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ON THE SYSTEM OF CONSERVATION LAWS AND ITS PERTURBATION IN THE BESOV SPACES

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Abstract. We prove that a system of conservation laws on $\mathbb{R}^N$ is locally well posed in the 'critical' Besov space $B^{\frac{N}{2}+1}_{2,1}$. The time of local existence depends only on the size of the inhomogeneous part of the initial data. We also obtain a blow-up criterion of the local solution. For the conservation system with a dissipation term added we prove global existence of solutions under the assumption of smallness of the homogeneous part of the arbitrary-sized $B^{\frac{N}{2}-1}_{2,1}$ norm of the initial data. For the proof of these results we essentially use the Littlewood-Paley decomposition of functions to derive the energy type of estimate.

1. Introduction

We are concerned with the following system of conservation laws:

$$A_0(u)\frac{\partial u}{\partial t}+\sum_{k=1}^{N}A_k(u)\frac{\partial u}{\partial x_k}=0, \quad (x,t)\in\mathbb{R}^N\times(0,\infty) \tag{1.1}$$

$$u(x,0)=u_0(x), \quad x\in\mathbb{R}^N \tag{1.2}$$

where $u=(u_1,\cdots,u_N)$, $u_j=u_j(x,t)$, $j=1,2,\cdots,N$. We assume that $A_0(u)$ is a positive-definite symmetric matrix satisfying the following uniform bounds

$$C^{-1}I\le A_0(u)\le CI \qquad \forall u\in\mathbb{R}^N \tag{1.3}$$

for some constant $C>1$, and the $A_k(u)$'s are symmetric matrices. Additional technical assumptions on $A_0(u)$ and $A_k(u)$'s will be specified in the statement of the main theorems below. There are many examples of partial

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differential equations that can be written in the form (1.1)(see e.g. [21]). For the Cauchy problem (1.1)-(1.2) with the initial data u_0 given in $H^m(\mathbb{R}^N)$, $m > \frac{N}{2} + 1$, the local unique existence of a solution is proved by Kato [18] and Lax [20] independently. More specifically, they proved unique existence of a solution belonging to $C([0,T]; H^m(\mathbb{R}^N))$ for some $T = T(\|u_0\|_{H^m})$. Recently there are many studies on the extension of the regularity class of the initial data for local existence/global existence for small data using the Besov, or Triebel-Lizorkin spaces (e.g. [4]-[10],[12]-[16], [19], [24] and references therein). Most of those results are concerned with specific equations. In this paper one of our aims is to prove the local unique existence and the continuation principle in the Besov spaces, $B^{\frac{N}{2}+1}_{2,1}$ for a rather general type of equation as in (1.1)-(1.2). Our other aim is to prove global existence for small data in $B^{\frac{N}{2}-1}_{2,1}$ with the system of the type (1.1)-(1.2) with a dissipation term added. The following is our main theorem concerning (1.1)-(1.2).

Theorem 1.1. *Let us assume that* $A_0(\cdot)$, $A_k(\cdot)$, $k = 1, 2, \cdots, N$ *satisfy* (1.3), *and the following conditions.*

$$A_k(0) = A_k'(0) = 0, \quad \forall k = 1, 2, \cdots, N. \tag{1.4}$$

$$A_0, A_0^{-1}, A_k \in W^{[\frac{N}{2}]+3,\infty}(\mathbb{R}^N), \quad \forall k = 1, 2, \cdots, N, \tag{1.5}$$

where $[\frac{N}{2}]$ *denotes the smallest integer exceeding* $\frac{N}{2}$.

(i) ***Local in time existence:*** *Suppose* $u_0 \in B^{\frac{N}{2}+1}_{2,1}$. *Then, there exists* $T = T(\|u_0\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}})$ *such that a unique solution* $u(t)$ *of the system* (1.1)-(1.2) *exists, which belongs to*

$$L^\infty([0,T]; B^{\frac{N}{2}+1}_{2,1}) \cap C([0,T]; B^s_{2,1}) \cap \mathrm{Lip}([0,T]; B^{\frac{N}{2}}_{2,1})$$

for all $s \in [0, \frac{N}{2} + 1)$.

(ii) ***Blow-up criterion:*** *The local in time solution blows up in* $B^{\frac{N}{2}+1}_{2,1}$ *at* $T_* > T$, *namely*

$$\limsup_{t \nearrow T_*} \|u(t)\|_{B^{\frac{N}{2}+1}_{2,1}} = \infty \tag{1.6}$$

if and only if

$$\int_0^{T_*} \|\nabla u(t)\|_{L^\infty} dt = \infty. \tag{1.7}$$

Remark 1.1. Regarding the continuity in time, although we proved below that $u(t) \in C([0,T]; B^s_{2,1})$ for $s \in [0, \frac{N}{2}+1)$, we could actually improve it showing $u(t) \in C([0,T]; B^{\frac{N}{2}+1}_{2,1})$ adapting the argument in the Sobolev space theory in Chapter 3 of [21].

Next, we consider the following hyperbolic conservation system with a dissipation term added.

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{N} A_k(u) \frac{\partial u}{\partial x_k} = \mu \Delta u, \quad (x,t) \in \mathbb{R}^N \times (0,\infty) \tag{1.8}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.9}$$

where $A_k(u)$, $k = 1, 2, \cdots, N$, satisfy the conditions specified in the statement of the theorem below. The following is our main theorem for the system (1.8)-(1.9).

Theorem 1.2 (global unique existence for small initial data)**.** *Here we set $N > 2$. Let us assume that $A_k(\cdot), k = 1, \cdots, N$ in* (1.8) *satisfy* (1.4)-(1.5). *Suppose $u_0 \in B^{\frac{N}{2}}_{2,1}$. Then, there exists an absolute constant $C_0 > 0$ with the following property:*

If $\|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}} < C_0 \mu$, then there exists a unique solution

$$u \in C([0,\infty); B^{\frac{N}{2}}_{2,1}) \cap L^1([0,\infty); \dot{B}^{\frac{N}{2}+1}_{2,1})$$

of the system (1.8)–(1.9), *which satisfies the estimate*

$$\sup_{0 \le t < \infty} \|u(t)\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}} + C_1 \mu \int_0^\infty \|u(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds \le 2 \|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}}, \tag{1.10}$$

where C_1 is an absolute constant.

Remark 1.2. We will find that the local well posedness for large initial data, $u_0 \in B^{\frac{N}{2}+1}_{2,1}$, holds for the system (1.8)-(1.9), by obvious modification of the proof of Theorem 1.1. For the case of large data in the less regular space considered in Theorem 1.2, $B^{\frac{N}{2}-1}_{2,1}$, however, the local well posedness of (1.8)-(1.9) is open to the the author's knowledge.

After this research was completed the author was informed about the work by D. Iftimie [17]*, where the author also considered the hyperbolic system in the Besov space, but with less general settings than those covered by Theorem* 1.1.

2. Preliminaries

In this section we set our notation, and recall definitions on the Besov spaces. We follow [22] and [23]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$ its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying $\operatorname{Supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$, and $\hat{\varphi}(\xi) > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$. Setting $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$ (in other words, $\varphi_j(x) = 2^{jN}\varphi(2^j x)$), we can adjust the normalization constant in front of $\hat{\varphi}$ so that (see e.g. Lemma 6.1.7, [3])

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j\ge k+1} \hat{\varphi}_j(\xi).$$

We observe

$$\operatorname{Supp}\ \hat{\varphi}_j \cap \operatorname{Supp}\ \hat{\varphi}_{j'} = \emptyset \text{ if } |j - j'| \ge 2. \tag{2.1}$$

Let $s \in \mathbb{R}$, $p, q \in [0, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \varphi_j * f$, and then the homogeneous Besov norm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \Big[\displaystyle\sum_{j=-\infty}^{\infty} 2^{jqs} \|\varphi_j * f\|^q_{L^p}\Big]^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\ \displaystyle\sup_{j\in\mathbb{Z}} \big[2^{js}\|\varphi_j * f\|_{L^p}\big] & \text{if } q = \infty \end{cases}.$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a semi-normed space with the semi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0, p, q \in [0, \infty]$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}.$$

The inhomogeneous Besov space is a Banach space equipped with the norm $\|\cdot\|_{B^s_{p,q}}$. Let $f \in [L^p(\mathbb{R}^N)]^N$, and $A_0(g)$ be the symmetric positive-definite matrix introduced in (1.1); then we introduce the modified L^p norm by

$$\|f\|_{L^p_{A_0(g)}} = \Big(\int_{\mathbb{R}^N} (f, A_0(g)f)^{\frac{p}{2}} dx\Big)^{\frac{1}{p}}. \tag{2.2}$$

We will also use the following modified Besov space norm.

$$\|f\|_{\dot{\beta}^s_{p,q}(g)} = \begin{cases} \Big[\sum_{j=-\infty}^{\infty} 2^{jqs} \|\varphi_j * f\|^q_{L^p_{A_0(g)}} \Big]^{\frac{1}{q}} & \text{if } q \in [1,\infty) \\ \sup_{j\in\mathbb{Z}} \big[2^{js} \|\varphi_j * f\|_{L^p_{A_0(g)}} \big] & \text{if } q = \infty \end{cases}$$

for homogeneous spaces.

For $s > 0, p, q \in [0,\infty]$ we also define the modified inhomogeneous Besov space norm $\|f\|_{\beta^s_{p,q}(g)}$ as

$$\|f\|_{\beta^s_{p,q}(g)} = \|f\|_{L^p_{A_0(g)}} + \|f\|_{\dot{\beta}^s_{p,q}(g)}.$$

Thanks to the condition (1.3) on A_0 we have the following equivalences

$$\|f\|_{L^p,g} \sim \|f\|_{L^p_{A_0(g)}}, \quad \|f\|_{\dot{B}^s_{p,q}} \sim \|f\|_{\dot{\beta}^s_{p,q}(g)}, \quad \|f\|_{B^s_{p,q}} \sim \|f\|_{\beta^s_{p,q}(g)}. \tag{2.3}$$

We now recall the following basic lemmas. *We note that we use the same constant C in the inequalities following, which might be different in each context.*

Lemma 2.1 (Bernstein's lemma)**.** *Assume that $f \in L^p$, $1 \le p \le \infty$, and supp $\hat{f} \subset \{2^{j-2} \le |\xi| < 2^j\}$, then there exists a constant C_k such that the following inequality holds*

$$C_k^{-1} 2^{jk} \|f\|_{L^p} \le \|D^k f\|_{L^p} \le C_k 2^{jk} \|f\|_{L^p}. \tag{2.4}$$

As an immediate corollary of the above lemma we have the equivalence of norms,

$$\|D^k f\|_{\dot{B}^s_{p,q}} \sim \|f\|_{\dot{B}^{s+k}_{p,q}}. \tag{2.5}$$

Lemma 2.2. *Let $s > 0$, $q \in [1,\infty]$, then there exists a constant C such that the following inequalities hold.*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\Big(\|f\|_{L^{p_1}} \|g\|_{\dot{B}^s_{p_2,q}} + \|g\|_{L^{r_1}} \|f\|_{\dot{B}^s_{r_2,q}} \Big), \tag{2.6}$$

where $p_1, r_1 \in [1,\infty]$ such that $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$. If $s_1, s_2 \le \frac{N}{p}$, $p \ge 2$, $s_1 + s_2 > 0$, $f \in \dot{B}^{s_1}_{p,1}$, $g \in \dot{B}^{s_2}_{p,1}$, then

$$\|fg\|_{B^{s_1+s_2-\frac{N}{p}}_{p,1}} \le C \|f\|_{\dot{B}^{s_1}_{p,1}} \|g\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.7}$$

Lemma 2.3. *(i) Let $s_1, s_2 \in \mathbb{R}$, $\theta \in [0,1]$. Then, the following interpolation inequality holds.*

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{2,1}} \le \|f\|^{\theta}_{\dot{B}^{s_1}_{2,1}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{2,1}}. \tag{2.8}$$

A similar inequality holds also for inhomogeneous norms.

(ii) Let $s>0$ and $f \in \dot{B}^s_{p,1}\cap L^\infty$. Suppose $G \in W^{[s]+2,\infty}(\mathbb{R}^N)$ is such that $G(0)=0$ and $s>0$. Then we have

$$\|G(f)\|_{\dot{B}^s_{p,1}} \le C\|f\|_{\dot{B}^s_{p,1}}. \tag{2.9}$$

(iii) Let $f,g \in \dot{B}^{\frac{N}{p}}_{p,1}$, $f-g \in \dot{B}^s_{p,1}$ for $s\in(-\frac{N}{p},\frac{N}{p}]$, and $G \in W^{[\frac{N}{2}]+3,\infty}$, $G'(0)=0$; then $G(f)-G(g)\in \dot{B}^s_{p,1}$, and the following inequality holds

$$\|G(f)-G(g)\|_{\dot{B}^s_{p,1}} \le C\Big(\|f\|_{\dot{B}^{\frac{N}{p}}_{p,1}}+\|g\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\Big)\|f-g\|_{\dot{B}^s_{p,1}}. \tag{2.10}$$

Lemma 2.4. *If s satisfies $s\in(-\frac{N}{p}-1,\frac{N}{p}]$, then we have*

$$\|[u,\Delta_j]w\|_{L^p} \le c_j 2^{-j(s+1)}\|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|w\|_{\dot{B}^s_{p,1}} \tag{2.11}$$

with $\sum_{j\in\mathbb{Z}} c_j \le 1$.

Lemma 2.1 is classical, and proved e.g. in [11]. The inequality (2.5) and (2.6) of Lemma 2.2 is proved e.g. in [8] and [10] respectively. In Lemma 2.3 (i) follows immediately from the definition of norms; (ii) is proved in [10], and (iii) is an immediate consequence of (ii), using the mean-value theorem. Lemma 2.4 is proved in the appendix of [15].

3. PROOF OF THEOREM 1.1

We define a sequence of functions $\{u^{(m)}\}_{m=1}^\infty$ as solutions of the sequence of linear systems recurrently defined by

$$A_0(u^{(m)})\frac{\partial u^{(m+1)}}{\partial t}+\sum_{k=1}^N A_k(u^{(m)})\frac{\partial u^{(m+1)}}{\partial x_k}=0 \quad (x,t)\in\mathbb{R}^n\times(0,\infty) \tag{3.1}$$

$$u^{(m+1)}(x,0)=S_{m+1}u_0(x), \quad x\in\mathbb{R}^N, \tag{3.2}$$

where $m=0,1,2,\cdots$. In particular we set $u^{(0)}=0$.

(i) Uniform bounds in $X_T := C([0,T];B^{\frac{N}{2}+1}_{2,1})$: We claim that there exist constants $T_1>0$, $M_1>0$ such that the solutions $\{u^{(m)}\}$ defined by (3.1) for

$m=1,\cdots$ satisfy

$$\sup_{0\le t\le T_1}\Big(\big\|u^{(m+1)}\big\|_{B^{\frac{N}{2}+1}_{2,1}}+\big\|\frac{\partial u^{(m+1)}}{\partial t}\big\|_{B^{\frac{N}{2}}_{2,1}}\Big)\le M_1. \tag{3.3}$$

Applying the operation $A_0^{-1}(u^{(m)})$, and then Δ_j, on both sides of (3.1), we have

$$\frac{\partial}{\partial t}\Delta_j u^{(m+1)}+\sum_{k=1}^{N}\Delta_j\Big\{A_0^{-1}(u^{(m)})A_k(u^{(m)})\frac{\partial u^{(m+1)}}{\partial x_k}\Big\}=0 \tag{3.4}$$

Applying $A_0(u^{(m)})$, and then taking the scalar product (3.4) with $\Delta_j u^{(m+1)}$, we obtain after elementary computations

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_j u^{(m+1)},A_0(u^{(m)})\Delta_j u^{(m+1)})&=(\Delta_j u^{(m+1)},div\mathbb{A}(u^{(m)})\Delta_j u^{(m+1)})\\&-\sum_{k=1}^{N}\frac{\partial}{\partial x_k}(\Delta_j u^{(m+1)},A_k(u^{(m)})\Delta_j u^{(m+1)})\\&+2\sum_{k=1}^{N}\Big(\Delta_j u^{(m+1)},A_0(u^{(m)})[\Delta_j,A_0^{-1}(u^{(m)})A_k(u^{(m)})]\frac{\partial u^{(m+1)}}{\partial x_k}\Big),\end{aligned} \tag{3.5}$$

after which we use the following notation:

$$\mathbb{A}=(A_0,A_1,\cdots,A_N),\quad div\mathbb{A}(u^{(m)})=\sum_{k=0}^{N}\frac{\partial A_k(u^{(m)})}{\partial x_k},\quad (x_0=t)$$

and

$$\|f\|_{L^2_{A_0}}=\Big(\int_{\mathbb{R}^N}(f,A_0(u^{(m)})f)dx\Big)^{\frac12},\qquad \|f\|_{\dot{\beta}^s_{2,1}}=\|f\|_{\dot{\beta}^s_{2,1}(u^{(m)})}.$$

Integrating (3.5) over $\mathbb{R}^N$, we have the estimates

$$\begin{aligned}&\frac{d}{dt}\|\Delta_j u^{(m+1)}\|^2_{L^2_{A_0}}=\int_{\mathbb{R}^N}(\Delta_j u^{(m+1)},div\mathbb{A}(u^{(m)})\Delta_j u^{(m+1)})dx\\&+2\sum_{k=1}^{N}\int_{\mathbb{R}^N}\Big(\Delta_j u^{(m+1)},A_0(u^{(m)})[\Delta_j,A_0^{-1}(u^{(m)})A_k(u^{(m)})]\frac{\partial u^{(m+1)}}{\partial x_k}\Big)dx\\&\le C\|\mathrm{div}\,\mathbb{A}(u^{(m)})\|_{L^\infty}\|\Delta_j u^{(m+1)}\|^2_{L^2_{A_0}}\\&+Cc_j2^{-j(\frac{N}{2}+1)}\sum_{k=1}^{N}\|\Delta_j u^{(m+1)}\|_{L^2_{A_0}}\|A_0^{-1}(u^{(m)})A_k(u^{(m)})\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|Du^{(m+1)}\|_{L^\infty}\end{aligned}$$

$$\leq C\big\|\mathbb{A}(u^{(m)})\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|\Delta_j u^{(m+1)}\big\|^2_{L^2_{A_0}}$$
$$+ Cc_j 2^{-j(\frac{N}{2}+1)}\big\|\Delta_j u^{(m+1)}\big\|_{L^2_{A_0}}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}$$
$$\leq C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|\Delta_j u^{(m+1)}\big\|^2_{L^2_{A_0}}$$
$$+ Cc_j 2^{-j(\frac{N}{2}+1)}\big\|\Delta_j u^{(m+1)}\big\|_{L^2_{A_0}}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}\big\|_{\dot{\beta}^{\frac{N}{2}+1}_{2,1}}, \tag{3.6}$$

where we used (2.7) and (2.9) for $s = \frac{N}{2}+1$ in the second estimate. Dividing both sides by $\|\Delta_j u^{(m+1)}\|_{L^2_{A_0}}$, we obtain

$$\frac{d}{dt}\big\|\Delta_j u^{(m+1)}\big\|_{L^2_{A_0}} \leq C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|\Delta_j u^{(m+1)}\big\|_{L^2_{A_0}}$$
$$+ Cc_j 2^{-j(\frac{N}{2}+1)}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}\big\|_{\dot{\beta}^{\frac{N}{2}+1}_{2,1}}. \tag{3.7}$$

Multiplying both sides of (3.7) by $2^{j(\frac{N}{2}+1)}$, and summing over $j \in \mathbb{Z}$, we have

$$\frac{d}{dt}\big\|u^{(m+1)}(t)\big\|_{\dot{\beta}^{\frac{N}{2}+1}_{2,1}} \leq C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}\big\|_{\dot{\beta}^{\frac{N}{2}+1}_{2,1}}. \tag{3.8}$$

Gronwall's lemma combined with the equivalence of norms in (2.2) implies

$$\big\|u^{(m+1)}(t)\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \leq C\big\|S_{m+1}u_0\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\exp\left(C\int_0^t \big\|u\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}ds\right)$$
$$\leq C_1\big\|u_0\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\exp\left(C_1 T\sup_{0\leq t\leq T}\big\|u^{(m)}(t)\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\right) \tag{3.9}$$

for some absolute constants C_1, where we used the fact

$$\begin{aligned}
\big\|S_{m+1}u_0\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} &\leq \sum_{j\in\mathbb{Z}}\sum_{j'\leq m+1} 2^{j(\frac{N}{2}+1)}\big\|\Delta_j\Delta_{j'}u_0\big\|_{L^2}\\
&\leq C\sum_{j'\leq m+1} 2^{j'(\frac{N}{2}+1)}\sum_{|j-j'|\leq 3} 2^{(j-j')(\frac{N}{2}+1)}\big\|\Delta_{j'}u_0\big\|_{L^2}\\
&\leq C\sum_{j'\leq m+1} 2^{j'(\frac{N}{2}+1)}\big\|\Delta_{j'}u_0\big\|_{L^2} \leq C\big\|u_0\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}.
\end{aligned}$$

We note parenthetically that the procedure from (3.7) to (3.9) is rather formal, but can be easily justifiable by integrating over $[0,t]$ of (3.7), and then using the Gronwall lemma in the version of integral inequality instead of one of differential inequality. By the standard induction argument, recalling $u^{(0)} = 0$, we find that for

$$T_1 = \frac{\ln 2}{2C_1\|u_0\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}} \tag{3.10}$$

we have

$$\sup_{0\le t\le T_1}\|u^{(m+1)}(t)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \le 2C_1\|u_0\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \qquad \forall m\ge 0. \tag{3.11}$$

In order to have uniform bounds in the $B^{\frac{N}{2}+1}_{2,1}$ norm of $\{u^{(m)}\}$ we start with estimates in the $L^2_{A_0}$ norm. We take the L^2 inner product (3.1) with $u^{(m+1)}$. Then, after integration by parts we have

$$\begin{aligned}\frac{d}{dt}\|u^{(m+1)}\|^2_{L^2_{A_0}} &= \int_{\mathbb{R}^N}(u^{(m+1)}, \operatorname{div}\mathbb{A}(u^{(m)})u^{(m+1)})dx\\ &\le \|\operatorname{div}\mathbb{A}(u^{(m)})\|_{L^\infty}\|u^{(m+1)}\|^2_{L^2} \le C\|\mathbb{A}(u^{(m)})\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|^2_{L^2_{A_0}}\\ &\le C\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|^2_{L^2_{A_0}}. \end{aligned}\tag{3.12}$$

Adding (3.12) to (3.8), we have

$$\frac{d}{dt}\|u^{(m+1)}(t)\|_{\beta^{\frac{N}{2}+1}_{2,1}} \le C\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|_{\beta^{\frac{N}{2}+1}_{2,1}}. \tag{3.13}$$

The same remark as the one below (3.9) holds here. By Gronwall's lemma we have

$$\|u^{(m+1)}(t)\|_{\beta^{\frac{N}{2}+1}_{2,1}} \le \|u_0^{(m+1)}\|_{\beta^{\frac{N}{2}+1}_{2,1}}\exp\Big(C\int_0^t\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}ds\Big), \tag{3.14}$$

and due to the equivalence of norms in (2.2),

$$\begin{aligned}\|u^{(m+1)}(t)\|_{B^{\frac{N}{2}+1}_{2,1}} &\le C\|u_0^{(m+1)}\|_{B^{\frac{N}{2}+1}_{2,1}}\exp\Big(C\int_0^t\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}ds\Big)\\ &\le C\|u_0\|_{B^{\frac{N}{2}+1}_{2,1}}\exp\Big(CT\sup_{0\le t\le T}\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\Big).\end{aligned}\tag{3.15}$$

This, combined with (3.11), implies

$$\sup_{0\le t\le T_1}\|u^{(m+1)}(t)\|_{B^{\frac{N}{2}+1}_{2,1}} \le 2C_1\|u_0\|_{B^{\frac{N}{2}+1}_{2,1}}\exp(C_2T_1\|u_0\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}) \qquad \forall m\ge 0, \tag{3.16}$$

where C_2 ia an absolute constant. From (3.4) we have

$$\|\Delta_j \frac{\partial u^{(m+1)}}{\partial t}\|_{L^2} \le \sum_{k=1}^N \|\Delta_j \Big\{ A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k} \Big\}\|_{L^2}. \qquad (3.17)$$

Multiplying by $2^{\frac{N}{2}j}$, and summing over $j \in \mathbb{Z}$ of (3.17), we find

$$\begin{aligned}
\|\frac{\partial u^{(m+1)}}{\partial t}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} &\le \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&\le C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \|\frac{\partial u^{(m+1)}}{\partial x_k}\|_{L^\infty} \\
&+ \quad C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{L^\infty} \|\frac{\partial u^{(m+1)}}{\partial x_k}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&\le C \|u^{(m)}\|_{B^{\frac{N}{2}}_{2,1}} \|u^{(m+1)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}, \qquad (3.18)
\end{aligned}$$

where we used (2.8), and the imbedding $\dot{B}^{\frac{N}{2}}_{2,1} \hookrightarrow L^\infty(\mathbb{R}^N)$. On the other hand, from (3.1) we have immediately

$$\|\frac{\partial u^{(m+1)}}{\partial t}\|_{L^2} \le \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)}) \frac{\partial u^{(m+1)}}{\partial x_k}\|_{L^2} \qquad (3.19)$$

$$\le C \sum_{k=1}^N \|A_0^{-1}(u^{(m)}) A_k(u^{(m)})\|_{L^\infty} \|\frac{\partial u^{(m+1)}}{\partial x_k}\|_{L^2} \le C \|u^{(m)}\|_{B^{\frac{N}{2}}_{2,1}} \|D u^{(m+1)}\|_{L^2}.$$

Adding (3.19) to (3.18) we find

$$\|\frac{\partial u^{(m+1)}}{\partial t}\|_{B^{\frac{N}{2}}_{2,1}} \le C \|u^{(m)}\|_{B^{\frac{N}{2}}_{2,1}} \|u^{(m+1)}\|_{B^{\frac{N}{2}+1}_{2,1}}. \qquad (3.20)$$

From (3.16), we obtain

$$\sup_{0 \le t \le T_1} \|\frac{\partial u^{(m+1)}}{\partial t}\|_{B^{\frac{N}{2}}_{2,1}} \le C. \qquad (3.21)$$

Combining (3.21) with (3.16), we complete the proof of (3.3).

(ii) Contraction in $Y_T := C([0,T]; B^s_{2,1}), s \in [0, \frac{N}{2}+1)$*:* We will show that $\|u^{(m+1)} - u^{(m)}\|_{Y_{T_0}} \to 0$ as $m \to \infty$ for some $T_0 \in (0, T_1]$. Taking the

difference between the $(m+1)$st equation and the (m)th equation, we obtain after rearrangement of terms

$$\frac{\partial}{\partial t}\Delta_j(u^{(m+1)}-u^{(m)})+\sum_{k=1}^{N}\Delta_j\Big\{(A_0^{-1}A_k)(u^{(m)})\frac{\partial(u^{(m+1)}-u^{(m)})}{\partial x_k}\Big\}$$

$$=-\sum_{k=1}^{N}\Delta_j\Big\{\Big((A_0^{-1}A_k)(u^{(m)})-(A_0^{-1}A_k)(u^{(m-1)})\Big)\frac{\partial u^{(m)}}{\partial x_k}\Big\}. \tag{3.22}$$

Applying $A_0(u^{(m)})$, and then taking the scalar product in $\mathbb{R}^N$ of (3.22) with $\Delta_j(u^{(m+1)}-u^{(m)})$ we obtain by computations similar to those above

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_0(u^{(m)})\Delta_j(u^{(m+1)}-u^{(m)})\Big)\\
&=\Big(\Delta_j(u^{(m+1)}-u^{(m)}),div\mathbb{A}(u^{(m)})\Delta_j(u^{(m+1)}-u^{(m)})\Big)\\
&-\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_k(u^{(m)})\Delta_j(u^{(m+1)}-u^{(m)})\Big)\\
&+2\sum_{k=1}^{N}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_0(u^{(m)})[\Delta_j,A_0^{-1}(u^{(m)})A_k(u^{(m)})]\\
&\qquad\times\frac{\partial}{\partial x_k}(u^{(m+1)}-u^{(m)})\Big)-\sum_{k=1}^{N}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_0(u^{(m)})\\
&\qquad\times\Delta_j\Big\{\Big((A_0^{-1}A_k)(u^{(m)})-(A_0^{-1}A_k)(u^{(m-1)})\Big)\frac{\partial u^{(m)}}{\partial x_k}\Big\}\Big).
\end{aligned}$$

Integrating both sides of this equation over $\mathbb{R}^n$, we have the following estimates

$$\begin{aligned}
&\frac{d}{dt}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}}^2\\
&=\int_{\mathbb{R}^N}(\Delta_j(u^{(m+1)}-u^{(m)}),div\mathbb{A}(u^{(m)})\Delta_j(u^{(m+1)}-u^{(m)}))dx\\
&+2\sum_{k=1}^{N}\int_{\mathbb{R}^N}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_0(u^{(m)})[\Delta_j,A_0^{-1}(u^{(m)})A_k(u^{(m)})]\\
&\quad\times\frac{\partial}{\partial x_k}(u^{(m+1)}-u^{(m)})\Big)dx-\sum_{k=1}^{N}\int_{\mathbb{R}^N}\Big(\Delta_j(u^{(m+1)}-u^{(m)}),A_0(u^{(m)})\Delta_j
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left\{ \left((A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)}) \right) \frac{\partial u^{(m)}}{\partial x_k} \right\} \Big) \\
&\le C\big\|div\mathbb{A}(u^{(m)})\big\|_{L^\infty}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}}^2 \\
&\quad \times + Cc_j 2^{-\frac{N}{2}j}\sum_{k=1}^N \big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}} \\
&\quad \times \big\|A_0^{-1}(u^{(m)})A_k(u^{(m)})\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}-u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&+ C\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}} \\
&\quad \times \sum_{k=1}^N \Big\|\Delta_j\Big[\Big\{(A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)})\Big\}\frac{\partial u^{(m)}}{\partial x_k}\Big]\Big\|_{L^2} \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}}^2 \\
&+ Cc_j 2^{-\frac{N}{2}j}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}-u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \\
&+ C\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}} \\
&\quad \times \sum_{k=1}^N \Big\|\Delta_j\Big[\Big\{(A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)})\Big\}\frac{\partial u^{(m)}}{\partial x_k}\Big]\Big\|_{L^2}.
\end{aligned}
$$

Dividing both sides by $\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}} \le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|\Delta_j(u^{(m+1)}-u^{(m)})\big\|_{L^2_{A_0}} \\
&+ Cc_j 2^{-\frac{N}{2}j}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}-u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \\
&+ C\sum_{k=1}^N \Big\|\Delta_j\Big[\Big\{(A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)})\Big\}\frac{\partial u^{(m)}}{\partial x_k}\Big]\Big\|_{L^2}.
\end{aligned}
$$

Multiplying by $2^{\frac{N}{2}j}$ and, then summing over $j \in \mathbb{Z}$ on both sides, we obtain

$$
\frac{d}{dt}\big\|u^{(m+1)}(t)-u^{(m)}(t)\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)}-u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}}
$$

$$
\begin{aligned}
&+ C\sum_{k=1}^{N}\Big\|\Big\{(A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)})\Big\}\frac{\partial u^{(m)}}{\partial x_k}\Big\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \\
&+ C\sum_{k=1}^{N}\big\|(A_0^{-1}A_k)(u^{(m)}) - (A_0^{-1}A_k)(u^{(m-1)})\big\|_{\dot{B}^{\frac{N}{2}}_{2,1}}\Big\|\frac{\partial u^{(m)}}{\partial x_k}\Big\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \\
&+ C(\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}}_{2,1}} + \big\|u^{(m-1)}\big\|_{\dot{B}^{\frac{N}{2}}_{2,1}})\big\|u^{(m)} - u^{(m-1)}\big\|_{\dot{B}^{\frac{N}{2}}_{2,1}}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \\
&\le C\big\|u^{(m+1)} - u^{(m)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} + C\big\|u^{(m)} - u^{(m-1)}\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \qquad (3.23)
\end{aligned}
$$

for $t \in [0, T_1]$, where we used (2.10) combined with the uniform estimate (3.11).

The corresponding estimate of $\|u^{(m+1)} - u^{(m)}\|_{L^2_{A_0}}$ is similar to the above, and simpler. Taking the L^2 inner product with the equation of $A_0(u^{(m)})\frac{\partial}{\partial t}(u^{(m+1)} - u^{(m)})$ by $u^{(m+1)} - u^{(m)}$, and then estimating using the Hölder inequality, and integrating by parts, and then dividing both sides by $\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2_{A_0}}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\big\|u^{(m+1)}(t) - u^{(m)}(t)\big\|_{L^2_{A_0}} \le C\big\|\mathrm{div}\mathbb{A}(u^{(m)})\big\|_{L^\infty}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2_{A_0}} \\
&+ C\sum_{k=1}^{N}\big\|[A_k(u^{(m)}) - A_k(u^{(m-1)})]\big\|_{L^2}\Big\|\frac{\partial u^{(m)}}{\partial x_k}\Big\|_{L^\infty} \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2_{A_0}} + C\big\|u^{(m)} - u^{(m-1)}\big\|_{L^2}\big\|u^{(m)}\big\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \\
&\le C\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2_{A_0}} + C\big\|u^{(m)} - u^{(m-1)}\big\|_{L^2_{A_0}} \qquad (3.24)
\end{aligned}
$$

for $t \in [0, T_1]$. Adding (3.24) to (3.23), we have

$$
\frac{d}{dt}\big\|u^{(m+1)}(t) - u^{(m)}(t)\big\|_{\dot{\beta}^{\frac{N}{2}}_{2,1}} \le C\big\|u^{(m+1)} - u^{(m)}\big\|_{\beta^{\frac{N}{2}}_{2,1}} + C\big\|u^{(m)} - u^{(m-1)}\big\|_{\beta^{\frac{N}{2}}_{2,1}} \qquad (3.25)
$$

on $[0,T_1]$. Using Gronwall's inequality, and the equivalence of norms in (2.2), we obtain

$$\begin{aligned}\|u^{(m+1)}-u^{(m)}\|_{Y_T} &\leq Ce^{CT}\Big(\|u_0^{(m+1)}-u_0^{(m)}\|_{B^{\frac{N}{2}}_{2,1}}+T\|u^{(m)}-u^{(m-1)}\|_{Y_T}\Big)\\ &\leq Ce^{CT}\Big(\|\Delta_m u_0\|_{B^{\frac{N}{2}}_{2,1}}+T\|u^{(m)}-u^{(m-1)}\|_{Y_T}\Big). \quad (3.26)\end{aligned}$$

for $T\leq T_1$ From this combined with the fact

$$\|\Delta_m u_0\|_{B^{\frac{N}{2}}_{2,1}} \leq C2^{-m}\|\Delta_m u_0\|_{B^{\frac{N}{2}+1}_{2,1}} \leq C2^{-m}\|u_0\|_{B^{\frac{N}{2}+1}_{2,1}} \leq C2^{-m}, \quad (3.27)$$

which follows from Bernstein's lemma, we can easily deduce that there exists $u\in L^\infty([0,T_1];B^{\frac{N}{2}+1}_{2,1})$ such that the sequence, $\{u^{(m)}\}$ converges to u in Y_T. Due to the interpolation inequality (2.7) for inhomogeneous spaces we have for all $s\in[\frac{N}{2},\frac{N}{2}+1)$

$$\begin{aligned}&\sup_{0\leq t\leq T}\|u(t)-u^{(m)}(t)\|_{B^s_{2,1}}\\ &\leq \sup_{0\leq t\leq T}\Big\{\|u(t)-u^{(m)}(t)\|^{1+\frac{N}{2}-s}_{B^{\frac{N}{2}}_{2,1}}(\|u(t)\|_{B^{\frac{N}{2}+1}_{2,1}}+\|u^{(m)}(t)\|_{B^{\frac{N}{2}+1}_{2,1}})^{s-\frac{N}{2}}\Big\}\\ &\leq C\sup_{0\leq t\leq T}\|u-u^{(m)}\|^{1+\frac{N}{2}-s}_{B^{\frac{N}{2}}_{2,1}}\to 0 \qquad \text{as } m\to\infty,\end{aligned}$$

and by the three ε-argument we find $u\in C([0,T];B^s_{2,1})$ for all $s\in[\frac{N}{2},\frac{N}{2}+1)$. Now, from the uniform estimate (3.21) we have

$$\|u^{(m)}(t_1)-u^{(m)}(t_2)\|_{B^{\frac{N}{2}}_{2,1}} \leq \int_{t_2}^{t_1}\Big\|\frac{\partial u^{(m+1)}}{\partial t}\Big\|_{B^{\frac{N}{2}}_{2,1}}dt \leq C|t_1-t_2|$$

for all $t_1,t_2\in[0,T_2]$. Thus, we have

$$\|u(t_1)-u(t_2)\|_{B^{\frac{N}{2}}_{2,1}} \leq \|u(t_1)-u^{(m)}(t_1)\|_{B^{\frac{N}{2}}_{2,1}}+C|t_1-t_2|+\|u(t_2)-u^{(m)}(t_2)\|_{B^{\frac{N}{2}}_{2,1}}. \quad (3.28)$$

Passing to the limit as $m\to\infty$ in (3.28), we obtain $u\in \mathrm{Lip}([0,T_1];B^{\frac{N}{2}}_{2,1})$. The proof of uniqueness of the solution is similar to the contraction part of the proof above, and we will be very brief. If u_1,u_2 are the two solutions corresponding to initial datum, $u_{1,0},u_{2,0}$ respectively, then following the same procedure leading to (3.26), we obtain

$$\|u_1-u_2\|_{Y_T} \leq Ce^{CT}\Big(\|u_{1,0}-u_{2,0}\|_{B^{\frac{N}{2}}_{2,1}}+T\|u_1-u_2\|_{Y_T}\Big),$$

from which we deduce that if $u_{1,0} = u_{2,0}$, then $u_1(t) = u_2(t)$ for all $t \in [0, T_1]$ if $Ce^{CT_2}T_2 < 1$. Thus the uniqueness of the solution is proved on $[0, T]$ for $T \le \min\{T_1, T_2\}$. In order to prove the continuation principle we just observe that starting from (1.1), and following the procedure leading to (3.15), we find that

$$\sup_{0\le t\le T} \|u(t)\|_{B^{\frac{N}{2}+1}_{2,1}} \le C\|u_0\|_{B^{\frac{N}{2}+1}_{2,1}} \exp\Big(C\int_0^T \|\nabla u\|_{L^\infty} ds\Big).$$

On the other hand, we have the obvious inequalities,

$$\int_0^T \|\nabla u\|_{L^\infty} dt \le C\int_0^T \|u(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds \le CT \sup_{0\le t\le T} \|u(t)\|_{B^{\frac{N}{2}+1}_{2,1}}, \qquad (3.29)$$

thanks to the embedding $\dot{B}^{\frac{N}{2}+1}_{2,1} \hookrightarrow Lip(\mathbb{R}^N)$. The blow-up criterion follows from (3.29) and (3.30) immediately. This completes the proof of Theorem 1.1. □

Remark after proof. The assumption of $A_k'(0) = 0$, $k = 1, 2, \cdots, N$ could be removed in Theorem 1.1. For the proof of this we use the following generalized version of (2.10):

$$\|G(f) - G(g)\|_{\dot{B}^s_{p,1}} \le C\Big(|G'(0)| + \|f\|_{\dot{B}^{\frac{N}{p}}_{p,1}} + \|g\|_{\dot{B}^{\frac{N}{p}}_{p,1}}\Big)\|f-g\|_{\dot{B}^s_{p,1}},$$

which can be derived easily. This leads to estimates of additional harmless terms in the above proof.

4. Proof of Theorem 1.2

We define a sequence $\{u^{(m)}\}$ by solving the linear system iteratively

$$\frac{\partial u^{(m+1)}}{\partial t} + \sum_{k=1}^N A_k(u^{(m)})\frac{\partial u^{(m+1)}}{\partial x_k} = \mu\Delta u^{(m+1)}, \quad (x,t) \in \mathbb{R}^N \times (0,\infty) \quad (4.1)$$

$$u^{(m+1)}(x,0) = S_m u_0(x), \quad x \in \mathbb{R}^N, \qquad (4.2)$$

where $m = 0, 1, 2, \cdots$, and we set $u^{(0)} = 0$.

(i) Uniform bounds in $C([0,\infty); B^{\frac{N}{2}-1}_{2,1}) \cap C([0,\infty); \dot{B}^{\frac{N}{2}}_{2,1}) \cap L^1([0,\infty); \dot{B}^{\frac{N}{2}+1}_{2,1})$. Applying the operation Δ_j on (4.1), we obtain

$$\frac{\partial}{\partial t}\Delta_j u^{(m+1)} + \sum_{k=1}^N \Delta_j\Big\{A_k(u^{(m)})\frac{\partial u^{(m+1)}}{\partial x_k}\Big\} = \mu\Delta\Delta_j u^{(m+1)}. \qquad (4.3)$$

Taking the L^2 inner product of (4.3) with $\Delta_j u^{(m+1)}$, we have after integration by parts

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta_j u^{(m+1)}\|_{L^2}^2+\mu\|\nabla\Delta_j u^{(m+1)}\|_{L^2}^2\\
&=-\sum_{k=1}^N\Big([\Delta_j,A_k(u^{(m)})]\frac{\partial u^{(m+1)}}{\partial x_k},\Delta_j u^{(m+1)}\Big)_{L^2}\\
&-\frac{1}{2}\sum_{k=1}^N\Big(\Delta_j u^{(m+1)}\frac{\partial A_k(u^{(m)})}{\partial x_k},\Delta_j u^{(m+1)}\Big)_{L^2}.
\end{aligned}\tag{4.4}
$$

Thus, using (2.3), (2.11), we estimate

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta_j u^{(m+1)}\|_{L^2}^2+C\mu 2^{2j}\|\Delta_j u^{(m+1)}\|_{L^2}^2\\
&\leq\ 2\sum_{k=1}^N\Big\|[\Delta_j,A_k(u^{(m)})]\frac{\partial u^{(m+1)}}{\partial x_k}\Big\|_{L^2}\|\Delta_j u^{(m+1)}\|_{L^2}\\
&\qquad+\sum_{k=1}^N\Big\|\frac{\partial A_k(u^{(m)})}{\partial x_k}\Big\|_{L^\infty}\|\Delta_j u^{(m+1)}\|_{L^2}^2\\
&\leq\ Cc_j2^{-j(\frac{N}{2}-1)}\sum_{k=1}^N\|A_k(u^{(m)})\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|_{\dot B^{\frac{N}{2}-1}_{2,1}}\|\Delta_j u^{(m+1)}\|_{L^2}\\
&\qquad+C\sum_{k=1}^N\|A_k(u^{(m)})\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\|\Delta_j u^{(m+1)}\|_{L^2}^2.
\end{aligned}\tag{4.5}
$$

Dividing both sides by $\|\Delta_j u^{(m+1)}\|_{L^2}$, and using (2.9), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta_j u^{(m+1)}\|_{L^2}+C\mu 2^{2j}\|\Delta_j u^{(m+1)}\|_{L^2}\\
&\leq Cc_j2^{-j(\frac{N}{2}-1)}\|u^{(m)}\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|_{\dot B^{\frac{N}{2}-1}_{2,1}}+C\|u^{(m)}\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\|\Delta_j u^{(m+1)}\|_{L^2}.
\end{aligned}\tag{4.6}
$$

Multiplying by $2^{j(\frac{N}{2}-1)}$, and summing over $j\in\mathbb{Z}$, we find

$$
\frac{d}{dt}\|u^{(m+1)}\|_{\dot B^{\frac{N}{2}-1}_{2,1}}+C_1\mu\|u^{(m+1)}\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\leq C_2\|u^{(m)}\|_{\dot B^{\frac{N}{2}+1}_{2,1}}\|u^{(m+1)}\|_{\dot B^{\frac{N}{2}-1}_{2,1}}.\tag{4.7}
$$

By Gronwall's lemma we have

$$\begin{aligned}
&\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)\right\|_{\dot B_{2,1}^{\frac N2-1}}+C_1\mu\int_0^\infty\left\|u^{(m+1)}(s)\right\|_{\dot B_{2,1}^{\frac N2+1}}ds\\
&\le\left\|u_0^{(m+1)}\right\|_{\dot B_{2,1}^{\frac N2-1}}\exp\left(C\int_0^\infty\left\|u^{(m)}(s)\right\|_{\dot B_{2,1}^{\frac N2+1}}ds\right)\qquad(4.8)\\
&\le C_2\left\|u_0\right\|_{\dot B_{2,1}^{\frac N2-1}}\exp\left(C_2\int_0^\infty\left\|u^{(m)}(s)\right\|_{\dot B_{2,1}^{\frac N2+1}}ds\right)
\end{aligned}$$

for $m=1,2,\cdots$. We set $K=\max\{C_1\mu,C_2\}$. Let the initial data u_0 satisfy $\left\|u_0\right\|_{\dot B_{2,1}^{\frac N2-1}}\le\frac{\ln(2/K)}{2K}$, namely

$$K\exp\left(2K\left\|u_0\right\|_{\dot B_{2,1}^{\frac N2-1}}\right)\le 2.\qquad(4.9)$$

Then, from our setting $u^{(0)}=0$, and using (4.8)-(4.9), we can see easily

$$\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)\right\|_{\dot B_{2,1}^{\frac N2-1}}+C_1\mu\int_0^\infty\left\|u^{(m+1)}(s)\right\|_{\dot B_{2,1}^{\frac N2+1}}ds\le 2\left\|u_0\right\|_{\dot B_{2,1}^{\frac N2-1}}\qquad(4.10)$$

for all $m=1,2,\cdots$ by an induction argument. Taking the L^2 inner product of (4.1) with $u^{(m+1)}$, we have after integration by parts

$$\begin{aligned}
&\frac12\frac{d}{dt}\left\|u^{(m+1)}\right\|_{L^2}^2+\mu\left\|\nabla u^{(m+1)}\right\|_{L^2}^2=-\frac12\sum_{k=1}^N\left(u^{(m+1)}\frac{\partial A_k(u^{(m)})}{\partial x_k},u^{(m+1)}\right)_{L^2}\\
&\le\frac12\sum_{k=1}^N\left\|\frac{\partial A_k(u^{(m)})}{\partial x_k}\right\|_{L^\infty}\left\|u^{(m+1)}\right\|_{L^2}^2\le\frac12\sum_{k=1}^N\left\|A_k(u^{(m)})\right\|_{\dot B_{2,1}^{\frac N2+1}}\left\|u^{(m+1)}\right\|_{L^2}^2\\
&\le C\left\|u^{(m)}\right\|_{\dot B_{2,1}^{\frac N2+1}}\left\|u^{(m+1)}\right\|_{L^2}^2.\qquad(4.11)
\end{aligned}$$

By Gronwall's lemma, we obtain

$$\begin{aligned}
\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)\right\|_{L^2}&\le\left\|u_0^{(m+1)}\right\|_{L^2}\exp\left(C_2\int_0^\infty\left\|u^{(m)}(t)\right\|_{\dot B_{2,1}^{\frac N2+1}}dt\right)\\
&\le C_2\left\|u_0\right\|_{L^2}\exp\left(C_2\int_0^\infty\left\|u^{(m)}(t)\right\|_{\dot B_{2,1}^{\frac N2+1}}dt\right).\qquad(4.12)
\end{aligned}$$

Combining (4.8) and (4.12), we have

$$\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)\right\|_{B_{2,1}^{\frac N2-1}}+C_1\mu\int_0^\infty\left\|u^{(m+1)}(s)\right\|_{\dot B_{2,1}^{\frac N2+1}}ds$$

$$\leq C_2 \|u_0\|_{B^{\frac{N}{2}-1}_{2,1}} \exp\Big(C_2 \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds\Big), \tag{4.13}$$

for $m = 1, 2, \cdots$. From the uniform estimate (4.13), and (4.9), we have

$$\sup_{0\leq t<\infty} \|u^{(m+1)}(t)\|_{B^{\frac{N}{2}-1}_{2,1}} + C_1\mu \int_0^\infty \|u^{(m+1)}(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds \leq 2\|u_0\|_{B^{\frac{N}{2}-1}_{2,1}} \tag{4.14}$$

for all $m = 1, 2, \cdots$, if $u_0 \in B^{\frac{N}{2}-1}_{2,1}$ satisfies (4.9).

Next, we derive the uniform estimate of $u^{(m)}$ in $C([0,\infty); \dot{B}^{\frac{N}{2}}_{2,1})$. Following the same procedure as in (4.1) to (4.5), but using (2.11) with $s = \frac{N}{2} - 1$ instead of $s = \frac{N}{2} - 2$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\Delta_j u^{(m+1)}\|^2_{L^2} + C\mu 2^{2j}\|\Delta_j u^{(m+1)}\|^2_{L^2} \\
&\leq Cc_j 2^{-j\frac{N}{2}} \sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|u^{(m+1)}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \|\Delta_j u^{(m+1)}\|_{L^2} \\
&\quad + C\sum_{k=1}^N \|A_k(u^{(m)})\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|\Delta_j u^{(m+1)}\|^2_{L^2}.
\end{aligned} \tag{4.15}$$

Dividing both sides by $\|\Delta_j u^{(m+1)}\|_{L^2}$, and using (2.9) again, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\Delta_j u^{(m+1)}\|_{L^2} &\leq Cc_j 2^{-j\frac{N}{2}} \|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|u^{(m+1)}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \\
&\quad + C\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|\Delta_j u^{(m+1)}\|_{L^2}.
\end{aligned} \tag{4.16}$$

Multiplying by $2^{j\frac{N}{2}}$, and summing over $j \in \mathbb{Z}$, we find

$$\frac{d}{dt}\|u^{(m+1)}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \leq C\|u^{(m)}\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} \|u^{(m+1)}\|_{\dot{B}^{\frac{N}{2}}_{2,1}}.$$

By Gronwall's lemma we have

$$\begin{aligned}
\sup_{0\leq t<\infty} \|u^{(m+1)}(t)\|_{\dot{B}^{\frac{N}{2}}_{2,1}} &\leq \|u_0^{(m+1)}\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \exp\Big(C\int_0^\infty \|u^{(m)}(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds\Big) \\
&\leq C\|u_0\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \exp\Big(C\int_0^\infty \|u^{(m)}(s)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} ds\Big)
\end{aligned} \tag{4.17}$$

for $m = 1, 2, \cdots$. Since we have the uniform bound,

$$\sup_{m\in\mathbb{N}} \int_0^\infty \|u^{(m)}(s)\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} ds < \infty;$$

by (4.10), we have the desired uniform estimate in $C([0,\infty); \dot{B}_{2,1}^{\frac{N}{2}})$.

(ii) Contraction in $C([0,\infty); B_{2,1}^{\frac{N}{2}-1}) \cap L^1([0,\infty); \dot{B}_{2,1}^{\frac{N}{2}+1})$. Taking the difference between the (m)th and $(m-1)$st equations of (4.1), we have

$$\frac{\partial(u^{(m+1)} - u^{(m)})}{\partial t} + \sum_{k=1}^N A_k(u^{(m)}) \frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k}$$
$$= -\sum_{k=1}^N (A_k(u^{(m)}) - A_k(u^{(m-1)})) \frac{\partial u^{(m)}}{\partial x_k} + \mu\Delta(u^{(m+1)} - u^{(m)}), \tag{4.18}$$

$$(u^{(m+1)} - u^{(m)})(0, x) = \Delta_m u_0(x). \tag{4.19}$$

Applying Δ_j to (4.15), and then taking the L^2 inner product with $\Delta_j(u^{(m+1)} - u^{(m)})$, we find after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 + \mu\|\nabla\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2$$
$$= -\sum_{k=1}^N \Big([\Delta_j, A_k(u^{(m)})]\frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k}, \Delta_j(u^{(m+1)} - u^{(m)})\Big)_{L^2}$$
$$+ \frac{1}{2}\sum_{k=1}^N \Big(\Delta_j(u^{(m+1)} - u^{(m)})\frac{\partial A_k(u^{(m)})}{\partial x_k}, \Delta_j(u^{(m+1)} - u^{(m)})\Big)_{L^2}$$
$$- \sum_{k=1}^N \Big(\Delta_j\Big\{\Big[A_k(u^{(m)}) - A_k(u^{(m-1)})\Big]\frac{\partial u^{(m)}}{\partial x_k}\Big\}, \Delta_j(u^{(m+1)} - u^{(m)})\Big)_{L^2}$$
$$\leq \sum_{k=1}^N \Big\|[\Delta_j, A_k(u^{(m)})]\frac{\partial(u^{(m+1)} - u^{(m)})}{\partial x_k}\Big\|_{L^2}\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$$
$$+ \frac{1}{2}\sum_{k=1}^N \Big\|\frac{\partial A_k(u^{(m)})}{\partial x_k}\Big\|_{L^\infty}\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2$$
$$+ \sum_{k=1}^N \Big\|\Delta_j\Big\{\Big[A_k(u^{(m)}) - A_k(u^{(m-1)})\Big]\frac{\partial u^{(m)}}{\partial x_k}\Big\}\Big\|_{L^2}\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$$

$$\leq Cc_j 2^{-j(\frac{N}{2}-1)} \sum_{k=1}^{N} \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$$
$$\times \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$$
$$+ C \sum_{k=1}^{N} \|A_k(u^{(m)})\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2$$
$$+ \sum_{k=1}^{N} \left\|\Delta_j\left\{ \left[A_k(u^{(m)}) - A_k(u^{(m-1)})\right] \frac{\partial u^{(m)}}{\partial x_k}\right\}\right\|_{L^2} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$$
$$\leq Cc_j 2^{-j(\frac{N}{2}-1)} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$$
$$+ C\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}^2 \tag{4.20}$$
$$+ \sum_{k=1}^{N} \left\|\Delta_j\left\{ \left[A_k(u^{(m)}) - A_k(u^{(m-1)})\right] \frac{\partial u^{(m)}}{\partial x_k}\right\}\right\|_{L^2} \|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}.$$

As previously, we divide by $\|\Delta_j(u^{(m+1)} - u^{(m)})\|_{L^2}$, multiply by $2^{j(\frac{N}{2}-1)}$, and sum over $j \in \mathbb{Z}$. Then we have

$$\frac{d}{dt}\|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + C\mu\|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}$$
$$\leq C\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$$
$$+ C\sum_{k=1}^{N} \left\|[A_k(u^{(m)}) - A_k(u^{(m-1)})]\frac{\partial u^{(m)}}{\partial x_k}\right\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$$
$$\leq C\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$$
$$+ C\|A_k(u^{(m)}) - A_k(u^{(m-1)})\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}$$
$$\leq C\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}$$
$$+ C\|u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \|u^{(m)} - u^{(m-1)}\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}, \tag{4.21}$$

where we used (2.10) and the uniform estimate of $\{u^{(m)}\}$ in $C([0,\infty); \dot{B}_{2,1}^{\frac{N}{2}})$ proved in the step (i) above. Taking the L^2 inner product (4.15) with

$u^{(m+1)} - u^{(m)}$, we have after integration by parts

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2}^2 + \mu\big\|\nabla(u^{(m+1)} - u^{(m)})\big\|_{L^2}^2 \\
&= -\frac{1}{2}\sum_{k=1}^{N}\Big((u^{(m+1)} - u^{(m)})\frac{\partial A_k(u^{(m)})}{\partial x_k}, (u^{(m+1)} - u^{(m)})\Big)_{L^2} \\
&+ \sum_{k=1}^{N}\Big(\Big\{\Big[A_k(u^{(m)}) - A_k(u^{(m-1)})\Big]\,\frac{\partial u^{(m)}}{\partial x_k}\Big\}, u^{(m+1)} - u^{(m)}\Big)_{L^2} \\
&\le \frac{1}{2}\sum_{k=1}^{N}\Big\|\frac{\partial A_k(u^{(m)})}{\partial x_k}\Big\|_{L^\infty}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2}^2 \\
&+ \sum_{k=1}^{N}\big\|A_k(u^{(m)}) - A_k(u^{(m-1)})\big\|_{L^2}\Big\|\frac{\partial u^{(m)}}{\partial x_k}\Big\|_{L^\infty}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2} \\
&\le C\sum_{k=1}^{N}\big\|A_k(u^{(m)})\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2}^2 \\
&+ C\sum_{k=1}^{N}\big\|A_k(u^{(m)}) - A_k(u^{(m-1)})\big\|_{L^2}\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2} \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2}^2 \\
&+ C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m)} - u^{(m-1)}\big\|_{L^2}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2}. \qquad (4.22)
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2} + \mu\big\|\nabla(u^{(m+1)} - u^{(m)})\big\|_{L^2} \qquad (4.23) \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{L^2} + C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m)} - u^{(m-1)}\big\|_{L^2}.
\end{aligned}
$$

Adding (4.20) to (4.18), we have

$$
\begin{aligned}
&\frac{d}{dt}\|u^{(m+1)} - u^{(m)}\|_{B_{2,1}^{\frac{N}{2}-1}} + C\mu\|u^{(m+1)} - u^{(m)}\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}} \qquad (4.24) \\
&\le C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m+1)} - u^{(m)}\big\|_{B_{2,1}^{\frac{N}{2}-1}} + C\big\|u^{(m)}\big\|_{\dot{B}_{2,1}^{\frac{N}{2}+1}}\big\|u^{(m)} - u^{(m-1)}\big\|_{B_{2,1}^{\frac{N}{2}-1}}.
\end{aligned}
$$

By Gronwall's lemma we obtain

$$\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)-u^{(m)}(t)\right\|_{B^{\frac{N}{2}-1}_{2,1}}+C_3\mu\int_0^\infty\left\|u^{(m+1)}-u^{(m)}\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt$$
$$\le\left\|u_0^{(m+1)}-u_0^{(m)}\right\|_{B^{\frac{N}{2}-1}_{2,1}}\exp\left(C\int_0^\infty\left\|u^{(m)}(t)\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt\right)$$
$$+C\sup_{0\le t<\infty}\left\|u^{(m)}(t)-u^{(m-1)}(t)\right\|_{B^{\frac{N}{2}-1}_{2,1}}\int_0^\infty\left\|u^{(m)}(t)\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt$$
$$\times\exp\left(C\int_0^\infty\left\|u^{(m)}(t)\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt\right)$$
$$\le C2^{(\frac{N}{2}-1)m}\left\|\Delta_m u_0\right\|_{L^2}\exp\left(\frac{C_4}{\mu}\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}\right)$$
$$+C_5\Big[\sup_{0\le t<\infty}\left\|u^{(m)}(t)-u^{(m-1)}(t)\right\|_{B^{\frac{N}{2}-1}_{2,1}}+C_3\mu\int_0^\infty\left\|u^{(m+1)}-u^{(m)}\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt\Big]$$
$$\times\frac{\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}}{\mu}\exp\left(\frac{C_4}{\mu}\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}\right).\tag{4.25}$$

Thus, if $\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}$ is so small (or, μ is so large) that

$$\frac{C_5}{\mu}\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}\exp\left(\frac{C_4}{\mu}\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}\right)\le\frac{1}{2};\tag{4.26}$$

then we have

$$\delta_{m+1}\le C\alpha_m+\frac{1}{2}\delta_m,\tag{4.27}$$

where we set

$$\delta_{m+1}=\sup_{0\le t<\infty}\left\|u^{(m+1)}(t)-u^{(m)}(t)\right\|_{B^{\frac{N}{2}-1}_{2,1}}+C_3\mu\int_0^\infty\left\|u^{(m+1)}-u^{(m)}\right\|_{\dot B^{\frac{N}{2}+1}_{2,1}}dt,$$

and

$$\alpha_m=2^{(\frac{N}{2}-1)m}\left\|\Delta_m u_0\right\|_{L^2}.$$

Since

$$\sum_{m=1}^\infty\alpha_m\le\sum_{m=-\infty}^\infty 2^{(\frac{N}{2}-1)m}\left\|\Delta_m u_0\right\|_{L^2}=\left\|u_0\right\|_{\dot B^{\frac{N}{2}-1}_{2,1}}<\infty,$$

we deduce easily that $\delta_m\to 0$ as $m\to\infty$, and thus $\{u^{(m)}\}$ is a Cauchy sequence, and there exists u in $L^\infty([0,\infty);B^{\frac{N}{2}-1}_{2,1})\cap L^1([0,\infty);\dot B^{\frac{N}{2}+1}_{2,1})$, which is the limit of the sequence $\{u^{(m)}\}$, and a solution of the system (1.12)-(1.13).

The fact that $u \in C([0,\infty); B^{\frac{N}{2}-1}_{2,1})$ follows by the convergence, $u^{(m)} \to u$ in $L^\infty([0,\infty); B^{\frac{N}{2}-1}_{2,1})$, and applying the three ε argument. For the proof of uniqueness we suppose two solutions

$$u, v \in L^\infty([0,\infty); B^{\frac{N}{2}-1}_{2,1}) \cap L^1([0,\infty); \dot{B}^{\frac{N}{2}+1}_{2,1})$$

correspond to the same initial data $u_0 = v_0$. We set

$$\delta = \sup_{0\le t<\infty} \|u(t) - v(t)\|_{B^{\frac{N}{2}-1}_{2,1}} + C_3\mu \int_0^\infty \|u(t) - v(t)\|_{\dot{B}^{\frac{N}{2}+1}_{2,1}} dt.$$

Following the same procedure as the proof of the contraction part of $\{u^{(m)}\}$ above, we obtain $\delta \le \frac{1}{2}\delta$, instead of (4.24), if $\|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}}$ is so small that

$$C_5 \|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}} \exp\Big(C_4 \|u_0\|_{\dot{B}^{\frac{N}{2}-1}_{2,1}}\Big) \le \frac{1}{2}.$$

Thus we have $\delta = 0$, and $u = v$. This completes the proof of Theorem 1.2. □

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