Advances in Differential Equations Volume 10, Number 9 (2005), 1053–1080

HIGHER INTEGRABILITY FOR PARABOLIC EQUATIONS OF $p(x,t)$-LAPLACIAN TYPE

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Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain withboundary $\partial\Omega$, and $Q = \Omega \times (0,T]$ be a cylinder of height $T < \infty$. We study local weak solutions of the parabolic equation

$$Lu \equiv \frac{\partial u}{\partial t} - div\left(|\nabla u|^{p(z)-2}\,\nabla u\right) = 0, \quad z = (x,t) \in \Omega \times (0,T),$$

with variable exponent of nonlinearity p. We assume that $p(z) \in C(\Omega)$ and is such that

$$\frac{2n}{n+2} < \alpha \leq p(z) \leq \beta < \infty, \quad z \in Q,$$

$$|p(z_1) - p(z_2)| \leq \omega(|z_1 - z_2|) \quad \forall\,(z_1,\, z_2) \in \overline{Q},$$

$$\overline{\lim_{\tau\to 0}} \quad \omega(\tau)\ln\frac{1}{\tau} < \infty.$$

We prove that the weak solution is bounded and establish Meyer's type estimates: there exists a positive constant $\varepsilon > 0$ such that for every subdomain Q', $\overline{Q'} \subset Q$,

$$\int_{Q'} |\nabla u|^{p(z)(1+\varepsilon)}\,dz < \infty.$$

1. Introduction

This paper is devoted to the proof of higher integrability (i.e. higher than was a priori assumed) for the gradients of weak solutions of parabolic equations of $p(x,t)$- Laplacian type with variable exponent of nonlinearity. The property of higher integrability was first discovered in [25, 26] for the solutions of elliptic equations, which is why the corresponding estimates for the gradients of weak solutions are often called Meyer's estimates.

Accepted for publication: June 2005.
AMS Subject Classifications: 35K55, 35K65, 35B05, 35B30.

By now, there exists an extensive literature devoted to deriving Meyer's estimates (alias higher integrability of the gradients) for weak solutions to elliptic and parabolic equations and systems of such equations with constant exponents of nonlinearity (see, for example, [1, 11, 12, 13, 14, 16, 22, 23, 26, 31]). The relevant regularity results concerning the stationary case can be found in [16, 22, 23, 26, 31].

A proof of Meyer's-type estimates (or higher integrability of gradients) for weak solutions of elliptic equations with variable (or non-standard) growth conditions can be found in [1, 15, 16, 17, 18, 19, 20].

Meyer's-type estimates for weak solutions of the stationary Stokes system for a non-Newtonian fluid with variable growth conditions were proved in [32].

The works [32, 33] contain applications of such estimates to the proof of existence of solutions of a coupled system describing the process of stationary thermo-convection in a non-Newtonian fluid, and for a model of thermistor. In [2] the higher integrability results were obtained for a system describing stationary flows of electro-rheological fluids.

In recent years, there has been an increasing interest in the study of such equations motivated by their applications to the mathematical modelling of non-Newtonian fluids, in particular, the electro-rheological fluids. These kinds of fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed in [27, 28].

The system of modified stationary Navier-Stokes equations was studied in [1, 2, 3, 27, 28]. In such systems the exponent of non-linearity p is usually a given function of some components of the unknown vector-valued solution. These components can be density, temperature, saturation, electric field, etc. The systems may degenerate with respect to the type or order at certain values of the solutions or its derivatives, and the solutions may possess special localization properties (dead core, finite time extinction, waiting time effect); see [4, 5, 6, 7, 9, 10].

For such systems, Meyer's-type estimates turn out to be one of the main tools in the study of existence and uniqueness of vector-valued solutions. Meyer's-type estimates were essentially used in [8] in the proof of existence and uniqueness theorems for a model of stationary thermo-convective flow of a non-Newtonian fluid.

In [3], various regularity properties of solutions of parabolic systems describing certain classes of non-Newtonian fluids were established: higher integrability, higher differentiability, partial regularity of the spatial gradient, etc.

We refer to the papers [1, 2, 3] also for a discussion of the regularity properties of weak solutions of the systems of equations with this type of nonlinearity (see also the references therein to the previous work on this issue). In Section 2 we give the definition of weak solutions and prove some energy relations for such solutions. In Section 3 we prove that weak solutions are locally bounded. In Section 4 we derive first a suitable Cacciopoli-type energy estimate for weak solutions, and then prove a lemma of Gehring type and the main theorem on higher integrability.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with boundary $\Gamma = \partial\Omega$, and $Q = \Omega \times (0, T]$ be a cylinder of height $T < \infty$. We study local weak solutions of the following parabolic equation:

$$Lu \equiv \frac{\partial u}{\partial t} - div\Big(|\nabla u|^{p(z)-2} \nabla u\Big) = 0, \quad z = (x,t) \in \Omega \times (0,T). \tag{2.1}$$

We assume that the weak solutions satisfy the energy estimate

$$\sup_{t\in(0,T]} \int_\Omega |u|^2 dx + \int_Q |\nabla u|^{p(z)}\, dz \leq C_0 < \infty. \tag{2.2}$$

It is also assumed that $p(z) \in C(\Omega)$ is such that

$$\frac{2n}{n+2} < \alpha \leq p(z) \leq \beta < \infty, \quad z \in Q, \tag{2.3}$$

$$|p(z_1) - p(z_2)| \leq \omega(|z_1 - z_2|)\forall \quad (z_1,\ z_2) \in Q, \tag{2.4}$$

where $\omega(\tau)$ is continuous for $\tau > 0$ and

$$\overline{\lim_{\tau\to 0^+}}\, \omega(\tau) \ln\frac{1}{\tau} = M < +\infty. \tag{2.5}$$

Under conditions (2.2), (2.5) we prove that every local weak solution is bounded and, moreover, that the estimate of Meyer type holds: there exists a positive constant $\varepsilon > 0$ such that for every subdomain Q', $\overline{Q'} \subset Q$,

$$\int_Q |\nabla u|^{p(z)(1+\varepsilon)}\, dz < \infty, \quad \varepsilon > 0, \quad \overline{Q'} \subset Q.$$

2.1. **Definition of local weak solution. Energy relations.**

Definition 1. *A locally integrable in Q function u is called a* local weak solution *of equation* (2.1) *if*

(i) $u \in C_{loc}(0,T;L^2_{loc}(\Omega)),\ \nabla u \in L^1_{loc}(Q),\ |\nabla u|^{p(\cdot)} \in L^1_{loc}(Q)$;
(ii) *for any test function $\zeta \in C^\infty(0,T;C_0^\infty(\Omega))$ the integral identity holds:*

$$\int_\Omega u\zeta dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega (-u\zeta_t + |\nabla u|^{p-2}\nabla u \nabla\zeta)dz = 0, \tag{2.6}$$

with $0 < t_1 < t_2 < T$.

From (2.6) we obtain that

$$\int_Q (-u\zeta_t + |\nabla u|^{p-2}\nabla u\nabla\zeta)dz = 0 \tag{2.7}$$

for $\zeta \in C_0^\infty(Q)$. Following ([24], Chapter 3, Section 1), it is easy to check that (2.6) and (2.7) are equivalent.

Lemma 1. *Let conditions* (2.3), (2.4), (2.5) *be fulfilled. Then the following energy relations hold:*

$$\int_Q \Big(-\frac{u^2}{2}\zeta_t + |\nabla u|^{p-2}\nabla u\nabla(u\zeta)\Big)dz = 0,\ \zeta \in C_0^\infty(Q), \tag{2.8}$$

$$\frac{1}{2}\int_\Omega u^2(x,t_2)\chi(x)dx + \int_0^{t_2}\int_\Omega \Big(|\nabla u|^{p-2}\nabla u\nabla(u\chi)\Big)dz = \frac{1}{2}\int_\Omega u^2(x,t_2)\chi(x)dx, \tag{2.9}$$

with $\chi(x) \in C_0^\infty(\Omega)$ *and* $0 \le t_1 \le t_2 \le T$.

Proof. Without loss of generality we may assume that $u(z)$ is defined on the whole of $\mathbb{R}^{n+1}$: it suffices to set $u \equiv 0$ for $z \notin Q$. By $f_h \in C_0^\infty$ we denote the standard mollifier of f (the averaging operator, see [21], [34]),

$$f_h(z) = \frac{1}{h^{n+1}}\int_{R^{n+1}} \rho(\frac{z-z'}{h})f(z')dz', \tag{2.10}$$

where

$$\rho \in C_0^\infty,\ \int_{R^{n+1}} \rho(z)dz = 1.$$

According to [21], [34],

$$\nabla f_h = \big(\nabla f\big)_h, \tag{2.11}$$

and

$$||f_h(z) - f(z)||_{p(\cdot)} \to 0,\ |h| \to 0, \tag{2.12}$$

if $f \in L^{p(z)}$ and $p(z)$ satisfies conditions (2.4), (2.5).

First we take in (2.7) the test function $(u_h\zeta)_h,\ \zeta \in C_0^\infty(Q)$. Using the well-known properties of mollifiers we may rewrite the first term in (2.7) in the following way:

$$\int_Q u\big((u_h\zeta)_h\big)_t dz = \int_Q u\big((u_h\zeta)_t\big)_h dz = \int_Q u_h\big(u_h\zeta\big)_t dz \tag{2.13}$$

$$= \int_Q \Big(\frac{(u_h)_t^2}{2}\zeta + (u_h)^2\zeta_t\Big) dz = \int_Q \frac{(u_h)^2}{2}\zeta_t dz.$$

It follows that

$$\lim_{h\to 0}\int_Q u\big((u_h\zeta)_h\big)_t dz = \int_Q \frac{u^2}{2}\zeta_t dz. \tag{2.14}$$

Let us consider the second term in (2.7):

$$\int_Q |\nabla u|^{p-2}\nabla u\nabla(u_h\zeta)_h dz = \int_Q |\nabla u|^{p-2}\nabla u\nabla(u\zeta)dz + I_h, \tag{2.15}$$

$$I_h = \int_Q |\nabla u|^{p-2}\nabla u\big(\nabla(u_h\nabla\zeta)_h - \nabla(u\zeta)\big)dz. \tag{2.16}$$

According to (2.11), (2.12),

$$||(\nabla u)_h - \nabla u||_{p(\cdot)} = ||\nabla u_h - \nabla u||_{p(\cdot)} \to 0,\ |h| \to 0, \tag{2.17}$$

whence we obtain, applying Hölder's inequality, that

$$|I_h| \le 2|||\nabla u|^{p-1}||_{\frac{p}{p-1},\,Q}\ ||\big(\nabla(u_h\zeta)_h - \nabla(u\zeta)\big)||_{p,\,Q} \to 0,\ h \to 0. \tag{2.18}$$

This completes the proof of relation (2.8). In fact, relation (2.8) is valid for a wider class of the functions ζ. It is sufficient to claim that $\zeta \in C_0(Q)$, $\zeta_t \in L^2(Q)$, $\nabla\zeta \in L^\infty(Q)$. To prove (2.9), we choose in (2.8)

$$\zeta(x,t) = \xi(t)\chi(x),\ \chi(x) \in C_0^\infty(\Omega),$$

$$\xi(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{\varepsilon}, & 0 \le t \le \varepsilon \\ 1, & \varepsilon < t \le T-\varepsilon \\ \frac{T-t}{\varepsilon}, & T-\varepsilon < t \le T \\ 0, & T < t \end{cases}.$$

Then

$$\int_Q (u)^2 \xi_t(t)\chi(x)dz = \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_\Omega u^2(x,s)\chi(x)dxds - \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\int_\Omega u^2(x,s)\chi(x)dx\,ds,$$

and by Lebesgue's theorem

$$\lim_{\varepsilon\to 0}\int_Q (u)^2\xi_t(t)\chi(x)dz = \int_\Omega u^2(x,0)\chi(x)dx - \int_\Omega u^2(x,\tau)\chi(x)dx,$$

$$\lim_{\varepsilon\to 0}\int_Q \Big(|\nabla u|^{p-2}\nabla(u\xi(t)\chi(x))\Big)dz = \int_0^{\tau}\int_\Omega \Big(|\nabla u|^{p-2}\nabla(u\chi(x))\Big)dz.$$

The proof of the lemma is completed. □

Remark 1. According to (2.8), the function

$$g(t) := \frac{1}{2}\int_\Omega u^2(x,t)\chi(x)dx$$

has the weak derivative $g_t = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u\chi)dx \in L^1(0,T)$, for every smooth function $\zeta(x) \in C_0^\infty(\Omega)$.

Proof. Letting in (2.8) $\zeta = \xi(t)\chi(x)$ we arrive at the relation

$$\int_0^T \Big[-\xi_t\Big(\int_\Omega \frac{u^2}{2}\chi(x)dx\Big) + \xi\Big(\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u\chi)dx\Big)\Big]dt = 0, \qquad (2.19)$$

whence the required assertion.

3. Boundedness of weak solutions

Theorem 1. (*Y. Alkhutov*) *Let $u(x,t)$ be a weak solution of equation* (2.1) *with uniformly continuous exponent $p(x,t)$ satisfying conditions* (2.3), (2.4), (2.5), ($n \geq 2$). *Then $u(x,t)$ is locally bounded in Q.*

Proof. Let $z_0 = (x_0,t_0) \in Q$ be arbitrary. We introduce the following notation:

$$s = \inf_{(x,t)\in Q\cap(|x-x_0|+|t-t_0|\leq R)} p(z), \quad l = \sup_{(x,t)\in Q\cap(|x-x_0|+|t-t_0|\leq R)} p(z), \qquad (3.1)$$

$$B_R(x_0) = \{x\in\Omega : |x-x_0| < R\},\ \Delta_R(t_0) = \{t\in\mathbb{R} : t_0-(R)^s < t < t_0)\}, \qquad (3.2)$$

$$Q_R = B_R(x_0)\times\Delta_R(t_0),\ \overline{Q}_{4R}\subset Q \qquad (3.3)$$

and

$$⨍_{Q_R} f dz = \frac{1}{|Q_R|}\int_{Q_R} f dz,\ |Q_R| = \omega_n R^{n+s}, \qquad (3.4)$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is sufficient to prove that there exists a constant $R \leq 1$ (which depends on the constants in (2.3), (2.4), (2.5), and the distance between (x_0, t_0) and parabolic boundary of Q), such that $u(x,t)$ is bounded in Q_R.

Let η be a cut-off function such that $\eta \in C^\infty(Q_{4R})$, $0 \leq \eta \leq 1$, and $\eta = 0$ in a neighborhood of the parabolic boundary of Q_{4R}, (that is, $\eta = 0$ on ∂B_{4R} and for $t = t_0 - (R)^s$). We introduce the function

$$w(x,t) = \max\,(u(x,t) + 1, 1),$$

where $u(x,t)$ is a weak solution of equation (2.1). It is easy to verify the validity of the following properties of w (see [24], Chapter 2, Section 4):

$$w(x,t) = u(x,t) + 1, \quad \text{if} \quad u \geq 0 \quad \text{and} \quad w(x,t) = 1 \quad \text{if} \quad u \leq 0$$

and

$$\nabla w = \nabla u, \quad \text{if} \quad u \geq 0 \quad \text{and} \quad \nabla w = 0, \quad \text{if} \quad u \leq 0.$$

We take for the test function in (2.6)

$$\zeta(x,t) = (w^\beta(x,t) - 1)\eta^l(x,t), \quad \beta \geq 1. \tag{3.5}$$

The scheme of proof is based on the Moser iterative method. On the first iteration step we chose $\beta = 1$ which makes the function (3.5) an admissible test function. The main objective is to derive a recurrence relation, which is the inverse Holder's inequality for the function $w(x,t)$. This relation would imply an increment of integrability of $w(x,t)$ on every iteration step. For the sake of simplicity, let us assume first that the weak solution is bounded and, respectively, that the function (3.5) is an admissible test function. The estimate on the maximum of the absolute value of (3.5) will be obtained at the end of the proof. A more rigorous justification of the arguments below requires the use of the cut-off functions $\min\{w(x,t), j\}$ instead of $w(x,t)$, with the subsequent passage to the limit when $j \to \infty$.

Taking into account the fact that

$$\zeta(x,t) = 0 \quad \text{if} \quad u \leq 0,$$

and applying the above-listed properties of $w(x,t)$, we arrive at the following relation:

$$\frac{1}{\beta+1}\int_{B_{4R}} w^{\beta+1}(\cdot,t_0)\eta^l\,(\cdot,t_0)dx + \beta\int_{Q_{4R}} |\nabla w|^p w^{\beta-1}\eta^l dxdt \tag{3.6}$$
$$= I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = l\int_{Q_{4R}} w^\beta |\nabla w|^{p-2}\nabla w\nabla\eta\eta^{l-1}\,dx\,dt,\ I_2 = \frac{l}{\beta+1}\int_{Q_{4R}} w^{\beta+1}\eta_t\eta^{l-1}\,dx\,dt.$$

$$I_3 = l\int_{Q_{4R}} w\eta_t\eta^{l-1}dx\,dt,\ I_4 = \int_{B_{4R}} w(\cdot,t_0)\eta^l(\cdot,t_0)\,dx.$$

Applying the inequalities $w \geq 1$ and $0 \leq \eta \leq 1$ and making use of Young's inequality, we evaluate the terms $I_i,\ i = 1,2,3,4$ in the following way:

$$|I_1| \leq \frac{\beta}{2}\int_{Q_{4R}} |\nabla w|^p w^{\beta-1}\eta^l dxdt + C(l)\int_{Q_{4R}} w^{\beta+p-1}\eta^{l-p}|\nabla\eta|^p\,dx\,dt,$$

$$|I_2| \leq \frac{1}{\beta+1}\int_{Q_{4R}} w^{\beta+1}|\eta_t|\eta^{l-1}\,dxdt,\ |I_3| \leq l\int_{Q_{4R}} w|\eta_t|\eta^{l-1}dx\,dt.$$

$$|I_4| \leq \frac{1}{2(\beta+1)}\int_{B_{4R}} w^{\beta+1}(\cdot,t_0)\eta^l(\cdot,t_0)\,dx + 2(\beta+1)\int_{B_{4R}} \eta^l(\cdot,t_0)dx$$

$$\leq \frac{1}{2(\beta+1)}\int_{B_{4R}} w^{\beta+1}(\cdot,t_0)\eta^l(\cdot,t_0)\,dx + C(\beta+1)R^{-s}\int_{Q_{4R}} w^{\beta+p-1}dxdt.$$

Here and throughout this section C denotes different constants which may depend on n,l,s,M. It will be specially indicated if C depends on other parameters. Gathering (3.6) with the last array of inequalities, we obtain the relation

$$\int_{B_{4R}} w^{\beta+1}(\cdot,t_0)\eta^l(\cdot,t_0)\,dx + \int_{Q_{4R}} |\nabla w|^p w^{\beta-1}\eta^l\,dx\,dt \leq C(\beta+1)^2 I,$$

where

$$I = \Big(\int_{Q_{4R}} \Big(w^{\beta+p-1}(R^{-s} + |\nabla\eta|^p + |\nabla\eta|^s + |\eta_t|) + w^{\beta+1}|\eta_t|\Big)\,dx\,dt\Big).$$

Since $\nabla w|^s \leq |\nabla w|^p + 1$ and $w \geq 1$, the last inequality can be given the form

$$\int_{B_{4R}} w^{\beta+1}(\cdot,t_0)\eta^l(\cdot,t_0)\,dx + \int_{Q_{4R}} |\nabla w|^s w^{\beta-1}\eta^l\,dx\,dt \leq C(n,l)(\beta+1)^2 I.$$

It follows that

$$\sup_{\tau\in\Delta_{4R}}\int_{B_{4R}} w^{\beta+1}(\cdot,\tau)\eta^l(\cdot,\tau)\,dx + \int_{Q_{4R}} |\nabla w|^s w^{\beta-1}\eta^l\,dx\,dt \leq C(\beta+1)^2 I, \tag{3.7}$$

whence

$$\sup_{\tau\in\Delta_{4R}}\int_{B_{4R}} \big(w\eta^l\big)^{\beta+1}\,dx + \int_{Q_{4R}} |\nabla\big(w\eta^l\big)^{\frac{\beta+s-1}{s}}|^s dx\,dt \leq C(\beta+1)^{l+2} I. \tag{3.8}$$

In particular,

$$\Big(\sup_{\tau\in\Delta_{4R}}\int_{B_{4R}}(w\eta^l)^{\beta+1}\,dx\Big)^{\frac{s}{n}}\Big(\int_{Q_{4R}}|\nabla(w\eta^l)^{\frac{\beta+s-1}{s}}|^s dxdt\Big)\le C((\beta+1)^{l+2}I)^{\frac{s+n}{n}}. \tag{3.9}$$

Let us apply the embedding inequality (compare with (4.28)):

$$\fint_{Q_r}|\varphi|^{qs}\,dx\,dt \tag{3.10}$$

$$\le C(s,n,m)r^s\Big(\fint_{Q_r}|\nabla\varphi|^s\,dx\,dt\Big)\Big(sup_{\tau\in\Delta_r}\fint_{B_r}|\varphi|^m\,dx\Big)^{\frac{s}{n}}$$

which is true for every $\varphi\in C^\infty(\overline{Q}_r)$, $\varphi=0$ on ∂B_r and $q=(n+m)/n$. The last inequality follows by means of integration of the inequality (see [24], pages 79-80)

$$\int_{B_r}|\varphi|^{qs}\,dx\le C\Big(\int_{B_r}|\nabla\varphi|^s\,dx\Big)\Big(\int_{B_r}|\varphi|^m\,dx\Big)^{\frac{s}{n}} \tag{3.11}$$

with respect to $t\in\Delta_r$.

Applying (3.10) with $q=(n+m)/n,\quad m=s(\beta+1)/(\beta+s-1)$ to the function

$$\varphi=(w\eta^l)^{\frac{\beta+s-1}{s}},$$

we have that

$$\fint_{Q_{2r}}\big(w\eta^l\big)^{q(\beta+s-1)}dx\,dt \tag{3.12}$$

$$\le CR^s\Big(\fint_{Q_{2r}}\Big|\nabla(w\eta^l)^{\frac{(\beta+s-1)}{s}}\Big|^s dx\,dt\Big)\Big(sup_{\tau\in\Delta_{2r}}\fint_{B_{2r}}(w\eta^l)^{\beta+1}dx\Big)^{\frac{s}{n}}$$

Gathering (3.9) with (3.12), we obtain the inequality

$$\fint_{Q_{2r}}(w\eta^l)^{q(\beta+s-1)}dx\,dt \tag{3.13}$$

$$\le CR^s\Big(\fint_{B_{2r}}\big(w^{\beta+p-1}(R^{-s}+|\nabla\eta|^p+|\eta_t|\ +w^{\beta+1}|\eta_t|)\big)\,dx\,dt\Big)^{(n+s)/n}.$$

Let $R\le\rho<r\le 2R$. We choose the function η in the last inequality as follows: $\eta=1$ in Q_ρ, $\eta=0$ in a neighborhood of the parabolic boundary of Q_r, (that is $\eta=0$ on ∂B_r and on $t=t_0-(2r)^s$). It is easy to check that

$$|\nabla\eta|\le Cr(R(r-\rho))^{-1},\ |\eta_t|\le C(r/R)^s(r^s-\rho^s)^{-1}. \tag{3.14}$$

Inequalities (3.13) and (3.14) lead to the inequality

$$\fint_{Q_\rho} (w\eta^l)^{q(\beta+s-1)} dx\,dt \tag{3.15}$$

$$\le C(n,s,l)(\beta+1)^{k(l+2)} R^{k(s-l)} \Big(\frac{r}{r-\rho}\Big)^{kl} \Big(\fint_{Q_{2r}} (w^{\beta+p-1} + w^{\beta+1})\, dx\,dt\Big)^k,$$

where

$$k = (n+s)/n,\ q = (n+m)/n. \tag{3.16}$$

Notice that $R^{s-l} \le C(M)$ according to (2.5), which is why the last inequality takes on the form

$$\fint_{Q_\rho} (w\eta^l)^{q(\beta+s-1)} dx\,dt \tag{3.17}$$

$$\le C(\beta+1)^{k(l+2)} \Big(\frac{r}{r-\rho}\Big)^{kl} \Big(\fint_{Q_{2r}} (w^{\beta+p-1} + w^{\beta+1})\, dx\,dt\Big)^k.$$

It follows from (2.3) that there exists a constant $\delta > 0$ such that

$$\max((2+\delta)n/(n+2), 1) < p_1 \le p(z), \quad z \in Q. \tag{3.18}$$

We will consider two different cases:

$$p_0 = p(x_0,\ t_0) \le 2 + 2\delta, \tag{3.19}$$

$$2 + 2\delta <\ p_0 = p(z_0) = p(x_0,\ t_0). \tag{3.20}$$

Let us consider first the case (3.19). There exists a constant R, depending on the constant M in (2.5) and on the distance between $(x_0,\ t_0)$ and the parabolic boundary of Q, such that

$$\max((2+\delta)n/(n+2), 1) < p_1 \le p(x,t) \le 2 + \delta, \quad \forall z \in Q_{4R} \subset Q. \tag{3.21}$$

It follows from (3.17) that

$$\fint_{Q_\rho} w^{q(\beta+s-1)} dx\,dt \tag{3.22}$$

$$\le C(\beta+1)^{k(l+2)} \Big(\frac{r}{r-\rho}\Big)^{kl} \Big(\fint_{Q_r} w^{\beta+1+\delta}\, dx\,dt\Big)^k.$$

as long as $w \ge 1$. Notice that due to (3.18) and the condition $\beta \ge 1$ the inequality

$$q(\beta+s-1) = \beta + s - 1 + \frac{(\beta+1)s}{n} > \beta + 1 + \delta$$

is true (see also (3.16)). We set

$$k = \frac{(n+s)}{n}, \quad \theta = \frac{s(n+1)}{n} - 1 \tag{3.23}$$

and rewrite (3.22) in the form

$$\Big(\fint_{Q_\rho} w^{k\beta+\theta}\,dx\,dt\Big)^{1/k} \le C(\beta+1)^{(l+2)}\Big(\frac{r}{r-\rho}\Big)^l \fint_{Q_r} w^{\beta+1+\delta}\,dx\,dt. \tag{3.24}$$

We may iterate this inequality. Let $j = 0, 1 \ldots$. Denote $r_j = R + 2^{-j}R$,

$$\chi_0 = 1, \quad \chi_1 = k, \quad \chi_j = k^j + (\theta - \delta - 1)(k^j - k)/(k-1), \quad j = 2, 3, \ldots$$

and take in (3.24) $r = r_j$, $\rho = r_{j+1}$, $\beta = \chi_j + \mu_j(\theta - \delta - 1)$, where $\mu_j = 0$ if $j = 0$ and $\mu_j = 1$ if $j \ge 1$. As a result, we obtain the recurrence relations

$$\Phi_{j+1} \le C^{k^{-j}}(2^j(1+\chi_j))^{(l+1)k^{-j-1}}\Phi_j$$

for the functions

$$\Phi_j = \Big(\fint_{Q_{r_j}} w^{\chi_j+\theta}\,dxdt\Big)^{k^{-j}}, \quad j \ge 1, \quad \Phi_0 = \Big(\fint_{Q_{2r}} w^{2+\delta}\,dxdt\Big).$$

Iterating the last inequality, we obtain the inequality

$$\sup_{Q_r} w \le C\Big(\fint_{Q_{2r}} w^{2+\delta}\,dx\,dt\Big)^\gamma, \tag{3.25}$$

with $\gamma = (k-1)/(k+\theta-\delta-2) = s/((n+2)s - (2+\delta)n)$. Because of estimate (3.9), proposition (3.18) and the embedding inequality (3.10), the last integral is convergent. We proceed to consider the case $p(x) \ge 2$ under the assumption that $x \in Q_{2r}$. In this case (3.22) can be rewritten as

$$\fint_{Q_\rho} w^{q(\beta+s-1)}\,dx\,dt \le C(\beta+1)^{k(l+2)}\Big(\frac{r}{r-\rho}\Big)^{kl}\Big(\fint_{Q_r} w^{\beta+p-1}\,dx\,dt\Big)^k, \tag{3.26}$$

with q defined in (3.16). We choose R so small that

$$2 \le p(x,t) \le s(n+1)/n, \quad (x,t) \in Q_{2r}. \tag{3.27}$$

Then (see (3.23))

$$q(\beta+s-1) = k\beta + s(n+1)/n - 1 \ge k\beta + p - 1$$

and it follows from (3.26) that

$$\Big(\fint_{Q_\rho} w^{k\beta+p-1}\,dx\,dt\Big)^{1/k} \le C(\beta+1)^{(l+2)}\Big(\frac{r}{r-\rho}\Big)^l \fint_{Q_\rho} w^{\beta+p-1}\,dx\,dt. \tag{3.28}$$

We iterate this relation letting $r_j = R + 2^{-j}R$, $\chi_j = k^j$, $\alpha_j = \chi_j + p - 1$ if $j = 0, 1, \dots$ and

$$\Phi_j = \Big(\fint_{Q_{r_j}} w^{\alpha_j}\, dxdt \Big)^{1/\chi_j}.$$

Taking in (3.28) $r = r_j$, $\rho = r_{j+1}$, $\beta = \chi_j$, we conclude that

$$\Phi_{j+1} \le C^{1/\chi_j} (2^j(1+\chi_j))^{(l+1)/\chi_{j+1}} \Phi_j,$$

and arguing by induction obtain the inequality

$$\Phi_j \le C\Phi_0.$$

For $w^{\chi_{j+s-1}} \le w^{\alpha_{j(x)}}$, the last inequality can be given the form

$$\Big(\fint_{Q_{r_j}} w^{\chi_j+s-1}\, dx\, dt \Big)^{1/(\chi_j+s-1)} \le C \Big(\fint_{Q_{2r}} w^p\, dx \Big)^{\chi_j/(\chi_j+s-1)}.$$

Passing to the limit when $j \to \infty$, we obtain the estimate

$$\sup_{Q_R} w \le C \fint_{Q_{2R}} w^p\, dx\, dt. \tag{3.29}$$

The last integral converges because of estimate (3.9), proposition (3.18), and the embedding inequality (3.10) (3.27). Inequalities (3.25), (3.29) hold for the function $w(x,t) = \max(-u(x,t)+1, 1)$, which yields boundedness of the weak solution u in Q_R. To prove that the weak solution $u(x,t)$ of equation (2.1) is locally bounded in the cylinder $Q' = \Omega' \times (\varepsilon, T]$, $\varepsilon > 0$, $\Omega' \subset Q$ one has to cover Q' by a finite number of small cylinders of the type Q_{2R}, assuming the validity of condition (3.18) or (3.27). The last step of the proof is to apply the embedding inequality (3.10) in order to evaluate the integrals on the right-hand sides of (3.25) and (3.29). This completes the proof of the theorem.

4. Higher integrability (Meyer's-type estimates) for the gradient of weak solutions

4.1. **Notation.** In what follows we use the following notation (compare with (3.1)-(3.4))

$$B_R(x_0) = \{x \in \Omega : |x - x_0| < R\}, \ \Lambda_{\theta R^2}(t_0) = \left\{ t \in \mathbb{R} : |t - t_0| < \theta R^2 \right\}, \tag{4.1}$$

$$Q_R = Q_{R,\theta R^2}(z_0) = B_R(x_0) \times \Lambda_{\theta R^2}(t_0), \tag{4.2}$$

$$\fint_{Q_R} f dz = \frac{1}{|Q_R|} \int_{Q_R} f dz, \ |Q_R| = 2\omega_n \theta R^{n+2}, \tag{4.3}$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$. We denote

$$p_1 = \min_{Q_{2R}} p(z),\ p_2 = \max_{Q_{2R}} p(z),\ p_0 = p(z_0). \tag{4.4}$$

4.2. **A priori estimates for weak solutions.** In this section, we derive several estimates on the integrals of weak solutions to equation (2.1).

Proposition 2. (A priori estimates). *Let $u(x,t)$ be a weak solution of equation* (2.1) *with $p(z)$ satisfying* (2.3). *For every Q_{2R}, $\overline{Q}_{2R} \subset Q$, the following estimate holds:*

$$\Psi \equiv \sup_{\Lambda_{\theta R^2}(t_0)} \int_{B_R(x_0)} |u - u_{_R}(t)|^2\, dx + \int_{Q_R(z_0)} |\nabla u|^p\, dz \le C(\Psi_1 + \Psi_2), \tag{4.5}$$

where

$$\Psi_1 = \frac{1}{\theta} \int_{Q_{2R}(z_0)} \Big| \frac{u - u_{2R}(t)}{R} \Big|^2 dz,\ \Psi_2 = \int_{Q_{2R}(z_0)} \Big| \frac{u - u_{2R}(t)}{R} \Big|^p dz, \tag{4.6}$$

$\theta \in (0,1)$, $C = C(n, \alpha, \beta)$ *is a positive constant, and* $u_{_R}(t) = \fint_{B_R} u dx$ *is the average of the function u in the ball $B_R(x_0)$.*

Proof. Estimate (4.5) follows from the integral identity (2.6) with $\zeta = (u - \widetilde{u}_{_{2R}}(t))\xi^\alpha(x)\tau^2(t),\ \alpha = p_2$. Here $\xi(x) \ge 0$ is the cut-off function for the ball $B_{2R}(x_0)$, $\tau(t)$ is the lower cut-off function for $\Lambda_{\theta R^2}(t_0)$, and

$$\widetilde{u}_{_{2R}}(t) = \Big(\int_{B_{2R}(x_0)} \xi^\alpha dx \Big)^{-1} \int_{B_{2R}(x_0)} u(x,t) \xi^\alpha(x) dx.$$

Substituting $\zeta = (u - \widetilde{u}_{_{2R}}(t))\xi^\alpha(x)\tau^2(t)$ into the integral identity (2.6), integrating the result over Q_{2R} and using the relation

$$\int_{B_{2R}(x_0)} \big(u(x,t) - \widetilde{u}_{_{2R}}(t)\big)\xi^\alpha(x) dx = 0,$$

we obtain

$$\Pi(t) \equiv \frac{1}{2} \int_{B_{2R}} |u(x,t) - \widetilde{u}_{_{2R}}(t)|^2\, \xi^\alpha(x)\tau^2(t) dx \tag{4.7}$$
$$+ \int_{Q_{2R}} \big(|\nabla u|^{p(z)}\, \xi^\alpha(x)\tau^2(t) \big) dz = \Phi,$$

$$\Phi \equiv -\alpha \int_{Q_{2R}} |\nabla u|^{p(z)-2}\, \nabla u\ (u - \widetilde{u}_{_{2R}}(t))\xi^{\alpha-1}\ \nabla\xi(x)\tau^2 dz \tag{4.8}$$
$$+ \int_{Q_{2R}} |u(x,t) - \widetilde{u}_{_{2R}}(t)|^2\, \xi^\alpha(x)\tau^{'}\tau(t) dz = 0,\ t \in \Lambda_{\theta 4R^2}(t_0).$$

Applying Young's inequality, we may write

$$|\nabla u|^{p(z)-1}\,\xi^{\alpha-1}|\nabla\xi||u-\widetilde{u}|\le\frac{1}{2}\,|\nabla u|^{p(z)}\,\xi^{\alpha}+C\xi^{(\alpha-p)}|\nabla\xi|^{p}|u-\widetilde{u}|^{p},$$

and then evaluate $|\Phi|$ in the following way:

$$|\Phi|\le\frac{1}{2}\int_{Q_{2R}}\Big(\,|\nabla u|^{p(z)}\,\xi^{\alpha}(x)\tau^{2}(t)\Big)dz+C\int_{Q_{2R}}\xi^{(\alpha-p)}|\nabla\xi|^{p}|u-\widetilde{u}|^{p}\tau^{2}dz$$
$$+\int_{Q_{2R}}|u(x,t)-\widetilde{u}_{2R}(t)|^{2}\,\xi^{\alpha}(x)|\tau^{'}\tau|dz.$$

It follows that

$$\Pi(t)=\int_{B_{2R}}|u-\widetilde{u}_{2R}|^{2}\,\xi^{\alpha}\tau^{2}dx+\int_{Q_{2R}}\Big(\,|\nabla u|^{p(z)}\,\xi^{\alpha}\tau^{2}\Big)dz \tag{4.9}$$
$$\le 2C\int_{Q_{2R}}\xi^{(\alpha-p)}|\nabla\xi|^{p}|u-\widetilde{u}|^{p}\tau^{2}dz+2\int_{Q_{2R}}|u(x,t)-\widetilde{u}_{2R}(t)|^{2}\,\xi^{\alpha}(x)|\tau^{'}\tau|dz.$$

By the choice of the cut-off functions ξ and τ

$$0\le\xi,\ \tau\le 1,\ |\nabla\xi|\le\frac{C}{R},\ |\tau^{'}|\le\frac{C}{\theta R^{2}}. \tag{4.10}$$

Using these properties and the inequalities

$$\int_{B_{R}(x_0)}|u-u_{R}(t)|^{2}\,dx \tag{4.11}$$
$$\le\int_{B_{R}(x_0)}|u-\widetilde{u}_{R}(t)|^{2}\,dx\le\int_{B_{2R}(x_0)}|u-\widetilde{u}_{2R}(t)|^{2}\,\xi^{\alpha}dx,$$

we arrive at the inequalities

$$\Big(\int_{B_{R}}|u(x,t)-u_{R}(t)|^{2}\,dx+\int_{Q_{2R}}\Big(\,|\nabla u|^{p(z)}\,\xi^{\alpha}(x)\tau^{2}(t)\Big)dz\Big) \tag{4.12}$$
$$\le\Pi(t)\le C\Big(\frac{1}{\theta}\int_{Q_{2R}(z_0)}\Big|\frac{u-u_{2R}(t)}{R}\Big|^{2}dz+\int_{Q_{2R}(z_0)}\Big|\frac{u-u_{2R}(t)}{R}\Big|^{p}dz\Big)$$

with $t\in\Lambda_{\theta 4R^{2}}(t_0)$. The last inequality gives us the required estimate (4.5), provided that the right-hand side is independent of t.

Proposition 3. *Let conditions* (2.3), (2.4), (2.5) *be fulfilled and* $u(x,t)$ *be a bounded weak solution of equation* (2.1). *Then*

$$C^{-1}\Psi_{2}^{(2)}\le\Psi_{2}=\int_{Q_{2R}}\Big|\frac{u-u_{2R}(t)}{R}\Big|^{p(\cdot)}dz\le C\Psi_{2}^{(1)}, \tag{4.13}$$

with $C = C(M, \alpha, \beta)$ *and*

$$\Psi_2^{(1)} = \int_{Q_{2R}(z_0)} \left| \frac{u - u_{2R}(t)}{R} \right|^{p_1} dz, \quad \Psi_2^{(2)} = \int_{Q_{2R}(z_0)} \left| \frac{u - u_{2R}(t)}{R} \right|^{p_2} dz. \quad (4.14)$$

Proof. To estimate Ψ_2 from above we make use of the following chain of inequalities:

$$\Psi_2 = \Big(\int_{Q_{2R}} \left| \frac{u - u_{2R}(t)}{R} \right|^{p_1} \left| \frac{u - u_{2R}(t)}{R} \right|^{p(\cdot)-p_1} dz \Big) \quad (4.15)$$

$$\leq \sup_{Q} \Big(|u - u_{2R}(t)|^{p(\cdot)-p_1} \Big) \max_{Q_{2R}} \Big(\frac{1}{R} \Big)^{p(\cdot)-p_1} \Big(\int_{Q_{2R}} \left| \frac{u - u_{2R}(t)}{R} \right|^{p_1} dz \Big).$$

According to (2.4), (2.5)

$$\max_{Q_{2R}} \Big(\frac{1}{R} \Big)^{p(\cdot)-p_1} \leq \max_{R \in [0,1]} \Big(\frac{1}{R} \Big)^{\omega(R+\theta R^2)} \leq C(M) < \infty. \quad (4.16)$$

Since the solution is bounded, the required inequality follows:

$$\Psi_2 = \int_{Q_{2R}} \left| \frac{u - u_{2R}(t)}{R} \right|^{p(\cdot)} dz \leq C \Psi_2^{(1)} = C \int_{Q_{2R}} \left| \frac{u - u_{2R}(t)}{R} \right|^{p_1} dz. \quad (4.17)$$

This proves the latter of inequalities (4.13). The former is proved likewise.

4.3. **Some special estimates.** We derive here the special estimates (4.29), (4.32) which will be the framework for the proof of a Gehring's-type lemma. Set

$$\lambda^{p_0} = \fint_{Q_{R,\,\theta R^2}(z_0)} \big(|\nabla u|^p + 1 \big) dz \geq 1, \; p_0 = p(z_0). \quad (4.18)$$

The parameter θ will be chosen in a special way for every R. For every fixed R we set $\lambda = \lambda(\theta)$. It is easy to check that the equation

$$\theta = \lambda^{2-p_0}(\theta) \quad (4.19)$$

always has a solution. Let us consider first the case $2 \leq p_0$. Taking into account (4.3), we may write

$$\int_{Q_{R,\,\theta R^2}(z_0)} \big(|\nabla u|^p + 1 \big) dz = 2\lambda^{p_0} \omega_n R^{n+2} \theta = 2\omega_n R^{n+2} \theta^{-\frac{2}{p_0-2}}. \quad (4.20)$$

The left-hand side of the last relation is smaller then the right-hand side of the same relation with small θ, but the situation is opposite if $\theta = 1$. This means that equation (4.19) has a solution θ. Let us assume now that $p_0 < 2$. The left-hand side of (4.19) is smaller than its right-hand side with small θ, and as in the previous case the situation changes to the opposite when

$\theta = 1$, whence the existence of a solution θ of equation (4.19). It is easy to verify that θ is a continuous function on R. In what follows, it is assumed that the numbers λ and θ satisfy (4.19). For the sake of simplicity we will use the notation $Q_{R,\theta R^2}(z_0) = Q_R$ with θ defined in (4.19). From (4.18), (4.19) we infer that

$$\lambda^{p_0} \le \frac{1}{|Q_R|}\int_Q \Big(|\nabla u|^p + 1\Big) dz = \frac{C}{w_n R^{n+2}\theta} = \frac{C\lambda^{p_0-2}}{R^{n+2}}.$$

The last inequality implies the estimate

$$\lambda^{p_0} \le \frac{C}{R^k}, \quad k = \frac{(n+2)\beta}{2} \tag{4.21}$$

with β given in (2.3).

We begin the consideration with the case $p_0 = p(z_0) \ge 2$. Let us assume that for some $R > 0$

$$⨍_{Q_{2R}} |\nabla u|^p dz \le C ⨍_{Q_R} |\nabla u|^p \, dz. \tag{4.22}$$

Proposition 4. *Let conditions* (4.4), (4.18), (4.19), (4.22) *be fulfilled and* $p_0 = p(z_0) \ge 2$. *Then under the conditions of Proposition* 3 *the following inequalities hold:*

$$\sup_{\Lambda_{\theta R^2}} \int_{B_R} |u - u_R(t)|^2 \, dx \le \tilde{C} R^{n+2}\lambda^2, \tag{4.23}$$

$$\int_{Q_R} |\nabla u|^p dz \le \tilde{C}\Big(\int_{Q_{2R}} \Big|\frac{u - u_{2R}(t)}{R}\Big|^{p_1} dz + |Q_{2R}|\Big), \tag{4.24}$$

with a constant $\tilde{C}$ *which depends on the constant* C *from* (4.22).

Proof. We will use inequalities (4.5) and (4.13). Applying Poincare's and Young's inequalities and taking into account (4.18), (4.19) and (4.20), we obtain the inequalities

$$\begin{aligned}\Psi_2^{(2)} \le \frac{1}{C}\Psi_2 \le \Psi_2^{(1)} = \int_{Q_{2R}} \Big|\frac{u - u_{2R}(t)}{R}\Big|^{p_1} dz \le C\int_{Q_{2R}} |\nabla u|^{p_1} \, dz \\ \le \tilde{C}\int_{Q_{2R}} (|\nabla u|^p + 1) \, dz \le \tilde{C} R^{n+2}\lambda^2\end{aligned} \tag{4.25}$$

with $\Psi_2^{(2)}$ and $\Psi_2^{(1)}$ from (4.14). To estimate Ψ_1 in (4.5) we apply Young's inequality in the form

$$\frac{1}{\theta}\Big|\frac{u - u_{2R}}{R}\Big|^2 \le \frac{2}{p_2}\varepsilon^{-\frac{p_2}{2}}\Big|\frac{u - u_{2R}}{R}\Big|^{p_2} + \frac{p_2 - 2}{p_2}\varepsilon^{\frac{p_2}{p_2-2}}\theta^{-\frac{p_2}{p_2-2}}. \tag{4.26}$$

Integrating (4.26) over Q_{2R}, with $\varepsilon = 1$ and then applying (4.19), we have that

$$\Psi_1 \le \frac{2}{p_2}\Psi_2^{(2)} + C\frac{p_2-2}{p_2}R^{n+2}\lambda^{\frac{2(p_0-2)}{p_2-2}} \le \tilde{C}\Big(\Psi_2^{(2)} + R^{n+2}\lambda^2\Big). \tag{4.27}$$

Gathering this inequality with (4.25), we obtain (4.23).

To prove (4.24) we integrate (4.26) with $\varepsilon > 0$ over Q_{2R} and apply the inequality

$$\theta^{-\frac{p_2}{p_2-2}} = \lambda^{\frac{p_2(p_0-2)}{p_2-2}} \le \lambda^{p_0}.$$

Then

$$\Psi_1 \le C\Big[\frac{2}{p_2}\varepsilon^{-\frac{p_2}{2}}\int_{Q_{2R}}\Big|\frac{u-u_{2R}}{R}\Big|^{p_2}dz + \frac{p_2-2}{p_2}\varepsilon^{\frac{p_2}{p_2-2}}\lambda^{p_0}|Q_{2R}|\Big]$$

$$\le C\Big[\frac{2}{p_2}\varepsilon^{-\frac{p_2}{2}}\int_{Q_{2R}}\Big|\frac{u-u_{2R}}{R}\Big|^{p_2}dz + \frac{p_2-2}{p_2}\varepsilon^{\frac{p_2}{p_2-2}}\Big(\int_{Q_{2R}}|\nabla u|^p\,dz + |Q_{2R}|\Big)\Big].$$

To obtain (4.24), it is sufficient to gather (4.5), (4.22), (4.13) and the last inequality with a sufficiently small ε. Given a number δ, ε should satisfy the inequality

$$\frac{p_2-2}{p_2}\varepsilon^{\frac{p_2}{p_2-2}} \le \delta.$$

This condition is satisfied for all $p_2 \ge 2$.

We will need the following proposition ([13, page 7]; compare with (3.10)).

Proposition 5. *For $s \ge \frac{n+2}{n}$ the following inequality holds:*

$$\int_{Q_{2R}}|u-u_{2R}(t)|^s\,dz \tag{4.28}$$

$$\le C(s,n)\Big(\sup_{\Lambda_{4\theta R^2}}\int_{B_{2R}}|u-u_{2R}(t)|^2\,dx\Big)^{\frac{s}{n+2}}\int_{Q_{2R}}|\nabla u|^{\frac{sn}{n+2}}\,dz.$$

Proposition 6. *Under the conditions of Proposition (4)*

$$\fint_{Q_R}|\nabla u|^p dz \le \tilde{C}\Big[\Big(\fint_{Q_{2R}}|\nabla u|^{\frac{np_1}{n+2}}dz\Big)^{\frac{n+2}{n}} + 1\Big]. \tag{4.29}$$

Proof. It follows from (4.24) that

$$\int_{Q_R}|\nabla u|^p dz \le \tilde{C}_1\Big(\int_{Q_{2R}}\Big|\frac{u-u_{2R}}{R}\Big|^{p_1}dz + |Q_{2R}|\Big).$$

Applying (4.28) with $s = p_1$ and (4.23), we obtain the inequalities

$$\begin{aligned}&\int_{Q_R} |\nabla u|^p\, dz \\ &\le C\Big(\frac{1}{R^{p_1}}\Big(\sup_{\Lambda_{4\theta R^2}} \int_{B_{2R}} |u-u_{2R}(t)|^2\, dx\Big)^{\frac{p_1}{n+2}}\Big(\int_{Q_{2R}} |\nabla u|^{\frac{p_1 n}{n+2}}\, dz\Big) + |Q_{2R}|\Big) \\ &\le C\Big(R^{-p_1}(R^{n+2}\lambda^2)^{\frac{p_1}{n+2}}\Big(\int_{Q_{2R}} |\nabla u|^{\frac{np_1}{n+2}}\, dz\Big) + |Q_{2R}|\Big) \\ &= C\Big(\lambda^{\frac{2p_1}{n+2}} \int_{Q_{2R}} |\nabla u|^{\frac{p_1 n}{n+2}}\, dz + |Q_{2R}|\Big),\end{aligned}$$

which yield the inequality

$$⨍_{Q_R} |\nabla u|^p dz \le C\Big(\lambda^{\frac{2p_1}{n+2}} ⨍_{Q_{2R}} |\nabla u|^{\frac{np_1}{n+2}}\, dz + 1\Big).$$

By virtue of Young's inequality

$$⨍_{Q_R} |\nabla u|^p dz \le C_\delta\Big(⨍_{Q_{2R}} |\nabla u|^{\frac{np_1}{n+2}}\, dz\Big)^{\frac{n+2}{n}} + \delta\lambda^{p_1} + C.$$

Applying (4.18), the inequality $\lambda^{p_1} \le \lambda^{p_0}$ and choosing δ sufficiently small, we obtain (4.29).

We proceed to study the case $\frac{2n}{n+2} < \alpha \le p_0 < 2$.

Proposition 7. *Let $u(x,t)$ be a bounded weak solution of equation* (2.1) *with $p(x,t)$ satisfying* (2.3), ($\frac{2n}{n+2} < \alpha \le p_0 < 2$). *Then*

$$X \equiv \sup_{\Lambda_{\theta R^2}} \int_{B_R} |u-u_R(t)|^2\, dx \le C\Big[A^{\frac{2}{p_1}}\ R^{n+2} + \theta\big(A^{\frac{2(p_2-1)}{p_1}} + 1\big)|Q_{2R}|\Big] \quad (4.30)$$

where $A = ⨍_{Q_{2R}} |\nabla u|^{p_1}\, dz$. Here the parameters R and θ need not satisfy (4.19).

Proof. We use the integral identity (2.6) with $\zeta = (u - u_{2R}(\widehat{t}))\xi^2(x)$, where $\xi(x) \ge 0$ is a cut-off function for the ball $B_{2R}(x_0)$, and $\widehat{t} \in [t_0 - \theta(2R)^2, t_0 - \theta R^2]$. Then

$$\frac{1}{2}\int_{B_{2R}} \big|u(x,s) - u_{2R}(\widehat{t})\big|^2 \xi^2(x)dx + \int_{\widehat{t}}^s \int_{B_{2R}} |\nabla u|^p\, \xi^2(x)dz$$

$$\le \frac{1}{2}\int_{B_{2R}} |u(x,\widehat{t}) - u_{2R}(\widehat{t})|^2\xi^2(x)dx + 2\int_{\widehat{t}}^s \int_{B_{2R}} |\nabla u|^{p-1}|u - u_{2R}(\widehat{t})||\xi||\nabla\xi|dz.$$

Let us make use of the inequality

$$|\nabla u|^{p-1}\left|u-u_{2R}(\widehat{t})\right|\xi\,|\nabla\xi|$$

$$\le \frac{\delta}{4\theta R^2}\left|u-u_{2R}(\widehat{t})\right|^2\xi^2+C_\delta\Big(|\nabla u|^{2(p_2-1)}+1\Big)|\nabla\xi|^2\,\theta R^2,$$

with $\delta\le 1/4$ and (4.10):

$$X\le \sup_{\Lambda_{\theta R^2}}\int_{B_R}\left|u-u_R(\widehat{t})\right|^2dx\le \sup_{\Lambda_{\theta R^2}}\int_{B_{2R}}\left|u-u_{2R}(\widehat{t})\right|^2\xi^2dx$$

$$\le 2\int_{B_{2R}}\left|u(x,\widehat{t})-u_{2R}(t)\right|^2dx+C_1\theta\int_{Q_{2R}}\Big(|\nabla u|^{2(p_2-1)}+1\Big)\,dz.$$

Using the embedding $W^{1,\,p_1}\subset L^2,\;\; 2n/(n+2)\le p_1,$ we have that

$$X\le C_2\Big(\fint_{B_{2R}}\left|\nabla u(\widehat{t})\right|^{p_1}dx\Big)^{\frac{2}{p_1}}R^{n+2}+\theta\int_{Q_{2R}}\Big(|\nabla u|^{2(p_2-1)}+1\Big)\,dz,$$

whence

$$X^{\frac{p_1}{2}}\le C_3\Big(\fint_{B_{2R}}\left|\nabla u(\widehat{t})\right|^{p_1}dx\Big)R^{\frac{(n+2)p_1}{2}}+\theta^{\frac{p_1}{2}}\Big(\int_{Q_{2R}}\Big(|\nabla u|^{2(p_2-1)}+1\Big)dz\Big)^{\frac{p_1}{2}}.$$

Integrating the last inequality with respect to $\widehat{t}\in\Lambda_{2R}\backslash\Lambda_R$ and dividing by $|\Lambda_{2R}\backslash\Lambda_R|$, we come to the inequality

$$X^{\frac{p_1}{2}}\le C_3\Big[\Big(\fint_{Q_{2R}}|\nabla u|^{p_1}dz\Big)R^{\frac{(n+2)p_1}{2}}+\theta^{\frac{p_1}{2}}\Big(\int_{Q_{2R}}\Big(|\nabla u|^{2(p_2-1)}+1\Big)dz\Big)^{\frac{p_1}{2}}\Big].$$

It follows that

$$X\le C_4\Big[\Big(\fint_{Q_{2R}}|\nabla u|^{p_1}dz\Big)^{\frac{2}{p_1}}R^{n+2}+\theta\Big(\int_{Q_{2R}}\Big(|\nabla u|^{2(p_2-1)}+1\Big)dz\Big)\Big].$$

The last integral can be estimated with the help of Hölder's inequality as follows:

$$\int_{Q_{2R}}|\nabla u|^{2(p_2-1)}dz\le\Big(\fint_{Q_{2R}}|\nabla u|^{p_1}dz\Big)^{\frac{2(p_2-1)}{p_1}}|Q_{2R}|\,,$$

if $(p_2-p_1)<1-\frac{p_1}{2}$. The last two inequalities imply (4.30).

Proposition 8. *Let the conditions of Proposition* 7 *be fulfilled. Then estimate* (4.23) *is true.*

Proof. If $A \le \lambda^{p_0}$, $\lambda \ge 1$, then it follows from (4.30) that

$$X \le c[\lambda^{\frac{2p_0}{p_1}} + \lambda^{\frac{2(p_2-1)p_0}{p_1}+2-p_0}]R^{n+2}. \tag{4.31}$$

By virtue of (4.21) and (2.5)

$$\lambda^{\frac{2p_0}{p_1}} = \lambda^2\Big(\lambda^{p_0}\Big)^{\frac{2(p_0-p_1)}{p_0p_1}} \le \lambda^2\Big(\frac{C}{R}\Big)^{\frac{2(p_0-p_1)k}{p_0p_1}} \le \tilde{C}\lambda^2.$$

Arguing in the same way we estimate the second term in (4.31) using the relation $\theta = \lambda^{2-p_0}$. This completes the proof of the proposition.

Proposition 9. *Under the conditions of Proposition* 7

$$⨍_{Q_R} |\nabla u|^p\, dz \le C_1\Big[\Big(⨍_{Q_{2R}} |\nabla u|^q dz\Big)^{\frac{p_0}{q}} + 1\Big], \quad q = \frac{2n}{n+2}. \tag{4.32}$$

Proof. By virtue of (4.5), (4.13)

$$\int_{Q_R} |\nabla u|^p\, dz \le C(\Psi_1 + \Psi_2^{(1)}). \tag{4.33}$$

We estimate first $\Psi_2^{(1)}$ through Ψ_1 using Young's inequality:

$$\frac{|u-u_{2R}|^{p_1}}{R^{p_1}} \le \varepsilon\theta^{\frac{p_1}{2-p_1}} + \varepsilon^{\frac{p_1-2}{2}}\frac{1}{\theta}\frac{|u-u_{2R}|^2}{R^2} \le \varepsilon\lambda^{p_0} + \varepsilon^{\frac{p_1-2}{2}}\frac{1}{\theta}\frac{|u-u_{2R}|^2}{R^2},$$

provided that $p_1/(2-p_1) \le p_0/(2-p_0)$. This inequality together with (4.6), (4.14) gives

$$\Psi_{(2)}^{(1)} \le \varepsilon\lambda^{p_0}|Q_R| + C_\varepsilon\Psi_1 \quad \forall \varepsilon > 0. \tag{4.34}$$

Using (4.28) with $s = 2$, we may write

$$\int_{Q_{2R}} |u-u_{2R}|^2\, dz \le C\Big(\sup_{\Lambda_{2R}}\Big(\int_{B_{2R}} |u-u_{2R}|^2\, dx\Big)\Big)^{\frac{2}{n+2}} \int_{Q_{2R}} |\nabla u|^{\frac{2n}{n+2}} dz. \tag{4.35}$$

It follows from proposition (8) (see formulas (4.23)), (4.33),(4.3), (4.35) that

$$⨍_{Q_R} |\nabla u|^p\, dz \le \varepsilon\lambda^{p_0} + \frac{C_\varepsilon}{\lambda^{2-p_0}R^2}(\lambda^2R^{n+2})^{\frac{2}{n+2}} ⨍_{Q_R} |\nabla u|^{\frac{2n}{n+2}}\, dz$$

$$= \varepsilon\lambda^{p_0} + C_\varepsilon\lambda^{p_0-q} ⨍_{Q_R} |\nabla u|^{\frac{2n}{n+2}}\, dz \le 2\varepsilon\lambda^{p_0} + \tilde{C}_\varepsilon\Big(⨍_{Q_R} |\nabla u|^{\frac{2n}{n+2}}\, dz\Big)^{\frac{p_0}{q}}.$$

Using (4.18) and choosing ε sufficiently small, we obtain the required inequalities.

It is very interesting to notice that the proof of (4.32) does not use (4.22).

4.4. **Gehring's-type lemma. The main theorem.** From now on we use estimates (4.29), (4.32) jointly, gathering the results obtained for the cases $p_0 \geq 2$ and $p_0 < 2$. To this end we introduce the functions

$$f(z) = \begin{cases} |\nabla u|^{\frac{2n}{n+2}} & if \quad p(z) < 2, \\ |\nabla u|^{\frac{p(z)n}{n+2}} & if \quad p(z) \geq 2, \end{cases} \qquad q(z) = \begin{cases} \frac{p(z)(n+2)}{2n} & if \quad p(z) < 2, \\ \frac{n+2}{n} & if \quad p(z) \geq 2. \end{cases} \tag{4.36}$$

Notice that $f^q = |\nabla u|^p$ and that the continuous function $q(z)$ satisfies (2.5).

Proposition 10. *Let us assume that for some* $R > 0$

$$\fint_{Q_{2R}} f^{q(z)} dz \leq C \fint_{Q_R} f^{q(z)} dz. \tag{4.37}$$

Then it follows from estimates (4.29) *and* (4.32) *that*

$$\fint_{Q_R} f^{q(z)} dz \leq \tilde{C}\Big[\Big(\fint_{Q_{2R}} f dz\Big)^{q(z_0)} + 1\Big]. \tag{4.38}$$

Proof. Condition (4.37) coincides with (4.22), therefore we have to derive (4.38) from (4.29) and (4.32).

Let us consider the case $p(z_0) \geq 2$. If $p_1 \geq 2$; that is, $p(x) \geq 2, x \in Q_{2R}$, then

$$\fint_{Q_{2R}} |\nabla u|^{\frac{p_1 n}{n+2}} dz \leq 1 + \fint_{Q_{2R}} |\nabla u|^{\frac{pn}{n+2}} dz = 1 + \fint_{Q_{2R}} f dz$$

whence, by virtue of (4.29),

$$\fint_{Q_R} f^q dz = \fint_{Q_R} |\nabla u|^{p(x)} dz \leq \tilde{C}\Big[\Big(\fint_{Q_{2R}} f dz\Big)^{q(z_0)} + 1\Big] \tag{4.39}$$

because $q(z_0) = \frac{n+2}{n}$.

Let us assume now that $p_1 < 2$. Then

$$\int_{Q_{2R}} |\nabla u|^{\frac{p_1 n}{n+2}} dz \leq |Q_{2R}| + \int_{Q_{2R} \cap \{p \geq 2\}} |\nabla u|^{\frac{pn}{n+2}} dz + \int_{Q_{2R} \cap \{p < 2\}} |\nabla u|^{\frac{n}{n+2}} dz$$
$$\leq |Q_{2R}| + 2 \int_{Q_{2R}} f dz,$$

and inequality (4.38) follows as in the previous case. The case $p(z_0) < 2$ can be considered analogously. The proposition is proved.

Now we pass to the proof of a Gehring's-type lemma. First of all, we prove two useful auxiliary propositions.

Let $d(z)$ be the parabolic distance between a point $z_0 \in Q$ and the boundary ∂Q. Instead of $f(z)$ we will consider the function

$$F(z) = d^{\frac{k}{q(z)}}(z) f(z),$$

where k is given by (4.21). Then $F^q = d^k f^q$.

Proposition 11. *Let us assume that*

$$\fint_{Q_{2R}} F^q dz \le c_1 \fint_{Q_R} F^q dz. \tag{4.40}$$

Then

$$\fint_{Q_R} F^q dz \le \widetilde{c}_1 \Big[\Big(\fint_{Q_{2R}} F dz \Big)^{q(z_0)} + 1 \Big]. \tag{4.41}$$

Proof. We show the equivalence of inequalities (4.40) and (4.37), and inequalities (4.41) and (4.38). Let us consider first inequalities (4.40) and (4.37). We write (4.37) in the form

$$\fint_{Q_R} F^{q(z)}(z) \Big(\frac{d(z_0)}{d(z)} \Big)^k dz \tag{4.42}$$

$$\le c \Big[\int_{Q_{2R}} F(z) \Big(\frac{d(z_0)}{d(z)} \Big)^{\frac{k}{q(z)}} \big(d(z_0) \big)^{k(\frac{1}{q(z_0)} - \frac{1}{q(z)})} dz + 1 \Big].$$

Notice that by the triangle inequality

$$\frac{3}{4} \le \frac{d(z_0)}{d(z)} \le \frac{5}{4} \text{ for } z \in Q_{2R}(z_0).$$

Moreover,

$$R \le d(z_0), \qquad k \Big| \frac{1}{q(z_0)} - \frac{1}{q(z)} \Big| \le c\tilde{w}(R + R^2),$$

where $\tilde{w}(t)$ is the modulus of continuity of the function q. By condition (2.5)

$$d(z_0)^{k(\frac{1}{q(z_0)} - \frac{1}{q(z)})} \le d(z_0)^{c\tilde{w}(R+R^2)} + R^{-c\tilde{w}(R+R^2)} \le C,$$

with C not depending on z_0. Then (4.41) follows from (4.42). This completes the proof of the proposition.

Proposition 12. *There exists $\gamma \ge 1$ such that for almost every $z_0 \in E(\lambda) = \{F(z) > \lambda\}$, $\lambda \ge \gamma$, one may find $R \in \big(0, \frac{d(z_0)}{4}\big)$ for which*

$$\fint_{Q_{4R}} F^q dz \le \fint_{Q_R} F^q dz = \lambda^{q(z_0)}. \tag{4.43}$$

Proof. The inequality $\sup_{Q_R} d(z) \le 5R$ holds for $R \in (\frac{d(z_0)}{4}, d(z_0))$. Applying (4.21), we conclude that

$$\fint_{Q_R} F^q dz \le 5^k R^k \fint_{Q_R} |\nabla u|^p dz \le c_1 \equiv \gamma^{q(z_0)}. \tag{4.44}$$

On the other hand, it follows from Lebesgue's theorem that

$$\lim_{R\to 0} \fint_{Q_R} F^q dz > \lambda^{q(z_0)} \ge \gamma^{q(z_0)}.$$

Therefore, the needed assertion follows from (4.44) and from the continuity of the integral $\fint_{Q_R} F^q dz$ with respect to R. The proposition is proved.

Now we prove the following

Lemma 2 (Gehring's-type Lemma)**.** *Let $q(z)$ be continuous and satisfy (2.5). Assume that $1 < \alpha \le q(z) \le \beta < \infty$, F^q, $\Phi^q \in L^1(Q)$, and that for some positive constants γ, c, l the following assertion holds: for almost every $z_0 \in E(\lambda) = \{z \in Q : F(z) > \lambda\}$ there exist $R = R(z_0) > 0$, $\theta = \theta(z_0) \in (0,1)$ such that $Q_{4R,\theta(z_0)} \subset Q$,*

$$\theta \ge R^l, \tag{4.45}$$

$$C^{-1} \fint_{Q_{4R,\, \theta(z_0)}} F^q dz \le \lambda^{q(z_0)} \tag{4.46}$$

$$\le C\Big[\Big(\fint_{Q_{2R,\, \theta(z_0)}} F dz\Big)^{q(z_0)} + \fint_{Q_{2R,\, \theta(z_0)}} \Phi^q dz\Big].$$

Then there exists $\varepsilon > 0$ such that $F^{q+\varepsilon} \in L^1(Q)$ if $\Phi^{q+\varepsilon} \in L^1(Q)$.

Proof. First of all, we observe that by virtue of (4.45), (4.46)

$$\lambda^{q(z_0)} \le \frac{c}{R^{n+2+l}}.$$

Hence, according to (2.5),

$$\lambda^{w(R)} \le C(M), \tag{4.47}$$

where C does not depend on z_0 and λ.

Now we start evaluating the integrals on the right-hand side of (4.46). Letting

$$q_1 = \min_{Q_{2R}} q(z), \quad q_2 = \max_{Q_{2R}} q(z), \quad q_0 = q(z_0)$$

and using the fact that since

$$\fint_{Q_{2R}} F^q dz \le 2^{n+2} \fint_{Q_{4R}} F^q dz \le C\lambda^{q_0}$$

(see (4.46)) we have that

$$I_1 \equiv \Big(\fint_{Q_{2R}} F dz \Big)^{q_0} = \Big(\fint_{Q_{2R}} F dz \Big)^{q_0-1} \cdot \fint_{Q_{2R}} F dz$$
$$\leq \Big(\fint_{Q_{2R}} F^{q_1} dz \Big)^{\frac{q_0-1}{q_1}} \cdot \fint_{Q_{2R}} F dz$$
$$\leq \Big(1 + \fint_{Q_{2R}} F^q dz \Big)^{\frac{q_0-1}{q_1}} \cdot \fint_{Q_{2R}} F dz \leq C\lambda^{\frac{q_0(q_0-1)}{q_1}} \fint_{Q_{2R}} F dz.$$

For every $\eta > 0$

$$\fint_{Q_{2R}} F dz \leq \lambda\eta + \frac{1}{|Q_{2R}|} \int_{Q_{2R} \cap E(\eta\lambda)} F dz.$$

Using (4.47), ($\lambda^\gamma \leq C,\ \gamma = q_0(q_0-1)/q_1 + 1 - q_0 = (q_0-1)(q_0-q_1)/q_1$), we have

$$I_1 \leq C\Big[\lambda^{\frac{q_0(q_0-1)}{q_1}+1}\eta + \lambda^{\frac{q_0(q_0-1)}{q_1}} \frac{1}{|Q_{2R}|} \int_{Q_{2R}\cap E(\eta\lambda)} F dz\Big] \tag{4.48}$$
$$\leq C\Big[\lambda^{q_0}\eta + \lambda^{q_0-1} \frac{1}{|Q_{2R}|} \int_{Q_{2R}\cap E(\eta\lambda)} F dz\Big].$$

Further, letting $D(\lambda) = \{x \in D :\ \Phi(x) > \lambda\}$ and $\eta \in (0,1)$, we obtain the inequality

$$I_2 \equiv \fint_{Q_{2R}} \Phi^q\, dz \leq \lambda^{p_2}\eta + \frac{1}{|Q_{2R}|} \int_{Q_{2R}\cap D(\eta\lambda)} \Phi^q dz \tag{4.49}$$
$$\leq C\Big[\lambda^{q_0}\eta + \frac{1}{|Q_{2R}|} \int_{Q_{2R}\cap D(\eta\lambda)} \Phi^q dz\Big].$$

Choosing now $\eta \in (0,1)$ sufficiently small and gathering (4.46), (4.48) and (4.49), we come to the estimate

$$\int_{Q_{4R}} F^q dz \leq c\Big[\lambda^{q_0-1} \int_{Q_{2R}\cap E(\eta\lambda)} F dz + \int_{Q_{2_R}\cap D(\eta\lambda)} \Phi^q dz\Big]. \tag{4.50}$$

Due to (4.47), this estimate remains true if in each of the balls Q_{2R} the exponent $q_0 = q(z_0)$ is replaced by an arbitrary $\hat{q}$ from the function $q|_{Q_{2R}}$. In particular, we can take for $\hat{q}$ the value suggested by the mean value theorem:

$$\int_{Q_{2R}\cap E(\eta\lambda)} \lambda^{q(z)-1} F(z)\, dz = \lambda^{\hat{q}-1} \int_{Q_{2R}\cap E(\eta\lambda)} F(z)\, dz.$$

The principal estimate (4.50) now takes on the form

$$\int_{Q_{4R}} F^q\,dz \le c\Big[\int_{Q_{2R}\cap E(\eta\lambda)} \lambda^{q(z)-1}F(z)\,dz + \int_{Q_{2R}\cap D(\eta\lambda)} \Phi^{q(z)}(z)\,dz\Big]. \quad (4.51)$$

Thus, for almost every $z_0 \in E(\lambda)$, $\lambda \ge \gamma$, there exists $R = R(z_0, \lambda)$ such that the estimate (4.51) is true. Consequently, by Vitali's theorem there exists a countable set of nonintersecting balls Q_{2R_j} such that $E(\lambda) \subset \cup Q_{4R_j}$. Using (4.51), we come to the inequality

$$\int_{E(\lambda)} F^q dz \le \sum_j \int_{Q_{4R_j}} F^q dz \quad (4.52)$$

$$\le C\Big[\sum_j \int_{Q_{2R_j}\cap E(\lambda\eta)} \lambda^{q(z)-1}F\,dz + \sum_j \int_{Q_{2R_j}\cap D(\lambda\eta)} \Phi^q\,dz\Big]$$

$$\le C\Big[\int_{E(\lambda\eta)} \lambda^{q(z)-1}F(z)\,dz + \int_{D(\lambda\eta)} \Phi^{q(z)}(z)\,dz\Big].$$

It is easy to verify the validity of the relation

$$I \equiv \int_{E(\gamma)} F^{q+\varepsilon} = -\int_\gamma^\infty \lambda^\varepsilon\,d_\lambda\Big(\int_{E(\lambda)} F^q\,dx\Big), \quad \varepsilon > 0,$$

if $q = const > 1$, (see E.M. Stein [30], Chapter I). Integration by parts gives

$$I = \varepsilon\int_\gamma^\infty \lambda^{\varepsilon-1}\int_{E(\lambda)} F^q\,dz\,d\lambda + \gamma^\varepsilon\int_{E(\gamma)} F^q dz. \quad (4.53)$$

Considering the step functions $q(x)$, one may show that the last relation is true for the variable exponent q. Therefore, it follows from (4.52) that

$$I \le \tilde{n}\varepsilon\Big[\int_\gamma^\infty \lambda^{\varepsilon-1}\Big(\int_{E(\lambda\eta)} \lambda^{q-1}F dz + \int_{D(\lambda\eta)} \Phi^q dz\Big)d\lambda\Big] + \gamma^\varepsilon\int_{E(\gamma)} F^q dz$$

$$\le \Big[c_1\varepsilon\int_{\gamma_0}^\infty \lambda^{\varepsilon-1}\int_{E(\lambda)} \lambda^{q-1}F\,dz\,d\lambda + C\varepsilon\int_{\gamma_0}^\infty \lambda^{\varepsilon-1}\int_{D(\lambda)} \Phi^q\,dz\,d\lambda\Big] \quad (4.54)$$

$$+\gamma_0^\varepsilon\int_{E(\gamma_0)} F^q\,dz, \quad \gamma_0 = \gamma\eta.$$

Let us evaluate each of the terms in the square brackets. Using (4.53) in the form

$$I_0 = \int_{E(\gamma_0)} F^{q+\varepsilon}\,dz = \int_{\gamma_0}^\infty \lambda^{\varepsilon-1}\int_{E(\lambda)} (\varepsilon+q-1)\lambda^{q-1}F\,dz\,d\lambda + \gamma_0^\varepsilon\int_{E(\gamma_0)} F\,dz,$$

we have that

$$C\varepsilon \int_{\gamma_0}^{\infty} \lambda^{\varepsilon-1} \int_{E(\lambda)} \lambda^{q-1} F \, dz \, d\lambda \leq \frac{C\varepsilon}{\varepsilon+\alpha-1} I_0 \leq \frac{C\varepsilon}{\varepsilon+\alpha-1}[I + \gamma^{\beta+\varepsilon}|Q|].$$

Moreover, relation (4.53) for the function Φ (in place of F) gives us the inequality

$$C\varepsilon \int_{\gamma_0}^{\infty} \lambda^{\varepsilon-1} \int_{D(\lambda)} \Phi^q \, dz \, d\lambda \leq C \int_{D(\gamma_0)} \Phi^{q+\varepsilon} \, dz.$$

Finally, inequality (4.54) becomes

$$I \leq \frac{c\varepsilon}{\varepsilon+\alpha-1}(I + \gamma^{\beta+\varepsilon}|Q|) + \gamma^{\varepsilon} \int_{E(\gamma)} F^q \, dz + \tilde{n} \int_{D(\gamma_0)} \Phi^{q+\varepsilon} \, dz.$$

Choosing ε so small that $\frac{C\varepsilon}{\varepsilon+\alpha-1} = \frac{1}{2}$, we get the needed estimates:

$$\int_{E(\gamma)} F^{q+\varepsilon} \, dz \leq C \int_Q (F^q + \Phi^{q+\varepsilon} + 1) \, dz,$$

$$\int_Q F^{q+\varepsilon} \, dz \leq C \int_Q (F^q + \Phi^{q+\varepsilon} + 1) \, dz.$$

This completes the proof of the lemma.

Now we can formulate

Theorem 13. *Let $u(x,t)$ be a bounded weak solution of equation* (2.1) (*in the sense of Definition* (1)) *with the uniformly continuous exponent $p(x,t)$ satisfying* (2.3)–(2.5), ($n \geq 2$). *Then there exists a positive constant $\varepsilon > 0$, such that in any subdomain Q', $\overline{Q'} \subset Q$,*

$$\int_{Q'} |\nabla u|^{p(z)(1+\varepsilon)} \, dz < \infty, \ \varepsilon > 0, \ \overline{Q'} \subset Q.$$

The positive constant ε depends only on C_0, M, α, β in (2.2), (2.3)–(2.5).

Proof. The proof of local higher integrability of the gradient of the weak solution $u(x,t)$ to equation (2.1) in a ball Q_R amounts to applying Proposition (6), (formula (4.29)) and Proposition (9), (formula (4.32)) with the consequent application of a Gehring's-type lemma. To prove the theorem in a cylinder $Q' = \Omega' \times (\varepsilon, T]$, $\varepsilon > 0$, $\Omega' \subset Q$ one has to cover Q' by a finite number of small domains of the type Q_R. This completes the proof of the theorem.

Acknowledgements. The authors would like to thank Professor A. Arkhipova for her suggestions concerning the employed techniques and for useful discussions. The authors express their special gratitude to Professor

Y. Alkhutov who contributed to this work by providing materials used in the proof of boundedness of weak solutions. The first author was partially supported by the Projects DECONT-Differential Equations of Continuum Mechanics and POCTI/MAT/61576/2004, FCT/MCES at the "Centro de Matemática," Universidade da Beira Interior, Portugal.

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