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EXISTENCE OF SOLUTIONS FOR STRONGLY NONLINEAR ELLIPTIC DIFFERENTIAL INCLUSIONS WITH UNILATERAL CONSTRAINTS

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Abstract. We study a nonlinear elliptic differential inclusion involving monotone and nonmonotone terms. Our formulation incorporates variational-hemivariational inequalities. The problem is strongly nonlinear in the sense that on the nonmonotone multivalued term we do not impose any growth condition and instead we use a generalized sign condition. Our approach is based on methods and techniques from multivalued analysis and from the theory of nonlinear operators of monotone type.

1. Introduction

In this paper we study the following nonlinear elliptic differential inclusion:

$$\begin{cases} -\operatorname{div} a(z,x(z),Dx(z)) + \beta(x(z)) + F(z,x(z)) \ni h(z) & \text{a.e. on } Z \\ x|_{\partial Z} = 0. \end{cases} \tag{1}$$

Here $Z \subset \mathbb{R}^N$ is a bounded domain with a C^1-boundary ∂Z, $a : Z\times\mathbb{R}\times\mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ and $F : Z \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ are multifunctions, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone map (hence β is the subdifferential in the sense of convex analysis of a lower semicontinuous, convex, $\mathbb{R} \cup \{+\infty\}$-valued function, $\not\equiv +\infty$), and $h \in L^p(Z)$, with $2 \le p < +\infty$. The multivalued term $-\operatorname{div} a(z,x(z),Dx(z))$ in (1) is understood in the following sense. For every $x \in W_0^{1,p}(Z)$, let

$$S^q_{a(\cdot,x(\cdot),Dx(\cdot))} = \{v \in L^q(Z,\mathbb{R}^N) \,:\, v(z) \in a(z,x(z),Dx(z)) \text{ a.e. on } Z\},$$

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where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\operatorname{div} a(\cdot, x(\cdot), Dx(\cdot)) = \{\operatorname{div} v \ : \ v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}\}.$$

From the representation theorem for $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$, we know that for every $v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}$ we have $\operatorname{div} v \in W^{-1,q}(Z)$.

Our hypotheses on $a(z, x, \xi)$ are such that we include several interesting generalizations of the p-Laplacian differential operator. The multifunction F can be of the form $F(z, x) = \partial k(z, x)$, where $k(z, x)$ is a function measurable in $z \in Z$ and locally Lipschitz in $x \in \mathbb{R}$, and $\partial k(z, \cdot)$ stands for the generalized (Clarke) subdifferential of the locally Lipschitz function $k(z, \cdot)$. So problem (1) incorporates as a special case the so-called "variational-hemivariational inequalities". Such problems have been studied recently, primarily for equations driven by the Laplacian differential operator and with β being the subdifferential of an indicator function; see Dinca-Panagiotopoulos-Pop [5], Goeleven, Motreanu [7], Goeleven, Motreanu, Panagiotopoulos [8], Kyritsi-Papageorgiou [10], Motreanu, Papageorgiou [13], and Filippakis, Papageorgiou [6] (the last three works involve the p-Laplacian differential operator). We should mention that the hemivariational inequalities are a new type of variational expressions, which arise in physical and engineering problems, when we deal with nonsmooth and nonconvex energy functionals. Such functionals appear if one wants to consider more realistic mechanical laws of nonmonotone and multivalued nature. For concrete applications we refer to Naniewicz-Panagiotopoulos [14]. Another related work is that of Carl-Motreanu [2], which studies a general nonlinear elliptic hemivariational inequality. However, their conditions on a (single-valued) function are more restrictive, $\operatorname{dom} \beta = \mathbb{R}$ (so their formulation excludes variational constraints) and there is a growth restriction on $j(z, x)$. In contrast, here we impose no growth restriction on the multifunction $F(z, x)$. We only employ a generalized sign condition and this makes problem (1) a strongly nonlinear problem. Our approach uses notions and techniques from nonlinear operator theory, multivalued analysis and nonsmooth analysis. For the general background on these subjects we refer to Barbu [1] and Denkowski, Migorski, and Papageorgiou [3] and [4].

2. MATHEMATICAL BACKGROUND

Let Y, V be two Hausdorff topological spaces. A multifunction $G : Y \to 2^V \setminus \{\emptyset\}$ is said to be upper semicontinuous (u.s.c., for short) if for every $C \subseteq V$ closed, the set $G^-(C) = \{y \in Y \ : \ G(y) \cap C \neq \emptyset\}$ is closed in Y (equivalently, for every $U \subseteq V$ open, the set $G^+(U) = \{y \in Y \ : \ G(y) \subseteq U\}$

is open in Y). Similarly, we say that G is lower semicontinuous (l.s.c., for short) if for every $C \subseteq V$ closed, the set $G^+(C)$ is closed in Y (equivalently, for every $U \subseteq V$ open, the set $G^-(U)$ is open in Y). If V is regular or if G has compact values, then the upper semicontinuity of G implies that G is closed; i.e., the graph $\operatorname{Gr} G = \{(y,v) \in Y \times V \,:\, v \in G(y)\}$ is closed. Also, if V is a metric space with metric d, then G is l.s.c. if and only if for all $v \in V$ the function $y \mapsto d(v, G(y))$ is upper semicontinuous on Y. When V is a metric space with d its metric, then for $C, E \subset V$ we set

$$h^*(C,E) = \sup\{d(c,E) \,:\, c \in C\}$$

and

$$h(C,E) = \max\{h^*(C,E), h^*(E,C)\}.$$

We recall that $h(C,E)$ is the Hausdorff distance between C and E. If V is complete, then the family of nonempty closed subsets of V endowed with the Hausdorff metric is a complete metric space. For V a metric space, we say that $G : Y \to 2^V \setminus \{\emptyset\}$ is h-l.s.c. (respectively, h-u.s.c.) if for every $y_0 \in Y$ the function $y \mapsto h^*(G(y_0), G(y))$ (respectively, the function $y \mapsto h^*(G(y), G(y_0))$) is continuous at $y_0 \in Y$. If G is both h-l.s.c. and h-u.s.c., then we say that G is h-continuous. In general, one has u.s.c. implies h-u.s.c. and h-l.s.c. implies l.s.c., and the reverse implications need not hold. However, if G has compact values, then u.s.c. if and only if h-u.s.c. and l.s.c. if and only if h-l.s.c.

In what follows, for Y a Hausdorff topological space, we will use the following notation:

$$P_f(Y) = \{C \subseteq Y \,:\, C \text{ is nonempty and closed}\}$$

and

$$P_k(Y) = \{C \subseteq Y \,:\, C \text{ is nonempty and compact}\}.$$

If Y is a normed space, then we set

$$P_{fc}(Y) = \{C \in P_f(Y) \,:\, C \text{ is convex}\}$$

and

$$P_{(w)kc}(Y) = \{C \subset Y \,:\, C \text{ is nonempty, (weakly-)compact and convex}\}.$$

Now let (Ω, Σ) be a measurable space and X a separable metric space. We recall that a multifunction $G : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be graph measurable if $\operatorname{Gr} G = \{(\omega, x) \in \Omega \times X \,:\, x \in G(\omega)\} \in \Sigma \times \mathcal{B}(X)$, with $\mathcal{B}(X)$ being the Borel σ-field, while a multifunction $G : \Omega \to P_f(X)$ is said to be measurable if for all $x \in X$, the function $\omega \mapsto d(x, G(\omega))$ is measurable. For $P_f(X)$-valued multifunctions, measurability implies graph measurability and the converse is true if there is a σ-finite measure μ on (Ω, Σ) such that Σ is μ-complete.

Let (Ω, Σ, μ) be a finite measure space, X a separable Banach space and $G : \Omega \to 2^X \setminus \{\emptyset\}$ a multifunction. For $1 \le p \le +\infty$ we define

$$S_G^p = \{g \in L^p(\Omega, X) \,:\, g(\omega) \in G(\omega) \ \mu\text{-a.e. on } \Omega\}.$$

This set may be empty. For a graph-measurable multifunction G, it is nonempty if and only if $\inf\{\|x\| \,:\, x \in G(\omega)\} \le \varphi(\omega)$ μ almost everywhere on Ω, with $\varphi \in L^p(\Omega)_+$.

Now let X be a reflexive Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). For an operator $A : D(A) \subseteq X \to 2^{X^*}$ the domain $D(A)$ is the set $D(A) = \{x \in X : A(x) \neq \emptyset\}$, the range is the set $R(A) = \bigcup_{x \in D(A)} A(x)$ and the graph is the set $\operatorname{Gr} A = \{(x, x^*) \in X \times X^* \,:\, x^* \in A(x)\}$. We recall that the operator $A : D(A) \subseteq X \to 2^{X^*}$ is said to be monotone if for all $(x, x^*), (y, y^*) \in \operatorname{Gr} A$ we have $\langle x^* - y^*, x - y \rangle \ge 0$. If $\langle x^* - y^*, x - y \rangle = 0$ implies that $x = y$, then we say that A is strictly monotone. The operator A is said to be maximal monotone if $\langle x^* - y^*, x - y \rangle \ge 0$ for all $(x, x^*) \in \operatorname{Gr} A$ implies that $(y, y^*) \in \operatorname{Gr} A$. This is equivalent to saying that the graph of A is maximal with respect to inclusion among the graphs of all monotone operators. The graph $\operatorname{Gr} A$ of a maximal monotone operator is sequentially closed in $X \times X^*_{\mathrm{w}}$ and in $X_{\mathrm{w}} \times X^*$ (here by X_{w} and X^*_{w} we denote the spaces X and X^* furnished with their respective weak topologies). A monotone operator is locally bounded in the interior of $D(A)$ (this interior may be empty) and $A|_{\operatorname{int} D(A)}$ is u.s.c. from X with the norm topology into X^*_{w}. An operator $A : D(A) \subseteq X \to 2^{X^*}$ is said to be coercive if $D(A)$ is bounded or $D(A)$ is unbounded and $\inf\{\|x^*\| \,:\, x^* \in A(x)\} \to +\infty$ as $\|x\| \to +\infty$. A maximal monotone, coercive operator is surjective. If A is monotone, $D(A) = X$ and for all $x, y \in X$, $\lambda \mapsto A(x + \lambda y)$ is u.s.c. from $[0,1]$ into X^*_{w} then A is maximal monotone.

If $X = H$ is a Hilbert space which is identified with its dual and $A : D(A) \subseteq H \to 2^H$ is a maximal monotone operator, then for every $\lambda > 0$ we can consider the following well-known operators: the resolvent of A

$$J_\lambda = (I + \lambda A)^{-1}$$

and the Yosida approximation of A

$$A_\lambda = \frac{1}{\lambda}\,(I - J_\lambda).$$

Both operators are single valued and everywhere defined and J_λ is nonexpansive, while A_λ is Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$ and monotone (hence maximal monotone).

Let $\Gamma_0(H)$ be the cone of all proper convex, lower semicontinuous functions $\varphi : H \to \mathbb{R} \cup \{+\infty\}$. The subdifferential in the sense of convex analysis of φ is the multifunction $x \mapsto \partial\varphi(x)$ defined by

$$\partial\varphi(x) = \{x^* \in H \,:\, (x^*, y - x)_H \le \varphi(y) - \varphi(x) \ \text{ for all } y \in H\}.$$

Here by $(\cdot,\cdot)_H$ we denote the inner product of H. It is well-known that $\partial\varphi$ is a maximal monotone operator. Given $\varphi \in \Gamma_0(H)$ and $\lambda > 0$, we recall that the Moreau-Yosida approximation φ_λ of φ is introduced by

$$\varphi_\lambda(x) = \inf\{\varphi(y) + \frac{1}{2\lambda}\,\|x - y\|^2 \,:\, y \in H\}, \quad \forall x \in H.$$

It is known that $\varphi_\lambda \uparrow \varphi$ as $\lambda \downarrow 0$ and φ_λ is convex and Fréchet differentiable. Moreover, we have $\partial\varphi_\lambda = (\partial\varphi)_\lambda$.

Next, we list some facts concerning pseudomonotone operators. For a reflexive Banach space X, an operator $A : X \to 2^{X^*}$ is said to be pseudomonotone if

(a) for every $x \in X$, $A(x) \in P_{wkc}(X^*)$;
(b) A is u.s.c. from every dimensional subspace W of X into X^*_{w};
(c) if $x_n \xrightarrow{\mathrm{w}} x$ in X, $x_n^* \in A(x_n)$ for all $n \ge 1$, and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \le 0$, then for every $y \in X$ we can find $x^*(y) \in A(y)$ such that

$$\langle x^*(y), x - y\rangle \le \liminf_{n\to\infty}\langle x_n^*, x_n - y\rangle.$$

If A is bounded (i.e., maps bounded sets to bounded sets) and satisfies condition (c), then it fulfills also condition (b). An operator $A : D(A) \subseteq X \to 2^{X^*}$ is said to be generalized pseudomonotone if for all $x_n \xrightarrow{\mathrm{w}} x$ in X and $x_n^* \xrightarrow{\mathrm{w}} x^*$ in X^* such that $(x_n, x_n^*) \in \operatorname{Gr} A$ and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \le 0$, we have $(x, x^*) \in \operatorname{Gr} A$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

The next proposition summarizes some important facts concerning nonlinear operators of monotone type.

Proposition 1. *If X is a reflexive Banach space and $A : D(A) \subseteq X \to 2^{X^*}$, then*

(a) *if A is maximal monotone, it is also generalized pseudomonotone;*
(b) *if A is pseudomonotone, it is also generalized pseudomonotone;*
(c) *if A is generalized pseudomonotone and bounded and for every $x \in X$, $A(x) \in P_{wkc}(X^*)$, then A is pseudomonotone;*

(d) *if A is pseudomonotone and coercive, then A is surjective* (*i.e.*, $R(A) = X^*$);

(e) *the sum of pseudomonotone operators is pseudomonotone.*

Finally, if $S : X \to X^*$, then we say that S is completely continuous if $x_n \xrightarrow{w} x$ in X implies that $S(x_n) \to S(x)$ in X^*. Clearly, this ensures that S is continuous and maps bounded subsets of X into relatively compact subsets of X^*.

3. Preliminary results

Our hypotheses on the data of problem (1) are the following:

H(a) $a : Z \times \mathbb{R} \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) $(z, x, \xi) \mapsto a(z, x, \xi)$ is graph measurable;

(ii) for almost all $z \in Z$ and all $x \in \mathbb{R}$ the mapping $\xi \mapsto a(z, x, \xi)$ is maximal monotone and $0 \in a(z, x, 0)$, for almost all $z \in Z$ and all $\xi \in \mathbb{R}^N$ the mapping $x \mapsto a(z, x, \xi)$ is l.s.c., and for almost all $z \in Z$ the mapping $(x, \xi) \mapsto a(z, x, \xi)$ has a closed graph;

(iii) for almost all $z \in Z$, all $(x, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and all $v \in a(z, x, \xi)$, we have

$$\|v\| \le a_1(z) + c_1(|x|^{p-1} + \|\xi\|^{p-1})$$

with $a_1 \in L^\infty(Z)_+$, $c_1 > 0$ and $2 \le p < +\infty$;

(iv) for almost all $z \in Z$, all $(x, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and all $v \in a(z, x, \xi)$, we have

$$(v, \xi)_{\mathbb{R}^N} \ge c_2\|\xi\|^p - a_2(z),$$

with $a_2 \in L^1(Z)_+$, $c_2 > 0$.

Remark. Suppose that $\theta : Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$) and assume that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$0 < \eta_1 \le \theta(z, x) \le \eta_2(z) \quad \text{with } \eta_2 \in L^\infty(Z)_+ .$$

If $\varphi(z, x, \xi) = \frac{1}{p}\,\theta(z, x)\|\xi\|^p$ and $a(z, x, \xi) = \partial_\xi \varphi(z, x, \xi) = \theta(z, x)\|\xi\|^{p-2}\xi$, then a satisfies hypotheses H(a) and we have a generalization of the p-Laplacian differential operator. Keeping $\theta(z, x)$ as above, other possibilities of $a(z, x, \xi) = \partial_\xi \varphi(z, x, \xi)$ are with the following different choices for φ:

$$\varphi(z, x, \xi) = \theta(z, x)\Big[\frac{1}{p}\,\|\xi\|^p + \frac{1}{r}\,\|\xi\|^r\Big], \quad 1 \le r < p,$$

$$\varphi(z, x, \xi) = \frac{\theta(z, x)}{p}\,\|\xi\|^p + \|\xi\| \ln(1 + \|\xi\|),$$

$$\varphi(z,x,\xi)=\frac{\theta(z,x)}{p}\Big[(1+\|\xi\|^2)^{\frac{p}{2}}-1\Big],$$

$$\varphi(z,x,\xi)=\frac{\theta(z,x)}{2}\,(A\xi,\xi)_{\mathbb{R}^N},\ A \text{ an } N\times N\text{-symmetric matrix, } A>0,\ p=2$$

$$\varphi(z,x,\xi)=\begin{cases}\frac{\theta(z,x)}{p}\,\|\xi\|^p & \text{if } \|x\|\le 1\\ \theta(z,x)\left[\frac{1}{p}\,\|\xi\|^p+\frac{1}{r}\,\|\xi\|^r\right]-\frac{\theta(z,x)}{r} & \text{if } \|x\|>1\end{cases},\quad 1\le r<p.$$

H(β) $\beta=\partial j$ with $j\in\Gamma_0(\mathbb{R})$, $j\ge 0$ and $j(0)=0$.

Remark. Hypothesis H(β) implies that $0\in\partial j(0)=\beta(0)$ and it is easy to check that for all $\lambda>0$ we have $j_\lambda(0)=0$ and $0=\partial j_\lambda(0)=j'_\lambda(0)$. Note that we do not require that $\operatorname{dom}\beta=\mathbb{R}$. This incorporates in our framework problems with convex constraints (variational inequalities).

H(F) $F: Z\times\mathbb{R}\to P_{kc}(\mathbb{R})$ is a multifunction such that
(i) $(z,x)\mapsto F(z,x)$ is Borel measurable;
(ii) for almost all $z\in Z$, $x\mapsto F(z,x)$ is u.s.c.;
(iii) for every $r>0$, there exists $\theta_r\in L^p(Z)$ such that for almost all $z\in Z$, all $|x|\le r$ and all $u\in F(z,x)$, we have $|u|\le\theta_r(z)$;
(iv) there exists $M>0$ such that for almost all $z\in Z$, we have

$$x\,\min\{u\,:\ u\in F(z,x)\}\ge 0\quad\text{if } x\le -M$$

and

$$x\,\max\{u\,:\ u\in F(z,x)\}\ge 0\quad\text{if } x\ge M.$$

Remark. Let $j: Z\times\mathbb{R}\to\mathbb{R}$ be a function such that for all $x\in\mathbb{R}$, $z\mapsto j(z,x)$ is measurable and for all $r>0$ there exists $k_r\in L^p(Z)$ such that $|j(z,x)-j(z,y)|\le k_r(z)|x-y|$ almost everywhere on Z for all $|x|,|y|\le r$ (in particular, for almost all $z\in Z$, $j(z,\cdot)$ is locally Lipschitz). Let $F(z,x)=\partial j(z,x)$, where $\partial j(z,\cdot)$ denotes the generalized (Clarke) subdifferential of $j(z,\cdot)$ (on the Lebesgue-null subset of Z where $j(z,\cdot)$ may fail to be locally Lipschitz, simply set $F(z,x)=\{0\}$ for all $x\in\mathbb{R}$). Recall that

$$\sigma(h;F(z,x))=\sup\{hu\,:\ u\in F(z,x)\}=j^0(z,x;h).$$

Here $j^0(z,x;h)$ is the generalized directional derivative at $x\in\mathbb{R}$ in the direction $h\in\mathbb{R}$ of the locally Lipschitz function $j(z,\cdot)$. From the definition of the generalized directional derivative, we have

$$j^0(z,x;h) = \inf_{m\geq 1} \sup_{\substack{r,s\in\mathbb{Q} \\ |r|,|s|\leq \frac{1}{m}}} \frac{j(z,x+r+sh)-j(z,x+r)}{s}.$$

Since $(z,x)\mapsto j(z,x)$ is measurable on $Z\times\mathbb{R}$, it follows that, for all $h\in\mathbb{R}$, the function $(z,x)\mapsto j^0(z,x;h)=\sigma(h;F(z,x))$ is measurable on $Z\times\mathbb{R}$. So from [3, page 435], we deduce that the multifunction $(z,x)\mapsto F(z,x)$ is measurable, hence (i) in H(F) is satisfied. Also it is well known that for almost all $z\in Z$, $x\mapsto \partial j(z,x)=F(z,x)$ is u.s.c., thus (ii) holds. Moreover, for almost all $z\in Z$, all $|x|\leq r$, and all $u\in F(z,x)=\partial j(z,x)$, we have $|u|\leq \hat{k}_r(z)$ for some $\hat{k}_r\in L^p(Z)_+$, which yields (iii). Assuming that there is $M>0$ such that, for almost all $z\in Z$, we have

$$0\leq -j^0(z,x;-x) \;\text{ if } x\leq -M\,,\quad 0\leq j^0(z,x;x) \;\text{ if } x\geq M,$$

then one obtains that $F(z,x)=\partial j(z,x)$ satisfies the generalized sign condition in hypothesis H(j)(iv). So the multifunction $F(z,x)=\partial j(z,x)$ fulfills all the hypotheses in H(F) and we see that our formulation incorporates as a special case hemivariational inequalities verifying this type of sign condition. We would like also to emphasize that we have not imposed any growth condition on the multifunction F. Instead we employ the generalized sign condition in hypothesis H(j)(iv). Problems with nonlinearities exhibiting this general behavior are known in the literature as "strong nonlinear problems" and were first studied by Hess [9].

We start with a result which establishes the existence of an approximate Carathéodory selection of $\operatorname{conv} F$, which still satisfies the generalized sign condition.

Proposition 2. *If $F: Z\times\mathbb{R}\to P_{kc}(\mathbb{R})$ is a multifunction which satisfies hypotheses* H(F)*, then, given $\varepsilon>0$, we can find $g_\varepsilon: Z\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function* (*i.e., for all $x\in\mathbb{R}$, $z\mapsto g_\varepsilon(z,x)$ is measurable and for almost all $z\in Z$, $x\mapsto g_\varepsilon(z,x)$ is continuous*) *such that*

(a) *for almost all $z\in Z$ and all $x\in\mathbb{R}$, we have*

$$g_\varepsilon(z,x)\in \operatorname{conv} F(z,x+\varepsilon\bar{B}_1)+\varepsilon\bar{B}_1$$

with $\bar{B}_1=[-1,1]$;

(b) *for every $r>0$, there exists $a_r\in L^p(Z)$ such that for almost all $z\in Z$ and all $|x|\leq r$, we have $|g_\varepsilon(z,x)|\leq a_r(z)$ and $a_r\in L^p(Z)$ can be chosen independent of $\varepsilon\in(0,1]$; and*

(c) *for almost all $z\in Z$ and all $|x|\geq M+1$, we have $xg_\varepsilon(z,x)\geq 0$.*

Proof. Let

$$\xi_1(z,x) = \min\{u \,:\, u \in F(z,x)\} \quad \text{and} \quad \xi_2(z,x) = \max\{u \,:\, u \in F(z,x)\}.$$

For every $\eta > 0$, we have that $F_1(z,x) = F(z,x) \cap (-\infty, \eta]$ is compact, possibly empty, and $\mathrm{Gr}\, F_1 \in \mathcal{B}(Z) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ with $\mathcal{B}(Z)$ (respectively, $\mathcal{B}(\mathbb{R})$) being the Borel σ-field of Z (respectively, of $\mathbb{R}$). This is a consequence of hypothesis H(F)(i). Then we have

$$\{(z,x) \in Z \times \mathbb{R} \,:\, \xi_1(z,x) \leq \eta\} = \mathrm{proj}_{Z\times\mathbb{R}} \mathrm{Gr}\, F_1 \in \mathcal{B}(Z \times \mathbb{R}) = \mathcal{B}(Z) \times \mathcal{B}(\mathbb{R})$$

(see [3, page 216]). So, $(z,x) \mapsto \xi_1(z,x)$ is Borel measurable on $Z \times \mathbb{R}$. In a similar fashion we can show the Borel measurability of $(z,x) \mapsto \xi_2(z,x)$. Moreover, for almost all $z \in Z$, we have that

$$x \mapsto \xi_1(z,x) \text{ is lower semicontinuous}$$

and

$$x \mapsto \xi_2(z,x) \text{ is upper semicontinuous.}$$

By redefining $\xi_1(\cdot,x)$ and $\xi_2(\cdot,x)$, $x \in \mathbb{R}$, on a Lebesgue-null set, we may assume that the above properties are true for all $z \in Z$.

Now with $M > 0$ in H(F)(iv) we define

$$\hat{\xi}_1(z,x) = \begin{cases} \xi_1(z,x) & \text{if } x \leq M \\ \xi_1(z,x)^+ & \text{if } x > M \end{cases}$$

and

$$\hat{\xi}_2(z,x) = \begin{cases} -\xi_2(z,x)^- & \text{if } x < -M \\ \xi_2(z,x) & \text{if } x \geq -M. \end{cases}$$

Evidently, both $\hat{\xi}_1$ and $\hat{\xi}_2$ are measurable functions and for all $z \in Z$, $x \mapsto \hat{\xi}_1(z,x)$ is lower semicontinuous, while $x \mapsto \hat{\xi}_2(z,x)$ is upper semicontinuous. We set

$$\hat{F}(z,x) = \left[\hat{\xi}_1(z,x), \hat{\xi}_2(z,x)\right] = \{y \in \mathbb{R} \,:\, \hat{\xi}_1(z,x) \leq y \leq \hat{\xi}_2(z,x)\}.$$

Note that $\hat{F}$ is measurable and, for all $z \in Z$, $x \mapsto \hat{F}(z,x)$ is u.s.c. Moreover, from the definitions of $\hat{\xi}_1$ and $\hat{\xi}_2$ and H(F)(iv), it is clear that

$$\xi_1(z,x) \leq \hat{\xi}_1(z,x) \leq \hat{\xi}_2(z,x) \leq \xi_2(z,x) \quad \text{for all } (z,x) \in Z \times \mathbb{R},$$

hence $\hat{F}(z,x) \subseteq F(z,x)$ for all $(z,x) \in Z \times \mathbb{R}$.

We fix $z \in Z$. We know from [3, page 440] that, given $\delta > 0$, we can find a continuous map $h_\delta : \mathbb{R} \to \mathbb{R}$ such that

$$h_\delta(x) \in \hat{F}(z, x + \delta B_1) + \delta B_1 \quad \text{for all } x \in \mathbb{R}, \text{ with } B_1 = (-1,1). \tag{2}$$

Choose $\delta \leq \min\{\varepsilon, \frac{1}{2}\}$. Then, we have

$$\begin{aligned}\sup\{u\,:\, u \in \hat{F}(z, x+\delta B_1)\} &= \sup\{u\,:\, u \in \hat{\xi}_2(z, x+\delta B_1)\}\\ &\leq 0 \quad \text{if } x < -\big(M+\frac{1}{2}\big)\end{aligned}$$

and

$$\begin{aligned}\inf\{u\,:\, u \in \hat{F}(z, x+\delta B_1)\} &= \inf\{u\,:\, u \in \hat{\xi}_1(z, x+\delta B_1)\}\\ &\geq 0 \quad \text{if } x > M+\frac{1}{2}\,.\end{aligned}$$

We show that there exists a continuous function $\hat{h}_\delta : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$x\hat{h}_\delta(x) \geq 0 \;\text{ for all } x \text{ with } |x| \geq M+1 \tag{3}$$

and

$$\hat{h}_\delta(x) \in \hat{F}(z, x+\delta B_1) + \delta B_1 \subseteq F(z, x+\delta B_1) + \delta B_1 \;\text{ for all } x \in \mathbb{R}\,. \tag{4}$$

The construction of the function $\hat{h}_\delta$ can be done as follows. First, we define

$$\begin{aligned}&\hat{h}_\delta(x) = -h_\delta(x)^- \;\text{ if } x \leq -(M+1)\\ &\hat{h}_\delta(x) = h_\delta(x) \;\text{ if } x \in \Big[-\big(M+\frac{1}{2}\big), M+\frac{1}{2}\Big]\\ &\hat{h}_\delta(x) = h_\delta(x)^+ \;\text{ if } x \geq M+1.\end{aligned}$$

It is clear that (3) is verified and for any $x \in \mathbb{R}$ there is $\zeta_x \in B_1$ such that $h_\delta(x) \in \hat{F}(z, x+\delta\zeta_x) + \delta B_1$ (see (2)). We also remark that if $x > M+\frac{1}{2}$ and $h_\delta(x) < 0$, then, since $\inf\{u\,:\, u \in \hat{F}(z, x+\delta\zeta_x)\} \geq 0$, one has

$$[h_\delta(x), 0] \subset \hat{F}(z, x+\delta\zeta_x) + \delta B_1\,. \tag{5}$$

Now, we define $\hat{h}_\delta$ on the interval $\big(M+\frac{1}{2}, M+1\big)$ in the following way. If $h_\delta(M+\frac{1}{2}) \geq 0$, put

$$\hat{h}_\delta(x) = h_\delta(x)^+ \;\text{ if } x \in \big(M+\frac{1}{2}, M+1\big)$$

(thus the continuity of $\hat{h}_\delta$ is assured and (4) is satisfied by (5) and (2)). Assume now $h_\delta(M+\frac{1}{2}) < 0$. If there is $x_0 \in \big(M+\frac{1}{2}, M+1\big)$ such that $h_\delta(x_0) = 0$, we define

$$\begin{aligned}&\hat{h}_\delta(x) = h_\delta(x) \;\text{ if } x \in \big(M+\frac{1}{2}, x_0\big)\\ &\hat{h}_\delta(x) = h_\delta(x)^+ \;\text{ if } x \geq x_0.\end{aligned}$$

Also in this situation, $\hat{h}_\delta$ satisfies the required properties. Finally, if $h_\delta(x) < 0$ for all $x \in \big(M+\frac{1}{2}, M+1\big)$, we define

$$\hat{h}_\delta(x) = \theta(x)h_\delta(x), \quad \forall x \in \big(M+\frac{1}{2}, M+1\big),$$

where $\theta : \big[M+\frac{1}{2}, M+1\big] \to \mathbb{R}$ is a continuous function with $\theta(M+\frac{1}{2}) = 0$, $\theta(M+1) = 1$, and $0 \le \theta \le 1$. Taking into account (5) and observing $\hat{h}_\delta(x) = \theta(x)h_\delta(x) \in [h_\delta(x), 0]$, we have (4) for $x \in (M+\frac{1}{2}, M+1)$. In addition, $\hat{h}_\delta$ is continuous. With a similar construction on the interval $\big(-(M+1), -\big(M+\frac{1}{2}\big)\big)$, we obtain $\hat{h}_\delta$ on $\mathbb{R}$ with the desired properties.

Next we will choose a continuous approximate selection of the multifunction F which will depend measurably in $z \in Z$. For this purpose, we introduce the multifunction $H_\varepsilon : Z \to 2^{C(\mathbb{R})} \setminus \{\emptyset\}$ defined by

$$\begin{aligned} H_\varepsilon(z) = \{h \in C(\mathbb{R}) \, : h(x) \in \operatorname{conv} F(z, x + \varepsilon\bar{B}_1) + \varepsilon\bar{B}_1 \text{ for all } x \in \mathbb{R} \\ \text{and } xh(x) \ge 0 \text{ for all } |x| \ge M+1\}. \end{aligned}$$

From the previous argument, we know that for all $z \in Z$, $H_\varepsilon(z) \neq \emptyset$ (we constructed above $\hat{h}_\delta \in H_\varepsilon(z)$). Let

$$F^*(z,x) = \operatorname{conv} F(z, x+\varepsilon\bar{B}_1) + \varepsilon\bar{B}_1 \quad \text{for all } (z,x) \in Z \times \mathbb{R}\,.$$

The multifunction $F^* : Z \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ has compact and convex values (recall that $F(z,\cdot)$ is u.s.c. with compact values and so it maps compact sets to compact sets) and, for all $z \in Z$, $x \mapsto F^*(z,x)$ is u.s.c. Moreover, for all $y \in \mathbb{R}$, we have

$$\sigma(y; F^*(z,x)) = \sup\big\{\sigma(y; F(z, x+e)) \, : \, e \in \bar{B}_\varepsilon\big\} + \varepsilon, \quad \text{where } \bar{B}_\varepsilon = [-\varepsilon, \varepsilon].$$

From [3, page 499], we know that

$$z \mapsto \sup\big\{\sigma(y; F(z, x+e)) \, : \, e \in \bar{B}_\varepsilon\big\} \text{ is Lebesgue measurable.}$$

Thus, for every $x \in \mathbb{R}$, the multifunction $z \mapsto F^*(z,x)$ is Lebesgue measurable. On the other hand, we have

$$\begin{aligned} \operatorname{Gr} H_\varepsilon = \{(z,h) \in Z \times C(\mathbb{R}) \, : d(h(x), F^*(z,x)) = 0 \text{ for all } x \in \mathbb{R} \\ \text{and } xh(x) \ge 0 \text{ for all } |x| \ge M+1\}. \end{aligned}$$

Let $\{r_n\}_{n\ge1}$ be an enumeration of the rationals in $\mathbb{R}$ and $\{s_m\}_{m\ge1}$ an enumeration of the rationals in $\{x \in \mathbb{R} \, : \, |x| \ge M+1\}$. Since $F^*(z,\cdot)$ is u.s.c. and $P_{kc}(\mathbb{R})$-valued, for every $y \in \mathbb{R}$ the function $x \mapsto d(y, F^*(z,x))$ is lower semicontinuous. Hence if $h \in C(\mathbb{R})$ and $x_n \to x$ in $\mathbb{R}$, we have

$$d(h(x), F^*(z,x)) \le \liminf_{n\to\infty} d(h(x), F^*(z,x_n))$$

and

$$|d(h(x), F^*(z, x_n)) - d(h(x_n), F^*(z, x_n))| \le |h(x) - h(x_n)| \to 0 \quad \text{as } n \to \infty.$$

So finally we obtain

$$d(h(x), F^*(z, x)) \le \liminf_{n\to\infty} d(h(x_n), F^*(z, x_n)),$$

which means that the function $x \mapsto d(h(x), F^*(z, x))$ is lower semicontinuous. Exploiting this fact, we have

$$\operatorname{Gr} H_\varepsilon = \bigcap_{m,n\ge 1} \{(z, h) \in Z \times C(\mathbb{R}) \,:\, d(h(r_n), F^*(z, r_n)) = 0,\ s_m h(s_m) \ge 0\}.$$

Note that, for every $n \ge 1$, the function $(z, h) \mapsto d(h(r_n), F^*(z, r_n))$ from $Z \times C(\mathbb{R})$ into $\mathbb{R}_+$ is a Carathéodory function if we furnish $C(\mathbb{R})$ with the topology of uniform convergence on compacta (compact-open topology). The space $C(\mathbb{R})$ topologized in this fashion becomes a separable Fréchet space. So, being Carathéodory, the function $(z, h) \mapsto d(h(r_n), F^*(z, r_n))$ is jointly measurable. It follows that $\operatorname{Gr} H_\varepsilon \in \mathcal{L}(Z) \times \mathcal{B}(C(\mathbb{R}))$, with $\mathcal{L}(Z)$ denoting the Lebesgue σ-field of Z and $\mathcal{B}(C(\mathbb{R}))$ the Borel σ-field of $C(\mathbb{R})$. Now we may use the Yankov-von Neumann-Aumann selection theorem (see [3, page 432]) and produce a Lebesgue measurable map $\hat{h}_\varepsilon : Z \to C(\mathbb{R})$ such that $\hat{h}_\varepsilon(z) \in H_\varepsilon(z)$ for all $z \in Z$. Set $g_\varepsilon(z, x) = \hat{h}_\varepsilon(z)(x)$ for all $(z, x) \in Z \times \mathbb{R}$. Then $g_\varepsilon : Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and it satisfies the assertions (a), (b), and (c), which completes the proof. □

Next let $A : W_0^{1,p}(Z) \to 2^{W^{-1,q}(Z)}$ be the multivalued operator defined by

$$A(x) = \{-\operatorname{div} v \,:\, v \in S^q_{a(\cdot, x(\cdot), Dx(\cdot))}\}, \quad \forall x \in W_0^{1,p}(Z).$$

By virtue of hypotheses H(a)(i) and (iii), we see that

$$S^q_{a(\cdot, x(\cdot), Dx(\cdot))} \in P_{wkc}(L^q(Z, \mathbb{R}^N)).$$

Therefore, for all $x \in W_0^{1,p}(Z)$, $A(x) \in P_{wkc}(W^{-1,q}(Z))$.

Fix $x \in W_0^{1,p}(Z)$ and let the operator $K_x : W_0^{1,p}(Z) \to P_{wkc}(W^{-1,q}(Z))$ be defined by

$$K_x(y) = \{-\operatorname{div} w \,: w \in S^q_{a(\cdot, x(\cdot), Dy(\cdot))}\}, \quad \forall y \in W_0^{1,p}(Z).$$

Lemma 3. *If hypotheses* H(a) *hold, then for every* $x \in W_0^{1,p}(Z)$, $K_x : W_0^{1,p}(Z) \to P_{wkc}(W^{-1,q}(Z))$ *is maximal monotone.*

Proof. Due to assumption H(a)(ii), it suffices to show that for every $y, v \in W_0^{1,p}(Z)$, the multifunction $\lambda \mapsto K_x(y + \lambda v)$ has closed graph in $[0, 1] \times$

$W^{-1,q}(Z)_{\mathrm{w}}$ (by $W^{-1,q}(Z)_{\mathrm{w}}$ we understand the dual Sobolev space $W^{-1,q}(Z)$ equipped with the weak topology). To this end, let $\{(\lambda_n, -\operatorname{div} w_n)\}_{n\geq 1} \subset [0,1] \times W^{-1,q}(Z)$ with $w_n \in S^q_{a(\cdot,x(\cdot),D(y+\lambda_n v)(\cdot))}$, $n \geq 1$, and assume that $\lambda_n \to \lambda$ in $[0,1]$ and $-\operatorname{div} w_n \xrightarrow{\mathrm{w}} w^*$ in $W^{-1,q}(Z)$ as $n \to \infty$. Because of hypothesis H(a)(iii) we have that $\{w_n\}_{n\geq 1} \subset L^q(Z, \mathbb{R}^N)$ is bounded. Hence, by passing to a suitable subsequence if necessary, we may assume that $w_n \xrightarrow{\mathrm{w}} w$ in $L^q(Z, \mathbb{R}^N)$. Then, from [3, page 484], we have

$$\begin{aligned} w(z) &\in \operatorname{conv} \limsup_{n\to\infty} a(z, x(z), D(y + \lambda_n v)(z)) \\ &\subseteq a(z, x(z), D(y + \lambda v)(z)) \quad \text{a.e. on } Z. \end{aligned}$$

The last inclusion is a consequence of the fact that, for almost all $z \in Z$, $a(z, x(z), \cdot)$ has closed graph (see hypothesis H(a)(ii)). So

$$w \in S^q_{a(\cdot,x(\cdot),D(y+\lambda v)(\cdot))} \quad \text{and} \quad -\operatorname{div} w_n \xrightarrow{\mathrm{w}} -\operatorname{div} w \text{ in } W^{-1,q}(Z).$$

Hence $w^* = -\operatorname{div} w \in K_x(y + \lambda v)$ and we have proved the maximal monotonicity of $y \mapsto K_x(y)$. □

Using this lemma we can determine the properties of the multivalued operator $A : W_0^{1,p}(Z) \to P_{wkc}(W^{-1,q}(Z))$ defined earlier.

Proposition 4. *If hypotheses* H(a) *hold, then*

$$A : W_0^{1,p}(Z) \to P_{wkc}(W^{-1,q}(Z))$$

is pseudomonotone.

Proof. By virtue of Proposition 1(c), it suffices to show that A is generalized pseudomonotone. To this end, let $\{(x_n, v_n^*)\}_{n\geq 1} \subset \operatorname{Gr} A$ and assume that

$$x_n \xrightarrow{\mathrm{w}} x \text{ in } W_0^{1,p}(Z), v_n^* \xrightarrow{\mathrm{w}} v^* \text{ in } W^{-1,q}(Z) \quad \text{and} \quad \limsup_{n\to\infty} \langle v_n^*, x_n - x\rangle \leq 0. \tag{6}$$

Hereafter the notation $\langle \cdot, \cdot \rangle$ stands for the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. By definition we can write that $v_n^* = -\operatorname{div} v_n$ with $v_n \in S^q_{a(\cdot,x_n(\cdot),Dx_n(\cdot))}$, $n \geq 1$. Because of hypothesis H(a)(iii), $\{v_n\}_{n\geq 1} \subset L^q(Z, \mathbb{R}^N)$ is bounded and so we may assume that $v_n \xrightarrow{\mathrm{w}} v$ in $L^q(Z, \mathbb{R}^N)$. Then $-\operatorname{div} v_n \xrightarrow{\mathrm{w}} -\operatorname{div} v$ in $W^{-1,q}(Z)$ and, consequently, $v^* = -\operatorname{div} v$. We will show that $v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}$, hence $v^* = -\operatorname{div} v \in A(x)$. Let $y \in W_0^{1,p}(Z)$ and $g \in S^q_{a(\cdot,x(\cdot),Dy(\cdot))}$. For every $n \geq 1$ we introduce the multifunction $G_n : Z \to 2^{\mathbb{R}^N}$ by

$$G_n(z) = \{u \in a(z, x_n(z), Dy(z)) : \|g(z) - u\| = d(g(z), a(z, x_n(z), Dy(z)))\}.$$

Note that $G_n(z) \neq \emptyset$ almost everywhere on Z. By redefining G_n on the exceptional Lebesgue-null set, we may assume without any loss of generality that $G_n(z) \neq \emptyset$ for all $z \in Z$. We have

$$\begin{aligned}\operatorname{Gr} G_n =& \operatorname{Gr} a(\cdot, x_n(\cdot), Dy(\cdot)) \cap \\ &\{(z,u) \in Z \times \mathbb{R}^N \ : \ \|g(z) - u\| - d(g(z), a(z, x_n(z), Dy(z))) = 0\}.\end{aligned}$$

Because of hypothesis H(a)(i), we have that

$$\operatorname{Gr} a(\cdot, x_n(\cdot), Dy(\cdot)) \in \mathcal{L}(Z) \times \mathcal{B}(\mathbb{R}^N).$$

The function $(z,u) \mapsto \|g(z) - u\| - d(g(z), a(z, x_n(z), Dy(z)))$, $n \geq 1$, is a Carathéodory function, hence it is jointly measurable. Therefore $\operatorname{Gr} G_n \in \mathcal{L}(Z) \times \mathcal{B}(\mathbb{R}^N)$ and we can apply the Yankov-von Neumann-Aumann selection theorem, which produces a Lebesgue measurable function $g_n : Z \to \mathbb{R}^N$ such that $g_n(z) \in G_n(z)$ for all $z \in Z$ and all $n \geq 1$. We have

$$\begin{aligned}\|g(z) - g_n(z)\| &= d(g(z), a(z, x_n(z), Dy(z))) \\ &\leq h^*(a(z, x(z), Dy(z)), a(z, x_n(z), Dy(z))). \end{aligned} \tag{7}$$

Because $x_n \xrightarrow{\text{w}} x$ in $W_0^{1,p}(Z)$, we have $x_n \to x$ in $L^p(Z)$ (recall that $W_0^{1,p}(Z)$ is embedded compactly into $L^p(Z)$). So, by passing to a suitable subsequence if necessary, we may assume that $x_n(z) \to x(z)$ almost everywhere on Z and $|x_n(z)| \leq k(z)$ almost everywhere on Z with $k \in L^p(Z)$. From the hypothesis H(a)(ii) we have that, for almost all $z \in Z$, $a(z, \cdot, Dy(\cdot))$ is l.s.c. and since the multifunction is $P_{kc}(\mathbb{R}^N)$-valued, it is also h^*-l.s.c. Hence it follows that $h^*(a(z, x(z), Dy(z)), a(z, x_n(z), Dy(z))) \to 0$ almost everywhere on Z as $n \to \infty$. Thus if we pass to the limit as $n \to \infty$ in (7), we obtain

$$\|g(z) - g_n(z)\| \to 0 \ \text{ a.e. on } Z.$$

From the Lebesgue dominated convergence theorem (see the hypothesis H(a)(iii)) and for the original sequence (by Urysohn's criterion), we have

$$g_n \to g \ \text{ in } L^q(Z, \mathbb{R}^N) \ \text{ as } n \to \infty.$$

Exploiting the monotonicity of $a(z, x_n(z), \cdot)$ for almost all $z \in Z$ and all $n \geq 1$, we have

$$\begin{aligned}0 \leq \int_Z (v_n(z) - g_n(z), Dx_n(z) - Dy(z))_{\mathbb{R}^N}\, dz &= \langle -\operatorname{div} v_n + \operatorname{div} g_n, x_n - y\rangle \\ &= \langle v_n^*, x_n - x\rangle + \langle v_n^*, x - y\rangle + \langle \operatorname{div} g_n, x_n - y\rangle. \end{aligned} \tag{8}$$

From (6) we know that $\limsup_{n\to\infty} \langle v_n^*, x_n - x\rangle \leq 0$ and $v_n^* \xrightarrow{\text{w}} -\operatorname{div} v = v^*$ in $W^{-1,q}(Z)$. Also, since $g_n \to g$ in $L^q(Z, \mathbb{R}^N)$, we have

$$\operatorname{div} g_n \to \operatorname{div} g \ \text{ in } W^{-1,q}(Z) \text{ as } n \to \infty.$$

So if in (8) we pass to the limit as $n \to \infty$ we obtain

$$0 \le \langle -\operatorname{div} v + \operatorname{div} g, x - y \rangle. \tag{9}$$

Note that $(y, -\operatorname{div} g) \in \operatorname{Gr} K_x$ was arbitrary and from Lemma 3 we know that K_x is maximal monotone. So from (9) we infer that $(x, -\operatorname{div} v) \in \operatorname{Gr} K_x$, hence $-\operatorname{div} v \in K_x(x)$ and we conclude that $v \in S^q_{a(\cdot, x(\cdot), Dx(\cdot))}$. It turns out that $v^* \in A(x)$.

It remains to show that $\langle v_n^*, x_n \rangle \to \langle v^*, x \rangle$ as $n \to \infty$, or, equivalently, $\langle -\operatorname{div} v_n, x_n \rangle \to \langle -\operatorname{div} v, x \rangle$ as $n \to \infty$. Arguing as above with the multifunction G_n constructed with $y = x$, we can find $\hat{g}_n \in S^q_{a(\cdot, x_n(\cdot), Dx(\cdot))}$, $n \ge 1$, such that $\hat{g}_n \to v$ in $L^q(Z, \mathbb{R}^N)$, hence $-\operatorname{div} \hat{g}_n \to -\operatorname{div} v = v^*$ in $W^{-1,q}(Z)$ as $n \to \infty$. Exploiting the fact that, for almost all $z \in Z$, $a(z, x_n(z), \cdot)$ is monotone (see hypothesis H(a)(ii)), we have

$$0 \le \int_Z (v_n(z) - \hat{g}_n(z), Dx_n(z) - Dx(z))_{\mathbb{R}^N}\, dz = \langle -\operatorname{div} v_n + \operatorname{div} \hat{g}_n, x_n - x \rangle,$$

which implies that

$$\langle -\operatorname{div} \hat{g}_n, x_n - x \rangle \le \langle -\operatorname{div} v_n, x_n - x \rangle,$$

thus

$$0 \le \liminf_{n\to\infty} \langle -\operatorname{div} v_n, x_n - x \rangle = \liminf_{n\to\infty} \langle v_n^*, x_n - x \rangle.$$

Combining this inequality with (6), we conclude that

$$\langle v_n^*, x_n - x \rangle \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$\langle v_n^*, x_n \rangle \to \langle v^*, x \rangle \quad \text{as } n \to \infty.$$

Therefore A is generalized pseudomonotone, hence it is pseudomonotone. □

Next we consider the Carathéodory approximate selection of $F(z, x)$ produced in Proposition 2 and we define the following sequence of truncations

$$g_{\varepsilon,k}(z, x) = \begin{cases} g_\varepsilon(z, x) & \text{if } |g_\varepsilon(z, x)| \le k \\ k \operatorname{sign}(g_\varepsilon(z, x)) & \text{if } |g_\varepsilon(z, x)| > k \end{cases}, \quad k \ge 1.$$

Let $N_{\varepsilon,k} : L^p(Z) \to L^q(Z)$ be the Nemitsky operator corresponding to the function $g_{\varepsilon,k}(\cdot, \cdot)$; i.e.,

$$N_{\varepsilon,k}(x)(\cdot) = g_{\varepsilon,k}(\cdot, x(\cdot)) \quad \text{for all } x \in L^p(Z).$$

Evidently, $N_{\varepsilon,k}$ is continuous and bounded and note that $N_{\varepsilon,k}(L^p(Z)) \subseteq L^\infty(Z) \subset L^q(Z)$. Let $G : L^2(Z) \to \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$G(x) = \int_Z j(x(z))\, dz \quad \text{for all } x \in L^2(Z),$$

where j is the function in H(β). Using Fatou's lemma we verify easily that $G \in \Gamma_0(L^2(Z))$ (hence $G \in \Gamma_0(L^p(Z))$, since $2 \le p$). For every $\lambda > 0$, the Moreau-Yosida approximation of G in $L^2(Z)$ is expressed by

$$\begin{aligned} G_\lambda(x) &= \inf\left\{ G(y) + \frac{1}{2\lambda}\,\|x-y\|_2^2 \,:\, y \in L^2(Z) \right\} \\ &= \inf\left\{ \int_Z (j(y(z)) + \frac{1}{2\lambda}\,|x(z)-y(z)|^2)\, dz \,:\, y \in L^2(Z) \right\} \\ &= \int_Z \inf\left\{ j(y) + \frac{1}{2\lambda}\,|y - x(z)|^2 \,:\, y \in \mathbb{R} \right\} dz \\ &= \int_Z j_\lambda(x(z))\, dz \quad \text{for all } x \in L^2(Z) \end{aligned}$$

(see [3, page 458]). Because j_λ is convex and differentiable, from the monotone convergence theorem we have that $G_\lambda \in C^1(L^2(Z))$ and

$$G'_\lambda(x) = N_{j'_\lambda}(x) \quad \text{for all } x \in L^2(Z).$$

Here $N_{j'_\lambda}$ is the Nemitsky operator corresponding to the function $j'_\lambda(\cdot)$; i.e.,

$$N_{j'_\lambda}(x)(\cdot) = j'_\lambda(x(\cdot)) \quad \text{for all } x \in L^2(Z).$$

In what follows, for notational simplicity, we set $N_\lambda = N_{j'_\lambda}$ for all $\lambda > 0$. For fixed $\varepsilon \in (0,1]$, $k \ge 1$, and $\lambda > 0$, we consider the following operator inclusion

$$h \in A(x) + N_\lambda(x) + N_{\varepsilon,k}(x). \tag{10}$$

Proposition 5. *If hypotheses* H(a), H(β), *and* H(F) *hold and* $h \in L^p(Z)$, *then problem* (10) *has at least one solution* $x \in W_0^{1,p}(Z)$.

Proof. Using the fact that $W_0^{1,p}(Z)$ is embedded compactly into $L^p(Z)$, we see that the operators $x \mapsto N_\lambda(x)$ and $x \mapsto N_{\varepsilon,k}(x)$ are completely continuous from $W_0^{1,p}(Z)$ into $W^{-1,q}(Z)$. This combined with Proposition 4 implies that $x \mapsto A(x) + N_\lambda(x) + N_{\varepsilon,k}(x)$ is a pseudomonotone operator (see Proposition 1(e)).

For every $x \in W_0^{1,p}(Z)$ and every $v^* \in A(x)$, we consider the sum

$$\langle v^*, x\rangle + \langle N_\lambda(x), x\rangle + \langle N_{\varepsilon,k}(x), x\rangle. \tag{11}$$

From the definition of A it is seen that $v^* = -\operatorname{div} v$ with $v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}$. So

$$\langle v^*, x\rangle = \langle -\operatorname{div} v, x\rangle = \int_Z (v(z), Dx(z))_{\mathbb{R}^N}\, dz \geq c_2\|Dx\|_p^p - \|a_2\|_1$$

(see hypothesis H(a)(iv)). Also we have

$$\langle N_\lambda(x), x\rangle = \int_Z j'_\lambda(x(z))x(z)\, dz.$$

Since, by H(β), $j'_\lambda(0) = 0$ and $j'_\lambda(\cdot)$ is monotone (see hypothesis H(β)), we see that

$$\langle N_\lambda(x), x\rangle \geq 0.$$

Finally, we have

$$|\langle N_{\varepsilon,k}(x), x\rangle| \leq \int_Z |g_{\varepsilon,k}(z, x(z))|\, |x(z)|\, dz \leq \eta_1(k)\|x\|_p$$

for some $\eta_1(k) > 0$. Returning to (11), and using the above inequalities, we obtain

$$\langle v^*, x\rangle + \langle N_\lambda(x), x\rangle + \langle N_{\varepsilon,k}(x), x\rangle \geq c_2\|Dx\|_p^p - \eta_2(k)\|Dx\|_p - \|a_2\|_1 \quad (12)$$

for some $\eta_2(k) > 0$. Here we have used Poincaré's inequality. Since $\|Dx\|_p$ is an equivariant norm on the Sobolev space $W_0^{1,p}(Z)$, from (12) we infer that the operator $x \mapsto A(x) + N_\lambda(x) + N_{\varepsilon,k}(x)$ is coercive, hence it is surjective (see Proposition 1(d)). So we can find $x \in W_0^{1,p}(Z)$ solving (10).

4. Existence Theorem

In this section, passing to the limit as $\lambda, \varepsilon \downarrow 0$ and $k \to \infty$ in (10), we will produce a solution for problem (1).

Theorem 6. *If hypotheses* H(a), H(β) *and* H(F) *hold and* $h \in L^p(Z)$, *then problem* (1) *has a solution* $x \in W_0^{1,p}(Z)$.

Proof. Let $\lambda_k, \varepsilon_k \downarrow 0$ as $k \to \infty$ and let $x_k \in W_0^{1,p}(Z)$, $k \geq 1$, be solutions of (10) corresponding to the parameters $(\lambda_k, \varepsilon_k, k)$ (see Proposition 5). We have

$$v_k^* + N_{\lambda_k}(x_k) + N_{\varepsilon_k,k}(x_k) = h \quad \text{with } v_k^* \in A(x_k), \quad (13)$$

which implies that

$$\langle v_k^*, x_k\rangle + \int_Z j'_{\lambda_k}(x_k(z))x_k(z)\, dz + \int_Z g_{\varepsilon_k,k}(z, x_k(z))x_k(z)\, dz = \int_Z h(z)x_k(z)\, dz$$

for all $k \geq 1$. From the proof of Proposition 5, we know that

$$\langle v_k^*, x_k \rangle \geq c_2 \|Dx_k\|_p^p - \|a_2\|_1 \quad \text{and} \quad \int_Z j'_{\lambda_k}(x_k(z)) x_k(z)\, dz \geq 0.$$

It follows that

$$c_2 \|Dx_k\|_p^p - \|a_2\|_1 + \int_Z g_{\varepsilon_k,k}(z, x_k(z)) x_k(z)\, dz \leq \|h\|_q \, \|x_k\|_p \,. \tag{14}$$

For $M > 0$ as in hypothesis H(F)(iv), set

$$Z(k) = \{z \in Z \,:\, |x_k(z)| \leq M + 1\}.$$

We have, for k sufficiently large, that

$$\begin{aligned} &\int_Z g_{\varepsilon_k,k}(z, x_k(z)) x_k(z)\, dz \\ &= \int_{Z(k)} g_{\varepsilon_k,k}(z, x_k(z)) x_k(z)\, dz + \int_{Z \setminus Z(k)} g_{\varepsilon_k,k}(z, x_k(z)) x_k(z)\, dz \\ &\geq \int_{Z(k)} g_{\varepsilon_k,k}(z, x_k(z)) x_k(z)\, dz \geq -\theta_1 \quad \text{for some } \theta_1 > 0 \end{aligned}$$

(see Proposition 2(c) and (b)). Using the estimate in (14), we obtain

$$c_2 \|Dx_k\|_p^p \leq \theta_2 \|Dx_k\|_p + \theta_3 \quad \text{for some } \theta_2, \theta_3 > 0 \text{ and all } k \geq 1,$$

which implies that

$$\{x_k\}_{k \geq 1} \subset W_0^{1,p}(Z) \text{ is bounded.} \tag{15}$$

From Ladyzhenskaya-Uraltseva [11, page 286], we have that $x_k \in L^\infty(Z)$ and there exists $M_1 > 0$ such that

$$\|x_k\|_\infty \leq M_1 \quad \text{for all } k \geq 1 \tag{16}$$

(the result applies in our case with a multivalued a because of the stronger growth conditions). Consider the locally Lipschitz continuous function $\eta_k : \mathbb{R} \to \mathbb{R}$ defined by

$$\eta_k(r) = |j'_{\lambda_k}(r)|^{p-2} j'_{\lambda_k}(r), \quad r \in \mathbb{R}\,.$$

By Theorem 2.2 of Marcus-Mizel [12] we know that

$$\eta_k(x_k(\cdot)) \in W_0^{1,r}(Z) \quad \text{for some } q \leq r \leq p \text{ and for all } k \geq 1.$$

Since $j'_{\lambda_k} = \beta_{\lambda_k}$ is Lipschitz continuous, $j''_{\lambda_k}(x)$ exists for all $x \in \mathbb{R} \setminus D$ with $|D|_1 = 0$ (here $|\cdot|_1$ denotes Lebesgue measure on $\mathbb{R}$). Moreover, by Stampacchia's theorem we know that $Dx_k(z) = 0$ almost everywhere on

$x_k^{-1}(D)$. Since η_k is Lipschitz continuous on $[-M_1, M_1]$ and one has (16), from the chain rule for Sobolev functions (see [3, page 348]), we derive

$$\begin{aligned} &D(\eta_k(x_k(\cdot)))(z) \\ &= \begin{cases} (p-1)|j'_{\lambda_k}(x_k(z))|^{p-2} j''_{\lambda_k}(x_k(z)) Dx_k(z) & \text{a.e. on } Z \setminus x_k^{-1}(D) \\ 0 & \text{a.e. on } x_k^{-1}(D). \end{cases} \end{aligned}$$

Note that $N_{\lambda_k}(x_k) \in L^p(Z) \subseteq L^{r'}(Z)$ (since $|j'_{\lambda_k}(x_k(z))| \le \frac{1}{\lambda_k} |x_k(z)|$), with $\frac{1}{r} + \frac{1}{r'} = 1$, and also $N_{\varepsilon_k,k}(x_k) \in L^\infty(Z)$ (from the definition of $g_{\varepsilon_k,k}$). Because of (13) we see that $v_k^* \in L^{r'}(Z) \subset W^{-1,r'}(Z)$. We can work in (13) with the pair $(W_0^{1,r}(Z), W^{-1,r'}(Z))$ and take the duality brackets $\langle \cdot, \cdot \rangle_r$ there. Using $\eta_k(x_k)$ as a test function in (13), we obtain

$$\begin{aligned} \langle v_k^*, \eta_k(x_k) \rangle_r + \int_Z |j'_{\lambda_k}(x_k(z))|^p \, dz + \int_Z g_{\varepsilon_k,k}(z, x_k(z)) \eta_k(x_k(z)) \, dz \\ = \int_Z h(z) \eta_k(x_k(z)) \, dz. \qquad (17) \end{aligned}$$

Using $v_k^* = -\operatorname{div} v_k$ it follows that

$$\begin{aligned} \langle v_k^*, \eta_k(x_k) \rangle_r &= \int_Z (v_k(z), D\eta_k(x_k(z)))_{\mathbb{R}^N} \, dz \\ &= (p-1) \int_{Z \setminus x_k^{-1}(D)} |j'_{\lambda_k}(x_k(z))|^{p-2} j''_{\lambda_k}(x_k(z)) (v_k(z), Dx_k(z))_{\mathbb{R}^N} \, dz. \end{aligned}$$

Because j_{λ_k} is convex and differentiable, we have that j'_{λ_k} is nondecreasing and so $j''_{\lambda_k}(x) \ge 0$ for all $x \in \mathbb{R} \setminus D$. Moreover, because $0 \in a(z, x_k(z), 0)$ almost everywhere on Z (see hypothesis H(a)(ii)), we have that

$$(v_k(z), Dx_k(z))_{\mathbb{R}^N} \ge 0$$

almost everywhere on Z. We infer that

$$\langle v_k^*, \eta_k(x_k) \rangle_r \ge 0. \qquad (18)$$

Using the definition of the truncations $g_{\varepsilon_k,k}$, Proposition 2(b), and (16), we have

$$\begin{aligned} \left| \int_Z g_{\varepsilon_k,k}(z, x_k(z)) \eta_k(x_k(z)) \, dz \right| &\le \int_Z |g_{\varepsilon_k}(z, x_k(z))| \, |\eta_k(x_k(z))| \, dz \\ \le \int_Z a_{M_1}(z) \, |j'_{\lambda_k}(x_k(z))|^{p-1} \, dz &\le \|a_{M_1}\|_p \, \|j'_{\lambda_k}(x_k)\|_p^{p-1} \qquad (19) \end{aligned}$$

for all k sufficiently large. Also from Hölder's inequality, we have

$$\left| \int_Z h(z)\eta_k(x_k(z))\, dz \right| \le \|h\|_p \, \|j'_{\lambda_k}(x_k)\|_p^{p-1} \,. \tag{20}$$

If we use (18), (19), and (20) in (17), we obtain

$$\|j'_{\lambda_k}(x_k)\|_p^p \le \beta_1 \|j'_{\lambda_k}(x_k)\|_p^{p-1}$$

for some $\beta_1 > 0$ and all k sufficiently large, which implies that

$$\{j'_{\lambda_k}(x_k(\cdot))\}_{k\ge 1} \subset L^p(Z) \text{ is bounded.} \tag{21}$$

Because of (15) and (21), and by passing to a suitable subsequence if necessary, we may assume that

$$x_k \xrightarrow{\text{w}} x \text{ in } W_0^{1,p}(Z),\ x_k \to x \text{ in } L^p(Z) \ \text{ and } \ j'_{\lambda_k}(x_k) \xrightarrow{\text{w}} u \text{ in } L^p(Z) \tag{22}$$

as $k \to \infty$. Since the Moreau-Yosida approximation $G_{\lambda_k}(\cdot)$ is convex and Fréchet differentiable on $L^2(Z)$, we have

$$(G'_{\lambda_k}(x_k), y - x_k)_2 = \int_Z j'_{\lambda_k}(x_k(z))(y - x_k)(z)\, dz \le G_{\lambda_k}(y) - G_{\lambda_k}(x_k)$$

for all $y \in L^2(Z)$ and all $k \ge 1$. Here by $(\cdot,\cdot)_2$ we denote the inner product in $L^2(Z)$. Since $G_{\lambda_k} \uparrow G$ we have $G_{\lambda_k} \xrightarrow{\Gamma} G$ as $k \to \infty$ (see [4, Proposition 3.7.10]). So we have

$$G(x) \le \liminf_{k\to\infty} G_{\lambda_k}(x_k) \tag{23}$$

(cf. [4, Theorem 3.7.19]). Passing to the limit superior as $k \to \infty$ and using (22) and (23) and the fact that $G_{\lambda_k} \uparrow G$, we obtain

$$(u, y - x)_2 \le G(y) - G(x) \ \text{ for all } y \in L^2(Z).$$

This means that $u \in \partial G(x) = S^2_{\partial j(x(\cdot))}$ (see, e.g., [3, page 589]). Then by (16) and Proposition 2(b), we have that $\{g_{\varepsilon_k,k}(\cdot, x_k(\cdot))\}_{k\ge1} \subset L^p(Z)$ is bounded. Since $p \ge 2$ we have $q \le p$, thus

$$\{g_{\varepsilon_k,k}(\cdot, x_k(\cdot))\}_{k\ge1} \subset L^q(Z) \text{ is bounded.}$$

So by passing to a subsequence if necessary, we may assume that

$$g_{\varepsilon_k,k}(\cdot, x_k(\cdot)) \xrightarrow{\text{w}} \hat{g} \ \text{ in } L^q(Z). \tag{24}$$

Recall that for all $k \ge 1$

$$g_{\varepsilon_k,k}(z, x_k(z)) \in \operatorname{conv} F(z, x_k(z) + \varepsilon_k \bar{B}_1) + \varepsilon_k \bar{B}_1 \ \text{ a.e. on } \{|g_{\varepsilon_k}(z, x_k(z))| \le k\}.$$

Using Proposition 4.7.44 of [3, page 484] and the fact that for almost all $z \in Z$, $F(z,\cdot)$ is u.s.c., in the limit as $k \to \infty$, we obtain

$$\hat{g}(z) \in F(z, x(z)) \quad \text{a.e. on } Z.$$

For every $k \geq 1$, we have

$$-\operatorname{div} v_k + N_{\lambda_k}(x_k) + N_{\varepsilon_k,k}(x_k) = h \quad \text{with } v_k \in S^q_{a(\cdot,x_k(\cdot),Dx_k(\cdot))}\,, \tag{25}$$

which implies that

$$\langle -\operatorname{div} v_k, x_k - x\rangle + \int_Z j'_{\lambda_k}(x_k(z))(x_k - x)(z)\,dz + \int_Z g_{\varepsilon_k,k}(z, x_k(z))(x_k - x)(z)\,dz$$
$$= \int_Z h(z)(x_k - x)(z)\,dz.$$

By virtue of the monotonicity of $j'_{\lambda_k} = \beta_{\lambda_k}$, it follows that

$$\langle -\operatorname{div} v_k, x_k - x\rangle + \int_Z j'_{\lambda_k}(x(z))(x_k - x)(z)\,dz + \int_Z g_{\varepsilon_k,k}(z, x_k(z))(x_k - x)(z)\,dz$$
$$\leq \int_Z h(z)(x_k - x)(z)\,dz$$

and thus by (22) and (24) one obtains

$$\limsup_{k\to\infty} \langle -\operatorname{div} v_k, x_k - x\rangle \leq 0. \tag{26}$$

Because of hypothesis H(a)(iii) and $v_k \in S^q_{a(\cdot,x_k(\cdot),Dx_k(\cdot))}$, one has that $\{v_k\}_{k\geq 1} \subset L^q(Z, \mathbb{R}^N)$ is bounded. So we may assume that

$$v_k \xrightarrow{\text{w}} v \quad \text{in } L^q(Z, \mathbb{R}^N),$$

which implies that

$$\operatorname{div} v_k \xrightarrow{\text{w}} \operatorname{div} v \quad \text{in } W^{-1,q}(Z). \tag{27}$$

From (26) and (27) and since $x_k \xrightarrow{\text{w}} x$ in $W^{1,p}_0(Z)$, $-\operatorname{div} v_k \in A(x_k)$ for all $k \geq 1$ and A is pseudomonotone (see Proposition 4), we have

$$-\operatorname{div} v \in A(x); \quad \text{i.e., } v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}\,.$$

Passing to the limit as $k \to \infty$ in (25), we obtain

$$-\operatorname{div} v + u + \hat{g} = h,$$

with $v \in S^q_{a(\cdot,x(\cdot),Dx(\cdot))}$, $u \in S^q_{\beta(x(\cdot))}$ and $\hat{g} \in S^q_{F(\cdot,x(\cdot))}$. Therefore $x \in W^{1,p}_0(Z)$ is a solution of problem (1). □

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