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NONLINEAR LIOUVILLE THEOREMS AND A PRIORI ESTIMATES FOR INDEFINITE SUPERLINEAR ELLIPTIC EQUATIONS

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Abstract. We establish two general nonlinear Liouville theorems for equations of the type

$$-\Delta u = h(x_1)f(u),\ u \geq 0 \ \text{ in } R^N,\ \sup_{R^N} u < +\infty.$$

We then show how these Liouville theorems can be used to obtain a priori estimates for positive solutions of indefinite superlinear elliptic equations for several new cases left open in previous research.

1. Introduction

In this paper, by proving some general nonlinear Liouville theorems, we establish a priori estimates for positive solutions of the indefinite superlinear elliptic problem

$$-\Delta u = \lambda u + a(x)u^p \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \tag{1.1}$$

where $p > 1$, λ is a parameter, a is a continuous sign-changing function, and Ω is a bounded smooth domain in R^N $(N \geq 2)$. As will be explained later, our results actually hold for much more general problems than (1.1).

Problem (1.1) and its many variants have been extensively studied recently; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 20, 22, 25, 28] and the references therein. In order to prove existence and multiplicity of positive solutions to (1.1), it is important to obtain a priori bounds for its positive

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solutions. If we denote $\Omega^+ = \{x \in \Omega : a(x) > 0\}$, $\Omega^0 = \{x \in \Omega : a(x) = 0\}$, $\Omega^- = \{x \in \Omega : a(x) < 0\}$, then by using results of Keller [19] and Osserman [24], it is easy to see that the positive solutions of (1.1) are bounded over compact subsets of Ω^- (for any $p > 1$, see Lemma 3.8 in Section 3 later). There also exist powerful techniques for ruling out the possibility that the maximum of the solution blows up in compact subsets of Ω^+. For example, the blow-up technique of Gidas and Spruck [17] can be used for such a purpose when $p < N^*$, where $N^* = (N+2)/(N-2)$ if $N \geq 3$, and $N^* = \infty$ when $N = 2$; while for $N \geq 3$ and $p = (N+2)/(N-2)$, various sufficient conditions on $a(x)$ are also known (see [8] Section 4 and the references cited there).

In sharp contrast, no effective method for obtaining a priori bounds over a neighborhood of Ω^0 had been known until [5], where Berestycki, Capuzzo-Docetta and Nirenberg provided a sufficient condition in terms of the exponent p and the behavior of $a(x)$ near Ω^0. Their condition on $a(x)$ is the following:

$$a \in C^2(\overline{\Omega}), \quad \Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega, \quad |\nabla a(x)| \neq 0 \quad \forall x \in \Omega^0. \tag{1.2}$$

Clearly this implies that Ω^0 is a C^2 manifold of dimension $(N-1)$ in the interior of Ω. Under condition (1.2), a blowing up argument reduces the a priori bound problem to the following entire space problem

$$-\Delta u = x_1 u^p, \quad u \geq 0 \quad \text{in } R^N. \tag{1.3}$$

If (1.3) has no bounded positive solution and $p \in (1, N^*)$, then a priori bounds for (1.1) near Ω^0 can be easily obtained. By proving a general Liouville theorem, it was shown in [5] that (1.3) has no positive solution when $p < (N+2)/(N-1)$, and hence a priori bounds near Ω^0 can be obtained for such p when (1.2) holds.

Later, W. Chen and C. Li in [8] proved, by a novel moving plane argument, that if (1.2) holds, then a priori bounds for (1.1) near Ω^0 can be established for any $p > 1$.

More recently, by refining the moving plane argument in [8], Chen and Li in [9] further relaxed the restriction on $a(x)$ near Ω^0. They established a priori bounds for (1.1) near Ω^0 with a general $p > 1$ by assuming that

(i) Ω^0 is a smooth manifold of dimension $(N-1)$ in the interior of Ω,
(ii) $a(x)$ is $C^{2,\alpha}$ and is monotone near Ω_0, and
(iii) $a(x)/|\nabla a(x)|$ vanishes on Ω^0 and is continuous near Ω^0.

Using these results, one can easily show that positive solutions of (1.1) are a priori bounded over Ω if $1 < p < N^*$ and if $a(x)$ satisfies conditions (i)-(iii) above.

On the other hand, by making use of the Liouville theorem in [5] and some new observations, Amann and Lopez-Gomez [4] improved the results of [5] in a different direction. Among other things they proved that if

$$\begin{cases} \Omega^0 \subset\subset \Omega,\ \partial(int\Omega^0),\ \partial\Omega^+ \text{ and } \partial\Omega^- \text{ are smooth, and} \\ a(x) = \alpha(x)[dist(x, \partial\Omega^+)]^\gamma \text{ in } \Omega^+ \end{cases} \tag{1.4}$$

for some $\gamma > 0$ and some continuous function $\alpha(x)$ which is positive near $\partial\Omega^+$, and if

$$p < \min\{\frac{N+1+\gamma}{N-1}, N^*\}, \tag{1.5}$$

then a priori bounds for (1.1) over Ω can be established.

Note that condition (1.4) allows Ω^0 to have nonempty interior in R^N, which is not allowed under (1.2) or the conditions in [9]. Moreover, in (1.4) there is no restriction on $a(x)$ when x approaches Ω^0 from Ω^-.

One naturally wonders whether the result of [4] remains true if (1.5) is relaxed to $p < N^*$.

The main purpose of this paper is to give a positive answer to this question, under the following extra condition:

$$a(x) = \alpha_1(x)[dist(x, \partial\Omega^+)]^{\gamma_1} \text{ in } \Omega^-, \tag{1.6}$$

for some $\gamma_1 > 0$ and some continuous function $\alpha_1(x)$ which is positive near $\partial\Omega^+$.

A key ingredient in our approach is a nonlinear Liouville theorem, which was motivated by the following Liouville theorem proved by C.S. Lin in [22] (see also a special case in [29]):

Theorem A. *If m is an odd positive integer, then the problem*

$$-\Delta u = x_1^m u^{N^*},\ u \geq 0 \quad in\ R^N (N \geq 3)$$

has only the trivial solution $u \equiv 0$.

It turns out that the proof of Lin for Theorem A can be further developed to establish a general nonlinear Liouville theorem, which will enable us to relax (1.5) to $p < N^*$ under (1.4) and (1.6). Note that we allow Ω^0 to have nonempty interior and $a(x)$ to be less smooth than $C^{2,\alpha}$, a situation not covered in [9]. In the variational approach, the so called "thick zero set" condition: $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$, has proven useful in establishing existence theorems for (1.1). In such a case, Ω^0 has nonempty interior. But our results here can deal with cases where Ω^0 has nonempty interior but the thick zero set condition is violated. Moreover, as explained below, our a priori bounds results hold for situations where the problem does not have a variational structure, for example, when Δ is replaced by a non-self-adjoint elliptic operator.

To give a taste of our general nonlinear Liouville theorem in Section 2, we state here a special case.

Theorem B. *Suppose that $p > 1$ and $h(t) = t|t|^\alpha$ or $h(t) = (t^+)^\alpha$ for some $\alpha > 0$, where $t^+ = \max\{t, 0\}$. Then the problem*

$$-\Delta u = h(x_1)u^p,\ u \geq 0 \ \text{ in } R^N,\ \sup_{R^N} u < +\infty$$

has only the trivial solution $u \equiv 0$.

In Section 3, we apply our Liouville theorem to establish a priori estimates for positive solutions of (1.1). By applying standard techniques, our method here can be extended to study problems more general then (1.1); for example, the Laplace operator Δ can be replaced by a general second-order elliptic operator, and u^p can be replaced by a class of general nonlinear functions $f(u)$, but in order to keep the exposition simple and to present the ideas more clearly, we have refrained from doing so. Moreover, we will not use our a priori bounds to discuss the existence and multiplicity problem here, as this is standard and follows the same arguments as in [5] and [4].

Though our Liouville theorems hold for any $p > 1$, for a priori bounds, we need the restriction $1 < p < N^*$. We believe that our method can be modified and further developed to discuss the critical exponent case $p = N^*$.

We refer to [11, 15, 21, 23, 26, 27] and the references therein for discussions of other Liouville theorems.

2. Nonlinear Liouville theorems

In this section, we prove two general nonlinear Liouville theorems.

Theorem 2.1. *Suppose that $h \in C^\alpha(R) \cap C^1(R \setminus \{0\})$ for some $\alpha \in (0, 1)$ and satisfies*

(H_1) *$h'(t) \geq 0$ for $t \neq 0$, $h'(t) > 0$ for $t > 0$, and*
(H_2) *$h(0) = 0$, $\overline{\lim}_{t \to 0^+} h(t)/h'(t) < +\infty$.*

Let $f(u)$ be a C^1 function of u for $u \in [0, +\infty)$, and satisfy

(F_1) *$f(0) = f'(0) = 0$, $\overline{\lim}_{u \to 0^+} f'(u)u/f(u) < +\infty$ and*
(F_2) *$f(u)$ is positive and nondecreasing for $u > 0$.*

Then any solution $u \in C^2(R^N)$ of the problem

$$-\Delta u = h(x_1)f(u),\ u \geq 0 \ \text{ in } R^N,\quad \sup_{R^N} u < +\infty \tag{2.1}$$

is either identically zero or strictly increasing in x_1. *Moreover, if we assume further that* $h(t) \to +\infty$ *as* $t \to +\infty$, *then* (2.1) *has only the trivial solution* $u \equiv 0$.

Clearly, if $h(t) = t|t|^{m-1}$ or $h(t) = (t^+)^m$ for some $m > 0$, and $f(u) = u^p$ for some $p > 1$, then all the conditions in Theorem 2.1 are satisfied and (2.1) has only the trivial solution $u \equiv 0$. Therefore Theorem B is a direct consequence of Theorem 2.1.

Proof. We use a moving plane argument inspired by [22]. Following conventional notations, we write, for any real number λ,

$$T_\lambda = \{x \in R^N : x_1 = \lambda\},\ \Sigma_\lambda = \{x \in R^N : x_1 < \lambda\},$$

$$x^\lambda = (x_1^\lambda, x_2, ..., x_N) = (2\lambda - x_1, x_2, ..., x_N),\ u^\lambda(x) = u(x^\lambda).$$

Clearly x^λ is the reflection of x in the hyperplane T_λ.

By the strong maximum principle, any solution of (2.1) is either identically zero or strictly positive in R^N. Let us assume that (2.1) has a positive solution u; we will make use of the family of functions $w_\lambda(x) = u(x^\lambda) - u(x)$ to show that it is strictly increasing in x_1. Our proof below is divided into several steps with each step proving a particular claim.

Claim 1: For $x \in \Sigma_\lambda$ and $\lambda \le 0$, $w_\lambda(x) \le 0$ implies $\Delta w_\lambda(x) \le 0$.

Indeed, for such x, $x_1 < \lambda \le 0$. If $x_1 \le 2\lambda$, then $x_1^\lambda \ge 0$ and hence $h(x_1^\lambda) \ge 0 \ge h(x_1)$. It follows that

$$\Delta w_\lambda(x) = h(x_1)f(u(x)) - h(x_1^\lambda)f(u(x^\lambda)) \le 0.$$

If $x_1 > 2\lambda$, then $x_1 < x_1^\lambda < 0$ and so

$$\Delta w_\lambda(x) = h(x_1)f(u(x)) - h(x_1^\lambda)f(u(x^\lambda)) \le h(x_1^\lambda)[f(u(x)) - f(u(x^\lambda))].$$

By assumption, $u(x) \ge u(x^\lambda)$ and hence $f(u(x)) - f(u(x^\lambda)) \ge 0$. As $h(x_1^\lambda) \le 0$, it follows that $\Delta w_\lambda(x) \le 0$. This proves Claim 1.

Claim 2: If $\lambda \le 0$, then $w_\lambda(x) \ge 0$ for every $x \in \Sigma_\lambda$.

Assume by way of contradiction that, for some $\lambda \le 0$,

$$\inf_{x \in \Sigma_\lambda} w_\lambda(x) < 0. \tag{2.2}$$

Define

$$g_\lambda(x) = \ln(\lambda + 3 - x_1) + \Sigma_{j=2}^N \ln[(\lambda + 3 - x_1)^2 + x_j^2],\ x \in \Sigma_{\lambda+1}.$$

A direct calculation shows that the positive function $g_\lambda(x)$ satisfies

$$\Delta g_\lambda(x) = -(\lambda + 3 - x_1)^{-2} < 0 \text{ in } \Sigma_{\lambda+1}.$$

Let $\overline{w}_\lambda(x) = w_\lambda(x)/g_\lambda(x)$. Since $g_\lambda(x) \to +\infty$ as $|x| \to +\infty$ in $\Sigma_{\lambda+1}$, by (2.2),

$$m := \inf_{x\in\Sigma_\lambda} \overline{w}_\lambda(x) < 0,$$

and m is achieved at some $x_0 \in \Sigma_\lambda$. Therefore,

$$\nabla\overline{w}_\lambda(x_0) = 0,\ \Delta\overline{w}_\lambda(x_0) \ge 0.$$

On the other hand, by Claim 1, $\Delta w_\lambda(x_0) \le 0$ and hence

$$\begin{aligned} 0 \le g_\lambda(x_0)\Delta\overline{w}_\lambda(x_0) &= \Delta w_\lambda(x_0) - 2\nabla g_\lambda(x_0)\cdot\nabla\overline{w}_\lambda(x_0) - \overline{w}_\lambda(x_0)\Delta g_\lambda(x_0) \\ &= \Delta w_\lambda(x_0) + m(\lambda + 3 - x_1^0)^{-2} < 0. \end{aligned}$$

This contradiction proves Claim 2.

Let us define $\lambda_* = \sup\{\lambda \in R : w_\mu(x) \ge 0 \text{ in } \Sigma_\mu \text{ for every } \mu \le \lambda\}$. By Claim 2, we find $\lambda_* \ge 0$. We are going to show that $\lambda_* = \infty$. But first, we prove the following result.

Claim 3: If $\lambda_* < +\infty$ and $\lambda_0 \in (-\infty, \lambda_*]$, or $\lambda_* = +\infty$ and $\lambda_0 \in (-\infty, +\infty)$, then

$$w_{\lambda_0}(x) > 0 \text{ in } \Sigma_{\lambda_0},\ \ \tfrac{\partial}{\partial x_1} w_{\lambda_0}(x) < 0 \text{ for } x \in T_{\lambda_0}.$$

By the definition of λ_* and by continuity (in case $\lambda_0 = \lambda_* < +\infty$), we have $w_{\lambda_0}(x) \ge 0$ in Σ_{λ_0}. We now show that

$$\Delta w_{\lambda_0}(x) + a(x) w_{\lambda_0}(x) \le 0 \text{ in } \Sigma_{\lambda_0},$$

where

$$a(x) = \begin{cases} h(2\lambda_0 - x_1)\frac{f(u(x)) - f(u^{\lambda_0}(x))}{u(x) - u^{\lambda_0}(x)}, & \text{if } w_{\lambda_0}(x) > 0, \\ h(2\lambda_0 - x_1) f'(u(x)), & \text{if } w_{\lambda_0}(x) = 0. \end{cases}$$

Indeed, for $x \in \Sigma_{\lambda_0}$, since $x_1 \le 2\lambda_0 - x_1$, we have

$$\begin{aligned} \Delta w_{\lambda_0}(x) &= h(x_1) f(u(x)) - h(2\lambda_0 - x_1) f(u^{\lambda_0}(x)) \\ &\le h(2\lambda_0 - x_1)[f(u(x)) - f(u^{\lambda_0}(x))] = -a(x) w_{\lambda_0}(x), \end{aligned}$$

as we wanted.

Suppose by way of contradiction that $w_{\lambda_0}(x_0) = 0$ for some $x_0 = (x_1^0, \tilde{x}^0) \in \Sigma_{\lambda_0}$; i.e., w_{λ_0} achieves its minimum in Σ_{λ_0} at some interior point x_0. Since $\Delta w_{\lambda_0}(x) - a(x) w_{\lambda_0}(x) \le 0$ in Σ_{λ_0}, by the strong maximum principle, we necessarily have $w_{\lambda_0}(x) \equiv 0$ in Σ_{λ_0}. We now choose $x^* = (x_1^*, \tilde{x}^*) \in \Sigma_{\lambda_0}$ such that $x_1^* < 0 < 2\lambda_0 - x_1^*$, and find

$$\begin{aligned} 0 = \Delta w_{\lambda_0}(x^*) &= h(x_1^*) f(u(x^*)) - h(2\lambda_0 - x_1^*) f(u^{\lambda_0}(x^*)) \\ &= [h(x_1^*) - h(2\lambda_0 - x_1^*)] f(u(x^*)) < 0. \end{aligned}$$

This contradiction shows that we always have $w_{\lambda_0}(x) > 0$ in Σ_{λ_0}.

To show the second inequality in Claim 3, we use the strong maximum principle again. Indeed, it follows from

$$\Delta w_{\lambda_0} - a(x)w_{\lambda_0} \le 0,\ w_{\lambda_0} > 0 \text{ in } \Sigma_{\lambda_0},\ w_{\lambda_0} = 0 \text{ on } \partial\Sigma_{\lambda_0} = T_{\lambda_0},$$

and the Hopf boundary lemma, that $\frac{\partial}{\partial x_1} w_{\lambda_0}(x) < 0$, for all $x \in T_{\lambda_0}$. This finishes the proof of Claim 3.

We are now ready to state and prove a key step in the proof of $\lambda_* = +\infty$. Suppose $\lambda_* < +\infty$. Then, by the definition of λ_*, there exists a sequence $\{\lambda_j\}$ satisfying $\lambda_j > \lambda_*$, $\lim_{j\to+\infty} \lambda_j = \lambda_*$, $\inf_{x\in\Sigma_{\lambda_j}} w_j(x) < 0$, where we have used the notation $w_j = w_{\lambda_j}$.

Claim 4: For all large j and each $x \in \Sigma_{\lambda_j}$, $w_j(x) \le 0$ implies $\Delta w_j(x) < 0$.

Suppose that Claim 4 is not true. Then, by passing to a subsequence, we may assume that for each j, there exists $x^j \in \Sigma_{\lambda_j}$ such that

$$w_j(x^j) \le 0,\ \Delta w_j(x^j) \ge 0. \tag{2.3}$$

Denote $x^j = (x_1^j, \tilde{x}^j)$. Since $\lambda_j > \lambda_* \ge 0$ and $x_1^j < \lambda_j$, we have $2\lambda_j - x_1^j > 0$. By (2.3),

$$h(x_1^j) f(u(x^j)) - h(2\lambda_j - x_1^j) f(u^{\lambda_j}(x^j)) = \Delta w_j(x^j) \ge 0.$$

It follows that $h(x_1^j) > 0$ and hence $x_1^j > 0$. Therefore, $x_1^j \in (0, \lambda_j)$ forms a bounded sequence. If $\{\tilde{x}^j\}$ is also bounded, then by passing to a further subsequence, we may assume $x^j \to x^0$ as $j \to +\infty$. There are two cases to consider now:

(a) $x^0 \in \Sigma_{\lambda_*}$, (b) $x^0 \in \partial\Sigma_{\lambda_*} = T_{\lambda_*}$.

In case (a) we find $w_{\lambda_*}(x^0) = \lim_{j\to+\infty} w_j(x^j) \le 0$, a contradiction to Claim 3. In case (b), it follows from $w_j(\lambda_j, \tilde{x}^j) = 0 \ge w_j(x_1^j, \tilde{x}^j)$ that $\frac{\partial}{\partial x_1} w_j(\overline{x}_1^j, \tilde{x}^j) \ge 0$ for some $\overline{x}_1^j \in (x_1^j, \lambda_j)$. Since $x_1^j \to x_1^0 = \lambda_*$ and $\lambda_j \to \lambda_*$ as $j \to \infty$, we necessarily have $\overline{x}_1^j \to \lambda_*$ as $j \to +\infty$. It follows that $\frac{\partial}{\partial x_1} w_{\lambda_*}(x^0) \ge 0$, a contradiction to Claim 3 too. Therefore $\{\tilde{x}^j\}$ must be unbounded.

Without loss of generality, we may assume that $|\tilde{x}^j| \to +\infty$ as $j \to +\infty$. Define $\overline{u}_j(x) = u(x + (0, \tilde{x}^j))$. By the boundedness of u and standard elliptic estimates, $\{\overline{u}_j\}$ is bounded in $C^{2,\alpha}_{loc}(R^N)$. Thus, by passing to a subsequence, $\overline{u}_j$ converges to a nonnegative function $\overline{u}$ in $C^2_{loc}(R^N)$ and $\overline{u}$ satisfies

$$-\Delta \overline{u} = h(x_1) f(\overline{u}) \text{ in } R^N.$$

By the strong maximum principle, we have either $\overline{u}(x) > 0$ in R^N, or $\overline{u} \equiv 0$. By Claim 3, we have $\overline{u}_j^{\lambda_*}(x) \ge \overline{u}_j(x)$ in Σ_{λ_*}, and hence

$$\overline{w}_{\lambda_*}(x) := \overline{u}^{\lambda_*}(x) - \overline{u}(x) \ge 0,\ \forall x \in \Sigma_{\lambda_*}.$$

Now the argument in the proof of Claim 3 can be applied to $\overline{u}$ to conclude that unless $\overline{u} \equiv 0$, we have

$$\overline{w}_{\lambda_*}(x) > 0 \ \forall x \in \Sigma_{\lambda_*}; \ \ \frac{\partial}{\partial x_1}\overline{w}_{\lambda_*}(x) < 0 \ \forall x \in T_{\lambda_*}. \tag{2.4}$$

We show next that (2.4) cannot happen and hence $\overline{u} \equiv 0$. In fact, if $\lim_{j\to+\infty} x_1^j = x_1^0 < \lambda_*$, then

$$\overline{w}_{\lambda_*}(x_1^0, 0, ..., 0) = \lim_{j\to+\infty} w_j(x^j) \le 0,$$

which violates the first part of (2.4). If $\lim_{j\to+\infty} x_1^j = x_1^0 = \lambda_*$, then due to $x_1^j < \lambda_j$ and $w_j(x_1^j, \tilde{x}^j) \le 0 = w_j(\lambda_j, \tilde{x}^j)$, we can find $t_j \in (x_1^j, \lambda_j)$ such that $\frac{\partial}{\partial x_1} w_j(t_j, \tilde{x}^j) \ge 0$. It follows that

$$\frac{\partial}{\partial x_1}\overline{w}_{\lambda_*}(\lambda_*, 0) \ge 0,$$

a contradiction to the second part of (2.4). Therefore we must have $\overline{u} = 0$; in other words, $\overline{u}_j \to 0$ in $C^2_{loc}(R^N)$. Let $v_j(x) = \overline{u}_j(x)/\overline{u}_j(0)$. Then v_j satisfies

$$-\Delta v_j = h(x_1) f(\overline{u}_j(x))/\overline{u}_j(0).$$

Using the equation for $\overline{u}_j$ and the Harnack inequality, we find that for any compact subset K of R^N, there exists a constant $C_K > 0$ such that $\overline{u}_j(x) \le C_K \overline{u}_j(0)$, $\forall x \in K$. It follows from our assumptions on f that

$$f(\overline{u}_j(x))/\overline{u}_j(0) \le f(C_K \overline{u}_j(0))/\overline{u}_j(0) \to C_K f'(0) = 0 \text{ as } j \to +\infty.$$

Thus, on any compact subset of R^N, $\{v_j\}$ is uniformly bounded and $\{\Delta v_j\}$ uniformly converges to 0. By standard elliptic estimates, it follows that, by passing to a subsequence, $v_j \to v$ in $C^2_{loc}(R^N)$, and v satisfies

$$\Delta v = 0, \ v \ge 0 \text{ in } R^N, \ v(0) = 1.$$

Hence, $v \equiv 1$; i.e., $v_j \to 1$ in $C^2_{loc}(R^N)$. In particular,

$$|\nabla \overline{u}_j(x)|/\overline{u}_j(x) = |\nabla v_j(x)|/v_j(x) \to 0 \text{ as } j \to +\infty$$

uniformly on any compact subset of R^N.

Since $\{x_1^j\}$ is bounded, the balls $B_j := \{x : |x - (x_1^j, 0)| < 1 + 2\lambda_*\}$, $j = 1, 2, ...$ are contained in some fixed compact subset of R^N. Therefore,

$$\sup_{x\in B_j} \frac{|\nabla \overline{u}_j(x)|}{\overline{u}_j(x)} \to 0 \text{ as } j \to +\infty. \tag{2.5}$$

For $x = (t, 0) \in R \times R^{N-1}$ with $t \in (x_1^j, \lambda_j) \subset (0, +\infty)$, let

$$l_j(t) = h(t) f(\overline{u}_j(x)) - h(t^{\lambda_j}) f(\overline{u}_j(x^{\lambda_j})).$$

We have

$$\begin{aligned} l_j'(t) &= h(t)f(\overline{u}_j(x))\Big(\frac{h'(t)}{h(t)}+\frac{f'(\overline{u}_j(x))\overline{u}_j(x)}{f(\overline{u}_j(x))}\frac{\frac{\partial}{\partial x_1}\overline{u}_j(x)}{\overline{u}_j(x)}\Big) \\ &\quad +h(t^{\lambda_j})f(\overline{u}_j(x^{\lambda_j}))\Big(\frac{h'(t^{\lambda_j})}{h(t^{\lambda_j})}+\frac{f'(\overline{u}_j(x^{\lambda_j}))\overline{u}_j(x^{\lambda_j})}{f(\overline{u}_j(x^{\lambda_j}))}\frac{\frac{\partial}{\partial x_1}\overline{u}_j(x^{\lambda_j})}{\overline{u}_j(x^{\lambda_j})}\Big). \end{aligned}$$

By (H_1) and (H_2), we can find $\sigma_0 > 0$ such that

$$\frac{h'(t)}{h(t)} \geq \sigma_0,\ \frac{h'(t^{\lambda_j})}{h(t^{\lambda_j})} \geq \sigma_0,\ \forall t \in (x_1^j, \lambda_j), \forall j \geq 1.$$

By (F_1) and (F_2), there exists $\tau_0 > 0$ such that

$$\frac{f'(\overline{u}_j(x))\overline{u}_j(x)}{f(\overline{u}_j(x))} \leq \tau_0,\ \frac{f'(\overline{u}_j(x^{\lambda_j}))\overline{u}_j(x^{\lambda_j})}{f(\overline{u}_j(x^{\lambda_j}))} \leq \tau_0,\ \forall t \in (x_1^j, \lambda_j), \forall j \geq 1.$$

In view of (2.5), we can now conclude that $l_j'(t) > 0$ for all large j and all $t \in (x_1^j, \lambda_j)$. Therefore, $l_j(x_1^j) < l_j(\lambda_j) = 0$ and

$$\Delta w_j(x^j) = l_j(x_1^j) < 0.$$

But this contradicts (2.3). The proof of Claim 4 is now complete.

Claim 5: $\lambda_* = \infty$.

Otherwise, $\lambda_* < +\infty$, and by Claim 4, we have a sequence $\lambda_j \to \lambda_*$ with the following properties:

(i) $\inf_{x\in\Sigma_{\lambda_j}} w_{\lambda_j}(x) < 0$,

(ii) For all large j, $x \in \Sigma_{\lambda_j}$ and $w_{\lambda_j}(x) \leq 0$ imply $\Delta w_{\lambda_j}(x) < 0$.

Let $g_\lambda(x)$ be as defined in the proof of Claim 2, and $\overline{w}_j(x) = w_{\lambda_j}(x)/g_{\lambda_j}(x)$. Then $\inf_{x\in\Sigma_{\lambda_j}} \overline{w}_j(x)$ is negative and is achieved at some $y^j \in \Sigma_{\lambda_j}$. By property (ii) above, we have $\Delta w_{\lambda_j}(y^j) < 0$, which implies $\Delta\overline{w}_j(y^j) < 0$. But as $\overline{w}_j$ achieves a minimum at y^j, we should have $\Delta\overline{w}_j(y^j) \geq 0$. This contradiction proves Claim 5.

We are finally ready to complete the proof of Theorem 2.1. By Claim 5 and Claim 3, we find $u(x) < u(x^\lambda)$, for all $x \in \Sigma_\lambda, \lambda \in R$. Therefore, $u(x) = u(x_1, \tilde{x})$ is strictly increasing in x_1. This proves the first part of the theorem.

Suppose now $h(t) \to +\infty$ as $t \to +\infty$. Let us define, for each $\sigma > 0$,

$$\Omega_\sigma = \{x = (x_1, \tilde{x}) \in R^N : \sigma < x_1 < \sigma + 1,\ |\tilde{x}| < 1\}.$$

Clearly the first eigenvalue μ_1 of the problem

$$-\Delta u = \mu u \text{ in } \Omega_\sigma,\ u|_{\partial\Omega_\sigma} = 0$$

is independent of σ.

If u is a positive solution of (2.1), then by the first part of the theorem that has just been proved, u is strictly increasing in x_1 and hence

$$-\Delta u = h(x_1)f(u) \geq h(\sigma)m_0 u,\ \forall x \in \Omega_\sigma, \tag{2.6}$$

where

$$m_0 = \frac{f(\xi_0)}{\sup_{R^N} u} > 0,\ \xi_0 = \min_{x_1=0,|x|\leq 1} u(x) > 0.$$

Since $u > 0$ in R^N, (2.6) implies $\mu_1 \geq h(\sigma)m_0$. It follows that

$$\overline{\lim}_{\sigma\to+\infty} h(\sigma) \leq \mu_1/m_0 < +\infty,$$

a contradiction to our assumption. Therefore (2.1) has only the trivial solution $u \equiv 0$. This completes the proof of Theorem 2.1. □

Remark 2.2. Apart from the example for $h(t)$ given in Theorem B, we list here a few more functions satisfying our conditions in Theorem 2.1:

(i) $h(t) = (t^+)^{\alpha_1} - (t^-)^{\alpha_2},\ \alpha_1, \alpha_2 > 0$;
(ii) $h(t) = \sigma_\beta(t),\ \beta > 0$;
(iii) $h(t) = \sigma_{\beta_1}(t) - \sigma_{\beta_2}(-t),\ \beta_1, \beta_2 > 0$.

Here $\sigma_\beta(t) = te^{-\beta/t}$ for $t > 0,\ \sigma_\beta(t) = 0$ for $t \leq 0$.

If $h(x_1)$ in (2.1) is nonpositive, we have a similar Liouville theorem, which holds even if the equality is replaced by an inequality. This result improves the main result in [19]. We will use it for our proof of the a priori bounds in Section 3, but we can also get the a priori bounds without using this result; see Remark 3.7.

Theorem 2.3. *Suppose that $h \in C(R)$ satisfies*

(H_3) $h(t) \leq 0$ for all t, $h(t) < 0$ for $t < 0$ and $\overline{\lim}_{t\to-\infty} h(t) < 0$.

Let $f(u)$ be a C^1 function of u for $u \in [0,\infty)$ that satisfies (F_2) and

(F_3) $f(0) = 0$, $\int_1^\infty F(u)^{-1/2}du < \infty$, where $F(u) = \int_0^u f(s)ds$.

Then the only function $u \in C^1(R^N)$ satisfying

$$-\Delta u \leq h(x_1)f(u),\ u \geq 0 \quad in \quad R^N,\ \sup_{R^N} u < +\infty \tag{2.7}$$

is $u \equiv 0$.

The first inequality in (2.7) is understood in the following sense,

$$\int_{R^N} \nabla u \cdot \nabla \phi dx \leq \int_{R^N} h(x_1)f(u)\phi dx,\ \forall \phi \geq 0,\ \phi \in C_0^\infty(R^N).$$

Proof. By (H_3), we can find $\tilde{h} \in C(R)$ such that

$$\tilde{h}(t) \geq h(t) \; \forall t \in R; \; \tilde{h}(t) < 0 \; \forall t < 0; \; \tilde{h}(t) = 0 \; \forall t \geq 0; \; \overline{\lim}_{t \to -\infty} \tilde{h}(t) < 0.$$

Suppose by way of contradiction that there exists some $u_* \in C^1(R)$ satisfying (2.7) with $M := \sup_{R^N} u_* > 0$. For $B_n := \{x \in R^N : |x| < n\}$, we consider the problem

$$-\Delta u = \tilde{h}(x_1) f(u) \text{ in } B_n, \; u|_{\partial B_n} = M. \tag{2.8}$$

Clearly $u = 0$ and $u = M$ are lower and upper solutions to (2.8). Therefore, (2.8) has at least one solution satisfying $0 \leq u \leq M$. Since $\tilde{h} \leq 0$ and $f(u)$ is nondecreasing in u, a standard comparison argument shows that (2.8) has a unique solution. Let us denote this solution by u_n. The same comparison argument also shows that $u_n \geq u_{n+1} \geq u_*$ on B_n for all $n \geq 1$. Therefore $\tilde{u}(x) := \lim_{n \to \infty} u_n(x)$ is well defined for every $x \in R^N$, and $\tilde{u} \geq u_*$ in R^N. Moreover, a standard regularity consideration shows that $\tilde{u} \in C^{2,q}_{loc}(R^N)$ is a weak solution of

$$-\Delta u = \tilde{h}(x_1) f(u) \text{ in } R^N. \tag{2.9}$$

We claim that $\tilde{u}$ is the maximal solution of (2.9) among all nonnegative solutions of (2.9) satisfying $u \leq M$ in R^N. Indeed, if u is such a solution of (2.9), then by the comparison argument used above, it follows from $u|_{\partial B_n} \leq M = u_n|_{\partial B_n}$ that $u \leq u_n$ in B_n and hence $u \leq \tilde{u}$ in R^N.

For any given $y_0 = (0, y_2^0, ..., y_N^0) \in R^N$, $u_{y_0}(x) := \tilde{u}(x + y_0)$ clearly solves (2.9). Therefore $\tilde{u}(x + y_0) \leq \tilde{u}(x)$. It follows that $\tilde{u}$ is a function of x_1 only; namely, $\tilde{u}(x) = \phi(x_1)$ and ϕ satisfies

$$-\phi'' = \tilde{h}(x_1) f(\phi), \; 0 \leq \phi \leq M \text{ in } R.$$

In particular, $\phi'' = 0$ and ϕ is a bounded function in $[0, \infty)$. Therefore, we necessarily have $\phi(x_1) \equiv m$ on $[0, \infty)$ with $0 \leq m \leq M$. We now find that ϕ is the unique solution of the initial-value problem

$$-\phi'' = \tilde{h}(x_1) f(\phi), \; \phi(0) = m, \; \phi'(0) = 0.$$

We must have $m > 0$ for otherwise the uniqueness of the initial-value problem implies $\phi \equiv 0$ in R, contradicting $\phi = \tilde{u} \geq u_*$. As $\phi'' = -\tilde{h}(x_1) f(\phi) \geq 0$, we find that $\phi'(x_1) \leq \phi'(0) = 0$ for $x_1 \leq 0$ and hence $\phi(x_1) \geq m$ for $x_1 \leq 0$. By our choice of $\tilde{h}$, there exists $\sigma > 0$ such that $\tilde{h}(x_1) \leq -\sigma$ for $x_1 \leq -1$. Thus

$$\phi'' \geq \sigma f(\phi), \; \phi \geq m \;\; \text{for} \;\; x_1 \leq -1.$$

It follows that

$$\left([\phi'(x_1)]^2/2 - \sigma F(\phi(x_1)) \right)' \leq 0, \; \forall x_1 \leq -1,$$

$$\frac{1}{2}[\phi'(x_1)]^2 - \sigma F(\phi(x_1)) \geq \frac{1}{2}[\phi'(-1)]^2 - \sigma F(\phi(-1)) \geq -\sigma F(\phi(-1)),$$

for all $x_1 \le -1$. From this we deduce, with $F_1(u) = F(u) - F(\phi(-1))$,

$$\frac{-\phi'(x_1)}{F_1(\phi(x_1))^{1/2}} \ge (2\sigma)^{1/2}, \ \forall x_1 < -1,$$

$$\int_{\phi(-2)}^{\infty} F_1(u)^{-1/2} du \ge \int_{\phi(-2)}^{\phi(x_1)} F_1(u)^{-1/2} du \ge (2\sigma)^{1/2}(-2 - x_1), \ \forall x_1 < -2.$$

It follows that $\int_{\phi(-2)}^{\infty} F_1(u)^{-1/2} du = \infty$. One easily sees that this implies $\int_1^{\infty} F(u)^{-1/2} du = \infty$, contradicting (F_3). □

3. A Priori Estimates

In this section, we show how our nonlinear Liouville theorems of the previous section can be used to obtain a priori bounds for positive solutions of (1.1). Our method in this section extends to more general situations than (1.1); for example, it works when Ω is replaced by some locally conformally flat manifolds as in [22], it works when the Laplace operator Δ is replaced by a general second-order uniformly elliptic operator and the nonlinear function u^p is replaced by a more general class of functions $f(u)$, and it works when the Dirichlet boundary condition is replaced by the Neumann or Robin boundary conditions. However, since these generalizations are rather standard and well known, we only present our results for (1.1).

For the lemmas to follow, we make the following assumptions.

(A_1) $\Omega^0 \subset\subset \Omega$ *and it has three connected components* D^+, D^-, *and* D^+_-, *where* D^+ *and* D^- *have nonempty interiors and smooth boundaries,* $\partial D^+ \subset \overline{\Omega^+}$, $\partial D^- \subset \overline{\Omega^-}$.

(A_2) D^+_- *may or may not have nonempty interior*;
- (a) *if* D^+_- *has empty interior, we assume that* $D^+_- = \Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ *is a smooth manifold of dimension* $(N-1)$;
- (b) *if* D^+_- *has nonempty interior, we assume that* $\partial D^+_- = \Gamma_- \cup \Gamma_+, \Gamma_- \cap \Gamma_+ = \emptyset$, *where* $\Gamma_- \subset \overline{\Omega^-}$ *and* $\Gamma_+ \subset \overline{\Omega^+}$ *are smooth manifolds of dimension* $(N-1)$.

Loosely speaking, these conditions imply that D^+ is surrounded by Ω^+, D^- is surrounded by Ω^-, and D^+_- is sandwiched between Ω^+ and Ω^-.

For any subdomain $D \subset \Omega$, let us denote by λ_1^D the first eigenvalue of

$$-\Delta u = \lambda u, \ u|_{\partial D} = 0.$$

We recall a necessary condition for (1.1) to have a positive solution established in [4].

Lemma 3.1. ([4], Theorem 3.3]) *If* (1.1) *has a positive solution, then* $\lambda < \sigma$, *where* $\sigma = \min\{\lambda_1^{\Omega^+ \cup D^+ \cup D^+_-}, \lambda_1^{D^-}\}$.

Lemma 3.2. *Suppose that* u_n *is a positive solution of* (1.1) *with* $\lambda = \lambda_n$, *where* $\{\lambda_n\}$ *is a bounded sequence. Let* $u_n(x_n) = \max_{\overline{\Omega}} u_n$. *Then we can always choose* $x_n \in \overline{\Omega^-} \cup \overline{\Omega^+}$.

Proof. Since u_n is a positive solution to (1.1) with $\lambda = \lambda_n$, we can apply Lemma 3.1 to conclude that $\lambda_n < \sigma$. This implies, by the strong maximum principle, that x_n cannot be in the interior of Ω^0 unless $u_n \equiv \max_{\overline{\Omega}} u_n$ in the component of the interior of Ω^0 that contains x_n. In the former case, $x_n \in \overline{\Omega^-} \cup \overline{\Omega^+}$, while in the latter case, we can also find $\tilde{x}_n \in \overline{\Omega^-} \cup \overline{\Omega^+}$ such that $u_n(\tilde{x}_n) = \max_{\overline{\Omega}} u_n$. □

Lemma 3.3. *Let* $\{x_n\}$ *be as in Lemma* 3.2 *and* $u_n(x_n) \to \infty$ *as* $n \to \infty$. *Suppose that* $p \in (1, N^*)$ *and case* $(A_2)(b)$ *happens, and*

$$a(x) = \alpha(x)[d(x, \Gamma_+)]^\gamma \text{ for } x \in \Omega^+ \text{ near } \Gamma_+,$$

where $\alpha(x)$ *is a positive continuous function defined in a small neighborhood of* Γ_+, *and* $\gamma > 0$. *Then* Γ_+ *does not contain any limiting points of* $\{x_n\}$.

Proof. Arguing indirectly, we suppose by passing to a subsequence that $x_n \to x_0 \in \Gamma_+$. By Lemma 3.2, we may assume that $\{x_n\} \subset \overline{\Omega^+}$. Put $M_n := u_n(x_n)$, $\rho_n := M_n^{-(p-1)/(2+\gamma)}$. We find that $M_n \to \infty$ and $\rho_n \to 0$ as $n \to \infty$. The change of variables $y := (x - x_n)/\rho_n$, $v_n := u_n/M_n$ transforms the differential equation for u_n into

$$-\Delta v_n = \rho_n^2 \lambda_n v_n + a_n(y) v_n^p, \tag{3.1}$$

where $a_n(y) = a(x_n + \rho_n y)\rho_n^{-\gamma}$. Let $\delta_n = d(x_n, \Gamma_+) = d(x_n, z_n)$, where $z_n \in \Gamma_+$. We may assume that $z_n \to z_0$. Denote by ν_n the unit normal of Γ_+ at z_n pointing inward of Ω^+. We obtain $d(x_n+\rho_n y) = |\delta_n + \rho_n \nu_n \cdot y + o(\rho_n)|$ and for all large n,

$$a_n(y) = \alpha(x_n + \rho_n y)[(\delta_n/\rho_n) + \nu_n \cdot y + o(1)]^\gamma \text{ provided } x_n + \rho_n y \in \Omega^+. \tag{3.2}$$

(i) Suppose that $\{\delta_n/\rho_n\}$ is bounded from above. By passing to a suitable subsequence, we may assume that $\delta_n/\rho_n \to s \geq 0$ as $n \to \infty$. Then by (3.2), $a_n(y) \to h(s + \nu_0 \cdot y)$ as $n \to \infty$, where $h(t) = \alpha(x_0)(t^+)^\gamma$.

By standard $W^{2,q}$ theory one may obtain estimates on $\{v_n\}$ ensuring that, by passing to a subsequence, $v_n \to v$ locally uniformly, with v defined in all of R^N and $v \in W^{2,q}_{loc}(R^N)$, for all $q > 1$. Moreover, v satisfies

$$-\Delta v = h(s + \nu_0 \cdot y)v^p, \ v \geq 0 \text{ in } R^N, \ v(0) = \max v = 1.$$

By a translation and rotation of the coordinates, we may assume that $s + \nu_0 \cdot y = y_1$ and hence v is a solution to

$$-\Delta v = h(y_1)v^p,\ v \geq 0 \text{ in } R^N,\ \max v = 1.$$

A standard regularity consideration shows $u \in C^2(R^N)$. But clearly this contradicts Theorem 2.1.

(ii) Suppose that $\{\delta_n/\rho_n\}$ is unbounded from above. By selecting a suitable subsequence, we may assume that $\beta_n := (\rho_n/\delta_n)^{\gamma/2} \to 0$ as $n \to \infty$. By introducing the variable $z := y/\beta_n$ and $w_n(z) := v_n(\beta_n z)$, equation (3.1) becomes

$$-\Delta w_n = (\beta_n \rho_n)^2 \lambda_n w_n + \beta_n^2 a_n(\beta_n z) w_n^p.$$

Clearly,

$$\beta_n^2 a_n(\beta_n z) = \alpha(x_n + \beta_n \rho_n z)[1 + \beta_n^{1+2/\gamma} \nu_n \cdot z + o(\beta_n)]^\gamma \to \alpha(x_0)$$

as $n \to \infty$. It follows that the problem

$$-\Delta w = \alpha(x_0) w^p,\ w \geq 0 \text{ in } R^N,\ w(0) = 1$$

has a solution, contradicting Theorem 1.2 of [17]. □

Lemma 3.4. *Let $\{x_n\}$ be as in Lemma 3.2 and $u_n(x_n) \to \infty$ as $n \to \infty$. Suppose that $p \in (1, N^*)$ and case $(A_2)(a)$ happens, and*

$$a(x) = \alpha_1(x)[d(x,\Gamma)]^{\gamma_1} \text{ for } x \in \Omega^+ \text{ near } \Gamma,$$

$$a(x) = -\alpha_2(x)[d(x,\Gamma)]^{\gamma_2} \text{ for } x \in \Omega^- \text{ near } \Gamma,$$

where $\alpha_1(x)$ and $\alpha_2(x)$ are positive continuous functions defined in a small neighborhood of Γ and $\gamma_1, \gamma_2 > 0$. Then Γ does not contain any limiting points of $\{x_n\}$.

Proof. By way of contradiction, we suppose by passing to a subsequence that $x_n \to x_0 \in \Gamma$. By passing to a further subsequence, we may assume that $\{x_n\} \subset \overline{\Omega^-}$ or $\{x_n\} \subset \overline{\Omega^+}$. For definiteness, we assume that the first case occurs; the second case can be proved in a similar fashion.

Put $M_n := u_n(x_n)$, $\rho_n := M_n^{-(p-1)/(2+\gamma_2)}$. We find that $M_n \to \infty$ and $\rho_n \to 0$ as $n \to \infty$. The change of variables $y := (x - x_n)/\rho_n$, $v_n := u_n/M_n$ transforms the differential equation for u_n into

$$-\Delta v_n = \rho_n^2 \lambda_n v_n + a_n(y) v_n^p, \tag{3.3}$$

where $a_n(y) = a(x_n + \rho_n y)\rho_n^{-\gamma_2}$. Let $\delta_n = d(x_n, \Gamma) = d(x_n, z_n)$, where $z_n \in \Gamma$. We may assume that $z_n \to z_0$. Denote by ν_n the unit normal of Γ at z_n pointing inward to Ω^-. We obtain

$$d(x_n + \rho_n y) = |\delta_n + \rho_n \nu_n \cdot y + o(\rho_n)|$$

and

$$a_n(y) = -\alpha_2(x_n + \rho_n y)|(\delta_n/\rho_n) + \nu_n \cdot y + o(1)|^{\gamma_2} \text{ provided } x_n + \rho_n y \in \Omega^-, \tag{3.4}$$

$$a_n(y) = \alpha_1(x_n+\rho_n y)|(\delta_n/\rho_n)+\nu_n \cdot y+o(1)|^{\gamma_1} \text{ provided } x_n+\rho_n y \in \Omega^+. \tag{3.5}$$

(i) Suppose that $\{\delta_n/\rho_n\}$ is bounded from above. By passing to a suitable subsequence, we may assume that $\delta_n/\rho_n \to s \geq 0$ as $n \to \infty$. Then by (3.4) and (3.5), $a_n(y) \to h(s + \nu_0 \cdot y)$ as $n \to \infty$, where $h(t) = \alpha_1(x_0)(t^-)^{\gamma_1} - \alpha_2(x_0)(t^+)^{\gamma_2}$.

By standard $W^{2,q}$ theory one may obtain estimates on $\{v_n\}$ ensuring that, by passing to a subsequence, $v_n \to v$ locally uniformly, with v defined in all of R^N and $v \in W^{2,q}_{loc}(R^N)$, for all $q > 1$. Moreover, v satisfies

$$-\Delta v = h(s + \nu_0 \cdot y)v^p, \ v \geq 0 \text{ in } R^N, \ v(0) = \max v = 1.$$

By a translation and rotation of the coordinates, we may assume that $s + \nu_0 \cdot y = -y_1$ and hence v is a solution to

$$-\Delta v = h(-y_1)v^p, \ v \geq 0 \text{ in } R^N, \ \max v = 1.$$

But clearly this contradicts Theorem 2.1.

(ii) Suppose that $\{\delta_n/\rho_n\}$ is unbounded from above. By selecting a suitable subsequence, we may assume that $\beta_n := (\rho_n/\delta_n)^{\gamma_2/2} \to 0$ as $n \to \infty$. By introducing the variable $z := y/\beta_n$ and $w_n(z) := v_n(\beta_n z)$, equation (3.3) becomes

$$-\Delta w_n = (\beta_n \rho_n)^2 \lambda_n w_n + \beta_n^2 a_n(\beta_n z) w_n^p.$$

Clearly,

$$\beta_n^2 a_n(\beta_n z) = -\alpha_2(x_n + \beta_n \rho_n z)[1 + \beta_n^{1+2/\gamma_2} \nu_n \cdot z + o(\beta_n)]^{\gamma_2} \to -\alpha(x_0)$$

as $n \to \infty$. It follows that the problem

$$-\Delta w = -\alpha_2(x_0)w^p, \ w \geq 0 \text{ in } R^N, \ w(0) = 1 \tag{3.6}$$

has a solution, contradicting Theorem II of [19].

If $\{x_n\} \subset \overline{\Omega^+}$ occurs, then the proof for case (i) is similar but for case (ii), instead of (3.6), we obtain

$$-\Delta w = \alpha_1(x_0)w^p, \ w \geq 0 \text{ in } R^N, \ w(0) = 1,$$

which contradicts Theorem 1.2 of [17]. □

An inspection of the proof of Lemma 3.3 shows that the same argument yields the following result.

Lemma 3.5. *Let* $\{x_n\}$ *be as in Lemma* 3.2 *and* $u_n(x_n) \to \infty$ *as* $n \to \infty$. *Suppose that* $p \in (1, N^*)$ *and*

$$a(x) = \alpha_3(x)[d(x, \partial D^+)]^{\gamma_3} \text{ for } x \in \Omega^+ \text{ near } \partial D^+,$$

where $\alpha_3(x)$ *is a positive continuous function defined in a small neighborhood of* ∂D^+ *and* $\gamma_3 > 0$. *Then* ∂D^+ *does not contain any limiting points of* $\{x_n\}$.

Lemma 3.6. *Let* $\{x_n\}$ *be as in Lemma* 3.2 *and* $u_n(x_n) \to \infty$ *as* $n \to \infty$. *Suppose that* $p > 1$ *and case* $(A_2)(b)$ *happens, and*

$$m \le -a(x)/[d(x, \Gamma_+)]^{\gamma_4} \le M \text{ for } x \in \Omega^- \text{ near } \Gamma_-,$$

where $M > m > 0$, $\gamma_4 > 0$. *Then* Γ_- *does not contain any limiting points of* $\{x_n\}$.

Similarly, if the above assumption holds for $a(x)$ *with* $x \in \Omega^-$ *near* ∂D^-, *then* ∂D^- *does not contain any limiting points of* $\{x_n\}$.

Proof. Arguing indirectly, we suppose by passing to a subsequence that $x_n \to x_0 \in \Gamma_-$. By Lemma 3.2, we may assume $\{x_n\} \subset \overline{\Omega^-}$. Put $M_n := u_n(x_n)$, $\rho_n := M_n^{-(p-1)/(2+\gamma_4)}$. We find that $M_n \to \infty$ and $\rho_n \to 0$ as $n \to \infty$. The change of variables $y := (x - x_n)/\rho_n$, $v_n := u_n/M_n$ transforms the differential equation for u_n into

$$-\Delta v_n = \rho_n^2 \lambda_n v_n + a_n(y) v_n^p, \tag{3.7}$$

where

$$a_n(y) = a(x_n + \rho_n y)\rho_n^{-\gamma_4}.$$

Let $\delta_n = d(x_n, \Gamma_-) = d(x_n, z_n)$, where $z_n \in \Gamma_-$. We may assume that $z_n \to z_0$. Denote by ν_n the unit normal of Γ_- at z_n pointing inward to Ω^-. We obtain

$$d(x_n + \rho_n y) = |\delta_n + \rho_n \nu_n \cdot y + o(\rho_n)|$$

and for all large n,

$$-a_n(y)/[(\delta_n/\rho_n) + \nu_n \cdot y + o(1)]^{\gamma_4} \in [m, M] \text{ provided } x_n + \rho_n y \in \Omega^-. \tag{3.8}$$

(i) Suppose that $\{\delta_n/\rho_n\}$ is bounded from above. By passing to a suitable subsequence, we may assume that $\delta_n/\rho_n \to s \ge 0$ as $n \to \infty$. Then by (3.8),

$$a_n(y) \le -m[(\delta_n/\rho_n) + \nu_n \cdot y + o(1)]^{\gamma_4} \to h(s + \nu_0 \cdot y) \text{ as } n \to \infty,$$

where $h(t) = -m(t^+)^{\gamma_4}$.

By (3.7), (3.8), and standard $W^{2,q}$ theory, one may obtain estimates on $\{v_n\}$ ensuring that, by passing to a subsequence, $v_n \to v$ locally uniformly, with v defined in all of R^N and $v \in W^{2,q}_{loc}(R^N)$, for all $q > 1$. Moreover, v satisfies

$$-\Delta v \le h(s + \nu_0 \cdot y)v^p, \ v \ge 0 \text{ in } R^N, \ v(0) = \max v = 1.$$

By a translation and rotation of the coordinates, we may assume that $s + \nu_0 \cdot y = -y_1$ and hence v is a solution to

$$-\Delta v \leq h(-y_1)v^p,\ v \geq 0 \text{ in } R^N,\ \max v = 1.$$

But clearly this contradicts Theorem 2.3.

(ii) Suppose that $\{\delta_n/\rho_n\}$ is unbounded from above. By selecting a suitable subsequence, we may assume that $\beta_n := (\rho_n/\delta_n)^{\gamma_4/2} \to 0$ as $n \to \infty$. By introducing the variable $z := y/\beta_n$ and $w_n(z) := v_n(\beta_n z)$, equation (3.7) becomes

$$-\Delta w_n = (\beta_n \rho_n)^2 \lambda_n w_n + \beta_n^2 a_n(\beta_n z) w_n^p.$$

Clearly,

$$\beta_n^2 a_n(\beta_n z) \leq -m[1 + \beta_n^{1+2/\gamma_4} \nu_n \cdot z + o(\beta_n)]^{\gamma_4} \to -m$$

as $n \to \infty$. It follows that the problem

$$-\Delta w \leq -mw^p,\ w \geq 0 \text{ in } R^N,\ w(0) = 1$$

has a solution, contradicting Theorem II of [19].

The proof for the fact that ∂D^- does not contain any limiting point of $\{x_n\}$ is exactly the same. □

Remark 3.7. In the above proof of Lemma 3.6, we can avoid using Theorem 2.3 for case (i) and avoid using Theorem II of [19] for case (ii) by the following consideration. In either case we derive the existence of a function in $W^{2,q}_{loc}(R^N)$ satisfying

$$-\Delta v \leq 0,\ v \geq 0 \text{ in } R^N,\ v(0) = \max v = 1.$$

By the strong maximum principle, this is possible only if $v \equiv 1$. In either case this leads to a contradiction.

Lemma 3.8. *Let $\{x_n\}$ be as in Lemma 3.2 and $u_n(x_n) \to \infty$. Suppose $p > 1$ and K is an arbitrary compact subset of Ω^-. Then K does not contain any limiting points of $\{x_n\}$.*

Proof. Since $K \subset \Omega^-$ is compact, we have $a(x) \leq \delta < 0$ for all $x \in K$ and some positive constant δ. Suppose by way of contradiction that $x_0 \in K$ is a limiting point of $\{x_n\}$. By passing to a subsequence, we may assume $x_n \to x_0$, $x_n \in \overline{B}_\eta(x_0) := \{x : |x - x_0| \leq \eta\} \subset \Omega^-$ for all n, and $a(x) \leq -\delta/2$ for all $x \in \overline{B}_\eta(x_0)$. Therefore we have

$$-\Delta u_n \leq \lambda^* u_n - (\delta/2)u_n^p \ \text{ in } B_\eta(x_0),$$

where $\lambda^* = \sup_{n \geq 1} \lambda_n$.

Let v denote a solution to

$$-\Delta v = \lambda^* v - (\delta/2)v^p \ \text{ in } B_\eta(x_0),\ v|_{\partial B_\eta(x_0)} = \infty,$$

whose existence is well known (see [19] and [24]). Then by the comparison principle, Lemma 2.3 in [16], we obtain $u_n \le v$ in $B_\eta(x_0)$ for all n. Hence $\{u_n(x_n)\}$ is bounded. This contradiction finishes the proof. □

Lemma 3.9. *Let $\{x_n\}$ be as in Lemma 3.2 and $u_n(x_n) \to \infty$. Suppose $p > 1$ and $a(x) < 0$ on $\partial\Omega$. Then $\partial\Omega$ does not contain any limiting points of $\{x_n\}$.*

Proof. By assumption $\{\lambda_n\}$ is bounded. Therefore there exists $\lambda^* > 0$ such that $\lambda_n \le \lambda^*$ for all $n \ge 1$. Let $a^*(x) = \min\{a(x), 0\}$. Then by (A_1) and (A_2), the set $\{x \in \Omega : a^*(x) = 0\}$ has two components: $D_1 := D^-$ and $D_2 := \Omega^+ \cup D_-^+ \cup D^+$. By Lemma 2.6 of [16], the problem

$$-\Delta u = \lambda^* u + a(x)u^p \text{ in } \Omega \setminus (D_1 \cup D_2),\ u|_{\partial\Omega} = 0,\ u|_{\partial D_1 \cup \partial D_2} = \infty$$

has a maximal positive solution u^*. Since

$$-\Delta u_n \le \lambda^* u_n + a^*(x)u_n^p \text{ in } \Omega \setminus (D_1 \cup D_2),\ u_n|_{\partial\Omega} = 0,\ u_n|_{\partial D_1 \cup \partial D_2} < \infty,$$

we can apply Lemma 2.3 in [16] to conclude that $u_n \le u^*$ in $\Omega \setminus (D_1 \cup D_2)$ for all $n \ge 1$. As $u_n(x_n) \to \infty$, it follows that $\partial\Omega$ does not contain any limiting points of $\{x_n\}$. □

Lemma 3.10. *Let $\{x_n\}$ be as in Lemma 3.2 and $u_n(x_n) \to \infty$. Suppose $p \in (1, N^*)$ and K is an arbitrary compact subset of Ω^+; then K does not contain any limiting points of $\{x_n\}$.*

Proof. This follows from the well-known blow up argument of [17]. □

Lemma 3.11. *Let $\{x_n\}$ be as in Lemma 3.2 and $u_n(x_n) \to \infty$. Suppose $p \in (1, N^*)$ and $a(x) > 0$ on $\partial\Omega$. Then $\partial\Omega$ does not contain any limiting points of $\{x_n\}$.*

Proof. This also follows from the well-known blow up argument of [17]. □

The above lemmas can be combined to yield various results on a priori estimates for positive solutions of (1.1). The following result is a direct consequence.

Theorem 3.12. *Suppose that (1.4) and (1.6) hold, $p \in (1, N^*)$, and λ belongs to a bounded interval Λ in R^1. Then there exists $C > 0$ such that any positive solution of (1.1) with $\lambda \in \Lambda$ satisfies $u \le C$ in Ω.*

Condition (1.6) can sometimes be relaxed. The following result describes such a case.

Theorem 3.13. *Suppose that* $(A_1), (A_2)$ *hold,* $p \in (1, N^*)$, λ *belongs to some bounded interval* Λ *in* R^1, *and* $a(x)$ *satisfies the condition in Lemma 3.5 near* ∂D^+. *Moreover, if case* $(A_2)(a)$ *happens, assume that* $a(x)$ *satisfies the conditions in Lemma 3.4 near* Γ, *and in case* $(A_2)(b)$, *assume that the conditions in Lemmas 3.3 and 3.6 are satisfied by* $a(x)$. *Then there exists* $C > 0$ *such that any positive solution of* (1.1) *with* $\lambda \in \Lambda$ *satisfies* $u \le C$ *in* Ω.

Proof. Arguing indirectly, we assume that there exists a sequence of positive solutions of (1.1) with $\lambda = \lambda_n \in \Lambda$ satisfying $u_n(x_n) = \max_{\overline{\Omega}} u_n \to \infty$ as $n \to \infty$. By Lemma 3.1, we have $\lambda_n < \sigma$. By Lemma 3.2, we may assume that $\{x_n\} \subset \overline{\Omega^-} \cup \overline{\Omega^+}$. By passing to a subsequence, we may assume that $x_n \to x_0$. But Lemmas 3.3-3.11 imply that $x_0 \notin \overline{\Omega^-} \cup \overline{\Omega^+}$. This contradiction finishes our proof. □

Remark 3.14. (i) Our a priori estimates can be used to obtain various existence and multiplicity results for (1.1) as in [4].

(ii) We can also apply our method to certain entire space problems. For example, in [13] and [14], the entire space problem

$$\Delta u = \lambda u + a(x)u^p, \ x \in R^N$$

was studied, where $p > 1$ and $a(x)$ is sign changing and satisfies $\lim_{|x|\to\infty} a(x) = a_\infty < 0$. In [13] we used the a priori estimates of [4] and hence had to restrict$p < \min\{(N+1+\gamma)/(N-1), N^*\}$, where $a(x)$ satisfies (1.4); while in [14] we used a variational approach and Liouville theorems in [26], which only required $p \in (1, N^*)$ but condition (1.2) was used. By using the results of this paper, we can prove the same results under the conditions of Theorem 3.12 above.

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