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GENERIC UNIQUENESS OF MINIMIZERS FOR A CLASS OF INFINITE HORIZON VARIATIONAL PROBLEMS ARISING IN CONTINUUM MECHANICS

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Abstract. In this paper we study infinite horizon variational problems arising in continuum mechanics and establish a generic uniqueness of an optimal solution.

1. Introduction

The study of variational problems and optimal control problems defined on infinite intervals has recently been a rapidly growing area of research. See, for example, [1, 4-7, 9, 10] and the references mentioned there. These problems arise in engineering [1], in models of economic growth [7, 11-14], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 20] and in the theory of thermodynamical equilibrium for materials [6, 10, 15-19, 20-23].

In this paper we consider the following variational problem

$$\int_0^T f(w(t), w'(t), w''(t))dt \to \min \qquad (P)$$

$$w \in W^{2,1}([0,T]),\ (w(0), w'(0)) = x,\ (w(T), w'(T)) = y,$$

where $T > 0$, $x, y \in R^2$, $W^{2,1}([0,T]) \subset C^1$ is the Sobolev space of functions possessing an integrable second derivative and f belongs to a space of functions to be described below.

The interest in variational problems of the form (P) stems from the theory of thermodynamical equilibrium for second-order materials developed in [6, 10, 15-19, 21-23].

Denote by $\mathfrak{A}$ the set of all continuous functions f: $R^3 \to R$ such that for each $N > 0$ the function $|f(x, y, z)| \to \infty$ as $|z| \to \infty$ uniformly on the set

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$\{(x,y) \in R^2\colon |x|,|y| \le N\}$. For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base:

$$E(N,\epsilon,\Gamma) = \{(f,g) \in \mathfrak{A} \times \mathfrak{A} : \tag{1.1}$$
$$|f(x_1,x_2,x_3) - g(x_1,x_2,x_3)| \le \epsilon \ (x_i \in R,\ |x_i| \le N,\ i = 1,2,3),$$
$$(|f(x_1,x_2,x_3)| + 1)(|g(x_1,x_2,x_3)| + 1)^{-1} \in [\Gamma^{-1},\Gamma]$$
$$((x_1,x_2,x_3) \in R^3,\ |x_1|,|x_2| \le N)\},$$

where $N > 0$, $\epsilon > 0$, $\Gamma > 1$. Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable (by a metric ρ). It is easy to verify that the uniform space $\mathfrak{A}$ is complete.

Let $a = (a_1,a_2,a_3,a_4) \in R^4$, $a_i > 0$ $(i = 1,2,3,4)$ and let α, β, γ be positive numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha,\beta,\gamma,a)$ the set of all functions $f \in \mathfrak{A}$ such that:

$$f(w,p,r) \ge a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4,\ (w,p,r) \in R^3; \tag{1.2}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w,p,r) > 0 \text{ for all } (w,p,r) \in R^3; \tag{1.3}$$

there is a monotone increasing function $M_f : [0,\infty) \to [0,\infty)$ such that for every $(w,p,r) \in R^3$

$$\sup\{f(w,p,r),\ |\partial f/\partial w(w,p,r)|,\ |\partial f/\partial p(w,p,r)|,\ |\partial f/\partial r(w,p,r)|\}$$
$$\le M_f(|w| + |p|)(1 + |r|^\gamma). \tag{1.4}$$

Denote by $\bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$ the closure of $\mathfrak{M}(\alpha,\beta,\gamma,a)$ in $\mathfrak{A}$. Note that the full description of integrands belonging to the space $\bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$ was obtained in [24]. Leizarowitz and Mizel [10] and Coleman, Marcus and Mizel [6] considered problems of type (P) with integrands $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ in order to study certain models in in the theory of thermodynamical equilibrium for materials. For these models integrands f have the minus signs in (1.2). A typical example is an integrand

$$f(w,p,r) = \psi(w) - bp^2 + cr^2,\ (w,p,r) \in R^3,$$

where b, c are positive constants and $\psi(\cdot)$ is a smooth function satisfying

$$\psi(w) \ge a|w|^\alpha - d,\ w \in R$$

for some $\alpha > 2$, $a, d > 0$ [10, 16]. In [18, 22, 23] we considered problems of type (P) with integrands $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$.

Consider any $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$. Since (1.2) has negative terms the function f can be unbounded. Nevertheless, as was shown in [18, Lemma 2.2] the corresponding integral functional is bounded from below on any bounded interval. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\Big\{ \liminf_{T\to+\infty} T^{-1}\int_0^T f(w(t), w'(t), w''(t))dt : \\ w \in W^{2,1}_{loc}([0,\infty)) \text{ and } (w(0), w'(0)) = x\Big\}, \quad (1.5)$$

where $x \in R^2$. Here $W^{2,1}_{loc}([0,\infty)) \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative. It was shown in [10] that $\mu(f)$ is well defined and is independent of the initial vector x.

A function $w \in W^{2,1}_{loc}([0,\infty))$ is called an (f)-good function if the function

$$\phi^f_w : T \to \int_0^T [f(w(t), w'(t), w''(t)) - \mu(f)]dt, \quad T \in (0,\infty)$$

is bounded. For every $w \in W^{2,1}_{loc}([0,\infty))$ the function ϕ^f_w is either bounded or diverges to $+\infty$ as $T \to +\infty$ and moreover, if ϕ^f_w is a bounded function, then

$$\sup\{|(w(t), w'(t))| : t \in [0,\infty)\} < \infty$$

[22, Proposition 3.5].

This fact is a continuous version of a result by Leizarowitz [9] established for discrete time control systems. Its proof is based on the result of Leizarowitz [9] applied to a function U^f_T which is defined below.

Leizarowitz and Mizel [10] established that for every $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ satisfying $\mu(f) < \inf\{f(w,0,s)\colon (w,s) \in R^2\}$ there exists a periodic (f)-good function. In [21] it was shown that this result is valid for every $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$.

Let $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$. For each $T > 0$ define a function $U^f_T\colon R^2 \times R^2 \to R$ by

$$U^f_T(x,y) = \inf\Big\{ \int_0^T f(v(t), v'(t), v''(t))dt\colon v \in W^{2,1}([0,T]), \\ (v(0), v'(0)) = x \text{ and } (v(T), v'(T)) = y\Big\}. \quad (1.6)$$

In [10] Leizarowitz and Mizel studied the function $U^f_T\colon R^2 \times R^2 \to R$, $T > 0$ and established the following representation formula

$$U^f_T(x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y),\ x,y \in R^2,\ T > 0, \quad (1.7)$$

where π^f: $R^2 \to R$ and $(T,x,y) \to \theta^f_T(x,y),\ x,y \in R^2,\ T>0$ are continuous functions,

$$\pi^f(x) = \inf\Big\{ \liminf_{T\to\infty} \int_0^T [f(w(t),w'(t),w''(t)) - \mu(f)]dt : \\ w \in W^{2,1}_{loc}([0,\infty)) \text{ and } (w(0),w'(0)) = x \Big\},\ x \in R^2, \tag{1.8}$$

$\theta^f_T(x,y) \geq 0$ for each $T > 0$, and each $x, y \in R^2$, and for every $T > 0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta^f_T(x,y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$, but their result also holds for every $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$ without change in the proofs.

In the sequel we use the following notation and definitions. We denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau > 0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v : [0,\tau] \to R^2$ as follows:

$$X_v(t) = (v(t), v'(t)),\ t \in [0,\tau].$$

We also use this definition for $v \in W^{2,1}_{loc}([0,\infty))$. Sometimes $(v(t),v'(t))$ is also denoted as $(v,v')(t)$. For each $x \in R^2$ and $A \subset R^2$

$$d(x,A) = \inf\{|x-y| :\ y \in A\}.$$

Set

$$\mathfrak{M} = \mathfrak{M}(\alpha,\beta,\gamma,a),\ \bar{\mathfrak{M}} = \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a).$$

Let $f \in \bar{\mathfrak{M}}$ and let $\delta > 0$. A function $v \in W^{2,1}_{loc}([0,\infty))$ is called an (f,δ)-minimal function if

$$\sup\{|(v,v')(t)| :\ t \in [0,\infty)\} < \infty$$

and if for each $T > 0$

$$\int_0^T f(v(t),v'(t),v''(t))dt \leq U^f_T(X_v(0), X_v(T)) + \delta.$$

Let $f \in \bar{\mathfrak{M}}$. A function $w \in W^{2,1}_{loc}[(0,\infty))$ is called an (f)-minimal if

$$\sup\{|X_w(t)| :\ t \in [0,\infty)\} < \infty$$

and if for each $T_1 \geq 0$, $T_2 > T_1$ and each function $u \in W^{2,1}([T_1,T_2])$ satisfying

$$X_w(T_1) = X_u(T_1),\ X_w(T_2) = X_u(T_2)$$

the following inequality holds:

$$\int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt \le \int_{T_1}^{T_2} f(u(t), u'(t), u''(t))dt.$$

Proposition 1.1. *Let $f \in \bar{\mathfrak{M}}$. For each $z \in R^2$ there exists an (f)-minimal function $v \in W^{2,1}_{loc}([0,\infty))$ such that $X_v(0) = z$.*

For the proof see Proposition 1.1 of [19] where this result was obtained for $f \in \mathfrak{M}$. It is not difficult to see that the result is true for any $f \in \bar{\mathfrak{M}}$ with the same proof.

In the sequel we consider a space $R^2 \times \bar{\mathfrak{M}}$ equipped with the metric

$$\rho_0((x,f),(y,g)) = |x-y| + \rho(f,g),\ x,y \in R^2,\ f,g \in \bar{\mathfrak{M}}.$$

The following two theorems are the main results of the paper. They will be proved in Section 5.

Theorem 1.1. *There exists a set $\mathcal{F} \subset R^2 \times \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense subsets of $R^2 \times \bar{\mathfrak{M}}$ such that for each $(x,f) \in \mathcal{F}$ there exists a unique (f)-minimal function $v \in W^{2,1}_{loc}([0,\infty))$ which satisfies $(v(0), v'(0)) = x$.*

Theorem 1.2. *Let $x \in R^2$. There exists a set $\mathcal{F}_x \subset \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}$ such that for each $f \in \mathcal{F}_x$ there is a unique (f)-minimal function $v \in W^{2,1}_{loc}([0,\infty))$ which satisfies $(v(0), v'(0)) = x$.*

The next theorem follows from Theorem 1.1 and a classical result by Kuratowski and Ulam [8].

Theorem 1.3. *There exists a set $\mathcal{F} \subset \bar{\mathfrak{M}}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}$ such that for each $f \in \mathcal{F}$ the following property holds:*

There is a set $F \subset R^2$ which is a countable intersection of open everywhere dense subsets of R^2 such that for each $x \in F$ there is a unique (f)-minimal function $v \in W^{2,1}_{loc}([0,\infty))$ satisfying $(v(0), v'(0)) = x$.

2. PRELIMINARIES

For a function $w \in W^{2,1}_{loc}([0,\infty))$ we denote by $\Omega(w)$ the set of all points $z \in R^2$ such that $X_w(t_j) \to z$ as $j \to \infty$ for some sequence of numbers $t_j \to \infty$.

We say that an integrand $f \in \bar{\mathfrak{M}}$ has an asymptotic turnpike property (or, briefly, (ATP)) if for each pair of (f)-good functions $w_1, w_2 \in W^{2,1}_{loc}([0,\infty))$ the equality $\Omega(w_1) = \Omega(w_2)$ holds.

Let $f \in \bar{\mathfrak{M}}$. For each $T_1 \in R$, each $T_2 > T_1$ and each $v \in W^{2,1}([T_1, T_2])$ set

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t), v''(t))dt$$

and put

$$\Gamma^f(T_1, T_2, v) = I^f(T_1, T_2, v) - T\mu(f) - \pi^f(X_v(T_1)) + \pi^f(X_v(T_2)). \quad (2.1)$$

A function $v \in W^{2,1}_{loc}([0,\infty))$ is called (f)-perfect [19] if the set $\{X_v(t) : t \in [0,\infty)\}$ is bounded and

$$\Gamma^f(0, T, v) = 0 \text{ for all } T \in (0,\infty).$$

Proposition 2.1. *Let $f \in \bar{\mathfrak{M}}$. Then for each $z \in R^2$ there exists an (f)-perfect function $w \in W^{2,1}_{loc}([0,\infty))$ such that $(w, w')(0) = z$.*

For the proof of Proposition 2.1 see Propositions 1.2 and 1.3 of [19]. In [19] this result was proved for $f \in \mathfrak{M}$ but it also holds for any $f \in \bar{\mathfrak{M}}$ with an analogous proof.

It is clear that any (f)-perfect function is (f)-minimal.

In [19, Theorem 1.1] we established the following important result.

Proposition 2.2. *Let $f \in \mathfrak{M}$ and let $v_1, v_2 \in W^{2,1}_{loc}([0,\infty))$ be (f)-minimal functions such that $(v_1, v_1')(0) = (v_2, v_2')(0)$. If there exist $t_1, t_2 \in [0,\infty)$ such that $(t_1, t_2) \neq (0,0)$ and $(v_1, v_1')(t_1) = (v_2, v_2')(t_2)$, then $v_1 = v_2$.*

Corollary 2.1. *Let $f \in \mathfrak{M}$ and let $v_1, v_2 \in W^{2,1}_{loc}([0,\infty))$ be (f)-minimal functions such that $(v_1, v_1')(0) = (v_2, v_2')(0)$. If v_1 is periodic, then $v_1 = v_2$.*

The following useful result was established in [2, Chapter 2, Section 3].

Proposition 2.3. *Let Ω be a closed subset of R^q. Then there exists a bounded nonnegative function $\phi \in C^\infty(R^q)$ such that $\Omega = \{x \in R^q : \phi(x) = 0\}$ and that for each integer $m \geq 1$ and each sequence of nonnegative integers $p_1, p_2, \dots, p_m$ the function $\partial^{|p|}\phi/\partial x_1^{p_1} \dots \partial x_m^{p_m} : R^q \to R^1$ is bounded, where $|p| = \sum_{i=1}^m p_i$.*

Proposition 2.4. *Let $f \in \bar{\mathfrak{M}}$, $\delta > 0$ and let a function $v \in W^{2,1}_{loc}([0,\infty))$ be (f,δ)-minimal. Then v is (f)-good.*

This proposition is proved analogously to Lemma 2.6 and Proposition 2.3 of [18].

Proposition 2.5. *Assume that an integrand $f \in \bar{\mathfrak{M}}$ has ATP,*

$$w_f \in W^{2,1}_{loc}(R),\ T_f > 0,\ w_f(t+T_f) = w_f(t) \text{ for all } t \in R$$

and

$$I^f(0, T_f, w_f) = T_f \mu(f).$$

Then

$$\sup\{\pi^f(z) :\ z \in \Omega(w_f)\} = 0.$$

Proof. Clearly $\pi^f(z) \le 0$ for all $z \in \Omega(w_f)$. Assume that

$$\Delta := \sup\{\pi^f(z) :\ z \in \Omega(w_f)\} < 0. \tag{2.2}$$

Since the function π^f is continuous and the set $\Omega(w_f)$ is compact there is $\epsilon > 0$ such that for each $z \in \Omega(w_f)$ and each $y \in R^2$ satisfying $|y - z| \le \epsilon$ the following inequality holds:

$$|\pi^f(z) - \pi^f(y)| \le |\Delta|/4. \tag{2.3}$$

Let $x \in R^2$. By (1.8) and the definition of an (f)-good function

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,w) - T\mu(f)] :\ w \text{ is}$$
$$(f)\text{-good and } X_w(0) = x\}. \tag{2.4}$$

Let $w \in W^{2,1}_{loc}([0,\infty))$ be (f)-good and satisfy

$$X_w(0) = x. \tag{2.5}$$

Since f has ATP the equality $\Omega(w) = \Omega(w_f)$ is valid. This equality implies that there is $s_0 > 0$ such that for each $s \ge s_0$

$$d(X_w(s), \Omega(w_f)) \le \epsilon/2. \tag{2.6}$$

Let $s \ge s_0$. Then (2.6) holds. By (1.6), (1.7) and (2.5)

$$I^f(0,s,w) - s\mu(f) \ge U^f_s(X_w(0), X_s(w)) - s\mu(f)$$
$$\ge \pi^f(X_w(0)) - \pi^f(X_w(s)) = \pi^f(x) - \pi^f(X_w(s)). \tag{2.7}$$

Inequality (2.6) implies that there is $z \in R^2$ such that

$$z \in \Omega(w_f) \text{ and } |X_w(s) - z| \le \epsilon/2. \tag{2.8}$$

In view of (2.8) and the choice of ϵ (see (2.3))

$$|\pi^f(X_w(s)) - \pi^f(z)| \le |\Delta|/4. \tag{2.9}$$

It follows from (2.8) and (2.2) that $\pi^f(z) \le \Delta$. Combined with (2.7), (2.2) and (2.9) this inequality implies that

$$I^f(0,s,w) - s\mu(f) \ge \pi^f(x) - \pi^f(X_w(s))$$

$$\geq \pi^f(x) - \pi^f(z) + \Delta/4 \geq \pi^f(x) - \Delta + \Delta/4 = \pi^f(x) - (3/4)\Delta.$$

By this inequality which holds for all real numbers $s \geq s_0$

$$\liminf_{T\to\infty}[I^f(0,T,w) - T\mu(f)] \geq \pi^f(x) - (3/4)\Delta.$$

Since this inequality holds for any (f)-good function w satisfying $X_w(0) = x$ it follows from (2.4) and (2.2) that

$$\pi^f(x) \geq \pi^f(x) - (3/4)\Delta > \pi^f(x),$$

a contradiction. The contradiction we have reached proves Proposition 2.5. □

In the sequel we use the following result obtained in [18, Lemma B5] which is an extension of Lemma 3.7 of [23].

Proposition 2.6. *Let $f \in \mathfrak{M}$, T, ϵ be positive numbers and let $w \in W^{2,1}_{loc}(R)$ satisfy*

$$w(t+T) = w(t) \text{ for all } t \in R \text{ and } I^f(0,T,w) = T\mu(f).$$

Then there exist numbers $\delta, q > 0$ such that the following assertion holds:

If $x, y \in R^2$ satisfy $d(x,\Omega(w)) \leq \delta$, $d(y,\Omega(w)) \leq \delta$ and if $\tau \geq q$, then there exists $v \in W^{2,1}([0,\tau])$ such that

$$X_v(0) = x,\ X_v(\tau) = y \text{ and } \Gamma^f(0,\tau,v) \leq \epsilon.$$

We also need the following result established in [23, Proposition 5.1]).

Proposition 2.7. *Let $f \in \bar{\mathfrak{M}}$. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ and a number $S > 0$ such that for every $g \in \mathcal{U}$ and every (g)-good function v,*

$$|X_v(t)| \leq S \text{ for all large enough numbers } t.$$

Proposition 2.8. *Let $f \in \bar{\mathfrak{M}}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ and $S > 0$ such that the following assertion holds:*

If $g \in \mathcal{U}$, $T_1 \geq 0$, $T_2 \geq T_1 + c$ and if $v \in W^{2,1}([T_1,T_2])$ satisfies

$$|X_v(T_1)|,\ |X_v(T_2)| \leq M_1,\ I^g(T_1,T_2,v) \leq U^g_{T_2-T_1}(X_v(T_1),X_v(T_2)) + M_2,$$

then

$$|X_v(t)| \leq S,\ t \in [T_1,T_2].$$

For this result we refer the reader to [10] (the proof of Proposition 4.4).

We also use the following two results of [23, Propositions 3.2 and 3.1].

Proposition 2.9. Assume that $f \in \bar{\mathfrak{M}}$, $0 < c_1 < c_2 < \infty$, $c_3 > 0$ and $\epsilon \in (0,1)$. Then there exists a neighborhood V of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{V}$, each $T \in [c_1, c_2]$ and each $x, y \in R^2$ satisfying $|x|$, $|y| \le c_3$ the inequality $|U_T^f(x,y) - U_T^g(x,y)| \le \epsilon$ holds.

Proposition 2.10. *Assume that $f \in \bar{\mathfrak{M}}$, $0 < c_1 < c_2 < \infty$, $\epsilon > 0$ and $D > 0$. Then there exists a neighborhood V of f in $\bar{\mathfrak{M}}$ such that for every $g \in V$, every $T \in [c_1, c_2]$ and every $w \in W^{2,1}([0,T])$ satisfying*

$$\min\{I^f(0,T,w), I^g(0,T,w)\} \le D$$

the inequality $|I^f(0,T,w) - I^g(0,T,w)| \le \epsilon$ holds.

3. Auxiliary results

The next result was obtained in [18, Theorems 3.1 and 3.2].

Proposition 3.1. *Let $f \in \mathfrak{M}$. Then there exists a nonnegative function $\phi \in C^\infty(R^1)$ such that $\phi(t) > 0$ if $|t|$ is large enough, $\phi^{(m)}$ is bounded for every integer $m \ge 0$ and the following statement holds: Denote*

$$f_r(x_1, x_2, x_3) = f(x_1, x_2, x_3) + r\phi(x_1),\ r, x_1, x_2, x_3 \in R. \tag{3.1}$$

Then for each $r \in (0,1)$, $f_r \in \mathfrak{M}$, $\mu(f_r) = \mu(f)$ and f_r possesses (ATP).

Proposition 3.2. *Let $f \in \mathfrak{M}$ have (ATP), $w_f \in W^{2,1}_{loc}(R)$ and $T_f > 0$ satisfy*

$$w_f(t + T_f) = w_f(t) \text{ for all } t \in R \text{ and } I^f(0, T_f, w_f) = T_f \mu(f) \tag{3.2}$$

and let

$$\bar{x} \in R^2 \setminus \{(w_f, w_f')(t) :\ t \in R^1\}. \tag{3.3}$$

Assume that $v_0 \in W^{2,1}_{loc}([0,\infty))$ is an (f)-perfect function which satisfies

$$(v_0, v_0')(0) = \bar{x}. \tag{3.4}$$

Then there exists a bounded nonnegative function $\phi \in C^\infty(R^2)$ such that

$$\{z \in R^2 :\ \phi(z) = 0\} = \{X_{v_0}(t) :\ t \in [0,\infty)\} \cup \{X_{w_f}(t) :\ t \in R\} \tag{3.5}$$

and that for each pair of nonnegative integers p_1, p_2 the function

$$\partial^{p_1+p_2}\phi/\partial x_1^{p_1}\partial x_2^{p_2} : R^2 \to R^1$$

is bounded and the following statement holds: Denote

$$f_r(x_1, x_2, x_3) = f(x_1, x_2, x_2) + r\phi(x_1, x_2),\ r, x_1, x_2, x_3 \in R. \tag{3.6}$$

Then for each $r \in (0,1)$, $f_r \in \mathfrak{M}$, $\mu(f_r) = \mu(f)$, v_0, w_f are (f_r)-minimal functions, f_r possesses (ATP) and if an (f_r)-minimal function $u \in W^{2,1}_{loc}([0,\infty))$ satisfies $X_u(0) = \bar{x}$, then $u = v_0$.

Proof. Since the function v_0 is (f)-perfect it is also (f)-minimal. Set

$$\Omega_0 = \{X_{v_0}(t):\ t \in [0,\infty)\} \cup \{X_{w_f}(t):\ t \in [0,\infty)\}. \tag{3.7}$$

Since v_0 is (f)-minimal it is (f)-good and the set Ω_0 is bounded. We show that Ω_0 is closed.

Let $\{z_i\}_{i=1}^{\infty} \subset \Omega_0$ and let $z = \lim_{i\to\infty} z_i$. Extracting if it is necessary a subsequence and re-indexing we may assume without loss of generality that one of the following cases holds:

(1) $\{z_i\}_{i=1}^{\infty} \subset \Omega(w_f)$;

(2) there is a bounded sequence $\{t_i\}_{i=1}^{\infty} \subset [0,\infty)$ such that $X_{v_0}(t_i) = z_0$ for all natural numbers i;

(3) there is a sequence $\{t_i\}_{i=1}^{\infty} \subset [0,\infty)$ such that

$$\lim_{i\to\infty} t_i = \infty,\ X_{v_0}(t_i) = z_i,\ i = 1,2,\dots$$

It is clear that in the first case $z \in \Omega(w_f) \subset \Omega_0$, in the second case

$$z \in \{X_{v_0}(t):\ t \in [0,\infty)\} \subset \Omega_0$$

and in the third case $z \in \Omega(v_0) = \Omega(w_f) \subset \Omega_0$. Thus Ω_0 is closed.

By Proposition 2.3 there exists a bounded nonnegative function $\phi \in C^{\infty}(R^2)$ which satisfies (3.5) and such that for each pair of integers p_1, p_2 the function $\partial^{p_1+p_2}\phi/\partial x_1^{p_1}\partial x_2^{p_2} : R^2 \to R^1$ is bounded. For each $r \in R$ define a function $f_r : R^3 \to R$ by (3.6).

Let $r \in (0,1)$. Clearly $f_r \in \mathfrak{M}$. Since ϕ is nonnegative we have

$$\mu(f_r) \ge \mu(f).$$

In view of (3.6), (3.5) and (3.2)

$$\begin{aligned} I^{f_r}(0,T_f,w_f) &= I^f(0,T_f,w_f) + r\int_0^T \phi(w_f(t), w_f'(t))dt \\ &= I^f(0,T_f,w_f) = \mu(f)T_f. \end{aligned}$$

Since w_f is periodic with a period T_f this equality implies that

$$\mu(f_r) = \mu(f) \tag{3.8}$$

and w_f is an (f_r)-minimal function which is periodic.

We show that v_0 is an (f_r)-minimal function. In order to meet this goal it is sufficient to show that for each $T > 0$

$$U_T^{f_r}(X_{v_0}(0), X_{v_0}(T)) = I^{f_r}(0,T,v_0).$$

Let $T > 0$ and let $v_1 \in W^{2,1}([0,T])$ satisfy

$$(v_0, v_0')=(v_1,v_1')(0),\ (v_0,v_0')(T)=(v_1,v_1')(T). \tag{3.9}$$

Since ϕ is nonnegative and v_0 is (f)-minimal it follows from (3.9), (3.6) and (3.5) that

$$I^{f_r}(0,T,v_1) \geq I^f(0,T,v_1) \geq I^f(0,T,v_0) = I^{f_r}(0,T,v_0).$$

Thus,

$$U_T^{f_r}(X_{v_0}(0), X_{v_0}(T)) = I^{f_r}(0,T,v_0)$$

for each $T > 0$ and v_0 is (f_r)-minimal. Now we show that f_r possess (ATP). Assume that a function $u \in W^{2,1}_{loc}([0,\infty))$ is (f_r)-good. Then

$$\sup\{|I^{f_r}(0,T,u) - \mu(f_r)| :\ T \in (0,\infty)\} < \infty.$$

Since ϕ is nonnegative it follows from (3.6) and (3.8) that

$$\sup\{I^f(0,T,u) - \mu(f) :\ T \in (0,\infty)\} < \infty$$

and u is (f)-good. Since f has (ATP) we conclude that $\Omega(u) = \Omega(w_f)$. Thus f_r has (ATP).

Assume now that $v \in W^{2,1}_{loc}([0,\infty))$ is an (f_r)-minimal function such that

$$(v,v')(0) = (v_0,v_0)(0) = \bar{x}. \tag{3.10}$$

We show that $v = v_0$. Let us assume the converse. Then it follows from Proposition 2.2, Corollary 2.1, (3.3), Proposition 2.2 of [19] and (3.7) that

$$(v,v')(t) \notin \Omega_0 \text{ for all } t \in (0,\infty). \tag{3.11}$$

Since ϕ is nonnegative it follows from (3.5), (3.11) and (3.7) that

$$\int_0^1 \phi(v(t),v'(t))dt > 0. \tag{3.12}$$

Choose a positive number

$$\Delta_0 < r\int_0^1 \phi(v(t),v'(t))dt. \tag{3.13}$$

Since v and v_0 are (f_r)-minimal it follows from Proposition 2.4 that they are (f_r)-good. Since f_r has (ATP) we conclude that

$$\Omega(v) = \Omega(v_0) = \Omega(w_f). \tag{3.14}$$

In view of Proposition 2.5

$$\sup\{\pi^f(z) :\ z \in \Omega(w_f)\} = \sup\{\pi^{f_r}(z) :\ z \in \Omega(w_f)\} = 0. \tag{3.15}$$

There is $\bar{z} \in R^2$ such that

$$\bar{z} \in \Omega(w_f) \text{ and } \pi^f(\bar{z}) = 0. \tag{3.16}$$

Since ϕ is nonnegative it follows from (3.6), (3.8) , (1.8) and (3.16) that

$$\pi^{f_r}(\bar{z}) \geq \pi^f(\bar{z}) \geq 0.$$

Combined with (3.16) and (3.15) this inequality implies that

$$\pi^{f_r}(\bar{z}) = 0. \tag{3.17}$$

The continuity of the functions π^f and π^{f_r} implies that there is $\delta_0 > 0$ such that

$$|\pi^f(z)| = |\pi^f(z) - \pi^f(\bar{z})|, \ |\pi^{f_r}(z)| = |\pi^{f_r}(z) - \pi^{f_r}(\bar{z})| \leq \Delta_0/16 \tag{3.18}$$

for each $z \in R^2$ satisfying $|z - \bar{z}| \leq 2\delta_0$. By Proposition 2.6 there exist numbers

$$q > 0, \ \delta \in (0, \delta_0) \tag{3.19}$$

such that the following property holds:

If

$$y, z \in R^2, \ d(y, \Omega(w_f)) \leq \delta, \ d(z, \Omega(w_f)) \leq \delta, \ \tau \geq q,$$

then there is $h \in W^{2,1}([0, \tau])$ such that

$$X_h(0) = y, \ X_h(\tau) = z, \ \Gamma^{f_r}(0, z, h) \leq \Delta_0/16. \tag{3.20}$$

We show that $\pi^{f_r}(\bar{x}) = \pi^f(\bar{x})$. Since ϕ is nonnegative it follows from (3.6), (1.8) and (3.8) that $\pi^{f_r}(\bar{x}) \geq \pi^f(\bar{x})$. Since v_0 is (f)-perfect relations (3.8), (3.6) and (3.5) imply that

$$\liminf_{T\to\infty}[I^{f_r}(0, T, v_0) - \mu(f_r)T] = \liminf_{T\to\infty}[I^{f_r}(0, T, v_0) - \mu(f)T]$$

$$= \liminf_{T\to\infty}[I^f(0, T, v_0) - \mu(f)T] = \liminf_{T\to\infty}[\pi^f(X_{v_0}(0)) - \pi^f(X_{v_0}(T))].$$

Combined with (3.14), (3.10) and (3.15) this equality implies that

$$\begin{aligned} \pi^{f_r}(\bar{x}) &\leq \liminf_{T\to\infty}[I^{f_r}(0, T, v_0) - \mu(f_r)T] \\ &= \inf\{\pi^f(\bar{x}) - \pi^f(z) : \ z \in \Omega(w_f)\} = \pi^f(\bar{x}). \end{aligned}$$

Therefore, we have shown that

$$\pi^{f_r}(\bar{x}) = \pi^f(\bar{x}). \tag{3.21}$$

In view of (3.14) and (3.16) there exist numbers s_1, s_2 such that

$$s_1 > 2, \ |X_{v_0}(s_1) - \bar{z}| \leq \delta, \ s_2 > s_1 + q, \ |X_v(s_2) - \bar{z}| \leq \delta. \tag{3.22}$$

By (3.16), (3.22) and the choice of δ, q (see (3.19), (3.20)) there is $h \in W^{2,1}([s_1, s_2])$ such that

$$X_h(s_1) = X_{v_0}(s_1),\ X_h(s_2) = X_v(s_2),\ \Gamma^{f_r}(s_1, s_2, h) \le \Delta_0/16. \tag{3.23}$$

Consider $u \in W^{2,1}([0, s_2])$ defined by

$$u(t) = v_0(t),\ t \in [0, s_1],\ u(t) = h(t),\ t \in [s_1, s_2]. \tag{3.24}$$

Relations (3.10), (3.24) and (3.23) imply that

$$X_u(0) = X_{v_0}(0) = X_v(0) = \bar{x},\ X_u(s_2) = X_v(s_2) \tag{3.25}$$

and

$$I^{f_r}(0, s_2, v) - I^{f_r}(0, s_2, u) = \Gamma^{f_r}(0, s_2, v) - \Gamma^{f_r}(0, s_2, u). \tag{3.26}$$

By (3.22), (3.23) and (3.24)

$$\begin{aligned} \Gamma^{f_r}(0, s_2, u) &= \Gamma^{f_r}(0, s_1, u) + \Gamma^{f_r}(s_1, s_2, u) \\ &= \Gamma^{f_r}(0, s_1, v_0) + \Gamma^{f_r}(s_1, s_2, h) \le \Gamma^{f_r}(0, s_1, v_0) + \Delta_0/16. \end{aligned} \tag{3.27}$$

It follows from the definition of Γ^f (see (2.1)), (3.5), (3.6) and (3.8) that

$$\begin{aligned} \Gamma^{f_r}(0, s_1, v_0) &= I^{f_r}(0, s_1, v_0) - s_1\mu(f_r) - \pi^{f_r}(X_{v_0}(0)) + \pi^{f_r}(X_{v_0}(s_1)) \\ &= I^{f}(0, s_1, v_0) - s_1\mu(f_r) - \pi^{f_r}(X_{v_0}(0)) + \pi^{f_r}(X_{v_0}(s_1)). \end{aligned} \tag{3.28}$$

In view of (3.22), (3.19) and the choice of δ_0 (see (3.18))

$$|\pi^{f_r}(X_{v_0}(s_1))| \le \Delta_0/16.$$

Since v_0 is (f)-perfect this inequality, (3.10), (3.8), (3.21) and (3.28) imply that

$$\begin{aligned} \Gamma^{f_r}(0, s_1, v_0) &\le I^{f}(0, s_1, v_0) - s_1\mu(f_r) - \pi^{f_r}(X_{v_0}(0)) + \Delta_0/16 \\ &= I^{f}(0, s_1, v_0) - s_1\mu(f) - \pi^{f_r}(\bar{x}) + \Delta_0/16 \qquad (3.29) \\ &= I^{f}(0, s_1, v_0) - s_1\mu(f) - \pi^{f}(\bar{x}) + \Delta_0/16 = -\pi^{f}(X_{v_0}(s_1)) + \Delta_0/16. \end{aligned}$$

By (3.22), (3.19) and the choice of δ_0 (see (3.18))

$$|\pi^{f}(X_{v_0}(s_1))| \le \Delta_0/16.$$

Combined with (3.29) this inequality implies that

$$\Gamma^{f_r}(0, s_1, v_0) \le \Delta_0/8. \tag{3.30}$$

It follows from (2.1), (3.6), (3.8), (3.10), (3.22), (3.13) and (3.21) that

$$\begin{aligned} \Gamma^{f_r}(0, s_2, v) &= I^{f_r}(0, s_2, v) - s_2\mu(f_r) - \pi^{f_r}(X_v(0)) + \pi^{f_r}(X_v(s_2)) \\ &= I^{f}(0, s_2, v) + r\int_0^{s_2} \phi(v(t), v'(t))dt - s_2\mu(f) - \pi^{f_r}(\bar{x}) + \pi^{f_r}(X_v(s_2)) \end{aligned}$$

$$\geq I^f(0, s_2, v) + \Delta_0 - s_2\mu(f) - \pi^f(\bar{x}) + \pi^{f_r}(X_v(s_2)). \tag{3.31}$$

In view of (3.22), (3.19) and the choice of δ_0 (see (3.18))

$$|\pi^f(X_v(s_2))|,\ |\pi^{f_r}(X_v(s_2))| \leq \Delta_0/16.$$

Combined with (3.31) and (3.10) this inequality implies that

$$\Gamma^{f_r}(0, s_2, v) \geq I^f(0, s_2, v) + \Delta_0 - s_2\mu(f) - \pi^f(\bar{x}) + \pi^f(X_v(s_2))$$

$$-\pi^f(X_v(s_2)) + \pi^{f_r}(X_v(s_2)) \geq \Delta_0 + \Gamma^f(0, s_2, v) - \Delta_0/8 \geq (7/8)\Delta_0. \tag{3.32}$$

Relations (3.26), (3.32), (3.27) and (3.30) imply that

$$I^{f_r}(0, s_2, v) - I^{f_r}(0, s_2, u) \geq (7/8)\Delta_0 - \Gamma^{f_r}(0, s_2, u) \geq (7/8)\Delta_0 - \Delta_0/16$$

$$-\Gamma^{f_r}(0, s_1, v_0) \geq (7/8)\Delta_0 - \Delta_0/16 - \Delta_0/8 > \Delta_0/2.$$

This contradicts the assumption that v is (f_r)-minimal (see (3.25)). The contradiction we have reached proves that $v = v_0$. Proposition 3.2 is proved. □

4. A stability result

Theorem 4.1. *Let $f \in \bar{\mathfrak{M}}$, $x \in R^2$ and let $\bar{v} \in W^{2,1}_{loc}([0, \infty))$ be a unique (f)-minimal function such that*

$$X_{\bar{v}}(0) = x. \tag{4.1}$$

Let $l > 0$ and $\epsilon > 0$. Then there exist $\delta > 0$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that the following assertion holds:

If $g \in \mathcal{U}$, $y \in R^2$ satisfies $|y - x| \leq \delta$ and if $u \in W^{2,1}_{loc}([0, \infty))$ is (g, δ)-minimal and satisfies $X_u(0) = y$, then

$$|X_u(t) - X_{\bar{v}}(t)| \leq \epsilon,\ t \in [0, l]. \tag{4.2}$$

Proof. Let us assume that the theorem does not hold. Then for each natural number n there exist $f_n \in \mathfrak{M}$ and $u_n \in W^{2,1}_{loc}([0, \infty))$ such that

$$\rho(f_n, f) \leq 1/n,\ |X_{u_n}(0) - x| \leq 1/n, \tag{4.3}$$

$$u_n \text{ is } (f_n, 1/n) - \text{minimal}, \tag{4.4}$$

$$\sup\{|X_{u_n}(t) - X_{\bar{v}}(t)| :\ t \in [0, l]\} \geq \epsilon. \tag{4.5}$$

By Proposition 2.4 and (4.4) u_n is (f_n)-good for each natural number n. In view of Proposition 2.7 and (4.3) we may assume that there is $M_0 > 0$ such that

$$\limsup_{t\to\infty} |X_{u_n}(t)| < M_0 \text{ for all natural numbers } n. \tag{4.6}$$

By (4.3), (4.4), (4.6) and Proposition 2.8 we may assume without loss of generality that there is $M_1 > M_0$ such that

$$|X_{u_n}(t)| \le M_1 \text{ for all } t \in [0, \infty) \text{ and all integers } n \ge 1. \tag{4.7}$$

Let k be a natural number. It follows from (4.7) and the continuity of U_k^f that the sequence $\{U_k^f(X_{u_n}(0), X_{u_n}(k))\}_{k=1}^{\infty}$ is bounded. There is $M_2 > 0$ such that

$$|U_k^f(X_{u_n}(0), X_{u_n}(k))| < M_2,\ n = 1, 2, \dots \tag{4.8}$$

Relations (4.7) and (4.3) and Proposition 2.9 imply that

$$|U_k^{f_n}(X_{u_n}(0), X_{u_n}(k)) - U_k^f(X_{u_n}(0), X_{u_n}(k))| \to 0 \text{ as } n \to \infty. \tag{4.9}$$

Together with (4.8) this relation implies that the sequence

$$\{U_k^{f_n}(X_{u_n}(0), X_{u_n}(k))\}_{n=1}^{\infty}$$

is bounded. Combined with (4.4) this implies that

$$\sup\{I^{f_n}(0, k, u_n) :\ n = 1, 2, \dots\} < \infty. \tag{4.10}$$

By (4.10), (4.3) and Proposition 2.10

$$I^{f_n}(0, k, u_n) - I^f(0, k, u_n) \to 0 \text{ as } n \to \infty. \tag{4.11}$$

Relations (4.10) and (4.11) imply that the sequence $\{I^f(0, k, u_n)\}_{n=1}^{\infty}$ is bounded. Combined with (4.7) and (1.2) this implies that

$$\sup\{\int_0^k |u_n''(t)|^\gamma :\ n = 1, 2, \dots\} < \infty. \tag{4.12}$$

Since (4.12) holds for all natural numbers k it follows from (4.7) that using a diagonalization process we can construct a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and $u_* \in W_{loc}^{2,\gamma}([0, \infty))$ such that for each natural number k

$$(u_{n_i}(t), u_{n_i}'(t)) \to (u_*(t), u_*'(t)) \text{ as } i \to \infty \text{ uniformly on } [0, k], \tag{4.13}$$

$$u_{n_i}'' \to u_*'' \text{ as } i \to \infty \text{ weakly in } L^\gamma[0, k]. \tag{4.14}$$

By the lower semicontinuity of integral functions, (4.12), (4.13) and (4.14), for each natural number k

$$I^f(0, k, u_*) \le \liminf_{k \to \infty} I^f(0, k, u_{n_i}). \tag{4.15}$$

Let k be a natural number. It follows from (4.15), (4.11), (4.4), (4.9), (4.13) and the continuity of U_k^f that

$$I^f(0, k, u_*) \le \liminf_{i \to \infty} I^f(0, k, u_{n_i}) \le \liminf_{i \to \infty} I^{f_{n_i}}(0, k, u_{n_i})$$

$$\leq \liminf_{i\to\infty}(U_k^{f_{n_i}}(X_{u_{n_i}}(0), X_{u_{n_i}}(k)) + 1/n_i) = \liminf_{i\to\infty} U_k^{f_{n_i}}(X_{u_{n_i}}(0), X_{u_{n_i}}(k))$$
$$= \liminf_{i\to\infty} U_k^f(X_{u_{n_i}}(0), X_{u_{n_i}}(k)) = U_k^f(X_{u_*}(0), X_{u_*}(k)).$$

Thus

$$I^f(0, k, u_*) = U_k^f(X_{u_*}(0), X_{u_*}(k)) \text{ for all natural numbers } k. \tag{4.16}$$

Relations (4.13) and (4.7) imply that $|X_{u_*}(t)| \leq M_1$ for all $t \geq 0$. Thus u_* is (f)-minimal.

By (4.13) and (4.3)

$$X_{u_*}(0) = x. \tag{4.17}$$

In view of (4.13) and (4.5)

$$\sup\{|X_{u_*}(t) - X_{\bar{v}}(t)| : \ t \in [0, l]\} \geq \epsilon/2.$$

On the other hand since $\bar{v}$ is a unique (f)-minimal function satisfying (4.1) it follows from (4.17) that $u_* = \bar{v}$, a contradiction. The contradiction we have reached proves Theorem 4.1. □

5. Proofs of Theorems 1.1 and 1.2

Set

$$\mathcal{F} = \{(x, f) \in R^2 \times \bar{\mathfrak{M}} : \quad \text{there is a unique } (f) - \text{minimal function}$$
$$v_{x,f} \in W^{2,1}_{loc}([0,\infty)) \text{ such that } (v_{x,f}, v'_{x,f})(0) = x\}. \tag{5.1}$$

For each $x \in R^2$ set

$$\mathcal{F}_x = \{f \in \bar{\mathfrak{M}} : \ (x, f) \in \mathcal{F}\}. \tag{5.2}$$

By Propositions 3.1 and 3.2 $\mathcal{F}$ is an everywhere dense subset of $R^2 \times \bar{\mathfrak{M}}$ and for each $x \in R^2$, $\mathcal{F}_x$ is an everywhere dense subset of $\bar{\mathfrak{M}}$.

Let $(x, f) \in \mathcal{F}$ and n be a natural number. By Theorem 4.1 there exist $\delta(x, f, n) > 0$ and an open neighborhood $\mathcal{U}(x, f, n)$ of f in $\bar{\mathfrak{M}}$ such that the following property holds:

(P1) If $g \in \mathcal{U}(x, f, n)$, $y \in R^2$ satisfies $|y - x| \leq \delta(x, f, n)$ and if $u \in W^{2,1}_{loc}([0,\infty))$ is (g)-minimal and satisfies $X_u(0) = y$, then

$$|X_u(t) - (v_{x,f}, v'_{x,f})(t)| \leq n^{-1}, \ t \in [0, n].$$

Set

$$\mathcal{F}_0 = \cap_{n=1}^{\infty} \cup \{\{z \in R^2 : \ |z - x| < \delta(x, f, n)\} \times \mathcal{U}(x, f, n) : \ (x, f) \in \mathcal{F}\} \tag{5.3}$$

and for $x \in R^2$ put

$$\mathcal{F}_{0x} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(x, f, n) : \ f \in \mathcal{F}_x\}. \tag{5.4}$$

Clearly $\mathcal{F}_0$ is a countable intersection of open everywhere dense subsets of $R^2 \times \bar{\mathfrak{M}}$,

$$\mathcal{F} \subset \mathcal{F}_0$$

and for each $x \in R^2$ $\mathcal{F}_{0x}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{M}}$, $\mathcal{F}_x \subset \mathcal{F}_{0x}$ and $(x, \mathcal{F}_{0x}) \subset \mathcal{F}_0$.

Let $(y, g) \in \mathcal{F}_0$. In order to complete the proofs of the theorems it is enough to show that $(y, g) \in \mathcal{F}$. It means that there exists a unique (g)-minimal function $v \in W^{2,1}_{loc}([0, \infty))$ satisfying $X_v(0) = y$. Let $v_1, v_2 \in W^{2,1}_{loc}([0, \infty))$ be (g)-minimal functions satisfying

$$X_{v_i}(0) = y, \ i = 1, 2. \tag{5.5}$$

Let n be a natural number. By (5.3) there is $(x, f) \in \mathcal{F}$ such that

$$|y - x| \le \delta(f, x, n), \ g \in \mathcal{U}(f, x, n). \tag{5.6}$$

In view of (P1), (5.5) and (5.6)

$$|X_{v_i}(t) - (v_{x,f}(t), v'_{x,f}(t))| \le 1/n, \ t \in [0, n], \ i = 1, 2,$$

$$|X_{v_1}(t) - X_{v_2}(t)| \le 2n^{-1}, \ t \in [0, n].$$

Since this inequality holds for all natural numbers n we conclude that $v_1 = v_2$. This completes the proofs of the theorems.

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