Advances in Differential Equations Volume 10, Number 9 (2005), 1035–1052

SYSTEMS OF SINGULAR POISSON EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We establish the existence of nontrivial solutions to systems of singular Poisson equations in unbounded domains, under some invariance conditions and singular subcritical growth. The proofs rely on a concentration-compactness argument and on a generalized linking theorem due to Krysewski and Szulkin.

1. Introduction

In this paper, we study the existence of solutions to the following systems of singular elliptic equations on $\mathbb{R}^N$:

$$\begin{cases} -\operatorname{div}(\nabla u|x|^{-2a}) = H_v(x,u,v), \\ -\operatorname{div}(\nabla v|x|^{-2a}) = H_u(x,u,v); \end{cases}$$

and

$$\begin{cases} -\operatorname{div}(\nabla u|x|^{-2a}) = \beta|x|^{-2(1+a)}v + H_v(x,u,v), \\ -\operatorname{div}(\nabla v|x|^{-2a}) = \eta|x|^{-2(1+a)}u + H_u(x,u,v); \end{cases}$$

where $N \geq 3$, $a \in [0,(N-2)/2)$, and $\beta, \eta \in\ [0, S(a,a+1))$, with $S(a,a+1)$ the best constant in the Caffarelli, Kohn, and Nirenberg inequality [2]. This type of problem comes from systems of anisotropic semilinear Schrödinger equations.

The first problem has been studied in the case $a = 0$ and on bounded domains by Felmer and Wang [7], Husolf, Mitidieri and van der Vorst [9],

Accepted for publication: April 2005.
AMS Subject Classifications: 35J50.
This work was partially supported by CRSNG-Canada.

and by Husholf and van der Vorst [8]. Recently, we [5] treated these problems (also with $a = 0$) for $H(x, u, v) := (|u|^{2^*} + |v|^{2^*})/2^*$ and in an unbounded domain. We established the existence of a nontrivial solution to the first problem in a domain $\Omega \subset \mathbb{R}^N \backslash \mathbb{Z}^N$ invariant by $\mathbb{Z}^N$ translations; and to the second one in an unbounded cylindrical domain and for $\beta, \eta \in (0, \lambda_1)$ with λ_1 the best constant in the Poincaré inequality.

To our knowledge, in the literature, there are no results establishing the existence of nontrivial solutions to these singular problems ($a \neq 0$) in unbounded domains. The aim of this paper is to establish the first results in this direction.

In this paper, for a fixed $R > 1$, we will assume that H has a singular subcritical growth and satisfies an invariance condition. More precisely, we will assume the following hypotheses:

(A0) The function $H \in C^1\big((\mathbb{R}^N \setminus \{0\}) \times \mathbb{R} \times \mathbb{R}\big)$ is such that $H(x, 0, 0) = 0$.

(A1) There exists $R > 1$ such that for each $m \in \mathbb{Z}$,

$$R^{mN} H(x, u, v) = H(R^{-m}x, R^{m\xi}u, R^{m\xi}v),$$

where $\xi := \frac{(N-2a-2)}{2}$.

(A2) There exist a constant $k > 0$ and $b, \hat{b} \in [a, a+1[$ such that

$$\Big(\big|H_u(x, u, v)\big| + \big|H_v(x, u, v)\big|\Big)\Big(|u| + |v|\Big) \leq k\Big(|x|^{-bp}|u|^p + |x|^{-\hat{b}\hat{p}}|v|^{\hat{p}}\Big),$$

where $p := \frac{2N}{N-2+2(b-a)}$ and $\hat{p} := \frac{2N}{N-2+2(\hat{b}-a)}$.

(A3) There exist

$$\alpha \in \Big[\frac{2N}{N+2}(p-1), 2(p-1)\Big), \quad \text{and} \quad \hat{\alpha} \in \Big[\frac{2N}{N+2}(\hat{p}-1), 2(\hat{p}-1)\Big)$$

such that, for every $u, v \neq 0$,

$$\frac{1}{\alpha} H_u(x, u, v)u + \frac{1}{\hat{\alpha}} H_v(x, u, v)v \geq H(x, u, v) > 0.$$

(A4) There exists $\sigma > 0$ such that

$$\inf\{H(x, u, v) : (x, u, v) \in S\} = \sigma,$$

where $S := \{(x, u, v) \in B(0, 1) \times \mathbb{R}^2 : |x|^{-d}|u|^\alpha + |x|^{-\hat{d}}|v|^{\hat{\alpha}} = 1\}$ with $d := N - \xi\alpha$ and $\hat{d} := N - \xi\hat{\alpha}$.

We will establish the existence of a nontrivial solution (u, v) such that $u, v = 0$ on $\{x \in \mathbb{R}^N : |x| = R^m, m \in \mathbb{Z}\}$.

Our proofs will rely on variational methods. Indeed, these problems have a variational formulation of the form

$$\varphi(u,v) := A(u,v) - \psi(u,v),$$

with the quadratic part A being a strongly indefinite operator. Therefore, classical min-max results can not be applied. Here our results will follow from a generalized linking theorem due to Krysewski and Szulkin [10] for suitable functionals defined on a Hilbert space $X = Y \oplus Z$ where both subspaces Y and Z could be infinite dimensional.

It is worthwhile to notice that the invariance condition imposed on H will imply that the functional φ will be invariant by dilatations of the form

$$\big(u_m(x), v_m(x)\big) := \Big(R^{m\xi}u(R^m x), R^{m\xi}u(R^m x)\Big).$$

Therefore, the Palais-Smale condition will not be satisfied at any critical levels. A concentration-compactness argument will permit us to overcome this difficulty.

The paper is organized as follows. Firstly, we present some preliminary results, then we study the functional φ. Our main results are established in section 4. Finally, we discuss the existence of radial and nonradial solutions.

2. Preliminaries

Let $N \geq 3$, $a \in [0,(N-2)/2)$ and $R \geq 1$; we consider the space $D^{1,2}_{a,R}(\mathbb{R}^N)$ (respectively $D^{1,2}_a(\mathbb{R}^N)$) the completion of $D_R(\mathbb{R}^N)$ (respectively $D(\mathbb{R}^N)$) with respect to the inner product:

$$(u,v) := \int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v \, dx,$$

where

$$D(\mathbb{R}^N) := \Big\{u \in C^\infty(\mathbb{R}^N) : u \text{ has compact support}\Big\},$$
$$D_R(\mathbb{R}^N) := \Big\{u \in D(\mathbb{R}^N) : u \text{ has compact support in } \bigcup_{m\in\mathbb{Z}} A_m\Big\},$$

with

$$A_m := \Big\{x \in \mathbb{R}^N : R^m < |x| < R^{m+1}\Big\}, \quad m \in \mathbb{Z}.$$

Let us recall the following result.

Lemma 2.1 (Wang and Willem [13]). *Let $N \geq 3$ and $0 \leq a < (N-2)/2$. If $u_n \rightharpoonup u$ in $D^{1,2}_a(\mathbb{R}^N)$, then $|x|^{-a}u_n \to |x|^{-a}u$ in $L^2_{loc}(\mathbb{R}^N)$.*

Let $b \in [a, a+1]$. We denote

$$L^p_b(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} : u \text{ is measurable and } \|u\|_{p,b} < \infty\},$$

where

$$\|u\|_{p,b} := \Big(\int_{\mathbb{R}^N} |x|^{-bp} |u(x)|^p \, dx \Big)^{\frac{1}{p}}.$$

It was established by Caffarelli, Kohn and Nirenberg [2] that if

$$p = p(a,b) := \frac{2N}{N-2+2(b-a)}, \tag{2.1}$$

$D^{1,2}_a(\mathbb{R}^N)$ can be continuously imbedded in $L^p_b(\mathbb{R}^N)$. This result is called the Caffarelli-Kohn-Nirenberg inequality (noted CKN inequality) with the best positive constant

$$S(a,b) := \inf_{\substack{u \in D^{1,2}_a(\mathbb{R}^N) \\ \|u\|_{p,b}=1}} \big| \, |x|^{-a} \nabla u \big|_2^2 > 0. \tag{2.2}$$

The following result gives sufficient conditions to ensure the convergence to 0 in $L^p_b(\mathbb{R}^N)$ of a bounded sequence of $D^{1,2}_{a,R}(\mathbb{R}^N)$. See Colin [C1] or [C2] for similar results on a cylindrical domain.

Lemma 2.2. *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}_{a,R}(\mathbb{R}^N)$. If $b < a+1$, and*

$$\sup_{m \in \mathbb{Z}} \int_{A_m} \Big(|x|^{-b} |u_n| \Big)^p \to 0, \quad \textit{when } n \to \infty,$$

then $u_n \to 0$ in $L^p_b(\mathbb{R}^N)$.

Proof. For each $m \in \mathbb{Z}$ and $u \in D_R(\mathbb{R}^N)$, the CKN inequality (2.2) leads to

$$\int_{A_m} \Big(|x|^{-b}|u| \Big)^p \, dx \le S(a,b)^{-1} \Big(\int_{A_m} |x|^{-2a} |\nabla u|^2 \, dx \Big)^{p/2}$$

since $u \in H^1_0(A_m)$ for all $m \in \mathbb{Z}$. By a density argument, the previous inequality holds for each $u \in D^{1,2}_{a,R}(\mathbb{R}^N)$. Consequently, for any $\mu \in (0,1)$, we have

$$\int_{A_m} \Big(|x|^{-b}|u| \Big)^p \, dx$$
$$\le S(a,b)^{-\mu} \Big(\int_{A_m} |x|^{-2a} |\nabla u|^2 \, dx \Big)^{(p/2)\mu} \Big(\int_{A_m} \Big(|x|^{-b}|u| \Big)^p \, dx \Big)^{1-\mu}.$$

Since $b \ne a+1$, we have that $p > 2$, and we can choose $\mu = 2/p$. Therefore,

$$\int_{\mathbb{R}^N} \left(|x|^{-b}|u|\right)^p \, dx$$
$$\leq S(a,b)^{-2/p}\Big(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, dx\Big) \sup_{m\in\mathbb{Z}} \Big(\int_{A_m} \left(|x|^{-b}|u|\right)^p \, dx\Big)^{1-(2/p)}.$$

The conclusion follows. □

Now, we recall for the sake of completeness two results which will be used in what follows. The first one is a corollary of a generalized linking theorem due to Krysewski and Szulkin [10] (see also [14]) that will play a crucial role in the proof of our main existence results.

Let $X = Y \oplus Z$ be a Hilbert space with Y a separable subspace of X which could be infinite dimensional and $Z = Y^\perp$. Let $P : X \to Y, Q : X \to Z$ be the orthogonal projections. Let $\rho > r > 0$, and $z \in Z$ such that $\|z\| = 1$. Define

$$\begin{aligned} M &:= \{u = y + \lambda z : \|u\| \leq \rho,\ \lambda \geq 0,\ y \in Y\}, \\ M_0 &:= \{u = y + \lambda z : y \in Y,\ \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}, \\ N &:= \{u \in Z : \|u\| = r\}. \end{aligned}$$

Theorem 2.3 (Kryszewski-Szulkin, 1998)**.** *Let $\psi \in C^1(X,\mathbb{R})$ be weakly sequentially lower semicontinuous, bounded below and such that ψ' is weakly sequentially continuous. If*

$$\phi(u) := \frac{\|Qu\|^2}{2} - \frac{\|Pu\|^2}{2} - \psi(u)$$

satisfies

$$m_0 := \inf_N \phi > 0 = \sup_{M_0} \phi, \quad m_1 := \sup_M \phi < \infty, \tag{2.3}$$

then there exists $c \in [m_0, m_1]$ and a sequence $\{u_n\} \subset X$ such that

$$\phi(u_n) \to c, \quad \phi'(u_n) \to 0.$$

We refer the reader to [14] for a proof of the following result.

Theorem 2.4 (Principle of symmetric criticality, Palais, 1979)**.** *Let X be a Hilbert space, and G a topological group such that its action on X is isometric. If $\phi \in C^1(X,\mathbb{R})$ is invariant and if u is a critical point for the restriction of ϕ to $FixG$, then u is a critical point of ϕ.*

3. Study of the functional φ

Let us denote by $X := D^{1,2}_{a,R}(\mathbb{R}^N) \times D^{1,2}_{a,R}(\mathbb{R}^N)$ the Hilbert space endowed with the scalar product:

$$\langle (u,v),(u_1,v_1)\rangle := \int_{\mathbb{R}^N} |x|^{-2a} \left(\nabla u \cdot \nabla u_1 + \nabla v \cdot \nabla v_1\right) dx.$$

We can write $X = Y \oplus Z$ with $Y := \{(-v,v) \in X\}$ and $Z := \{(u,u) \in X\}$. Let us denote by P (respectively Q) the projection of X onto Y (respectively Z).

We define the functional $\varphi : X \to \mathbb{R}$ by

$$\begin{aligned}\varphi(u,v) &:= \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v - H(x,u,v)\, dx \\ &= \frac{\|Q(u,v)\|^2}{2} - \frac{\|P(u,v)\|^2}{2} - \psi(u,v),\end{aligned}$$

where

$$\psi(u,v) := \int_{\mathbb{R}^N} H(x,u,v)\, dx,$$

and $H \in C^1((\mathbb{R}^N \backslash \{0\}) \times \mathbb{R} \times \mathbb{R})$ satisfies assumptions (A0)–(A4) given in the introduction.

In this section, we study the functional φ. We start with some observations on the map H.

Remark 3.1. Assumptions (A0) and (A2) imply the following growth conditions:

$$\begin{aligned}|H(x,u,v)| &= \Big| \int_0^1 \frac{d}{dt} H(x,tu,tv)\, dt \Big| \\ &\leq \int_0^1 |H_u(x,tu,tv)u + H_v(x,tu,tv)v|\ dt \qquad (3.1)\\ &\leq k \int_0^1 |x|^{-bp}|u|^p + |x|^{-\hat{b}\hat{p}}|v|^{\hat{p}}\, dt = k\big(|x|^{-bp}|u|^p + |x|^{-\hat{b}\hat{p}}|v|^{\hat{p}}\big);\end{aligned}$$

$$\max\Big\{\big|H_u(x,u,v)\big|, \big|H_u(x,u,v)\big|\Big\} \leq k\Big(|x|^{-bp}|u|^{p-1} + |x|^{-\hat{b}\hat{p}}|u|^{\hat{p}-1}\Big). \qquad (3.2)$$

As in classical results, Assumption (A4) permits us to deduce that H is bounded below by an appropriate function for x in the closed unit ball.

Lemma 3.2. *Assumptions* (A0), (A3)*, and* (A4) *imply that*

$$H(x,u,v) \geq \sigma\big(|x|^{-d}|u|^{\alpha} + |x|^{-\hat{d}}|v|^{\hat{\alpha}}\big) - \sigma \qquad (3.3)$$

for all $|x| \leq 1$.

Proof. For every $(x,u,v)\in S$, we define $h:[1,\infty[\to\mathbb{R}$ by

$$h(t):=H(x,t^{1/\alpha}u,t^{1/\hat{\alpha}}v).$$

In integrating $h'(s)/h(s)$ from 1 to t, we deduce by (A3) that $h(t)\geq h(1)t$; i.e.,

$$H(x,t^{1/\alpha}u,t^{1/\hat{\alpha}}v)\geq tH(x,u,v)\geq\sigma t. \tag{3.4}$$

Now, consider $(x,u,v)\in B(0,1)\times\mathbb{R}^2$ such that

$$t:=|x|^{-d}|u|^{\alpha}+|x|^{-\hat{d}}|v|^{\hat{\alpha}}\geq 1.$$

So, $(x,u_1,v_1)\in S$ for $u_1:=t^{-1/\alpha}u$, and $v_1:=t^{-1/\hat{\alpha}}v$. From (3.4), we deduce that

$$H(x,u,v)=H(x,t^{1/\alpha}u_1,t^{1/\hat{\alpha}}v_1)\geq\sigma t=\sigma\left(|x|^{-d}|u|^{\alpha}+|x|^{-\hat{d}}|v|^{\hat{\alpha}}\right).$$

Finally, on the set $\{(x,u,v)\in B(0,1)\times\mathbb{R}^2:|x|^{-d}|u|^{\alpha}+|x|^{-\hat{d}}|v|^{\hat{\alpha}}\leq 1\}$,

$$H(x,u,v)\geq 0\geq\sigma\left(|x|^{-d}|u|^{\alpha}+|x|^{-\hat{d}}|v|^{\hat{\alpha}}\right)-\sigma,$$

the desired conclusion is obtained. □

Remark 3.3. Assumption (A4) is satisfied if we assume (A0), (A1), (A3), and

$$\liminf_{m\to\infty}c_m>0,$$

where, for every $m\in\mathbb{N}\cup\{0\}$,

$$c_m:=\inf\{R^mH(x,R^{-Nm/\alpha}u,R^{-Nm/\hat{\alpha}}v):(x,u,v)\in S_0\},$$

with $S_0:=\{(x,u,v)\in S:x\in\ (1/R,1]\}$, and S is defined in (A4).

Indeed, for $(x,u,v)\in S$,

$$(x_0,u_0,v_0):=(R^mx,R^{md/\alpha}u,R^{m\hat{d}/\hat{\alpha}}v)\in S_0.$$

Hence,

$$\begin{aligned}H(x,u,v)&=H(R^{-m}x_0,R^{-md/\alpha}u_0,R^{-m\hat{d}/\hat{\alpha}}v_0)\\&=R^{mN}H(x_0,R^{-mN/\alpha}u_0,R^{-mN/\hat{\alpha}}v_0)\geq R^N\tilde{c},\end{aligned}$$

where $0<\tilde{c}\leq c_m$ for every $m\in\mathbb{N}\cup\{0\}$. Such a $\tilde{c}$ exists since $c_m>0$ by (A3).

The functional φ and its derivative satisfies some nice continuity properties as is shown in the two following lemmas.

Lemma 3.4. *The function φ is C^1. Moreover, for every $(u,v),(w,z)\in X$,*

$$\langle \varphi'(u,v),(w,z)\rangle = \int_{\mathbb{R}^N} |x|^{-2a}\,(\nabla u\cdot\nabla z+\nabla v\cdot\nabla w) \\ -\,(H_u(x,u,v)w+H_v(x,u,v)z)\;dx.$$

Proof. Let $(u,v),(w,z)\in X$. For $x\in\mathbb{R}^N\backslash\{0\}$, and $|t|\in\ (0,1)$, there exists $\lambda\in\ (0,1)$ such that

$$\frac{H(x,u(x)+tw(x),v(x)+tz(x))-H(x,u(x),v(x))}{t} \\ = H_u(x,u(x)+\lambda tw(x),v(x)+\lambda tz(x))w(x) \\ +H_v(x,u(x)+\lambda tw(x),v(x)+\lambda tz(x))z(x).$$

Assumption (A2) and (3.2) imply that

$$\frac{\big|H(x,u(x)+tw(x),v(x)+tz(x))-H(x,u(x),v(x))\big|}{|t|} \\ \le\Big(|x|^{-bp}|u(x)+\lambda tw(x)|^{p-1}+|x|^{-\hat b\hat p}|v(x)+\lambda tz(x)|^{\hat p-1}\Big)\Big(|w(x)|+|z(x)|\Big) \\ \le\Big(|x|^{-b}(|u(x)|+|w(x)|)\Big)^{p-1}|x|^{-b}\big(|w(x)|+|z(x)|\big) \\ +\Big(|x|^{-\hat b}(|v(x)|+|z(x)|)\Big)^{\hat p-1}|x|^{-\hat b}\big(|w(x)|+|z(x)|\big).$$

The term on the right-hand side is in L^1 by the Hölder inequality, since $w,z\in L_b^p(\mathbb{R}^N)\cap L_{\hat b}^{\hat p}(\mathbb{R}^N)$. The Lebesgue dominated convergence theorem implies that

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^N} H(x,u+tw,v+tz)-H(x,u,v)\,dx \\ =\int_{\mathbb{R}^N} H_u(x,u,v)w+H_v(x,u,v)z\,dx.$$

Now, assume that $(u_n,v_n)\to(u,v)$ in X. From (A0) and Lemma 2.1, we deduce that

$$H_u(x,u_n(x),v_n(x))-H_u(x,u(x),v(x))\to 0,$$

and

$$H_v(x,u_n(x),v_n(x))-H_v(x,u(x),v(x))\to 0,$$

almost everywhere in $\mathbb{R}^N$. Also, (A2), (3.2) and the continuous imbeddings $D^{1,2}_{a,R}(\mathbb{R}^N) \hookrightarrow L^p_b(\mathbb{R}^N)$ and $D^{1,2}_{a,R}(\mathbb{R}^N) \hookrightarrow L^{\hat{p}}_{\hat{b}}(\mathbb{R}^N)$ imply that

$$|x|^b\Big(H_u(x,u_n(x),v_n(x)) - H_u(x,u(x),v(x))\Big)$$

and

$$|x|^{\hat{b}}\Big(H_v(x,u_n(x),v_n(x)) - H_v(x,u(x),v(x))\Big)$$

are bounded in $L^{p/(p-1)}$ and $L^{\hat{p}/(\hat{p}-1)}$ respectively. So,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} H_u(x,u_n,v_n)w + H_v(x,u_n,v_n)z - H_u(x,u,v)w - H_v(x,u,v)z\,dx\Big| \\
&\quad = \Big|\int_{\mathbb{R}^N} |x|^b\Big(H_u(x,u_n,v_n) - H_u(x,u,v)\Big)|x|^{-b}w \\
&\qquad + |x|^{\hat{b}}\Big(H_v(x,u_n,v_n) - H_v(x,u,v)\Big)|x|^{-\hat{b}}z\,dx\Big| \\
&\quad \le \|w\|_{L^{p,b}}\Big(\int_{\mathbb{R}^N}\Big||x|^b\big(H_u(x,u_n,v_n) - H_u(x,u,v)\big)\Big|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}} \\
&\qquad + \|z\|_{L^{\hat{p},\hat{b}}}\Big(\int_{\mathbb{R}^N}\Big||x|^{\hat{b}}\big(H_v(x,u_n,v_n) - H_v(x,u,v)\big)\Big|^{\frac{\hat{p}}{\hat{p}-1}}dx\Big)^{\frac{\hat{p}-1}{\hat{p}}} \\
&\quad \le C\Big(\big(\int_{\mathbb{R}^N}\Big||x|^b\big(H_u(x,u_n,v_n) - H_u(x,u,v)\big)\Big|^{\frac{p}{p-1}}dx\big)^{\frac{p-1}{p}} \\
&\qquad + \big(\int_{\mathbb{R}^N}\Big||x|^{\hat{b}}\big(H_v(x,u_n,v_n) - H_v(x,u,v)\big)\Big|^{\frac{\hat{p}}{\hat{p}-1}}dx\big)^{\frac{\hat{p}-1}{\hat{p}}}\Big),
\end{aligned}$$

for every $(w,z) \in X$ such that $\|(w,z)\| \le 1$. This shows that the Gâteau derivative of ψ is continuous and hence ψ is C^1.

On the other hand,

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^N}|x|^{-2a}\nabla(u+tw)\cdot\nabla(v+tz) - \nabla u\cdot\nabla v\,dx = \langle (u,v),(z,w)\rangle.$$

This Gâteau derivative is obviously continuous; so φ is C^1 and

$$\begin{aligned}
\langle \varphi'(u,v),(w,z)\rangle = \int_{\mathbb{R}^N}|x|^{-2a}&\left(\nabla u\cdot\nabla z + \nabla v\cdot\nabla w\right) \\
&- \left(H_u(x,u,v)w - H_v(x,u,v)z\right)\,dx.
\end{aligned}$$

□

Lemma 3.5. *The maps ψ and ψ' are weakly sequentially continuous.*

Proof. Suppose that $(u_n, v_n) \rightharpoonup (u, v)$ in X. So, $\{u_n\}$ and $\{v_n\}$ are bounded in $D^{1,2}_{a,R}(\mathbb{R}^N)$. Arguing as in the proof of the previous lemma, and using (A0), (A2), and Lemma 2.1, we deduce that

$$\psi(u_n, v_n) \to \psi(u, v),$$

and for every $w, z \in D_R(\mathbb{R}^N)$,

$$\langle \psi'(u_n, v_n), (w, z)\rangle \to \langle \psi'(u, v), (w, z)\rangle.$$

Moreover, $\{\psi'(u_n, v_n)\}$ is bounded in X, so $\psi'(u_n, v_n) \rightharpoonup \psi'(u, v)$. □

Now, we want to show that φ satisfies appropriate inequalities relative to a linking situation.

Lemma 3.6. *Let $z \in D_R(\mathbb{R}^N)$ be such that $\|(z, z)\| = 1$ with compact support in $B(0, 1)\backslash\{0\}$. Then there exists $r > 0$ such that*

$$m_0 := \inf_{\substack{(u,u)\in Z \\ \|(u,u)\|=r}} \varphi(u, u) > 0. \tag{3.5}$$

Moreover, there exists $\rho > r$ such that

$$\max_{M_0} \varphi = 0 \quad \textit{and} \quad \sup_{M} \varphi =: m_1 < \infty, \tag{3.6}$$

where

$$M := \{u = y + \lambda z : \|u\| \le \rho,\ \lambda \ge 0,\ y \in Y\}\,, \tag{3.7}$$

$$M_0 := \{u = y + \lambda z : y \in Y,\ \|u\| = \rho \text{ and } \lambda \ge 0,\ \text{ or } \|u\| \le \rho \text{ and } \lambda = 0\}\,. \tag{3.8}$$

Proof. Inequality (3.1) and the CKN inequality (2.2) imply directly (3.5) since there exists $K > 0$ such that, for $(u, u) \in Z$,

$$\varphi(u, u) \ge \frac{\|(u, u)\|^2}{2} - K\left(\|(u, u)\|^p + \|(u, u)\|^{\hat{p}}\right). \tag{3.9}$$

Observe that, on Y, we have

$$\varphi(-v, v) = \frac{-\|(-v, v)\|^2}{2} - \int_{\mathbb{R}^N} H(x, -v, v)\, dx \le 0.$$

On the other hand, by (A3) and Lemma 3.2, on $Y \oplus \mathbb{R}(z, z)$, we have

$$\begin{aligned}
&\varphi\big((-v, v) + \lambda(z, z)\big) \\
&\quad = -\frac{1}{2}\|(-v, v)\|^2 + \frac{\lambda^2}{2}\|(z, z)\|^2 - \int_{\mathbb{R}^N} H(x, -v + \lambda z, v + \lambda z)\, dx
\end{aligned}$$

$$\leq -\frac{1}{2}\|(-v,v)\|^2 + \frac{\lambda^2}{2}\|(z,z)\|^2 - \int_{\mathrm{supp}(z)} H(x,-v+\lambda z, v+\lambda z)\,dx$$
$$\leq -\frac{1}{2}\|(-v,v)\|^2 + \frac{\lambda^2}{2}$$
$$- \sigma \int_{\mathrm{supp}(z)} |x|^{-d}|-v+\lambda z|^{\alpha} + |x|^{-\hat{d}}|v+\lambda z|^{\hat{\alpha}} - 1\,dx$$
$$\leq -\frac{1}{2}\|(-v,v)\|^2 + \frac{\lambda^2}{2} - c_1 \int_{\mathrm{supp}(z)} |-v+\lambda z|^{\alpha} + |v+\lambda z|^{\hat{\alpha}}\,dx - c_2.$$

Denote by W the closure of $Y \oplus \mathbb{R}(z,z)$ in $L^{\alpha}\big(\mathrm{supp}(z)\big) \times L^{\hat{\alpha}}\big(\mathrm{supp}(z)\big)$. Since there exists a continuous projection of W onto $\mathbb{R}(z,z)$ and all the norms are equivalent on the latter space, it follows that

$$\varphi\big((-v,v)+\lambda(z,z)\big) \leq -\frac{1}{2}\|(-v,v)\|^2 + \frac{\lambda^2}{2} - c_3(\lambda^{\alpha}+\lambda^{\hat{\alpha}}) - c_2,$$

with $c_2, c_3 > 0$. Therefore, we deduce that

$$\varphi(w) \to -\infty \quad \text{whenever } \|w\| \to \infty, \text{ and } w \in Y \oplus \mathbb{R}(z,z).$$

So, for some $\rho > r$, $\max_{M_0} \varphi = 0$. Finally, the CKN inequality (2.2) and (3.1) imply that φ maps bounded sets into bounded sets, hence $\sup_M \varphi < \infty$. □

Lemma 3.7. *There exists $c \in [m_0, m_1]$ and a bounded sequence $\{(u_n, v_n)\}$ in X such that*

$$\varphi(u_n,v_n) \to c > 0, \quad \varphi'(u_n,v_n) \to 0. \tag{3.10}$$

Proof. It follows from Theorem 2.3 and Lemmas 3.4–3.6 that there exist $c \in [m_0, m_1]$ and a sequence $\{(u_n, v_n)\}$ in X satisfying (3.10). Moreover, since φ is invariant under a rescaling of the form (4.2), we may assume that the sequence $\{(u_n, v_n)\}$ is such that

$$\|(u_n,v_n)\| \leq \Big(\int_{|x|\leq 1} |x|^{-2a}\Big(|\nabla u_n|^2 + |\nabla v_n|^2\Big)\,dx\Big)^{1/2} + 1. \tag{3.11}$$

Let us define

$$(s_n,t_n) := \frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}}\Big(\frac{1}{\alpha}u_n, \frac{1}{\hat{\alpha}}v_n\Big).$$

Assumption (A3), Lemma 3.2, and (3.10) imply that, for n large enough,

$$c + 1 + \|(s_n,t_n)\| \geq \varphi(u_n,v_n) - \langle \varphi'(u_n,v_n), (s_n,t_n)\rangle$$
$$= \frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}} \int_{\mathbb{R}^N} \frac{1}{\alpha} H_u(x,u_n,v_n)u_n + \frac{1}{\hat{\alpha}} H_v(x,u_n,v_n)v_n dx - \int_{\mathbb{R}^N} H(x,u_n,v_n)dx$$
$$\geq \Big(\frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}} - 1\Big) \int_{\mathbb{R}^N} H(x,u_n,v_n)\,dx$$

$$\geq \sigma\Big(\frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}}-1\Big)\int_{|x|\leq 1}|x|^{-d}|u_n|^{\alpha}+|x|^{-\hat{d}}|v_n|^{\hat{\alpha}}-c_1\,dx.$$

Hence,

$$C_1+C_2\|(u_n,v_n)\|\geq \int_{|x|\leq 1}|x|^{-d}|u_n|^{\alpha}+|x|^{-\hat{d}}|v_n|^{\hat{\alpha}}\,dx. \tag{3.12}$$

On the other hand, it follows from (3.10) that for n large enough,

$$\|(u_n,v_n)\|^2\leq \Big|\int_{\mathbb{R}^N}H_u(x,u_n,v_n)v_n+H_v(x,u_n,v_n)u_n\,dx\Big|+\varepsilon\|(v_n,u_n)\|. \tag{3.13}$$

Using Assumption (A2), the CKN inequality (2.2), and (3.11), we deduce the existence of a constant $C_3>0$ such that

$$\int_{|x|\geq 1}|H_u(x,u_n,v_n)v_n|+|H_v(x,u_n,v_n)u_n|\,dx\leq C_3. \tag{3.14}$$

Again, by (A2), we have

$$\begin{aligned}\int_{|x|\leq 1}&|H_u(x,u_n,v_n)v_n|+|H_v(x,u_n,v_n)u_n|\,dx\\ &\leq k\int_{|x|\leq 1}|x|^{-bp}|u_n|^{p-1}|v_n|+|x|^{-\hat{b}\hat{p}}|v_n|^{\hat{p}-1}|u_n|\,dx\\ &= k\int_{|x|\leq 1}|x|^{-bp+\theta}|u_n|^{p-1}|x|^{-\theta}|v_n|+|x|^{-\hat{b}\hat{p}+\hat{\theta}}|v_n|^{\hat{p}-1}|x|^{-\hat{\theta}}|u_n|\,dx,\end{aligned}$$

where $\theta := N\big(\frac{\alpha-p+1}{\alpha}\big)-\xi$ and $\hat{\theta} := N\big(\frac{\hat{\alpha}-\hat{p}+1}{\hat{\alpha}}\big)-\xi$. So, since

$$d=(bp-\theta)\frac{\alpha}{p-1}\quad\text{and}\quad \hat{d}=(\hat{b}\hat{p}-\hat{\theta})\frac{\hat{\alpha}}{\hat{p}-1},$$

the Hölder inequality yields

$$\begin{aligned}\int_{|x|\leq 1}&|H_u(x,u_n,v_n)v_n|+|H_v(x,u_n,v_n)u_n|\,dx\\ &\leq k\Big[\Big(\int_{|x|\leq 1}|x|^{-d}|u_n|^{\alpha}\,dx\Big)^{\frac{p-1}{\alpha}}\Big(\int_{|x|\leq 1}(|x|^{-\theta}|v_n|)^{\frac{\alpha}{(\alpha-p+1)}}\,dx\Big)^{\frac{\alpha-p+1}{\alpha}}\\ &\quad+\Big(\int_{|x|\leq 1}|x|^{-\hat{d}}|v_n|^{\hat{\alpha}}\,dx\Big)^{\frac{\hat{p}-1}{\hat{\alpha}}}\Big(\int_{|x|\leq 1}(|x|^{-\gamma}|u_n|)^{\frac{\hat{\alpha}}{\hat{\alpha}-\hat{p}+1}}\,dx\Big)^{\frac{\hat{\alpha}-\hat{p}+1}{\hat{\alpha}}}\Big].\end{aligned}$$

Observe that $\theta,\hat{\theta}\in[a,a+1)$, and

$$\frac{\alpha}{\alpha-p+1}=\frac{2N}{N-2+2(\theta-a)}\quad\text{and}\quad\frac{\hat{\alpha}}{\hat{\alpha}-\hat{p}+1}=\frac{2N}{N-2+2(\hat{\theta}-a)}.$$

Therefore, the CKN inequality (2.2) implies that

$$\begin{aligned}\int_{|x|\le 1} &|H_u(x,u_n,v_n)v_n| + |H_v(x,u_n,v_n)u_n|\,dx\\ &\le k_1\Big[\Big(\int_{|x|\le 1}|x|^{-d}|u_n|^{\alpha}dx\Big)^{\frac{p-1}{\alpha}}\|(0,v_n)\| + \Big(\int_{|x|\le 1}|x|^{-\hat d}|v_n|^{\hat\alpha}dx\Big)^{\frac{\hat p-1}{\hat\alpha}}\|(u_n,0)\|\Big]\\ &\le k_1\Big[\Big(\int_{|x|\le 1}|x|^{-d}|u_n|^{\alpha}\,dx\Big)^{\frac{p-1}{\alpha}} + \Big(\int_{|x|\le 1}|x|^{-\hat d}|v_n|^{\hat\alpha}\,dx\Big)^{\frac{\hat p-1}{\hat\alpha}}\Big]\|(u_n,v_n)\|.\end{aligned}$$

Consequently, by (3.13), (3.14), since $\|(u_n,v_n)\| = \|(v_n,u_n)\|$,

$$\begin{aligned}\|(u_n,v_n)\|^2 \le k_1\Big[&\Big(\int_{|x|\le 1}|x|^{-d}|u_n|^{\alpha}\,dx\Big)^{\frac{p-1}{\alpha}}\\ &+ \Big(\int_{|x|\le 1}|x|^{-\hat d}|v_n|^{\hat\alpha}\,dx\Big)^{\frac{\hat p-1}{\hat\alpha}} + \tilde\varepsilon\Big]\|(u_n,v_n)\| + C_3. \quad (3.15)\end{aligned}$$

Now, it follows from (3.12) and (3.15) that

$$\begin{aligned}\|(u_n,v_n)\|^2 \le k_1\Big[&\Big(C_1 + C_2\|(u_n,v_n)\|\Big)^{\frac{p-1}{\alpha}}\\ &+ \Big(C_1 + C_2\|(u_n,v_n)\|\Big)^{\frac{\hat p-1}{\hat\alpha}} + \tilde\varepsilon\Big]\|(u_n,v_n)\| + C_3.\end{aligned}$$

Finally, since $(p-1)/\alpha < 1$ and $(\hat p-1)/\hat\alpha < 1$, we conclude that the sequence $\{(u_n,v_n)\}$ is bounded. □

4. Main results

We first consider the following system of semilinear singular Poisson equations:

$$\begin{cases}-\mathrm{div}(\nabla u|x|^{-2a}) = H_v(x,u,v),\\ -\mathrm{div}(\nabla v|x|^{-2a}) = H_u(x,u,v),\end{cases} \qquad u,v\in D^{1,2}_{a,R}(\mathbb{R}^N). \quad (4.1)$$

This problem allows a variational formulation and we look for (weak) solutions of (4.1) which are critical points of the functional φ studied in the previous section and defined by

$$\varphi(u,v) := \int_{\mathbb{R}^N}|x|^{-2a}\nabla u\cdot\nabla v - H(x,u,v)\,dx.$$

Notice that if H satisfies the invariance assumption (A1), then, for any $(u,v)\in X$ and for any $m\in\mathbb{Z}$, the dilated functions

$$\tilde u_m(x) := R^{m\xi}u(R^m x) \quad\text{and}\quad \tilde v_m(x) := R^{m\xi}v(R^m x) \quad (4.2)$$

satisfy

$$\big||x|^{-a}\nabla\tilde{u}_m\big|_2 = \big||x|^{-a}\nabla u\big|_2, \quad \big||x|^{-a}\nabla\tilde{v}_m\big|_2 = \big||x|^{-a}\nabla v\big|_2,$$

and

$$\int_{\mathbb{R}^N} H(x,\tilde{u}_m,\tilde{v}_m)\,dx = \int_{\mathbb{R}^N} H(x,u,v)\,dx.$$

Consequently, the functional φ is invariant by dilation, and hence the Palais-Smale condition fails at each critical level c.

Here is our first main result.

Theorem 4.1. *Under the assumptions* (A0)–(A4), *the problem* (4.1) *has a nontrivial solution.*

Proof. Lemma 3.7 guarantees the existence of a bounded sequence $\{(u_n,v_n)\}$ in X such that

$$\varphi(u_n,v_n)\to c>0, \quad \text{and} \quad \varphi'(u_n,v_n)\to 0. \tag{4.3}$$

Now, let us assume that

$$\delta_1 := \limsup_{n\to\infty}\sup_{m\in\mathbb{Z}}\int_{A_m}|x|^{-bp}|u_n|^p = 0,$$
$$\delta_2 := \limsup_{n\to\infty}\sup_{m\in\mathbb{Z}}\int_{A_m}|x|^{-\hat{b}\hat{p}}|v_n|^{\hat{p}} = 0.$$

Lemma 2.2 implies that

$$u_n\to 0 \text{ in } L^p_b(\mathbb{R}^N), \quad \text{and} \quad v_n\to 0 \text{ in } L^{\hat{p}}_{\hat{b}}(\mathbb{R}^N). \tag{4.4}$$

As in the proof of Lemma 3.7, we fix

$$(s_n,t_n) = \frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}}\Big(\frac{1}{\alpha}u_n, \frac{1}{\hat{\alpha}}v_n\Big).$$

Combining (A2), (A3), (4.3), and (4.4), we deduce that

$$\begin{aligned}
c &= \lim_{n\to\infty}\varphi(u_n,v_n) - \big\langle\varphi'(u_n,v_n),(s_n,t_n)\big\rangle\\
&= \lim_{n\to\infty}\frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}}\int_{\mathbb{R}^N}\frac{1}{\alpha}H_u(x,u_n,v_n)u_n + \frac{1}{\hat{\alpha}}H_v(x,u_n,v_n)v_n\,dx\\
&\qquad -\int_{\mathbb{R}^N}H(x,u_n,v_n)\,dx\\
&\le \lim_{n\to\infty}\frac{\alpha\hat{\alpha}}{\alpha+\hat{\alpha}}\int_{\mathbb{R}^N}\frac{1}{\alpha}H_u(x,u_n,v_n)u_n + \frac{1}{\hat{\alpha}}H_v(x,u_n,v_n)v_n\,dx\\
&\le \lim_{n\to\infty} C\int_{\mathbb{R}^N}|x|^{-bp}|u_n|^p + |x|^{-\hat{b}\hat{p}}|v_n|^{\hat{p}}\,dx = 0,
\end{aligned}$$

which leads to a contradiction since $c > 0$.

Therefore, we must have $\delta := \max\{\delta_1, \delta_2\} > 0$. Taking a subsequence if necessary, we deduce the existence of $m_n \in \mathbb{Z}$ such that

$$\int_{A_{m_n}} |x|^{-bp}|u_n|^p + |x|^{-\hat{b}\hat{p}}|v_n|^{\hat{p}}\, dx > \frac{\delta}{2}.$$

The sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ defined by

$$\tilde{u}_n(x) := R^{-m_n\xi}u_n(R^{-m_n}x) \quad \text{and} \quad \tilde{v}_n(x) := R^{-m_n\xi}v_n(R^{-m_n}x)$$

is such that

$$\int_{A_0} |x|^{-bp}|\tilde{u}_n|^p + |x|^{\hat{b}\hat{p}}|\tilde{v}_n|^{\hat{p}}\, dx > \frac{\delta}{2}, \tag{4.5}$$

it satisfies (4.3), and is bounded by dilation invariance. Taking again a subsequence if needed, there exists $(u, v) \in X$ such that

$$(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v) \quad \text{in } X.$$

Lemma 2.1 implies that $|x|^{-a}\tilde{u}_n \to |x|^{-a}u$, $|x|^{-a}\tilde{v}_n \to |x|^{-a}v$, in $L^2_{loc}(\mathbb{R}^N)$. This, combined with (4.5), insures that $(u, v) \neq 0$. Finally, the weakly sequential continuity of φ' gives

$$\|\varphi'(u, v)\| \le \liminf_{n\to\infty} \|\varphi'(\tilde{u}_n, \tilde{v}_n)\| = 0.$$

Consequently (u, v) is a nontrivial solution of (4.1). □

Corollary 4.2. *For every $\lambda, \gamma > 0$, the problem*

$$\begin{cases} -div(\nabla u|x|^{-2a}) = \lambda|x|^{-\hat{b}\hat{p}}|v|^{\hat{p}-2}v, \\ -div(\nabla v|x|^{-2a}) = \gamma|x|^{-bp}|u|^{p-2}u, \end{cases} \qquad u, v \in D^{1,2}_{a,R}(\mathbb{R}^N) \tag{4.6}$$

has a nontrivial solution.

Now, we consider another system which is a slight perturbation of the preceding problem (4.1):

$$\begin{cases} -\mathrm{div}(\nabla u|x|^{-2a}) = \beta|x|^{-2(1+a)}v + H_v(x, u, v), \\ -\mathrm{div}(\nabla v|x|^{-2a}) = \eta|x|^{-2(1+a)}u + H_u(x, u, v), \end{cases} \quad u, v \in D^{1,2}_{a,R}(\mathbb{R}^N). \tag{4.7}$$

Again, the problem (4.7) has a variational formulation since the critical points of φ_1 are solutions of (4.7), where

$$\begin{aligned} \varphi_1(u, v) &:= \int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v - \frac{1}{2}|x|^{-2(1+a)}\left(\eta u^2 + \beta v^2\right) - H(x, u, v)\, dx \\ &= \frac{\|Q(u, v)\|^2}{2} - \frac{\|P(u, v)\|^2}{2} - \psi(u, v) - \Phi(u, v) = \varphi(u, v) - \Phi(u, v), \end{aligned}$$

and

$$\Phi(u,v) := \int_{\mathbb{R}^N} |x|^{-2(1+a)} \Big(\eta \frac{u^2}{2} + \beta \frac{v^2}{2}\Big)\, dx,$$

and where φ and ψ are given above in this article. Here is our second main result.

Theorem 4.3. *Let $\beta, \eta \in (0, S(a, a+1))$ and H satisfy* (A0)–(A4) *with $\alpha = \hat{\alpha} > 2$. Then the problem* (4.7) *has a nontrivial solution.*

Proof. Arguing as in Lemmas 3.4 and 3.5 we can show that φ_1 is C^1, weakly sequentially lower semicontinuous, Φ' is also weakly sequentially continuous, and

$$\begin{aligned}\langle \varphi_1'(u,v),(w,z)\rangle &= \int_{\mathbb{R}^N} |x|^{-2a} \left(\nabla u\cdot\nabla z + \nabla v\cdot\nabla w\right)\, dx \\ &\quad - \int_{\mathbb{R}^N} |x|^{-2(a+1)} \left(\eta u w + \beta v z\right) - H_u(x,u,v)w - H_v(x,u,v)z\, dx.\end{aligned}$$

Set $\zeta := (\beta+\eta)/2$, then for $(u,u) \in Z$ it follows from the CKN inequality (2.2), and (3.9), that

$$\varphi_1(u,u) \ge \Big(\frac{1}{2} - \frac{\zeta}{2S(a,a+1)}\Big)\|(u,u)\|^2 - K\big(\|(u,u)\|^p + \|(u,u)\|^{\hat{p}}\big).$$

Therefore, there exists $r > 0$ such that

$$m_0 := \inf_{\substack{(u,u)\in Z\\ \|(u,u)\|=r}} \varphi_1(u,u) > 0.$$

Again, we fix $z \in D_R(\mathbb{R}^N)$ with compact support in $B(0,1)\backslash\{0\}$ and such that $\|(z,z)\| = 1$. The fact that for $(u,v) \in X$ $\varphi_1(u,v) \le \varphi(u,v)$, and the arguments used in the proof of Lemma 3.6 lead to the existence of $\rho > r$ such that $\max_{M_0} \varphi_1 = 0$ and $m_1 := \sup_M \varphi_1 < \infty$, where M and M_0 are given respectively by (3.7) and (3.8). Theorem 2.3 guarantees the existence of a sequence $\{(u_n, v_n)\} \subseteq X$ such that

$$\varphi_1(u_n,v_n) \to c \in [m_0, m_1], \quad \varphi_1'(u_n,v_n) \to 0.$$

To show that the sequence $\{(u_n, v_n)\}$ is bounded, we proceed as in the proof of Lemma 3.7 in fixing $(s_n, t_n) := (u_n, v_n)/2$. To conclude, we argue as in the proof of Theorem 4.1. □

Corollary 4.4. *For every $\lambda, \gamma > 0$, $\beta, \eta \in (0, S(a, a+1))$, the problem*

$$\begin{cases} -div(\nabla u|x|^{-2a}) = \beta|x|^{-2(1+a)}v + \lambda|x|^{-\hat{b}\hat{p}}|v|^{\hat{p}-2}v, \\ -div(\nabla v|x|^{-2a}) = \eta|x|^{-2(1+a)}u + \gamma|x|^{-bp}|u|^{p-2}u, \end{cases} \quad u,v \in D^{1,2}_{a,R}(\mathbb{R}^N) \tag{4.8}$$

has a nontrivial solution.

5. Radial and nonradial solutions

In this paragraph, we want to observe that if H is invariant under some group action, then we can deduce more information about a solution of (4.1) or (4.7). In particular, we will see that the problems (4.6) and (4.8) have a nontrivial radial solution and a nontrivial nonradial solution.

Theorem 5.1. *Assume* (A0)–(A4)*, and*

(A5) $H(gx,u,v) = H(x,u,v), \quad \forall g \in O(N), \forall x \in \mathbb{R}^N \setminus \{0\}, \forall (u,v) \in \mathbb{R}^2.$

Then the problems (4.1) *and* (4.7) *have a nontrivial radially symmetric solution.*

Proof. Let us define the action of the group $O(N)$ on X by

$$g\big(u(x),v(x)\big) := \big(u(g^{-1}x), v(g^{-1}x)\big), \quad g \in O(N).$$

We consider the following space

$$X_{O(N)} := \mathrm{Fix}(O(N)) = \{(u,v) \in X : g(u,v) = (u,v) \quad \forall g \in G\},$$

whose elements are the pairs of radially symmetric functions. Replacing X by $X_{O(N)}$ in the above proofs, Theorem 4.1 (respectively Theorem 4.3) give us a nontrivial critical point (u,v) of φ (respectively φ_1) restricted to $X_{O(N)}$. But by the principle of symmetric criticality (Theorem 2.4), (u,v) is a nontrivial and radially symmetric critical point of φ (respectively φ_1). □

Similarly, if H satisfies another invariance condition, then we can deduce the existence of a nonradial solution.

Theorem 5.2. *Assume* (A0)–(A4)*, and*

(A6) $H(g_0(x), -u, -v) = H(x,u,v), \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \forall (u,v) \in \mathbb{R}^2$*, and* g_0 *be the involution on* $\mathbb{R}^N$ *defined by*

$$g_0(x_1, x_2, x_3, \dots, x_n) = (x_2, x_1, x_3, \dots, x_n).$$

Then the problems (4.1) *and* (4.7) *have a nonradially symmetric solution.*

Proof. As in the proof of the previous theorem, we deduce the existence of $(u,v) \in X_G$ a solution of (4.1) or (4.7), where

$$X_G := \mathrm{Fix}(G) = \{(u,v) \in X : g(u,v) = (u,v) \quad \forall g \in G\},$$

with the action of the group $G = \{1, g_0\}$ on X defined by

$$g(u(x),v(x)) = \begin{cases} (u(x), v(x)), & \text{if } g = 1, \\ (-u(g_0(x)), -v(g_0(x))), & \text{if } g = g_0. \end{cases}$$

It is also clear that $(0, 0)$ is the only radial function of X_G. □

As a corollary, we obtain the existence of a nontrivial radial and a nontrivial nonradial solution to problems (4.6) and (4.8).

Corollary 5.3. *For every* $\lambda, \gamma > 0$ (*respectively and* $\beta, \eta \in (0, S(a, a+1))$), *the problem* (4.6) (*respectively* (4.8)) *has a nontrivial radial and a nontrivial nonradial solutions.*

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