^2_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))}\\ &\leq C_T\Big(\|\phi_t\|^2_{L^2(\Sigma_1)} \int_0^T \Big\| P_\gamma^{\frac{1}{2}}(a\omega_t)\Big\|^2_{L^2(\Gamma_0)} dt\Big). \end{aligned} \quad (4.1)$$

Proof. Suppose the given inequality is false. Then there exists a sequence of initial data $\{[\vec{\phi}_0^{(n)},\vec{\omega}_0^{(n)}]\} \subset \mathbf{H}$ and a corresponding solution sequence $\{[\vec{\phi}^{(n)},\vec{\omega}^{(n)}]\}$ of the PDE (1.13), for which $\|\phi_t^{(n)}\|_{L^2(\Sigma_1)} < +\infty$ for all n, and which moreover satisfies

$$\begin{aligned}&\Big\|\Big[\vec{\phi}^{(n)},\vec{\omega}^{(n)}\Big]\Big\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} = 1 \quad \forall n;\\ &\Big\|\phi_t^{(n)}\Big\|^2_{L^2(\Sigma_1)} + \int_0^T \Big\| P_\gamma^{\frac{1}{2}}(a\omega_t^{(n)})\Big\|^2_{L^2(\Gamma_0)} dt \to 0 \quad \text{as } n\to\infty. \end{aligned} \quad (4.2)$$

Consequently, the existence of the inequality (3.24), for T large enough, and the convergence in (4.2), imply that the sequence $\{\|[\vec{\phi}_0^{(n)},\vec{\omega}_0^{(n)}]\|_{\mathbf{H}}\}_{n=1}^{\infty}$ is bounded uniformly in n. Consequently, there is a subsequence, still denoted by$\{[\vec{\phi}_0^{(n)},\vec{\omega}_0^{(n)}]\}_{n=1}^{\infty}$, and $[\vec{\phi}_0^{*},\vec{\omega}_0^{*}] \in \mathbf{H}$ such that

$$[\vec{\phi}_0^{(n)},\vec{\omega}_0^{(n)}] \to [\vec{\phi}_0^{*},\vec{\omega}_0^{*}] \quad \text{in } \mathbf{H} \text{ weakly.} \quad (4.3)$$

If we denote $[\vec{\phi}^{*},\vec{\omega}^{*}]$ as the solution pair of (1.13), corresponding to terminal data $[\vec{\phi}_0^{*},\vec{\omega}_0^{*}]$, then *a fortiori*

$$\{[\vec{\phi}^{(n)},\vec{\omega}^{(n)}]\} \to [\vec{\phi}^{*},\vec{\omega}^{*}] \text{ in } L^\infty(0,T;\mathbf{H}) \text{ weak star.} \quad (4.4)$$

Moreover, reading off the ϕ-wave equation in (1.13), and using the weak convergence of $\{\phi^{(n)}\}$, we deduce the estimate

$$\left\|\phi_{tt}^{(n)}\right\|_{C([0,T];[H^1(\Omega)]')} \leq C, \text{ uniformly in } n. \tag{4.5}$$

Similarly, using the plate equation in (1.13) in conjunction with elliptic theory, Lemma 8, and the uniform boundedness of $\{\|[\vec{\phi}_0^{(n)},\vec{\omega}_0^{(n)}]\|_{\mathbf{H}}\}_{n=1}^{\infty}$ and $\{[\vec{\phi}^{(n)},\vec{\omega}^{(n)}]\}$, we have

$$\left\|\omega_{tt}^{(n)}\right\|_{L^2(0,T;L^2(\Gamma_0))} \leq C, \text{ uniformly in } n. \tag{4.6}$$

From (4.4)–(4.6) and a classic compactness result of J. Simon in [44], we conclude that $[\vec{\phi}^{(n)},\vec{\omega}^{(n)}] \to [\vec{\phi}^*,\vec{\omega}^*]$ strongly, in $C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))$. In consequence, we have from (4.2) the equality

$$\left\|\vec{\phi}^*,\vec{\omega}^*\right\|_{C([0,T];H^{1-\epsilon}(\Omega)\times H^{-\epsilon}(\Omega)\times H^{2-\epsilon}(\Gamma_0)\times H^{1-\epsilon}(\Gamma_0))} = 1. \tag{4.7}$$

Furthermore, given the convergence in (4.2), we conclude that

$$\phi_t^*|_{\Sigma_1} = 0 \text{ on } \Sigma_1;\ \ \omega_t^* = 0 \text{ on } \operatorname{supp}(a). \tag{4.8}$$

But these relations immediately yield up a contradiction by Holmgren's uniqueness theorem. In fact, setting $p \equiv \phi_t^*$, then p solves the overdetermined system

$$\begin{aligned} p_{tt} &= \Delta p \text{ on } Q \\ \left.\frac{\partial p}{\partial \nu}\right|_{\Gamma_1} &= 0 \text{ on } \Sigma_1 \\ p|_{\Gamma_1} &= 0 \text{ on } \Sigma_1 \\ a(x)\left.\frac{\partial p}{\partial \nu}\right|_{\Gamma_0} &= 0 \text{ on } \Sigma_0. \end{aligned}$$

By Holmgren's uniqueness theorem, we infer then that $\phi_t^* = p = 0$, for $T > 0$ large enough. Furthermore, the variable $\tilde{p} \equiv \omega_t^*$ solves the plate equation

$$\begin{aligned} (I - P_\gamma)\,\tilde{p}_{tt} + \Delta^2\tilde{p} &= 0 \text{ on } (0,T)\times\Gamma_0 \\ \tilde{p}|_{\partial\Gamma_0} &= \Delta\tilde{p}|_{\partial\Gamma_0} = 0 \text{ on } (0,T)\times\partial\Gamma_0 \\ a(x)\tilde{p} &= 0 \text{ on } (0,T)\times\operatorname{supp}(a). \end{aligned}$$

Again, for $T > 0$, we have that necessarily $\omega_t^* = \tilde{p} = 0$ From this and the ellipticity of A_D^2, we conclude further that $\omega^* = 0$ on Σ_0. Thus, from elliptic theory $\vec{\phi}^* = 0$ and $\vec{\omega}^* = 0$, which contradicts the relation (4.7). This concludes the proof of Lemma 17. □

The proof of Theorem 6 (or equivalently, the derivation of the estimate (1.15)) is now completed by combining the estimate (3.24) with Lemma 17. In turn, by the argument we gave at the beginning of the paper, Theorem 6 and Lemma 4 will yield the exact controllability of the original "purely Neumann" problem (1.1).

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