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EXISTENCE AND REGULARITY RESULTS FOR SOLUTIONS TO NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper we prove some existence and regularity results for solutions to a class of nonlinear parabolic equations whose prototype is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = f(x,t) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega\times]0,T[$, $T > 0$, Γ the lateral surface $\partial\Omega\times]0,T[$, $\triangle_p$ is the so-called $p-$Laplace operator, $p > 1$ and f belongs to some space $L^r(0,T;L^q(\Omega))$, $r \geq 1$, $q \geq 1$.

1. Introduction

In this paper we consider a class of nonlinear parabolic problems whose prototype is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = f(x,t) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega\times(0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega\times(0,T)$, $\triangle_p$ is the so-called $p-$Laplace operator, $p > 1$ and f belongs to some space $L^r(0,T;L^q(\Omega))$, $r \geq 1$, $q \geq 1$.

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We are interested in proving existence and regularity results for solutions to the problem (1.1). We begin by recalling some well-known results. In the linear case $a(x,t,u,\nabla u) = A(x,u)\nabla u$, it is proved in [11] that, if the datum f belongs to $L^r(0,T;L^q(\Omega))$ with $q>1$ and

$$\begin{cases} 2 < \frac{N}{q} + \frac{2}{r} \leq \frac{N}{r} + 2, & \text{if } 1 \leq r < 2, \\ 2 < \frac{N}{q} + \frac{2}{r} \leq \frac{N}{2} + 2, & \text{if } r \geq 2, \end{cases} \tag{1.2}$$

then any weak solution $u \in L^2(0,T;W_0^{1,2}(\Omega))$ belongs to $L^\sigma(Q)$, where

$$\sigma = \frac{(N+2)qr}{Nr - 2q(r-1)}.$$

We explicitly observe that, when $r \geq 2$, (1.2) implies $q \geq (2^*)'$, where $2^* = \frac{2N}{N-2}$. Therefore, the datum f belongs to $L^2(0,T;W^{-1,2}(\Omega))$ and the existence of a weak solution u belonging to $L^2(0,T;W_0^{1,2}(\Omega))$ is a consequence of classical results (cf. [13]). The existence of such a solution in the case where $1 \leq r < 2$ can be also proved by using duality techniques.

The nonlinear case has been studied in [5], where the authors extend the result proved in [11] to a class of problems whose prototype is (1.1). They prove that, if the datum f belongs to $L^r(0,T;L^q(\Omega))$ with $1<p<N$, $q>1$ and

$$\begin{cases} p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np+2p-N}{2}, & \text{if } 1 \leq r < p', \\ p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{p} + p, & \text{if } r \geq p', \end{cases} \tag{1.3}$$

then any weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^s(Q)$ where

$$s = \frac{rq(N+p) + N(p-2)(q(r-1)+r)}{rN - pq(r-1)}. \tag{1.4}$$

The existence of a solution belonging to $L^p(0,T;W^{1,p}(\Omega))$ is a consequence of classical results when $r \geq p'$, since, in this case, (1.3) implies $q \geq (p^*)' = \frac{Np}{N(p-1)+p}$ and therefore f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$. If $r < p'$ the existence of such a solution u is proved in [5] by approximating the problem (1.1) by a sequence of problems whose data belong to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$. Actually, under the assumption that the datum f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with $1<p<N$, $r>1$, and $q>1$ such that (1.3) holds true, the authors proved an a priori estimate of the solution u of (1.1) in $L^s(Q)$ and in $L^p(0,T;W_0^{1,p}(\Omega))$. In the same paper the authors also consider the case $p \geq N$.

Observe that some cases are not allowed in [5]. For example, if $1 \le r < p'$, it follows that

$$\frac{N}{r}(1-\frac{p}{2}) + \frac{Np+2p-N}{2} \le \frac{N}{p} + p,$$

so that the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r > 1$ and $q > 1$ such that

$$\frac{N}{r}(1-\frac{p}{2}) + \frac{Np+2p-N}{2} < \frac{N}{q} + \frac{p}{r} \le \frac{N}{p} + p, \qquad 1 \le r < p',\ q > 1 \quad (1.5)$$

is not considered in [5]. Moreover, the condition

$$\frac{N}{q} + \frac{p}{r} \le \frac{N}{r}(1-\frac{p}{2}) + \frac{Np+2p-N}{2},$$

in (1.3) implies $q \ge 2$ if $r = 1$. Therefore the case where f belongs to $L^r(0,T;L^q(\Omega))$ with r and q such that

$$p < \frac{N}{q} + \frac{p}{r} \le \frac{N}{p} + p, \qquad 1 \le r < p',\ 1 \le q < 2, \quad (1.6)$$

is not allowed in [5].

The existence of a solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ and some regularity results in the case where the datum f belongs to $L^r(0,T;L^q(\Omega))$ with r and q such that $p = \frac{N}{q} + \frac{p}{r}$, are proved in [12] and [9].

The case where $p > \frac{N}{q} + \frac{p}{r}$ is first studied in [1] (in the linear case) and then in [9] (in the nonlinear case); under such assumptions any solutions in the distributional sense are in $L^\infty(Q)$ (other results concerning the boundedness of weak solutions are proved in [15]).

Finally, let us consider the case where f belongs to $L^1(Q)$ (i.e., $f \in L^r(0,T;L^q(\Omega))$ with $r = q = 1$). The existence of a solution in the distributional sense, which in general does not belongs to $L^p(0,T;W_0^{1,p}(\Omega))$, is proved in [7]; the authors proved that such a solution belongs to $L^m(0,T;W_0^{1,m}(\Omega))$ with $m < p - \frac{1}{N+1}$, when $p > 2 - \frac{1}{N+1}$. Regularity results in the case where f belongs to $L^1(Q)$ are proved in [6]. When $p \le 2 - \frac{1}{N+1}$ the existence of a renormalized solution or an entropy solution is proved in [2], [3], [18], [16], [10]. The notion of renormalized solution has been introduced in [14] in the case of elliptic equations, while the equivalent notion of entropy solution has been introduced in [4]. Regularity results for these types of solutions are proved in [16], [20].

In the present paper we study the cases not considered before. We first prove some regularity results and some a priori estimates in the case where

f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$; then we use such results to obtain existence and regularity results for a renormalized solution in the case where f belongs to $L^r(0,T;L^q(\Omega))$ (and not necessarily to $L^{p'}(0,T;W^{-1,p'}(\Omega))$).

Let us begin with the results proved in the case where f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$.

Theorems 2.1 and 2.2 below concern the case where f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with $1 < p < N$. In Theorem 2.1 we assume that

$$p < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r} + p, \quad q > 1,\ r \geq 1. \tag{1.7}$$

In such a case the existence of a solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$, which satisfies (1.1) in the distributional sense, is a consequence of classical results. We prove that such a solution belongs to $L^s(Q)$, where s is given by (1.4). We also prove that it belongs to some space $L^{s_1}(0,T;L^{s_2}(\Omega))$, where s_1 and s_2 will be specified in the statement of Theorem 2.1. A priori estimates in such spaces are proved.

Observe that Theorem 2.1 gives a priori estimates also in the cases (1.5) and (1.6) not considered in [5].

In Theorem 2.2 below we assume that the datum f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with r and q such that

$$\frac{N}{q} + \frac{p}{r} > \frac{N}{r} + p, \quad q > 1,\ r \geq 1. \tag{1.8}$$

We prove that any solutions in the distributional sense $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belong to $L^{\hat{s}}(Q)$ and also to the space $L^{\hat{s}_1}(0,T;L^{\hat{s}_2}(\Omega))$ where $\hat{s}$, $\hat{s}_1$, and $\hat{s}_2$ will be specified in the statement of Theorem 2.2. We find that the values $\hat{s}$ and $\hat{s}_2$ do not depend on r; this means that the summability of u does not increase by increasing the summability of the datum f with respect to the "time variable".

Analogous results concerning the cases where $p = N$ and $p > N$ are given in Theorems 2.3 and 2.4 respectively.

Let us consider the case where f belongs to $L^r(0,T;L^q(\Omega))$ (and not necessarily to $L^{p'}(0,T;W^{-1,p'}(\Omega))$).

In Section 3 we state some existence and regularity results for renormalized solutions to the problem (1.1) according to the three values of p, i.e. $1 < p < N$, $p = N$ and $p > N$.

Theorems 2.5 and 2.6 concern the case where $1 < p < N$. In Theorem 2.5 we assume that f belongs to $L^r(0,T;L^q(\Omega))$ with r and q such that (1.7) holds true and we prove that a renormalized solution u exists and it belongs to $L^s(Q)$, where s is given by (1.4).

Observe that values of r and q close to 1 are allowed in Theorems 2.5 and 2.6. This leads us to consider renormalized solutions.

The existence results are proved by approximating problem (1.1) with problems whose data belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with r and q specified in Theorems 2.5, 2.6, 2.7, and 2.8. The a priori estimates given by Theorems 2.1-2.4 for the solutions of such approximated problems, which belong to $L^p(0,T;W_0^{1,p}(\Omega))$, hold true. This allows us to pass to the limit in the approximated problems by using the same arguments used in [2] (cf. [5], [6]).

Theorems 2.7 and 2.8 concern the existence and regularity for renormalized solutions in the cases $p = N$ and $p > N$ respectively.

2. Statements of the main results

In the present paper we consider a nonlinear parabolic problem which can be formally written as

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\, a(x,t,u,\nabla u) = f(x,t) & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ is the lateral surface $\partial\Omega \times (0,T)$ and p is a real number such that $p > 1$.

Moreover $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function which satisfies the conditions

$$a(x,t,s,\xi) \cdot \xi \geq \lambda |\xi|^p, \quad \lambda > 0, \tag{2.2}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta))\,(\xi - \eta) > 0, \quad \xi \neq \eta, \tag{2.3}$$

$$|a(x,t,s,\xi)| \leq \beta[|\xi|^{p-1} + |s|^{p-1} + \eta(x,t)], \quad \beta > 0, \ \eta(x,t) \in L^{p'}(\Omega), \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, for almost every $(x,t) \in Q$, for every $s \in \mathbb{R}$ and for every $\xi, \eta \in \mathbb{R}^N$.

Finally, we assume that

$$f \in L^r(0,T;L^q(\Omega)), \tag{2.5}$$

where r and q are constants to be specified in the statements of Theorems 2.1-2.8 below.

2.1. **Regularity results and a priori estimates in the case where $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$.** In the present section we state some regularity results and we give some a priori estimates concerning the case where the datum f of the problem (2.1) belongs to the space $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, where r and q are constants to be specified in the statements of Theorems 2.1, 2.2, 2.3 and 2.4 below.

Definition 2.1. We say that u is a weak solution of (2.1) if it satisfies the following condition

$$u \in L^p(0,T;W_0^{1,p}(\Omega)); \tag{2.6}$$

and moreover,

$$-\int_Q u\frac{\partial \phi}{\partial t} + \int_Q a(x,t,u,\nabla u)\nabla\phi = \int_Q \phi f(x,t), \tag{2.7}$$

for every $\phi \in C^\infty(\bar{Q})$ which is zero in a neighborhood of $\Gamma \cup (\Omega \times \{T\})$.

Since the datum f is assumed to be in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ the existence of a weak solution to (2.1) is a consequence of classical results (cf. [13]).

Theorem 2.1. *Let $1 < p < N$. Assume that hypotheses (2.2)-(2.4) hold true and that $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with r and q such that*

$$p < \frac{N}{q} + \frac{p}{r} \le \frac{N}{r} + p, \quad q > 1, \ r \ge 1. \tag{2.8}$$

Then every weak solution u belongs to $L^s(Q)$, with

$$s = \frac{rq(N+p) + N(p-2)(q(r-1)+r)}{rN - pq(r-1)}, \tag{2.9}$$

and

$$\|u\|_{L^s(Q)} \le C_1, \tag{2.10}$$

where C_1 is a constant depending only on N, p, q, r, T, λ, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$. Moreover, $u \in L^{s_1}(0,T;L^{s_2}(\Omega))$, with

$$0 < s_1 = \alpha' r', \ 0 < s_2 = \alpha' q', \ \alpha' = \frac{Nr(q-1) + q(r-1)[p(N+1) - 2N]}{Nr - qp(r-1)}, \tag{2.11}$$

and

$$\int_0^T \Big(\int_\Omega |u(t)|^{s_2}\, dx\Big)^{\frac{s_1}{s_2}} dt \le C_2, \tag{2.12}$$

where C_2 is a constant depending only on N, p, q, r, T, λ, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$.

Remark 2.1. Observe that the assumption $\frac{N}{q} + \frac{p}{r} \le \frac{N}{r} + p$ implies that $\alpha' > 0$ and therefore that s_1 and s_2 are positive constants.

Remark 2.2. Theorem 2.1 is already proved in [5] in the case where f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with r and q satisfying (2.8) and $r \ge p'$. It is an improvement of the analogous result contained in [5] when $1 \le r < p'$. Indeed in [5], under the assumptions (2.2)-(2.4) and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with q and r such that

$$p < \frac{N}{q} + \frac{p}{r} \le \frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np + 2p - N}{2}, \quad q > 1, \ 1 \le r < p', \tag{2.13}$$

it is proved that every weak solution u belongs to $L^s(Q)$, where s is given by (2.9).

Observe that condition (2.13) is less general than condition (2.8) in the case $1 \le r < p'$. Indeed, since $r < p'$, it follows that

$$\frac{N}{r}\left(1 - \frac{p}{2}\right) + \frac{Np + 2p - N}{2} < \frac{N}{r} + p.$$

Moreover, (2.13) implies that, if $r = 1$, then $q \ge 2$ and therefore the case where f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with $r = 1$ and $1 < q < 2$ is not allowed in [5]; this case is allowed in Theorem 2.1.

Theorem 2.2. *Let $1 < p < N$. Assume that hypotheses* (2.2)-(2.4) *hold true and that $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with r and q such that*

$$\frac{N}{q} + \frac{p}{r} > \frac{N}{r} + p, \quad q > 1, \ r > 1. \tag{2.14}$$

Then every weak solution u belongs to $L^{\hat{s}}(Q)$ with

$$\hat{s} = \frac{[N(p-1)(q-1) + N - pq](N+p)}{N(N-pq)} + p - 2, \tag{2.15}$$

and

$$\|u\|_{L^{\hat{s}}(Q)} \le C_3, \tag{2.16}$$

where C_3 is a constant depending only on N, p, q, r, T, λ, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$. *Moreover, $u \in L^{\hat{s}_1}(0,T;L^{\hat{s}_2}(\Omega))$, with*

$$0 < \hat{s}_1 = \hat{\alpha} r', \quad 0 < \hat{s}_2 = \hat{\alpha} q', \quad \hat{\alpha} = \frac{N(p-1)(q-1)}{N - pq}, \tag{2.17}$$

and

$$\int_0^T \Big(\int_\Omega |u(t)|^{\hat{s}_2}\, dx \Big)^{\frac{\hat{s}_1}{\hat{s}_2}} dt \le C_4, \tag{2.18}$$

where C_4 is a constant depending only on N, p, q, r, T, λ, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$.

Remark 2.3. Observe that the assumptions $1 < p < N$ and $\frac{N}{q} + \frac{p}{r} > \frac{N}{r} + p$ imply $\frac{N}{q} > p$ and therefore $\hat{\alpha} > 0$. We deduce that $\hat{s}_1$ and $\hat{s}_2$ are positive constants.

Remark 2.4. Observe that the indexes of summability of u, i.e. $\hat{s}$ and $\hat{s}_2$, do not depend on r. In other words the summability of u in the cylinder Q or the summability of u with respect to the space variable does not increase by increasing the summability of the datum f with respect to the time.

Theorem 2.3. *Let $p = N$. Assume that hypotheses* (2.2)-(2.4) *hold true and that $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with r and q such that*

$$1 < \frac{1}{q} + \frac{1}{r}, \quad q > 1, \ r \ge 1. \tag{2.19}$$

Then every weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^{\tilde{s}}(Q)$ for every $\tilde{s}$ such that

$$\tilde{s} < \tilde{s}_0, \quad \tilde{s}_0 = \frac{2rq + (N-2)(q(r-1)+r)}{r - q(r-1)}, \tag{2.20}$$

and

$$\|u\|_{L^{\tilde{s}}(Q)} \le C_5, \tag{2.21}$$

where C_5 is a constant depending only on λ, N, p, q, r, T, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$. Moreover, $u \in L^{\tilde{s}_1}(0,T;L^{\tilde{s}_2}(\Omega))$, for every $\tilde{s}_1$ and $\tilde{s}_2$ such that

$$0 < \tilde{s}_1 < \tilde{\alpha} r', \quad 0 < \tilde{s}_2 < \tilde{\alpha} q', \quad \tilde{\alpha} = \frac{r(q-1) + q(r-1)(N-1)}{r - q(r-1)}, \tag{2.22}$$

and

$$\int_0^T \Big(\int_\Omega |u(t)|^{\tilde{s}_2}\, dx \Big)^{\frac{\tilde{s}_1}{\tilde{s}_2}} dt \le C_6, \tag{2.23}$$

where C_6 is a constant depending only on λ, N, p, q, r, T, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$.

Remark 2.5. Observe that the assumption $1 < \frac{1}{q} + \frac{1}{r}$ implies $\tilde{\alpha} > 0$ and therefore that $\hat{s}_1$ and $\hat{s}_2$ are positive constants.

Theorem 2.4. *Let $p > N$. Assume that hypotheses* (2.2)-(2.4) *hold true and that $f \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$, with r and q such that* (2.19) *holds true. Then every weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^{\tilde{s}_0}(Q)$ where $\tilde{s}_0$ is defined in* (2.20) *and*

$$\|u\|_{L^{\tilde{s}_0}(Q)} \le C_7, \tag{2.24}$$

where C_7 is a constant depending only on λ, N, p, q, r, T, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$. Moreover, $u \in L^{\tilde{s}'_1}(0,T;L^{\tilde{s}'_2}(\Omega))$, with

$$0 < \tilde{s}'_1 = \tilde{\alpha} r', \quad 0 < \tilde{s}'_2 = \tilde{\alpha} q', \tag{2.25}$$

and

$$\int_0^T \Big(\int_\Omega |u(t)|^{\tilde{s}'_2}\, dx \Big)^{\frac{\tilde{s}'_1}{\tilde{s}'_2}} dt \le C_8, \tag{2.26}$$

where $\tilde{\alpha}$ is defined in (2.22) *and C_8 is a constant depending only on λ, N, p, q, r, T, $|\Omega|$ and $\|f\|_{L^r(0,T;L^q(\Omega))}$.*

Remark 2.6. We explicitly remark that the cases

$$p > \frac{N}{q} + \frac{p}{r}, \quad 1 < p < N \tag{2.27}$$

and

$$\frac{1}{q} + \frac{1}{r} \le 1, \quad p \ge N \tag{2.28}$$

are not contained in Theorems 2.1-2.4. We are able to prove that in such cases every weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^\infty(Q)$, but this result is already proved in [12] and in [9] in a more general setting.

2.2. **Existence and regularity results in the case where $f \in L^r(0,T;L^q(\Omega))$.** In the present section we state some existence and regularity results for renormalized solutions under the assumption that the datum f of the problem (2.1) belongs to $L^r(0,T;L^q(\Omega))$ (and not necessarily to the dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$), where r and q will be specified in Theorems 2.5, 2.6, 2.7, and 2.8 below.

Denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k; that is

$$T_k(s) = \begin{cases} s & |s| \le k, \\ k\ \mathrm{sign}(s) & |s| > k, \end{cases} \qquad \forall s \in \mathbb{R}.$$

We now give the definition of renormalized solution of (2.1).

Definition 2.2. We say that u is a renormalized solution of (2.1) if it satisfies the following conditions:

$$u \in L^\infty(0,T;L^1(\Omega)), \tag{2.29}$$

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)), \quad k \geq 0, \tag{2.30}$$

$$\int_{\{(t,x)\in Q:\ n\leq |u(t,x)|\leq n+1\}} a(x,t,u,\nabla u)\nabla u \to 0, \quad \text{as} \quad n \to +\infty; \tag{2.31}$$

moreover, it follows that

$$\frac{\partial S(u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u)\nabla u = fS'(u) \quad \text{in } \mathcal{D}'(Q) \tag{2.32}$$

and

$$S(u)(x,0) = 0 \quad \text{in } \Omega, \tag{2.33}$$

for every S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support.

Theorem 2.5. *Let* $1 < p < N$. *Assume that hypotheses* (2.2)-(2.5) *holds true with* r *and* q *such that* (2.8) *is satisfied with* $1 \leq r < p'$. *Then there exists at least a renormalized solution* u *to* (2.1) *such that*

$$u \in L^s(Q), \tag{2.34}$$

where s *is defined by* (2.9).

Remark 2.7. Observe that, if $1 < p < N$ and (2.8) holds with the additional condition $r \geq p'$, the existence of a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$, such that (2.34) is satisfied, is proved in [5]. In such a case we can prove that u also belongs to $L^{s_1}(0,T;L^{s_2}(\Omega))$ with $s_1 > 1$ and $s_2 > 1$.

Remark 2.8. This remark concerns the bounds on r, $1 \leq r < p'$, assumed in Theorem 2.5. In [11] (in the linear case $a(x,t,u,\nabla u) = A(x,u)\nabla u$), and in [5] (in the nonlinear case), it is already proved that every weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ belongs to $L^s(Q)$ when f belongs to $L^r(0,T;L^q(\Omega))$ with r and q which satisfy (2.8) with $r \geq p'$ under the assumption $1 < p < N$. Moreover in [5], under the assumptions (2.2)-(2.5) with q and r such that (2.13) holds true, it is proved that there exists at least one solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$ such that u belongs to $L^s(Q)$, where s is given by (2.9).

Observe that condition (2.13) is less general then condition (2.8) in the case $1 \leq r < p'$, in Theorem 2.5. Moreover (2.13) implies that, if $r = 1$, then $q \geq 2$ and therefore the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r = 1$

and $1 < q < 2$ is not considered in [5]; this case is covered by Theorem 2.5 (cf. Remark 2.2).

Remark 2.9. Observe that, by Theorem 2.5, we deduce that if f belongs to $L^\sigma(Q)$ (i.e., $f \in L^r(0,T;L^q(\Omega))$ with $r = q = \sigma$) with $1 < \sigma < \frac{N+p}{p}$, then u belongs to $L^{\tilde{s}}$ with $\tilde{s} = \frac{[N(p-1)+p]\sigma}{N+p-\sigma p}$. This result has been proved under the more restricted assumption $\sigma \geq \frac{(N+2)p}{(N+2)p-N}$ in [6].

Remark 2.10. Observe that the case $f \in L^r(0,T;L^1(\Omega))$, $r \geq 1$, is not considered under the assumptions of Theorem 2.5. The case where f belongs to $L^1(Q)$ (i.e. $f \in L^r(0,T;L^q(\Omega))$ with $r = q = 1$) is studied in [6]. By using the same arguments contained in [6], it can be proved that, if $f \in L^r(0,T;L^1(\Omega))$, $r > 1$, there exists a weak solution u of problem (2.1) which belongs to $L^s(Q)$ for every $1 \leq s < \frac{N(p-1)+p}{N}$.

Theorem 2.6. *Let $1 < p < N$. Assume that hypotheses* (2.2)-(2.5) *hold true with r and q such that* (2.14) *is satisfied. Then there exists at least a renormalized solution u of* (2.1) *such that*

$$u \in L^{\hat{s}}(Q) \tag{2.35}$$

with $\hat{s}$ defined by (2.15).

Theorem 2.7. *Let $p = N$. Assume that hypotheses* (2.2)-(2.5) *hold true with r and q such that* (2.19) *is satisfied. Then there exists at least a renormalized solution u of* (2.1) *such that $u \in L^{\tilde{s}}(Q)$ for any $\tilde{s}$ such that* (2.20) *is satisfied.*

Theorem 2.8. *Let $p > N$. Assume that hypotheses* (2.2)-(2.5) *hold true with r and q such that* (2.19) *is satisfied. Then there exists at least a renormalized solution u of* (2.1) *such that $u \in L^{\tilde{s}_0}(Q)$ with $\tilde{s}_0$ defined by* (2.20).

Remark 2.11. Under the assumptions (2.2)-(2.5) with

$$p = \frac{N}{q} + \frac{p}{r}, \quad q > 1, \ r > 1, \tag{2.36}$$

we are able to prove that there exists a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega))$, which belongs to $L^q(Q)$, for every $1 \leq q < +\infty$ (see Remark 3.1 below). This result is already proved in [12] and in [9] in a more general setting.

3. Proofs of Theorems 2.1- 2.4

In the present section we will prove Theorems 2.1, 2.2, 2.3, and 2.4. We first recall two well-known results which will be useful in the following.

The following well-known interpolation inequality holds true

$$||f||_{L^{\mu}(\Omega)} \leq ||f||^{\delta}_{L^{\mu_1}(\Omega)} ||f||^{1-\delta}_{L^{\mu_2}(\Omega)}, \tag{3.1}$$

for every f belonging to $L^{\mu_1}(\Omega) \cap L^{\mu_2}(\Omega)$ with $1 \leq \mu_1, \mu_2 \leq +\infty$, where $\frac{1}{\mu} = \frac{\delta}{\mu_1} + \frac{1-\delta}{\mu_2}$ and $0 \leq \delta \leq 1$.

By (3.1) it is easy to deduce that every function v belonging to $L^p(0,T; L^{\rho_1}(\Omega)) \cap L^{\infty}(0,T; L^{\rho_2}(\Omega))$ with $p \geq 0$, $\rho_1 \geq 1$, and $\rho_2 \geq 1$, also belongs to $L^{\rho}(Q)$, where $\rho = \rho_1 \frac{\rho_2 - p}{\rho_2} + p$. Moreover, it follows that

$$\int_Q |v|^{\rho} \leq ||v||^{\frac{\rho_1(\rho_2-p)}{\rho_2}}_{L^{\infty}(0,T;L^{\rho_1}(\Omega))} \int_0^T \|v\|^p_{L^{\rho_2}(\Omega)}\, d\tau. \tag{3.2}$$

3.1. **Proof of Theorem 2.1. Preliminary step.** For $n > 0$, denote by $T_n : \mathbb{R} \to \mathbb{R}$ the usual truncation at level n; that is

$$T_n(s) = \begin{cases} s, & |s| \leq n, \\ n \text{ sign } s, & |s| > n. \end{cases}$$

Define, for any $n > 0$ and $0 < \varepsilon < n$ fixed, the function $\Phi_{\varepsilon,n} : \mathbb{R} \to \mathbb{R}$

$$\Phi_{\varepsilon,n}(s) = \begin{cases} \varepsilon^{\alpha-1} s, & |s| \leq \varepsilon, \\ |T_n(s)|^{\alpha-1} T_n(s), & |s| > \varepsilon, \end{cases}$$

where $\alpha > 0$ is a real number which will be specified later (and which can be less than 1 (cf. Remark 3.2 below)).

Let us consider the function

$$\varphi_{\varepsilon,n} = \Phi_{\varepsilon,n}(u)\chi_{]0,t[}(\tau), \quad \tau \in (0,T), \tag{3.3}$$

for every $t \in (0,T)$ fixed.

Observe that $\Phi_{\varepsilon,n}(s)$ is a Lipschitz-continuous function such that $\Phi_{\varepsilon,n}(0) = 0$, for every $\alpha > 0$ and $\varepsilon > 0$ fixed. Moreover, since u belongs to $L^p(0,T; W^{1,p}_0(\Omega))$, it follows that $T_n(u) \in L^{\infty}(Q) \cap L^p(0,T; W^{1,p}_0(\Omega))$. Therefore, since u is a solution of (2.1) in the distributional sense, we can choose $\varphi_{\varepsilon,n}$ as a test function in order to obtain

$$\int_{\Omega} \Psi_{\varepsilon,n}(u)(t)\, dx + \int_0^t \int_{\Omega} a(x,\tau,u,\nabla u)\nabla \Phi_{\varepsilon,n}(u) = < f, \Phi_{\varepsilon,n}(u) >, \tag{3.4}$$

where $\Psi_{\varepsilon,n}(s)$ is the primitive function of $\Phi_{\varepsilon,n}(s)$ given by

$$\Psi_{\varepsilon,n}(s) = \int_0^s \Phi_{\varepsilon,n}(\sigma) d\sigma, \qquad s \in \mathbb{R},$$

and $< \, , \, >$ denotes the duality pairing between $L^{p'}(0,T; W^{-1,p'}(\Omega))$ and $L^p(0,T; W^{1,p}_0(\Omega))$.

By definition of $\Phi_{\varepsilon,n}$, we get

$$\int_\Omega \Psi_{\varepsilon,n}(u)(t)dx + \alpha \int_0^t \int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} a(x,\tau,u,\nabla u)\cdot\nabla T_n(u)|T_n(u)|^{\alpha-1}$$
$$+ \varepsilon^{\alpha-1}\int_0^t \int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}} a(x,\tau,u,\nabla u)\cdot\nabla u$$
$$\le \int_0^t \int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} |f||T_n(u)|^\alpha + \varepsilon^{\alpha-1}\int_0^t \int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}} |f||u|. \tag{3.5}$$

Now we estimate the terms in the left-hand side of (3.5).

First, we evaluate the term $\int_\Omega \Psi_{\varepsilon,n}(u)(t)dx$. A simple calculation shows that

$$\begin{cases} \Psi_{\varepsilon,n}(s) = \dfrac{\varepsilon^{\alpha-1}}{2}s^2, & |s|\le\varepsilon, \\ \Psi_{\varepsilon,n}(s) \ge \dfrac{|T_n(s)|^{\alpha+1}}{\alpha+1} + \dfrac{(\alpha-1)}{2(\alpha+1)}\varepsilon^{\alpha+1}, & |s|>\varepsilon. \end{cases} \tag{3.6}$$

Therefore, we have the following estimate

$$\int_\Omega \Psi_{\varepsilon,n}(u)(t)dx \ge \frac{1}{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}} |T_n(u)(t)|^{\alpha+1}dx \tag{3.7}$$
$$+ \frac{(\alpha-1)}{2(\alpha+1)}\varepsilon^{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}} 1dx + \frac{\varepsilon^{\alpha-1}}{2}\int_{\Omega\cap\{|u(t)|\le\varepsilon\}} |u(t)|^2dx.$$

We now evaluate the second term in the left-hand side of (3.5). Using the ellipticity condition (2.2), we have

$$\alpha\int_0^t \int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} a(x,\tau,u,\nabla u)\cdot\nabla T_n(u)|T_n(u)|^{\alpha-1}$$
$$= \alpha\int_0^t \int_{\Omega\cap\{\varepsilon<|u(\tau)|<n\}} a(x,\tau,u,\nabla u)\cdot\nabla u\,|T_n(u)|^{\alpha-1}$$
$$\ge \alpha\lambda\int_0^t \int_{\Omega\cap\{\varepsilon<|u(\tau)|<n\}} |\nabla u|^p|T_n(u)|^{\frac{(\alpha-1)p}{p}}$$
$$= \alpha\lambda\Big(\frac{p}{\alpha-1+p}\Big)^p\int_0^t \int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} |\nabla(|T_n(u)|^{\frac{\alpha-1}{p}+1})|^p. \tag{3.8}$$

Analogously, using the ellipticity condition (2.2), we can evaluate the third term in the left-hand side of (3.5) in the following way

$$\varepsilon^{\alpha-1}\int_0^t\int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}} a(x,\tau,u,\nabla u)\cdot\nabla u \ge \lambda\varepsilon^{\alpha-1}\int_0^t\int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}}|\nabla u|^p. \tag{3.9}$$

Combining (3.5) and (3.7)-(3.9), we get:

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}}|T_n(u)(t)|^{\alpha+1}dx+\frac{(\alpha-1)}{2(\alpha+1)}\varepsilon^{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}}1dx\\
&\quad+\frac{\varepsilon^{\alpha-1}}{2}\int_{\Omega\cap\{|u(t)|\le\varepsilon\}}|u(t)|^2dx\\
&\quad+\alpha\lambda\left(\frac{p}{\alpha+p-1}\right)^p\int_0^t\int_{\Omega\cap\{|u(\tau)|>\varepsilon\}}|\nabla(|T_n(u)|^{\frac{\alpha-1}{p}+1})|^p\\
&\quad+\lambda\varepsilon^{\frac{(\alpha-1)p}{p}}\int_0^t\int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}}|\nabla u|^p\\
&\le\int_0^t\int_{\Omega\cap\{|u(\tau)|>\varepsilon\}}|f||T_n(u)|^\alpha+\varepsilon^{\alpha-1}\int_0^t\int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}}|f||u|,
\end{aligned}\tag{3.10}$$

for every $t\in(0,T)$.

For $\tau\in]0,t[$ fixed, denote by $v_{\varepsilon,n}(\tau)\ :\ \Omega\to\mathbb{R}$ the function defined by

$$v_{\varepsilon,n}(\tau)=\begin{cases}\varepsilon^{\frac{\alpha-1}{p}}u(\tau), & |u(\tau)|\le\varepsilon\\ |T_n(u)(\tau)|^{\frac{\alpha-1}{p}}T_n(u)(\tau), & |u(\tau)|>\varepsilon.\end{cases}$$

Observe that $v_{\varepsilon,n}(\tau)$ belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ for almost every $\tau\in[0,t[$. Therefore the sum of the fourth and fifth terms in the left-hand side of (3.10) can be estimated by

$$C_{\lambda,p,\alpha}\int_0^t\int_\Omega|\nabla v_{\varepsilon,n}(\tau)|^p,$$

where

$$C_{\lambda,p,\alpha}=\lambda\ \min\Big\{1,\alpha\Big(\frac{p}{\alpha+p-1}\Big)^p\Big\}.$$

By the Sobolev inequality, there exists a constant $S_{p,N}$, which depends only on N and p, such that for every open set $\Omega\subset\mathbb{R}^N$ and $V\in W_0^{1,p}(\Omega)$, one has

$$\|V\|_{L^{p^*}(\Omega)}\le S_{p,N}\|\nabla V\|_{W_0^{1,p}(\Omega)}, \tag{3.11}$$

where $p^* = \frac{Np}{N-p}$. Therefore we have

$$\begin{aligned}
&C_{\lambda,p,\alpha}\int_0^t\int_\Omega |\nabla v_{\varepsilon,n}(\tau)|^p = C_{\lambda,p,\alpha}\int_0^t\int_{\Omega\cap\{|u(\tau)|\geq\varepsilon\}} |\nabla\left(|T_n(u)|^{\frac{\alpha-1}{p}}T_n(u)\right)|^p \\
&\quad + C_{\lambda,p,\alpha}\varepsilon^{\frac{\alpha-1}{p}}\int_0^t\int_{\Omega\cap\{|u(\tau)|<\varepsilon\}} |\nabla u|^p \\
&\geq C_{\lambda,N,p,\alpha}\int_0^t\left(\int_\Omega |v_{\varepsilon,n}(\tau)|^{p^*}dx\right)^{\frac{p}{p^*}}d\tau \\
&= C_{\lambda,N,p,\alpha}\int_0^t\Big(\int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} |T_n(u)|^{p^*(\frac{\alpha-1}{p}+1)}dx \\
&\quad + \varepsilon^{\frac{(\alpha-1)p^*}{p}}\int_{\Omega\cap\{|u(\tau)|\leq\varepsilon\}} |u|^{p^*}dx\Big)^{\frac{p}{p^*}}d\tau,
\end{aligned}$$

where $C_{\lambda,N,p,\alpha} = S_{p,N}C_{\lambda,p,\alpha}$ is a positive constant which depends on λ, N, p and α. By substituting in (3.10), we have

$$\begin{aligned}
&\frac{1}{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}} |T_n(u)(t)|^{\alpha+1}dx + \frac{(\alpha-1)}{2(\alpha+1)}\varepsilon^{\alpha+1}\int_{\Omega\cap\{|u(t)|>\varepsilon\}} 1dx \\
&+ \frac{\varepsilon^{\alpha-1}}{2}\int_{\Omega\cap\{|u(t)|\leq\varepsilon\}} |u(t)|^2dx \\
&+ C_{\lambda,N,p,\alpha}\int_0^t\Big(\int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} |T_n(u)|^{p^*(\frac{\alpha-1}{p}+1)}dx \\
&+ \varepsilon^{\frac{(\alpha-1)p^*}{p}}\int_{\Omega\cap\{|u(\tau)|\leq\varepsilon\}} |u|^{p^*}dx\Big)^{\frac{p}{p^*}}d\tau \\
&\leq \int_0^t\int_{\Omega\cap\{|u(\tau)|>\varepsilon\}} |f||T_n(u)|^\alpha + \varepsilon^{\alpha-1}\int_0^t\int_{\Omega\cap\{|u(\tau)|\leq\varepsilon\}} |f||u|.
\end{aligned} \tag{3.12}$$

Now we want to pass to the limit for ε going to 0 in (3.12).
It is obvious that the second term in the left-hand side of (3.12) goes to zero when ε goes to zero.

The third term in the left-hand side of (3.12) can be estimated by $\frac{1}{2}\varepsilon^{\alpha+1}|\Omega|$, which goes to zero when ε tends to zero. The fourth term goes to

$$C_{\lambda,N,p,\alpha}\int_0^t\Big(\int_\Omega |T_n(u(\tau))|^{\frac{p^*(\alpha+p-1)}{p}}dx\Big)^{\frac{p}{p^*}}d\tau,$$

since

$$\varepsilon^{\frac{(\alpha-1)p^*}{p}} \int_{\Omega\cap\{|u(\tau)|\le\varepsilon\}} |u(\tau)|^{p^*} \le \varepsilon^{\frac{p^*(\alpha+p-1)}{p}} |\Omega|,$$

which goes to zero when ε tends to zero.

Finally, the second term in the right-hand side of (3.12) can be estimated by $\varepsilon^{\alpha}||f||_{L^1(0,T;L^1(\Omega))}$, which goes to zero when ε tends to zero. Therefore, letting ε go to zero in (3.12), we get the following inequality

$$\frac{1}{\alpha+1}\int_\Omega |T_n(u)(t)|^{\alpha+1}dx + C_{\lambda,N,p,\alpha}\int_0^t \Big(\int_\Omega |T_n(u)|^{\frac{p^*(\alpha+p-1)}{p}}dx\Big)^{\frac{p}{p^*}}d\tau \le \int_0^t\int_\Omega |f||T_n(u)|^{\alpha},$$

for every $t\in(0,T)$. Finally, we obtain

$$\sup_{t\in[0,T]}\frac{1}{\alpha+1}\int_\Omega |T_n(u)|^{\alpha+1}dx + C_{\lambda,N,p,\alpha}\int_0^T \Big(\int_\Omega |T_n(u)|^{\frac{p^*(\alpha+p-1)}{p}}dx\Big)^{\frac{p}{p^*}}d\tau \le \int_0^T\int_\Omega |f||T_n(u)|^{\alpha}. \tag{3.13}$$

Denote by w the following function

$$w = |T_n(u)|^{\frac{\alpha+p-1}{p}}. \tag{3.14}$$

Then the inequality (3.13) is equivalent to

$$||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)}d\tau \le C\int_0^T\int_\Omega |f||w|^{\frac{\alpha p}{\alpha+p-1}}, \tag{3.15}$$

where C is a positive constant depending only on λ, N, p and α.

From now on C will denote a positive constant which depends only on the data and which can change from line to line.

Case $f\in L^r(0,T;L^q(\Omega))$ with $r>1$, $q>1$. First step. In this step we prove an estimate of each term of the left-hand side of (3.15) when f belongs to $L^r(0,T;L^q(\Omega))$ with $r>1$ and $q>1$.

By the Hölder inequality and assumption (2.5), we have

$$\int_0^T\int_\Omega |f||w|^{\frac{\alpha p}{\alpha+p-1}} \le \int_0^T\Big(\int_\Omega |f|^q dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |w(\tau)|^{\frac{\alpha p q'}{\alpha+p-1}}dx\Big)^{\frac{1}{q'}}d\tau$$

$$\le ||f||_{L^r(0,T;L^q(\Omega))}\Big(\int_0^T\Big(\int_\Omega |w(\tau)|^{\frac{\alpha p q'}{\alpha+p-1}}dx\Big)^{\frac{r'}{q'}}d\tau\Big)^{\frac{1}{r'}}, \tag{3.16}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Let us assume that α and δ satisfy the condition

$$\frac{\alpha + p - 1}{\alpha p q'} = \frac{\delta}{p^*} + (1 - \delta)\frac{\alpha + p - 1}{p(\alpha + 1)}, \quad 0 \le \delta \le 1. \tag{3.17}$$

Observe that, since $\alpha > 0$ and $p > 1$, it follows that $\frac{p(\alpha+1)}{\alpha+p-1} > 1$.

We can apply the interpolation inequality (3.1) with $\mu_1 = p^*$ and $\mu_2 = \frac{p(\alpha+1)}{\alpha+p-1}$; that is,

$$||w(\tau)||_{L^{\frac{\alpha p q'}{\alpha+p-1}}(\Omega)} \le ||w(\tau)||^{\delta}_{L^{p^*}(\Omega)} ||w(\tau)||^{1-\delta}_{L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega)},$$

for $\tau \in (0, T)$ fixed. Therefore we get:

$$\begin{aligned}
\Big(\int_0^T \Big(\int_\Omega |w(\tau)|^{\frac{\alpha p q'}{\alpha+p-1}} dx\Big)^{\frac{r'}{q'}} d\tau\Big)^{\frac{1}{r'}} &= \Big(\int_0^T \|w(\tau)\|^{\frac{\alpha p r'}{\alpha+p-1}}_{L^{\frac{\alpha p q'}{\alpha+p-1}}(\Omega)} d\tau\Big)^{\frac{1}{r'}} \qquad (3.18)\\
&\le \Big(\int_0^T ||w(\tau)||^{\frac{\delta\alpha p r'}{\alpha+p-1}}_{L^{p^*}(\Omega)} ||w(\tau)||^{(1-\delta)\frac{\alpha p r'}{\alpha+p-1}}_{L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega)} d\tau\Big)^{\frac{1}{r'}}\\
&\le \sup_{\tau\in[0,T]} ||w(\tau)||^{(1-\delta)\frac{\alpha p}{\alpha+p-1}}_{L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega)} \Big(\int_0^T ||w(\tau)||^{\frac{\delta\alpha p r'}{\alpha+p-1}}_{L^{p^*}(\Omega)} d\tau\Big)^{\frac{1}{r'}}\\
&= ||w||^{(1-\delta)\frac{\alpha p}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} \Big(\int_0^T ||w(\tau)||^{\frac{\delta\alpha p r'}{\alpha+p-1}}_{L^{p^*}(\Omega)} d\tau\Big)^{\frac{1}{r'}},
\end{aligned}$$

if we are able to choose α and δ in such a way that (3.17) is satisfied.

Observe that, if $\delta = 0$, (3.18) reduces to (3.20) below (with $C = T^{\frac{1}{r'}}$). Therefore let $\delta > 0$. Let us assume also that

$$\frac{\alpha + p - 1}{\delta\alpha r'} \ge 1. \tag{3.19}$$

Then, from (3.18), since we can apply the Hölder inequality, it follows that

$$\begin{aligned}
&\Big(\int_0^T \|w(\tau)\|^{\frac{\alpha p r'}{\alpha+p-1}}_{L^{\frac{\alpha p q'}{\alpha+p-1}}(\Omega)} d\tau\Big)^{\frac{1}{r'}} \qquad (3.20)\\
&\le C||w||^{(1-\delta)\frac{\alpha p}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} \Big(\int_0^T ||w(\tau)||^{p}_{L^{p^*}(\Omega)} d\tau\Big)^{\frac{\delta\alpha}{\alpha+p-1}},
\end{aligned}$$

where C is a constant which depends only on T, p, r, α, and δ if we are able to choose α and δ in such a way that (3.17) and (3.19) are satisfied.

Combining (3.15), (3.16), and (3.20), we obtain

$$\begin{aligned} &||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \\ &\leq C||f||_{L^r(0,T;L^q(\Omega))} ||w||^{\frac{(1-\delta)\alpha p}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} \left(\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \right)^{\frac{\delta\alpha}{\alpha+p-1}}, \end{aligned} \tag{3.21}$$

if we are able to choose α and δ in such a way that (3.17) and (3.19) are satisfied.

Observe that if $\delta = 1$, (3.21) reduces to (3.22) below. Then assume that $\delta < 1$. Since $\frac{\alpha+1}{\alpha(1-\delta)} > 1$, we can apply Young's inequality with the exponents $\frac{\alpha+1}{\alpha(1-\delta)}$ and $(\frac{\alpha+1}{\alpha(1-\delta)})' = \frac{\alpha+1}{1+\alpha\delta}$, and we get

$$\begin{aligned} &||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \\ &\leq C||f||^{\frac{\alpha+1}{1+\delta\alpha}}_{L^r(0,T;L^q(\Omega))} \left(\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \right)^{\frac{\delta\alpha(\alpha+1)}{(\alpha+p-1)(1+\alpha\delta)}}, \end{aligned} \tag{3.22}$$

where C is a constant which depends only on λ, N, p, T, r, δ, and α.

Since $p > 1$ and $\delta \leq 1$, then $\alpha(1-\delta) + (p-1)(1+\delta\alpha) > 0$; i.e.,

$$\frac{(\alpha+p-1)(1+\delta\alpha)}{\delta\alpha(\alpha+1)} > 1.$$

Therefore, we can apply Young's inequality with the exponents

$$\frac{(\alpha+p-1)(1+\alpha\delta)}{\delta\alpha(\alpha+1)}$$

and

$$\left(\frac{(\alpha+p-1)(1+\delta\alpha)}{\delta\alpha(\alpha+1)} \right)' = \frac{(\alpha+p-1)(1+\alpha\delta)}{\alpha(1-\delta)+(p-1)(1+\delta\alpha)}.$$

We obtain

$$||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \leq C||f||^{\frac{(\alpha+p-1)(\alpha+1)}{\alpha(1-\delta)+(p-1)(1+\delta\alpha)}}_{L^r(0,T;L^q(\Omega))}, \tag{3.23}$$

where C is a constant which depends only on λ, N, p, T, r, δ, and α.

Finally, by (3.23), we deduce the following estimates

$$||w||_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} \leq C, \tag{3.24}$$

and

$$\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \le C, \tag{3.25}$$

where C is a positive constant which depends only on λ, p, N, T, r, q, $||f||_{L^r(0,T;L^q(\Omega))}$, α, and δ, if we are able to choose α and δ such that (3.17) and (3.19) are satisfied; that is,

$$1+\frac{p-1}{\alpha} = pq'[\frac{\delta}{p^*}+(1-\delta)\frac{\alpha+p-1}{p(\alpha+1)}], \quad 1+\frac{p-1}{\alpha} \ge \delta r', \quad 0 \le \delta \le 1. \tag{3.26}$$

Case $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$. Second step. In the previous step we have proved that the a priori estimates (3.24) and (3.25) hold true if we are able to choose α and δ such that (3.26) is satisfied.

In this step we show that such a choice of α and δ is possible and we show the values of α and δ which satisfy (3.26).

First we observe that, in the case where

$$\frac{1}{p^*} = \frac{\alpha+p-1}{p(\alpha+1)}; \tag{3.27}$$

i.e., $\alpha = \frac{2N-p(N+1)}{p}$, the equality in (3.26) is satisfied for $\alpha = q-1$ and for any value of $\delta \in [0,1]$.

Therefore, if (3.27) is satisfied, we choose $\alpha = q-1 \equiv \frac{2N-p(N+1)}{p}$. Now, let us assume

$$\frac{1}{p^*} \ne \frac{\alpha+p-1}{p(\alpha+1)}. \tag{3.28}$$

Therefore, (3.26) is equivalent to

$$\delta = \frac{(1+\frac{p-1}{\alpha})(1-\frac{\alpha q'}{\alpha+1})}{\frac{pq'}{p^*} - \frac{(\alpha+p-1)q'}{\alpha+1}}, \quad 1+\frac{p-1}{\alpha} \ge \delta r', \quad 0 \le \delta \le 1. \tag{3.29}$$

A simple calculation shows that the values of α and δ which optimize (3.29) are

$$\begin{cases} \alpha' = \dfrac{Nr(q-1)+q(r-1)[p(N+1)-2N]}{Nr-qp(r-1)}, \\ \delta' = \dfrac{\frac{\alpha'+p-1}{p(\alpha'+1)}}{\frac{\alpha'+p-1}{p(\alpha'+1)}+r'-\frac{pq'}{p^*}} \end{cases} \tag{3.30}$$

(the value of α' is obtained by substituting in the second condition in (3.29) the value of δ given by the first condition in (3.29); the value δ' is obtained by observing that, if $\alpha = \alpha'$, the equality holds true in the second condition in (3.29), and therefore by (3.26) we have $\delta' r' = pq'[\frac{\delta'}{p^*}+(1-\delta')\frac{\alpha'+p-1}{p(\alpha'+1)}]$).

Observe that, since we assume (2.8), we have $p < \frac{N}{q} + \frac{p}{r}$; i.e., $Nr' > qp$; moreover, we have $\frac{N}{q} + \frac{p}{r} \leq \frac{N}{r} + p$, which implies that

$$Nr'(q-1) + q[p(N+1) - 2N] > 0.$$

Therefore, the value α' is positive.

Observe also that the assumption (2.8) implies that $r' \geq \frac{pq'}{p^*}$ and therefore the value δ' belongs to $[0,1]$.

Finally, observe that, in the case where (3.27) is satisfied, it follows that

$$\alpha' \equiv q - 1 \equiv \frac{2N - p(N+1)}{p}.$$

The values α' and δ', given by (3.30), are the values of α and δ, which satisfy all conditions in (3.26). Therefore, inequalities (3.24) and (3.25) are satisfied with $\alpha = \alpha'$ and $\delta = \delta'$; that is,

$$||w||_{L^\infty(0,T;L^{\frac{p(\alpha'+1)}{\alpha'+p-1}}(\Omega))} \leq C, \tag{3.31}$$

and

$$\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \leq C, \tag{3.32}$$

where C is a positive constant which depends only on λ, p, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$.

Case $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$. Third step: proof of (2.10). By inequalities (3.31) and (3.32), we deduce that

$$w \in L^\infty(0,T;L^{\frac{p(\alpha'+1)}{\alpha'+p-1}}(\Omega)) \cap L^p(0,T;L^{p^*}(\Omega)).$$

Therefore, we can apply the inequality (3.2), with $\rho_1 = \frac{p(\alpha'+1)}{\alpha'+p-1}$ and $\rho_2 = p^*$. Using estimates (3.31) and (3.32), we get

$$\int_Q |w|^\rho \leq C, \tag{3.33}$$

where

$$\rho = \frac{p(\alpha'+1)}{\alpha'+p-1} - \frac{p}{p^*}\frac{p(\alpha'+1)}{\alpha'+p-1} + p = \frac{p(N + \frac{p(\alpha'+1)}{\alpha'+p-1})}{N}, \tag{3.34}$$

and C is a constant depending only on λ, p, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$.

By definition (3.14) of w and by (3.33), we obtain the estimate

$$\int_Q |T_n(u)|^{\frac{\alpha'+p-1}{N}\left(N + \frac{p(\alpha'+1)}{\alpha'+p-1}\right)} \leq C,$$

and letting n go to infinity,

$$\int_Q |u|^{\frac{\alpha'+p-1}{N}\left(N+\frac{p(\alpha'+1)}{\alpha'+p-1}\right)} \leq C, \tag{3.35}$$

where C is a constant depending only on λ, p, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$.

By substituting in (3.35) the values α' given by (3.30), we obtain (2.10).

Observe that we choose $\alpha = \alpha'$, since this value, which is the maximum that α can attain, gives the higher summability of $|u|$.

Case $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$. Fourth step: proof of (2.12). By inequalities (3.20), (3.31) and (3.32), we obtain

$$\int_0^T \|w(\tau)\|_{L^{\frac{\alpha' pq'}{\alpha'+p-1}}(\Omega)}^{\frac{\alpha' pr'}{\alpha'+p-1}} d\tau \leq C \tag{3.36}$$

and by definition (3.14) of w

$$\int_0^T \||T_n(u(\tau))|^{\frac{\alpha'+p-1}{p}}\|_{L^{\frac{\alpha' pq'}{\alpha'+p-1}}(\Omega)}^{\frac{\alpha' pr'}{\alpha'+p-1}} d\tau \leq C.$$

Letting n go to infinity,

$$\int_0^T \||u(\tau)|^{\frac{\alpha'+p-1}{p}}\|_{L^{\frac{\alpha' pq'}{\alpha'+p-1}}(\Omega)}^{\frac{\alpha' pr'}{\alpha'+p-1}} d\tau \leq C, \tag{3.37}$$

where C is a constant depending only on λ, p, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$.

By substituting in (3.37) the value α' given by (3.30), we obtain (2.12).

Case $f \in L^r(0,T;L^q(\Omega))$ with $r = 1$, $q > 1$.

In this step we prove an apriori estimate of each term of the left-hand side of (3.15) when f belongs to $L^1(0,T;L^q(\Omega))$ with $q > 1$.

As in the previous case, we begin by estimating the right-hand side of (3.15). By the Hölder inequality and assumption (2.5), we have

$$\int_0^T \int_\Omega |f||w|^{\frac{\alpha p}{\alpha+p-1}} \leq ||f||_{L^1(0,T;L^q(\Omega))} \sup_{\tau\in[0,T]} \left(\int_\Omega |w(\tau)|^{\frac{\alpha pq'}{\alpha+p-1}} dx\right)^{\frac{1}{q'}}. \tag{3.38}$$

Let us assume that α satisfies the condition $\frac{\alpha+1}{\alpha q'} \geq 1$; that is,

$$\alpha \leq q - 1. \tag{3.39}$$

Then we can apply the Hölder inequality and we get

$$\sup_{\tau\in[0,T]} \left(\int_\Omega |w(\tau)|^{\frac{\alpha pq'}{\alpha+p-1}} d\tau\right)^{\frac{1}{q'}} \leq |\Omega|^{1-\frac{\alpha q'}{\alpha+1}} \sup_{\tau\in[0,T]} \left(\int_\Omega |w(\tau)|^{\frac{p(\alpha+1)}{\alpha+p-1}} dx\right)^{\frac{\alpha}{\alpha+1}}$$

$$= |\Omega|^{1-\frac{\alpha q'}{\alpha+1}} ||w||^{\frac{p\alpha}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))}. \tag{3.40}$$

Combining (3.15), (3.38), and (3.40), we obtain

$$||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau$$
$$\leq C||f||_{L^1(0,T;L^q(\Omega))} ||w||^{\frac{p\alpha}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))}.$$

Since, for every $\alpha > 0$, it follows that $\frac{\alpha+1}{\alpha} > 1$, we can apply Young's inequality with the exponents $\frac{\alpha+1}{\alpha}$and $(\frac{\alpha+1}{\alpha})' = \alpha + 1$. Arguing as in the previous case, we have

$$||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} + \int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \leq C, \tag{3.41}$$

where C is a constant depending only on λ, p, N, T, r, q, $|\Omega|$, and $||f||_{L^1(0,T;L^q(\Omega))}$ and moreover α has to satisfy the condition (3.39).

By (3.41) we deduce the following estimates

$$||w||^{\frac{p(\alpha+1)}{\alpha+p-1}}_{L^\infty(0,T;L^{\frac{p(\alpha+1)}{\alpha+p-1}}(\Omega))} \leq C, \tag{3.42}$$

and

$$\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \leq C, \tag{3.43}$$

where C is a constant depending only on λ, p, N, T, r, q, $|\Omega|$, and, $||f||_{L^1(0,T;L^q(\Omega))}$ and moreover α has to satisfy the condition (3.39).

We choose $\alpha = q - 1$, the maximum value which can be attained. By substituting this value of α in the inequalities (3.42) and (3.43), we have

$$||w||^{\frac{pq}{q+p-2}}_{L^\infty(0,T;L^{\frac{pq}{q+p-2}}(\Omega))} \leq C, \tag{3.44}$$

and

$$\int_0^T ||w(\tau)||^p_{L^{p^*}(\Omega)} d\tau \leq C, \tag{3.45}$$

where C is a constant depending only on λ, p, N, T, r, q, $|\Omega|$, and $||f||_{L^1(0,T;L^q(\Omega))}$.

By inequalities (3.44) and (3.45), we deduce that

$$w \in L^\infty(0,T; L^{\frac{pq}{q+p-2}}(\Omega)) \cap L^p(0,T; L^{p^*}(\Omega)).$$

Therefore, we can apply inequality (3.2), with $\rho_1 = \frac{pq}{q+p-2}$ and $\rho_2 = p^*$. Using estimates (3.44) and (3.45), we get

$$\int_Q |w|^\sigma \le C, \tag{3.46}$$

where

$$\sigma = \frac{pq}{q+p-2} - \frac{p}{p^*}\frac{pq}{q+p-2} + p = \frac{p\left(N + \frac{pq}{q+p-2}\right)}{N}, \tag{3.47}$$

and C is a constant depending only on $\lambda, p, N, T, r, q, |\Omega|$, and $\|f\|_{L^1(0,T;L^q(\Omega))}$

By definition (3.14) of w, choosing $\alpha = q-1$ and letting n go to infinity, we conclude that $u \in L^s(Q)$ with s given by (2.9).

In an analogous way, by inequality (3.44), by definition (3.14) of w, we conclude that $u \in L^\infty(0,T;L^{s_2}(\Omega))$.

Remark 3.1. Observe that, if

$$\frac{N}{q} + \frac{p}{r} = \frac{N}{r} + p; \tag{3.48}$$

i.e., $\frac{r'}{q} = \frac{p}{N}$, every $\alpha > 0$ satisfies (3.26). Therefore, if we assume that hypotheses (2.2)-(2.5) hold true with r and q such that (3.48) is satisfied, we deduce ,by the proof made of Theorem 2.1, that u belongs to $L^s(Q)$ for every $1 \le s < +\infty$ (see Remark 2.11 above).

Remark 3.2. Let us observe that if we impose the more restrictive condition $\alpha \ge 1$ we obtain the same results proved in [5], where (2.10) is proved under the assumption (2.8) with the condition $r \ge p'$.

Remark 3.3. Under the assumption that $p > 2 - \frac{1}{N+1}$ and $1 \le r < p'$, as in [5], we can prove that $u \in L^m(0,T;W^{1,m}(\Omega))$ where

$$1 < m = \frac{N(p-1) + p + (N+p)\alpha'}{N+1+\alpha'} < p.$$

Moreover, we have

$$\int_0^T \|\nabla u(\tau)\|^m_{L^m(\Omega)} d\tau \le c, \tag{3.49}$$

where c is a positive constant which depends only on λ, p, N, r, q, T, $|\Omega|$, and $\|f\|_{L^r(0,T;L^q(\Omega))}$.

3.2. **Proof of Theorem 2.2.** The proof made in the preliminary step of Theorem 2.1 and in the first step of the case $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$ is exactly the same when r and q satisfy (2.14). Therefore, the estimates (3.24) and (3.25) hold true, if we are able to choose α and δ in such a way that (3.26) is satisfied.

As made in the second step of the case $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$, of Theorem 2.1, we show that it is possible to choose α and δ in such a way that (3.26) is satisfied. A simple calculation shows that, in this case, such values of α and δ are given by

$$\hat{\alpha} = \frac{N(p-1)}{(N-p)q' - N}, \quad \hat{\delta} = 1. \tag{3.50}$$

Therefore, the estimates (3.24) and (3.25) hold true with $\alpha = \hat{\alpha}$ and $\delta = \hat{\delta}$, defined by (3.50).

The proof proceeds now exactly as in the third and fourth steps of the proof of Theorem 2.1 in the case where $f \in L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$, with the value of $\hat{\alpha}$ given by (3.50).

3.3. **Proofs of Theorems 2.3 and 2.4.** The proofs of Theorem 2.3 and 2.4 are very similar to the proof of Theorem 2.1. Here we give a sketch of the proof of Theorem 2.3, in the case where $p = N$ and f belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) \cap L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$.

Since $p = N$, from the Sobolev inequality, there exists a constant S'_N, which depends only on N, such that for every open set $\Omega \subset \mathbb{R}^N$ and $V \in W_0^{1,N}(\Omega)$, one has

$$\|V\|_{L^{\tilde{p}}(\Omega)} \le S'_N \||\nabla V|\|_{L^N(\Omega)}, \tag{3.51}$$

for every $1 \le \tilde{p} < +\infty$.

Observe that the proof of Theorem 2.1 made in the preliminary step of Theorem 2.1 holds true when we replace the Sobolev inequality (3.11) with the inequality (3.51). Therefore we obtain the analogue of inequality (3.15) (where $p = N$ and p^* is replaced by $\tilde{p}$); that is,

$$||w||^{\frac{N(\alpha+1)}{\alpha+N-1}}_{L^\infty(0,T;L^{\frac{N(\alpha+1)}{\alpha+N-1}}(\Omega))} + \int_0^T ||w(\tau)||^N_{L^{\tilde{p}}(\Omega)} d\tau \le C \int_0^T \int_\Omega |f||w|^{\frac{\alpha N}{\alpha+N-1}}, \tag{3.52}$$

for every $1 \le \tilde{p} < +\infty$, where C is a positive constant depending only on λ, N, and α.

Starting from (3.52), we make the same proof of Theorem 2.1 in the first step in the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r > 1$, $q > 1$ and we obtain the inequality

$$\left(\int_0^T \|w(\tau)\|^{\frac{\alpha N r'}{\alpha+N-1}}_{L^{\frac{\alpha N q'}{\alpha+N-1}}(\Omega)} d\tau \right)^{\frac{1}{r'}} \tag{3.53}$$

$$\leq C||w||^{(1-\delta)\frac{\alpha N}{\alpha+N-1}}_{L^\infty(0,T;L^{\frac{N(\alpha+1)}{\alpha+N-1}}(\Omega))} \left(\int_0^T ||w(\tau)||^N_{L^{\tilde{p}}(\Omega)} d\tau \right)^{\frac{\delta\alpha}{\alpha+N-1}},$$

where C is a constant which depends only on T, N, r, α, and δ, if we are able to choose α and δ in such a way that

$$1+\frac{N-1}{\alpha} = Nq'[\frac{\delta}{\tilde{p}}+(1-\delta)\frac{\alpha+N-1}{N(\alpha+1)}],\ \ 1+\frac{N-1}{\alpha} \geq \delta r',\ \ 0 \leq \delta \leq 1. \tag{3.54}$$

As in the previous case, we deduce the following estimates

$$||w||_{L^\infty(0,T;L^{\frac{N(\alpha+1)}{\alpha+N-1}}(\Omega))} \leq C, \tag{3.55}$$

and

$$\int_0^T ||w(\tau)||^N_{L^{\tilde{p}}(\Omega)} d\tau \leq C, \tag{3.56}$$

where C is a positive constant which depends only on λ, N, T, r, α, δ, q, and $||f||_{L^r(0,T;L^q(\Omega))}$, if we are able to choose α and δ such that (3.54) is satisfied.

As in the proof of Theorem 2.1 made in the second step of the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r>1$, $q>1$, a simple calculation shows that the values of α and δ which optimize (3.54) are the following

$$\tilde{\alpha} = \frac{N-1+\frac{r'}{q'}-\frac{N}{\tilde{p}}}{\frac{r'}{q}-1+\frac{N}{\tilde{p}}}, \quad \tilde{\delta} = \frac{\frac{\tilde{\alpha}+N-1}{N(\tilde{\alpha}+1)}}{\frac{\tilde{\alpha}+N-1}{N(\tilde{\alpha}+1)}+r'-\frac{Nq'}{\tilde{p}}}, \tag{3.57}$$

for every $\tilde{p} \geq \frac{Nq'}{r'}$.

First observe that , since (2.19) holds true, it follows that $r' > q$; moreover, since $\tilde{p} \geq \frac{Nq'}{r'} > \frac{Nq'}{(N-1)q'+r'}$, then $\tilde{\alpha} > 0$.

Second, since $\tilde{p} \geq \frac{Nq'}{r'}$, then $\tilde{\delta}$ belongs to [0,1].

As in the proof of Theorem 2.1 made in the third step of the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r>1$, $q>1$, we get

$$\int_Q |w|^\rho \leq C, \tag{3.58}$$

where

$$\rho = \frac{N(\tilde{\alpha}+1)}{\tilde{\alpha}+N-1} - \frac{N^2}{\tilde{p}}\frac{(\tilde{\alpha}+1)}{\tilde{\alpha}+N-1}+N, \tag{3.59}$$

for every $\tilde{p} \geq \frac{Nq'}{r'}$. By definition (3.14) of w and by letting n go to infinity

$$\int_Q |u|^s \leq C, \tag{3.60}$$

where

$$s = \frac{\tilde{\alpha}+N-1}{N}\Big[\frac{N(\tilde{\alpha}+1)}{\tilde{\alpha}+N-1} - \frac{N^2}{\tilde{p}}\frac{(\tilde{\alpha}+1)}{\tilde{\alpha}+N-1} + N\Big],$$

for every $\tilde{p} \geq \frac{Nq'}{r'}$ and C is a constant depending only on λ, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$. Letting $\tilde{p}$ go to infinity, we obtain (2.21).

To prove (2.23), we proceed as in the proof of Theorem 2.1 made in the fourth step of the case where f belongs to $L^r(0,T;L^q(\Omega))$ with $r>1$, $q>1$.

By inequalities (3.53), we obtain

$$\int_0^T \|w(\tau)\|_{L^{\frac{\tilde{\alpha}Nq'}{\tilde{\alpha}+N-1}}(\Omega)}^{\frac{\tilde{\alpha}Nr'}{\tilde{\alpha}+N-1}} d\tau \leq C. \tag{3.61}$$

By definition (3.14) of w, we get

$$\int_0^T \||u(\tau)|\|_{L^{\tilde{s}_2}(\Omega)}^{\tilde{s}_1} d\tau \leq C, \tag{3.62}$$

where $\tilde{s}_1 = \tilde{\alpha}r'$, $\tilde{s}_2 = \tilde{\alpha}q'$, and C is a constant depending only on λ, N, T, r, q, and $||f||_{L^r(0,T;L^q(\Omega))}$. Letting $\tilde{p}$ go to infinity, we obtain (2.23).

Remark 3.4. The proof of Theorem 2.4 is analogous to the proof of Theorem 2.3; we have to replace the Sobolev inequality (3.51) with the inequality

$$\|V\|_{L^\infty(\Omega)} \leq S_N''\||\nabla V|\|_{L^p(\Omega)},$$

for every $V \in W_0^{1,p}(\Omega)$.

4. PROOFS OF THEOREMS 2.5-2.8

In order to prove existence and regularity results of Theorems 2.5 -2.8, we approximate the datum f by a sequence $(f_n)_n$ of functions belonging to $L^\infty(Q)$ (and therefore to $L^{p'}(0,T;W^{-1,p'}(\Omega))$) that converges to f in $L^r(0,T;L^q(\Omega))$ strongly and we consider the sequence of the solutions $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ of the approximated problems whose data are f_n.

The a priori estimates given by Theorems 2.1- 2.4 for such solutions u_n allow us to reduce the proof to results proved in [2]. Such results concern the possibility to pass to the limit in the approximated problems.

Here we just give a sketch of the proof of Theorem 2.5.

4.1. **Proof of Theorem 2.5.** Let us consider a sequence $(f_n)_n$ of approximated data such that $f_n \in L^\infty(\Omega)$, $f_n \to f$ in $L^r(0,T;L^q(\Omega))$ strongly, where r and q satisfy the assumption (2.8).

Let $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ be a weak solution of the problem

$$\begin{cases} \frac{\partial u_n}{\partial t} - \text{div } a(x,t,u_n,\nabla u_n) = f_n(x,t) & \text{in } Q \\ u_n(x,0) = 0 & \text{in } \Omega \\ u_n(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Since f_n also belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$, such a solution u_n exists (cf. [13]). By Theorem 2.1 we get

$$\|u_n\|_{L^s(Q)} \le C_1. \tag{4.2}$$

Therefore there is a subsequence, still denoted by u_n, which converges in $L^s(Q)$ weakly to some function u belonging to $L^s(Q)$ (see [19]). Moreover the sequence $\frac{\partial u_n}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$, therefore $u_n \to u$ in $L^1(Q)$ strongly, and hence $u_n \to u$ almost everywhere in Q. Then from (4.2) we deduce that $u \in L^s(Q)$, and that

$$\|u\|_{L^s(Q)} \le C_1. \tag{4.3}$$

Furthermore, by classical results (cf. [19]), $u(0) = 0$.

The proof that u is a renormalized solution proceeds now as in [2]. As in [2] we can prove that u satisfies (2.29), (2.30), and (2.31). Moreover by the equation in (4.1), we can deduce that for any S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, it follows that

$$\begin{aligned} \frac{\partial S(u_n)}{\partial t} &- \text{div}(S'(u_n)a(x,t,u_n,\nabla u_n)) + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \\ &= fS'(u_n), \quad \text{in } \mathcal{D}'(Q). \end{aligned} \tag{4.4}$$

Proceeding exactly as in [2], we can prove that it is possible to pass to the limit in (4.4) and therefore u satisfies (2.32).

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