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THE CAUCHY PROBLEM FOR QUASILINEAR HYPERBOLIC EQUATIONS WITH NON-ABSOLUTELY CONTINUOUS COEFFICIENTS IN THE TIME VARIABLE

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Abstract. We consider the Cauchy problem

$$\begin{cases} P\big(t,x,D^{m-1}u,D_t,D_x\big)u(t,x)=f(t,x,D^{m-1}u) \\ \partial_t^j u(0,x)=u_j(x), \quad j=0,...,m-1, \end{cases}$$

in $[0,T]\times\mathbb{R}^n$ for a quasilinear weakly hyperbolic operator

$$P\big(t,x,D^{m-1}u,D_t,D_x\big)=D_t^m+\sum_{j=0}^{m-1}\sum_{|\alpha|=m-j}a_\alpha^{(j)}(t,x,D^{m-1}u)D_x^\alpha D_t^j$$

with coefficients $a_\alpha^{(j)}$ having the first time derivative with singular behavior of the type t^{-q}, $q>1$, as $t\to 0$.

We show that for $t\le T_0^*$, T_0^* sufficiently small, given Cauchy data in a Gevrey class G^σ there exists a unique solution $u\in C^{m-1}([0,T_0^*];G^\sigma)$ provided that $\sigma<\frac{qr}{qr-1}$ where r denotes the largest multiplicity of the characteristic roots.

1. Introduction

Let us start by considering the linear Cauchy problem

$$\begin{cases} P\big(t,x,D_t,D_x\big)u(t,x)=f(t,x), & (t,x)\in[0,T]\times\mathbb{R}^n, \\ \partial_t^j u(0,x)=u_j(x), & j=0,...,m-1, \end{cases} \tag{1.1}$$

for a weakly hyperbolic operator of order $m\ge 2$

$$P=D_t^m+\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-j}a_\alpha^{(j)}(t,x)D_x^\alpha D_t^j,$$

$D_t=-i\partial_t$, $D_x=-i\nabla_x$, and suppose $a_\alpha^{(j)}\in C^1([0,T];G^\sigma(\mathbb{R}^n))$, where

$$G^\sigma(\mathbb{R}^n)=\{f(x):|\partial^\alpha f(x)|\le c_\alpha A^{|\alpha|}\alpha!^\sigma,\ A>0,\ \alpha\in\mathbb{Z}_+^n\}$$

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is the Gevrey class of index $\sigma > 1$.

In that case we know that the Cauchy problem (1.1) is well posed in G^σ, without any Levi condition on the lower order terms, for every

$$\sigma < \frac{r}{r-1},$$

where r denotes the largest multiplicity of the characteristic roots (see for example [1], [7]). We recall that the Cauchy problem (1.1) is said to be well posed in a space X of functions on $\mathbb{R}^n$ if for every $u_j \in X$, $f \in C([0,T];X)$ it has a unique solution $u \in C^{m-1}([0,T];X)$.

The problem (1.1) has been widely studied also in the case of coefficients which are not regular in time, both as regards the modulus of continuity (starting from [6]) and the singular behavior of the first time derivative of the coefficients (see [5], [3], [4]). In particular in this second situation the case of coefficients satisfying

$$a_\alpha^{(j)} \in C^1(]0,T];B^\infty(\mathbb{R}^n)), \quad |\partial_t a_\alpha^{(j)}| \le \frac{c_{\alpha j}}{t^q}, \quad q \ge 1,\ t \in]0,T],$$

for $|\alpha| + j = m$, has been considered. In [4] it has been proved that if the characteristic roots are regular in time, then problem (1.1) is well posed in Gevrey classes of index

$$\sigma < \frac{qr}{qr-1}.$$

Here our aim is to extend this result also to the case of a quasilinear hyperbolic Cauchy problem:

$$\begin{cases} P\big(t,x,D^{m-1}u,D_t,D_x\big)u(t,x) = f(t,x,D^{m-1}u), & (t,x) \in [0,T]\times\mathbb{R}^n, \\ \partial_t^j u(0,x) = u_j(x), & j = 0,...,m-1, \end{cases} \tag{1.2}$$

for an operator

$$P = D_t^m + \sum_{j=0}^{m-1} \sum_{|\alpha|=m-j} a_\alpha^{(j)}(t,x,D^{m-1}u) D_x^\alpha D_t^j, \tag{1.3}$$

where $D^{m-1}u = (\partial_{t,x}^\alpha u)_{|\alpha|\le m-1}$ is a vector in $\mathbb{R}^l$, $l = \sharp\{\alpha:\ |\alpha| \le m-1\}$, and the functions $a_\alpha^{(j)}$, f are supposed to have a Gevrey behavior with respect to the variables x and $y = D^{m-1}u$, following [2]. Hereafter we will take the Cauchy data $u_j = 0$, $j = 0,...,m-1$, without any loss of generality.

We can prove also in this case the existence and uniqueness of a local in time solution in Gevrey classes of index

$$1 < \sigma < \sigma_0 = \frac{qr}{qr-1}, \tag{1.4}$$

with a loss of derivatives.

In the case $r = 1$, $m = 2$, $q = 1$, the Cauchy problem (1.2) has already been considered by [9].

Remark 1.1. Notice that in condition (1.4):

- for $q = 1$, $\sigma_0 = \frac{r}{r-1}$. This is the bound by [7], [1]; it cannot be improved without assuming any Levi condition on the lower order terms;
- for $r = 1$, $\sigma_0 = \frac{q}{q-1}$. This is the bound by [5] for strictly hyperbolic equations, and it is sharp as they proved by means of counterexamples.

It is interesting to notice that in any case the life time T_0^* of the solution is not influenced by the breakdown of $\partial_t a_\alpha^{(j)}$ at $t = 0$, but only on the non-linearity of the problem, as we are going to see later on. If the coefficients are defined also in a left neighborhood of 0, and here they have the same behavior as in the right neighborhood, then the solution can be extended for $t \leq 0$ into a small interval.

2. Main result

We will state our results on Sobolev-Gevrey-type spaces: for $\epsilon > 0$, $\sigma \geq 1$ we denote

$$H^{s,\epsilon,\sigma} = \{u \in H^s(\mathbb{R}^n) : e^{\epsilon \langle D_x \rangle^{\frac{1}{\sigma}}} u(x) \in H^s(\mathbb{R}^n)\}, \tag{2.1}$$

where $\langle \xi \rangle$ stands for $\sqrt{1 + |\xi|^2}$, and $H^s(\mathbb{R}^n)$ is the usual Sobolev space.

The norm is here defined by

$$||u||_{s,\epsilon,\sigma} = ||e^{\epsilon \langle D_x \rangle^{\frac{1}{\sigma}}} u||_s.$$

The space $H^{s,\epsilon,\sigma}$ is a Banach algebra if $s > \frac{n}{2}$, so starting from here we always consider Sobolev spaces of index $s > \frac{n}{2}$. These kinds of spaces are such that $H^{s,\epsilon,\sigma} \subset G^\sigma, \epsilon > 0$. We will have to deal for $t \in [0, T]$ with Sobolev-Gevrey functions depending on time; so for $k \in \mathbb{N}$ and for a nonnegative function $\epsilon(t)$ of the form $\epsilon(t) = \epsilon_0 - ct^\delta$, $t \in [0, T]$, $c, \epsilon_0 > 0$, we define

$C^k([0, T]; H^{s,\epsilon(t),\sigma})$

$$= \{u(t,x) : t \to e^{\epsilon(t)\langle D_x\rangle^{\frac{1}{\sigma}}}\partial_t^j u \in C([0,T]; H^{s-j}),\ j = 0,...,k\}.$$

In these spaces the norm is

$$||u||_{C^k([0,T];H^{s,\epsilon(t),\sigma})} = \sup_{j=0,...,k} ||e^{\epsilon(t)\langle D_x\rangle^{\frac{1}{\sigma}}}\partial_t^j u||_{C([0,T];H^{s-j})}.$$

In this paper we will use the pseudodifferential calculus, so following the notation of [10], we introduce for $m \in \mathbb{R}$ the class $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ which is the space of all symbols $a(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq c_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}, \tag{2.2}$$

for every $\alpha, \beta \in \mathbb{Z}_+^n$, $x, \xi \in \mathbb{R}^n$; this is the limit space as $\ell \to +\infty$ of the Banach spaces S_ℓ^m that consist of all symbols $a(x,\xi)$ such that

$$|a|_\ell^{(m)} = \sup_{x,\xi} \sup_{|\alpha|+|\beta|\leq\ell} \frac{|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|}{\langle\xi\rangle^{m-|\alpha|}} < +\infty. \tag{2.3}$$

Moreover, we need to introduce the following classes of Gevrey symbols: for $m \in \mathbb{R}$, $\sigma \geq 1$, $\epsilon > 0$ we denote

$$S_{s,\epsilon}^{m,\sigma}(\mathbb{R}^{2n}) = \{a(x,\xi) : ||\partial_\xi^\alpha a(\cdot,\xi)||_{s,\epsilon,\sigma} \leq c_\alpha\langle\xi\rangle^{m-|\alpha|}, \forall\alpha\},$$

while for $m \in \mathbb{R}$, $\sigma, \sigma' \geq 1$, $\epsilon, \epsilon' > 0$ we denote

$$\begin{aligned} & S_{s,\epsilon,s',\epsilon'}^{m,\sigma,\sigma'}(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^n) \\ & = \ \{a(x,y,\xi) : ||\partial_\xi^\alpha a(\cdot,\cdot,\xi)||_{H^{s,\epsilon,\sigma}\times H^{s',\epsilon',\sigma'}} \leq c_\alpha\langle\xi\rangle^{m-|\alpha|}, \forall\alpha\}, \end{aligned}$$

where $H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}$ is the tensor product of the spaces $H^{s,\epsilon,\sigma}$ and $H^{s',\epsilon',\sigma'}$.

Now, let us consider the Cauchy problem

$$\begin{cases} P\big(t,x,D^{m-1}u,D_t,D_x\big)u(t,x) = f(t,x,D^{m-1}u), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \partial_t^j u(0,x) = 0, & j = 0,...,m-1, \end{cases} \tag{2.4}$$

for the operator (1.3), and assume that the linear operator $P\big(t,x,y,D_t,D_x\big)$ depending on the parameter y has coefficients $a_\alpha^{(j)}(t,x,y)$ defined on $[0,T] \times \mathbb{R}^n \times Y$, where $Y \subset \mathbb{R}^l$ is a neighborhood of the origin.

Suppose that $P\big(t,x,y,D_t,D_x\big)$ is a hyperbolic operator with real characteristic roots $\{\lambda_j(t,x,y,\xi)\}_{j=1,..,m}$ and with principal symbol given by

$$P_m(t,x,y,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,x,y,\xi)). \tag{2.5}$$

The roots can always be collected into $r \geq 1$ groups G_j, $j = 1, ..., r$, each one with $m_j \geq 1$ elements:

$$\begin{cases} G_1 = \{\lambda_1, ..., \lambda_{m_1}\} \\ G_2 = \{\lambda_{m_1+1}, ..., \lambda_{m_1+m_2}\} \\ \vdots \\ G_r = \{\lambda_{m-m_r+1}, ..., \lambda_m\} \\ m_1 \geq m_2 \geq ... \geq m_r \geq 1, \qquad m_1 + ... + m_r = m, \end{cases} \tag{2.6}$$

such that if we take λ_j, $\lambda_{j'}$ in the same group it holds that

$$|\lambda_j(t,x,y,\xi) - \lambda_{j'}(t,x,y,\xi)| \geq c|\xi|, \quad c > 0, j \neq j'. \tag{2.7}$$

Remark 2.1. When $G_1 \supseteq G_2 \supseteq \ldots \supseteq G_r$, we have an operator with roots of constant multiplicity, and r is the largest multiplicity. The particular case of a strictly hyperbolic operator is obtained for $r = 1$ (so, only one group of separated roots). If it is not possible to separate the roots, we have $r = m$ and $m_j = 1$ for all j.

As to the coefficients, we assume

$$a_\alpha^{(j)}(t,x,y) \in B([0,T]; H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}), \tag{2.8}$$

where, for a space of functions X, $B([0,T]; X)$ denotes the space of all functions defined on $[0,T]$ with values in X that are bounded as functions of time.

For the characteristic roots we suppose

$$\begin{cases} \lambda_j(t,x,y,\xi) \in C([0,T]; S^{1,\sigma,\sigma'}_{s,\epsilon,s',\epsilon'}) \cap C^1(]0,T]; S^{1,\sigma,\sigma'}_{s,\epsilon,s',\epsilon'}), \\ t^q \partial_t \lambda_j(t,x,y,\xi) \in B([0,T]; S^{1,\sigma,\sigma'}_{s,\epsilon,s',\epsilon'}), \qquad q > 1. \end{cases} \tag{2.9}$$

Remark 2.2. Condition (2.9) is in general stronger than

$$a_\alpha^{(j)}(t,x,y) \in C([0,T]; H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}) \cap C^1(]0,T]; H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}),$$
$$t^q \partial_t a_\alpha^{(j)}(t,x,y) \in B([0,T]; H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}), \quad j + |\alpha| = m,$$

but these conditions become equivalent in the case of characteristic roots of constant multiplicity.

Then we have the main result of this paper:

Theorem 2.3. *Let the operator P in* (1.3) *satisfy all the hypotheses* (2.5)–(2.9) *and the condition* (1.4) :

$$1 < \sigma < \frac{qr}{qr-1}.$$

With

$$\sigma' < \sigma, \tag{2.10}$$

let us take

$$f(t,x,y) \in C([0,T]; H^{s-m+1,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}) \tag{2.11}$$

such that

$$f(t,x,0) \in C([0,T]; H^{s-m+1+\eta,\epsilon,\sigma}), \tag{2.12}$$

where

$$\eta = \frac{r-1}{qr}, \tag{2.13}$$

and moreover suppose that $f(t,x,0)$ has compact support.

Then there exists a $T_0^* > 0$ such that if $t \le T_0^*$, then the Cauchy problem (2.4) *has a unique solution $u \in C^m([0,T_0^*]; H^{s,\epsilon-w(t),\sigma})$, where*

$$w(t) = \frac{\lambda}{\delta} t^\delta, \tag{2.14}$$

for a $\delta \in (0,1)$ and a large enough parameter λ.

3. Outline of the proof

Firstly we notice that we can always write

$$f(t,x,D^{m-1}u) = f(t,x,0) + \sum_{|\alpha| \le m-1} b_\alpha(t,x,D^{m-1}u) D^\alpha_{t,x} u,$$

so without loss of generality we can reduce (2.4) to a new Cauchy problem of the type

$$\begin{cases} P\big(t,x,D^{m-1}u,D_t,D_x\big)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \partial_t^j u(0,x) = 0, & j = 0,...,m-1, \end{cases} \tag{3.1}$$

where

$$P = D_t^m + \sum_{j=0}^{m-1} \sum_{|\alpha| \le m-j} a_\alpha^{(j)}(t,x,D^{m-1}u) D_x^\alpha D_t^j$$

and

$$a_\alpha^{(j)}(t,x,y) \in C([0,T]; H^{s,\epsilon,\sigma} \times H^{s',\epsilon',\sigma'}),\ j + |\alpha| \le m.$$

By Corollary 2.3, Theorem 2.11 and Corollary 2.8 in [2] the coefficients of the linear operator $P\big(t,x,D^{m-1}u,D_t,D_x\big)$ are in $H^{s,\epsilon-w(t),\sigma}$, with w the function defined in (2.14), provided that (2.10) holds and that t is small enough:

$$t \le T^* = \sqrt[\delta]{\frac{\epsilon(2-2^{1/\sigma})^\delta}{3\lambda}}, \tag{3.2}$$

where T^* is independent of the data. Moreover, the composition operator $D^{m-1}u \to a_\alpha^{(j)}(t,x,D^{m-1}u)$ maps balls of $\left(H^{s-m+1,\epsilon-w(t),\sigma}\right)^l$ into balls of $H^{s-m+1,\epsilon-w(t),\sigma}$.

Let us fix $\mu > \frac{n}{2}$, a positive number M to be chosen later on, and consider

$$s = \mu + M + m.$$

Take the function $w(t)$ in (2.14), $\epsilon > 0$, $\sigma \geq 1$ and let $r_1 > r_0 > 0$ be real numbers to be fixed at the end of the proof. With E defined by

$$E = \{u \in C^m([0,T]; H^{\mu+m+M,\epsilon-w(t),\sigma}); \\ ||u||_{C^{m-1}([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \leq r_0,\ ||u||_{C^m([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \leq r_1\}, \tag{3.3}$$

given $u \in E$ we consider the linear Cauchy problem for the unknown v:

$$\begin{cases} P\big(t,x,D^{m-1}u(t,x),D_t,D_x\big)v(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \partial_t^j v(0,x) = 0, & j = 0,...,m-1. \end{cases} \tag{3.4}$$

We want to prove Theorem 2.3 by showing that the map

$$\begin{array}{rccc} S: & E & \longrightarrow & E \\ & u & \to & v \end{array} \tag{3.5}$$

defined by (3.4) is well defined and has a fixed point provided that $t \leq T_0^*$, with T_0^* depending only on the data and the operator $P(t,x,y,D_t,D_x)$.

So we have to prove the following:

Proposition 3.1. *Given $u \in E$, consider problem* (3.4) *under condition* (1.4). *Take $f \in C([0,T]; H^{\mu+M+1+\eta,\epsilon,\sigma})$, with η given by* (2.13).

Then there are positive constants M, λ, δ, T^* not depending on the data such that if $t \leq T^*$, then problem (3.4) *has a unique solution*

$$v \in C^m([0,T]; H^{\mu+M+m,\epsilon-w(t),\sigma}).$$

Moreover, we can fix $r_1 > r_0 > 0$ depending on the data and find a T_0 depending on the data such that if $t \leq min\{T^*, T_0\} = T_0^*$, then the unique solution v belongs to E.

The above result of well posedness in Gevrey-Sobolev spaces is equivalent to the well posedness in the usual Sobolev space of the Cauchy problem for the conjugate operator

$$P_\Lambda = e^{\Lambda(t,D_x)} P\ e^{-\Lambda(t,D_x)}$$

with

$$\Lambda(t,D_x) = (\epsilon - w(t))\,\langle D_x \rangle^{\frac{1}{\sigma}}. \tag{3.6}$$

So we need to recall the following result from [8]:

Proposition 3.2. *Let* $a(t,x,\xi) \in C^k([0,T];S^{m,\sigma}_{s,\epsilon-w(t)})$ *and* $\Lambda = \Lambda(t,D_x)$ *as in* (3.6). *Then* $a_\Lambda(t,x,\xi) \in C^k([0,T];S^m)$*; moreover, for every positive integer* ℓ' *there are a positive integer* $\ell_0 = \ell_0(\ell',\sigma,n)$, $\ell_0 \geq \ell'$ *and a constant* $c = c(s,\ell',\sigma,k,n) > 0$ *such that*

$$|a_\Lambda|_{C^k([0,T];S^m_{\ell'})} \leq c|a|_{C^k([0,T];S^{m,\sigma}_{s,\epsilon-w(t)})}, \quad \ell \geq \ell_0. \tag{3.7}$$

The organization of the proof is as follows: we factorize the operator $P(t,x,\ D^{m-1}u(t,x),D_t,D_x)$ using mollified characteristic roots, we reduce problem (3.4) to an equivalent first-order system by means of such a factorization, then we establish an energy estimate for the system and consequently for (3.4). Finally we choose r_1, r_0 to make the solution v be an element of E. An application of a usual fixed-point scheme completes the proof of Theorem 2.3.

4. The linear problem

Let us consider $u \in E$ defined in (3.3). The first step in the proof of Proposition 3.1 is to give a factorization of the principal part of the operator $P\big(t,x,\ D^{m-1}u,D_t,D_x\big)$. If we compose (2.9) with $y = D^{m-1}u(t,x)$ and use the results by [2] under condition (2.10) we find that the characteristic roots of $P\big(t,x,D^{m-1}u,D_t,D_x\big)$ are

$$\lambda_j(t,x,D^{m-1}u,\xi) \in C\Big([0,T];S^{1,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big) \cap C^1\Big(]0,T];S^{1,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big),$$

such that

$$t^q\partial_t\lambda_j(t,x,D^{m-1}u,\xi) \in B\Big([0,T];S^{1,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big)$$

provided that $T \leq T^*$, T^* given by (3.2). We extend the roots on $(-\infty,0]$ by setting

$$\lambda_j(t,x,D^{m-1}u(t,x),\xi) = \lambda_j(0,x,0,\xi)$$

if $t < 0$, and then we introduce the following mollified roots:

$$\tilde{\lambda}_j(t,x,D^{m-1}u(t,x),\xi) = \int \lambda_j(s,x,D^{m-1}u(s,x),\xi)\cdot\rho((t-s)\langle\xi\rangle)\langle\xi\rangle\ ds,$$

where $\rho \in C_0^\infty(\mathbb{R})$, *supp* $\rho \subset \mathbb{R}^+$, $0 \leq \rho \leq 1$, $\int \rho = 1$. Obviously

$$\begin{cases} \tilde{\lambda}_j - \lambda_j \in C\Big([0,T];S^{1,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big), \\ \partial_t^k\tilde{\lambda}_j \in C\Big([0,T];S^{k+1,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big),\ k \in \mathbb{N}, \end{cases} \tag{4.1}$$

but it is easy to see that, for

$$||u||_{C^m([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \le r_1,$$

we have

$$\begin{cases} t^q\Big(\tilde{\lambda}_j - \lambda_j\Big) \in B\Big([0,T]; S^{0,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big), \\ t^q \partial_t^k \tilde{\lambda}_j \in B\Big([0,T]; S^{k,\sigma}_{\mu+M+1,\epsilon-w(t)}\Big),\ k \in \mathbb{N}, \end{cases} \tag{4.2}$$

with norms uniformly with respect to $u \in E$. A comparison between (4.1) and (4.2) makes clear that it is possible to decrease the orders of $\tilde{\lambda}_j - \lambda_j$ and $\partial_t^k \tilde{\lambda}_j$, but this causes a worsening in the seminorms of these symbols: they are integrable on $[0,T]$ in (4.1) but not in (4.2). This double behavior of the regularized roots becomes very important in the reduction of problem (3.4) to a first-order system, using the two properties (4.1) and (4.2) in different regions of the phase space.

Let us consider now the operator

$$Q = (D_t - \tilde{\lambda}_m) \cdots (D_t - \tilde{\lambda}_1).$$

By (4.1) and (4.2) one has the following factorization of P:

$$P = Q + R,$$

$$R = \sum_{j=0}^{m-1} R_j(t,x,D^{M_0+m}u, D_x)\langle D_x\rangle^{m-1-j} D_t^j,$$

where for $j = 0, ..., m-1$ we have both

$$R_j \in C\Big([0,T]; S^{1,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big) \tag{4.3}$$

and

$$t^q R_j \in B\Big([0,T]; S^{0,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big), \tag{4.4}$$

and where $D^{M_0+m}u$ denotes the vector $(\partial_t^k \partial_x^\beta u)_{k\le m,\ k+|\beta|\le M_0+m}$, with M_0 a positive integer depending only on $m_j = 1, ..., r$ and on the dimension n.

We want to interpolate between (4.3) and (4.4) in order to have R_j, $j = 0, ..., m-1$ which globally satisfy the estimate

$$||\partial_\xi^\alpha R_j||_{\mu+M-M_0,\epsilon-w(t),\sigma} \le \frac{c_{r_1,\alpha}}{t^{1-\delta}} \langle\xi\rangle^{h-|\alpha|},\ \forall\alpha, \tag{4.5}$$

for every $\delta \in (0,1)$, with $h = h(\delta) \in (0,1)$. To do that, we fix a $\delta \in (0,1)$ and we introduce a separation in the phase space: we use (4.3) when $t^{1-\delta}\langle\xi\rangle^\gamma \le 1$ and (4.4) when $t^{1-\delta}\langle\xi\rangle^\gamma \ge 1$, then we choose γ to find the optimal h in (4.5).

For $t^{1-\delta}\langle\xi\rangle^\gamma \leq 1$ we have

$$||\partial_\xi^\alpha R_j||_{\mu+M-M_0,\epsilon-w(t),\sigma} \leq c_{r_1,\alpha}\langle\xi\rangle^{1-|\alpha|} \leq \frac{c_{r_1,\alpha}}{t^{1-\delta}}\langle\xi\rangle^{1-\gamma-|\alpha|},$$

whereas for $t^{1-\delta}\langle\xi\rangle^\gamma \geq 1$ we have

$$||\partial_\xi^\alpha R_j||_{\mu+M-M_0,\epsilon-w(t),\sigma} \leq \frac{c_{r_1,\alpha}}{t^q}\langle\xi\rangle^{-|\alpha|} \leq \frac{c_{r_1,\alpha}}{t^{1-\delta}}\langle\xi\rangle^{\frac{\gamma(q-1+\delta)}{1-\delta}-|\alpha|}.$$

The best choice of γ is given by $1-\gamma = \frac{\gamma(q-1+\delta)}{1-\delta}$; that is, $\gamma = \dfrac{1-\delta}{q}$. So

$$h = 1 - \frac{1-\delta}{q} = \frac{q-1+\delta}{q}, \tag{4.6}$$

and for every $\delta \in (0,1)$ we have

$$t^{1-\delta}R_j \in B\Big([0,T]; S^{h,\sigma}_{\mu+M-M_0,\epsilon-w(t),\sigma}\Big),$$

with h given by (4.6).

The second step in our proof is to reduce problem (3.4) to an equivalent one for a first-order system. Let us consider $\rho \in (0,1)$ to be fixed later on and define the vector $Z = (z_0, ..., z_{m-1})^t$ by:

$$\begin{cases}
z_0 = \langle D_x\rangle^{m-r+\rho(r-1)}v \\
z_1 = \langle D_x\rangle^{m-r-1+\rho(r-1)}(D_t-\tilde\lambda_1)v \\
\vdots \\
z_{m_1-1} = \langle D_x\rangle^{m-r-m_1+1+\rho(r-1)}(D_t-\tilde\lambda_{m_1-1})\cdots(D_t-\tilde\lambda_1)v \\
z_{m_1} = \langle D_x\rangle^{m-r-m_1+1+\rho(r-2)}(D_t-\tilde\lambda_{m_1})\cdots(D_t-\tilde\lambda_1)v \\
z_{m_1+1} = \langle D_x\rangle^{m-r-m_1+\rho(r-2)}(D_t-\tilde\lambda_{m_1+1})\cdots(D_t-\tilde\lambda_1)v \\
\vdots \\
z_{m_1+m_2-1} = \langle D_x\rangle^{m-r-m_1-m_2+2+\rho(r-2)}(D_t-\tilde\lambda_{m_1+m_2-1})\cdots(D_t-\tilde\lambda_1)v \\
\vdots \\
z_{m-m_r-1} = \langle D_x\rangle^{m_r-1+\rho}(D_t-\tilde\lambda_{m-m_r-1})\cdots(D_t-\tilde\lambda_1)v \\
z_{m-m_r} = \langle D_x\rangle^{m_r-1}(D_t-\tilde\lambda_{m-m_r})\cdots(D_t-\tilde\lambda_1)v \\
z_{m-m_r+1} = \langle D_x\rangle^{m_r-2}(D_t-\tilde\lambda_{m-m_r+1})\cdots(D_t-\tilde\lambda_1)v \\
\vdots \\
z_{m-1} = (D_t-\tilde\lambda_{m-1})\cdots(D_t-\tilde\lambda_1)v.
\end{cases} \tag{4.7}$$

By induction on $j = 0, ..., m-1$ we get $D_t^j v$ from (4.7), then we apply ∂_x^β for $|\beta| = m-1-j$ and we find that

$$D^{m-1}v = S(t, x, D^{M_0+m-1}u, D_x)Z, \tag{4.8}$$

where

$$S \in B\Big([0,T]; S^{(r-1)(1-\rho),\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big).$$

We have

$$\begin{cases} (D_t - \tilde{\lambda}_1)z_0 = \langle D_x\rangle z_1 + a_0 z_0 \\ \vdots \\ (D_t - \tilde{\lambda}_{m_1})z_{m_1-1} = \langle D_x\rangle^\rho z_{m-1} + a_{m_1-1}z_{m_1-1} \\ \vdots \\ (D_t - \tilde{\lambda}_m)z_{m-1} = f - \displaystyle\sum_{j=0}^{m-1} R_j(t, x, D^{m+M_0}u, D_x)\langle D_x\rangle^{m-1-j}D_t^j v \end{cases}$$

with a_j of order zero for $j = 0, ..., m-1$. Thus, the Cauchy problem (3.4) is equivalent to

$$\begin{cases} (\partial_t - iK + A)Z = F, \\ Z(0,x) = 0, \end{cases} \tag{4.9}$$

where $K(t, x, D^{m-1}u, \xi)$ is a block diagonal matrix, with blocks K_j, $j = 1, ..., r$,

$$K_j = \begin{pmatrix} \tilde{\lambda}_{m-(m_j+...+m_r)+1} & \langle D_x\rangle & \cdots & 0 \\ 0 & \tilde{\lambda}_{m-(m_j+...+m_r)+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \langle D_x\rangle \\ 0 & \cdots & 0 & \tilde{\lambda}_{m-(m_j+...+m_r)+m_j} \end{pmatrix},$$

$A(t, x, D^{m+M_0}u, \xi) = A_1 + A_2$ is such that

$$A_1 = (a_{ik}^{(1)}), \quad a_{ik}^{(1)} = \begin{cases} \langle D_x\rangle^\rho & (i,k) = (jr+1, jr) \\ 0 & (i,k) \neq (jr+1, jr), \end{cases} \tag{4.10}$$

$$t^{1-\delta}A_2 \in B\Big([0,T]; S^{h+(r-1)(1-\rho),\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big), \tag{4.11}$$

and $F = (0, ..., 0, if)^t$. Here and in the following, M_0 still denotes a possibly large integer, but still depending only on n and m_j, $j = 1, ..., r$.

We minimize now the order of A looking at (4.10) and (4.11); we have to choose ρ as the solution of the equation $\rho = h + (r-1)(1-\rho)$, so

$$\rho = \frac{h+r-1}{r} = \frac{qr-1+\delta}{qr}. \tag{4.12}$$

Thus, in problem (4.9) the remainder A is such that

$$t^{1-\delta} A \in B\Big([0,T]; S^{\rho,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big),$$

for ρ as in (4.12).

Consider now the matrix $M(t,x,D^{m-1}u, D_x)$ that diagonalizes K and the new variable $V = MZ$. Notice that because of the interpolations done we can consider M, $\partial_t M$, M^{-1} to be all in $B([0,T]; S^{h,\sigma}_{\mu+M-M_0,\epsilon-w(t)})$. Problem (4.9) is equivalent to

$$\begin{cases} LV = \bar{F}, \\ V(0,x) = 0, \end{cases} \tag{4.13}$$

where

$$\begin{cases} L = \partial_t - i\bar{K} + \bar{A}, \\ \bar{K} = \begin{pmatrix} \tilde{\lambda}_1 & & \\ & \ddots & \\ & & \tilde{\lambda}_m \end{pmatrix}, \end{cases} \tag{4.14}$$

and $\bar{A}(t,x,D^{m+M_0}u,\xi)$ satisfies again

$$t^{1-\delta}\bar{A} \in B\Big([0,T]; S^{\rho,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big). \tag{4.15}$$

It is well known that the assumption $\sigma \leq \dfrac{1}{\rho}$ is necessary to have existence and uniqueness of a local in time solution $V(t,\cdot)$ of problem (4.13) in a Gevrey space of index σ . We have

$$\sigma \leq \frac{1}{\rho} = \frac{qr}{qr-1+\delta} < \frac{qr}{qr-1},$$

so condition (1.4) appears in a natural way. Moreover, for any choice of σ we have the corresponding choice of $\delta \in (0,1)$:

$$\delta = 1 - \big(1 - \frac{1}{\sigma}\big)qr. \tag{4.16}$$

5. Energy estimates

To give an energy estimate for (4.13) we need to prove the following:

Proposition 5.1. *Given $u \in E$, consider problem* (4.13) *for the operator* (4.14) *under the hypothesis* (4.15) *and under condition* (1.4). *There are positive constants M, λ, T^* (λ the parameter in* (2.14)*) such that for every $V \in C([0,T]; H^{\mu,\epsilon-w(t),\sigma})$, $T \le T^*$ we have for all $\mu \ge n/2$ the following energy inequality of strictly hyperbolic type:*

$$||V(t)||^2_{\mu,\epsilon-w(t),\sigma} \le e^{\tilde{c}_{r_1} t}\Big[||V(0)||^2_{\mu,\epsilon,\sigma} + \int_0^t ||\bar{F}(\tau)||^2_{\mu,\epsilon-w(\tau),\sigma}\, d\tau\Big] \tag{5.1}$$

if $t \le T \le T^*$.

Proof. It is sufficient to prove (5.1) only for $\mu = 0$ since $\langle D_x\rangle^{\mu} L \langle D_x\rangle^{-\mu}$ satisfies the same hypotheses as L for every μ. Let us introduce the conjugation of L by

$$L_{\epsilon-w(t)} := e^{(\epsilon-w(t))\langle D_x\rangle^{\frac{1}{\sigma}}} L e^{-(\epsilon-w(t))\langle D_x\rangle^{\frac{1}{\sigma}}}.$$

From Proposition 3.2 we have

$$L_{\epsilon-w(t)} = \partial_t - i\bar{K} + \bar{A}_1 + \lambda t^{\delta-1}\langle D_x\rangle^{\frac{1}{\sigma}},$$

where $\bar{A}_1(t,x,D^{m+M_1}u,\xi)$ is such that

$$t^{1-\delta}\bar{A}_1 \in B\Big([0,T]; S^{1/\sigma,\sigma}_{\mu+M-M_1,\epsilon-w(t)}\Big),$$

and where $M_1 \ge M_0$ depends only on a finite number of derivatives we have to consider to perform all the operations we need.

In this way the estimate (5.1) for the operator L is equivalent to the following estimate for $L_{\epsilon-w(t)}$:

$$||V(t)||_0^2 \le e^{\tilde{c}_{r_1} t}\Big[||V(0)||_0^2 + \int_0^t ||L_{\epsilon-w(\tau)}V(\tau)||_0^2\, d\tau\Big], \tag{5.2}$$

$t \in [0,T]$, for every $V \in C^1([0,T]; H^1(\mathbb{R}^n))$.

To prove (5.2), let us consider

$$\begin{aligned}
\frac{d}{dt}||V(t)||_0^2 &= 2Re\langle V'(t), V(t)\rangle_0 \\
&= 2Re\langle i\bar{K}V, V\rangle_0 - 2Re\langle\Big(\bar{A}_1 + \frac{\lambda}{t^{1-\delta}}\langle D_x\rangle^{\frac{1}{\sigma}}\Big)V, V\rangle_0 \\
&+ 2Re\langle L_{\epsilon-w(t)}V, V\rangle_0.
\end{aligned}$$

Since $\bar{K}$ is real diagonal and thanks to (3.7) we have

$$|Re\langle i\bar{K}V, V\rangle_0| \leq c_{r_1}||V||_0^2,$$
$$|\langle \bar{A}_1 V, V\rangle_0| \leq \frac{\lambda_{r_1}}{t^{1-\delta}}\langle\langle D_x\rangle^{\frac{1}{\sigma}}V, V\rangle_0,$$

provided that we have fixed $M > M_1$ with a larger M_1 depending also on μ. Next we fix the constant

$$\lambda > \lambda_{r_1}, \tag{5.3}$$

so that $\bar{A}_1 + \frac{\lambda}{t^{1-\delta}}\langle D_x\rangle^{\frac{1}{\sigma}}$ is a positive operator for $t > 0$. For such a λ we have:

$$\frac{d}{dt}||V(t)||_0^2 \leq \tilde{c}_{r_1}||V||_0^2 + ||L_{\epsilon-w(t)}V||_0^2.$$

Conditions (5.3) and (3.2) together give $t \leq T^*$. An application of Gronwall's inequality immediately gives estimate (5.2). □

Now, let us come back to problem (3.4). By Proposition 5.1 and looking at (4.8) (where $(r-1)(1-\rho) = \frac{(r-1)(1-\delta)}{qr} < \eta$) we find for (3.4) a solution

$$v \in C^m([0,T]; H^{\mu+m,\epsilon-w(t),\sigma}).$$

The solution is less regular than $u \in C^m([0,T]; H^{\mu+m+M,\epsilon-w(t),\sigma})$, so it is impossible to use a fixed-point scheme.

To obtain a v as regular as u, let us take derivatives ∂_x^β in (3.4) for all $|\beta| \leq M$. For $v^{(\beta)} = \partial_x^\beta v$ one obtains the equations:

$$\begin{cases} Pv^{(\beta)} + [\partial_x^\beta, P]v^{(0)} = \partial_x^\beta f, & (t,x) \in [0,T]\times\mathbb{R}^n, \\ \partial_t^j v^{(\beta)}(0,x) = 0, & j = 0,...,m-1, \end{cases} \qquad |\beta| \leq M. \tag{5.4}$$

Defining the functions $v_k^{(\beta)}$ in the same way as the functions z_k in (4.7), $k = 0,...,m-1$, but using the new function $v^{(\beta)}$ instead of v, $|\beta| \leq M$, we obtain

$$V^{(\beta)} = (v_0^{(\beta)}, ..., v_{m-1}^{(\beta)})^t, \qquad W = \{V^{(\beta)}; |\beta| \leq M\}, \tag{5.5}$$

and reducing problem (5.4) to a first-order system we find

$$\begin{cases} L_0 W = F_0, \\ W(0,x) = 0, \end{cases} \tag{5.6}$$

where

$$L_0 = \partial_t - iK_0 + A_0 + HQ,$$

$-iK_0 + A_0$ is a block diagonal matrix with all blocks equal to $-i\bar{K} + \bar{A}$ in (4.14), and the term HQW represents the commutators $[\partial_x^\beta, P]v$ by means of (4.8): H is a matrix of functions,

$$H(t, x, D^{M+m-1}u) \in C^1([0,T]; H^{\mu+1}),$$

whereas Q is a matrix of pseudodifferential operators of order η,

$$Q(t, x, D^{M_0+m}u, \xi) \in C\Big([0,T]; S^{\eta,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big).$$

To have by (5.6) a solution v of (3.4) in $C^m([0,T]; H^{\mu+m+M,\epsilon-w(t),\sigma})$ like u, one has to prove the following:

Proposition 5.2. *Given $u \in E$, let us consider the Cauchy problem* (5.6) *under condition* (1.4)*. There are positive constants λ, T^* such that for every $W \in C([0,T]; H^{\mu,\epsilon-w(t),\sigma})$, $T \le T^*$, $\mu > n/2$, we have the following estimate of strictly hyperbolic type:*

$$||W(t)||^2_{\mu,\epsilon-w(t),\sigma} \le e^{\tilde{c}_{r_1} t}\Big[||W(0)||^2_{\mu,\epsilon,\sigma} + \int_0^t ||F_0(\tau)||^2_{\mu,\epsilon-w(\tau),\sigma}\, d\tau\Big] \qquad (5.7)$$

if $t \le T \le T^*$.

Proof. In repeating the proof of Proposition 5.1, the only new term to control is HQ. It has to be a bounded operator from $H^{\mu+1/\sigma,\epsilon-w(t),\sigma}$ to $H^{\mu,\epsilon-w(t),\sigma}$ uniformly with respect to $u \in E$ defined in (3.3) and $t \in [0,T]$; but this is true for $t \le T^*$ (see (3.2)) because $H^{\mu+1/\sigma,\epsilon-w(t),\sigma}$ is a Banach algebra:

$$\begin{aligned} ||HQW||_{\mu+1/\sigma,\epsilon-w(t),\sigma} &\le ||H||_{\mu+1/\sigma,\epsilon-w(t),\sigma} \cdot ||QW||_{\mu+1/\sigma,\epsilon-w(t),\sigma} \\ &\le \tilde{\tilde{C}}_{r_1} ||W||_{\mu,\epsilon-w(t),\sigma} \end{aligned}$$

(we recall that we have fixed $\mu > n/2$ from the beginning, and moreover that $\eta < 1/\sigma$). Now the proof of Proposition 5.2 follows from the one of Proposition 5.1. □

6. Construction of a fixed-point scheme

Proof of Proposition 3.1. To complete the proof of Proposition 3.1 using Proposition 5.2, we only have to notice that from (4.8) and (5.5) we have

$$\begin{cases} D^{m-1}v = \bar{S}(t, x, D^{M+M_0+m}u, D_x)W, \\ \bar{S} \in B\Big([0,T]; S^{\eta,\sigma}_{\mu+M-M_0,\epsilon-w(t)}\Big). \end{cases} \qquad (6.1)$$

So for the function $v \in C^m([0,T]; H^{\mu+m+M,\epsilon-w(t),\sigma})$ corresponding to W in (5.7) we have the following energy estimate:

$$\sum_{j=0}^{m-1} ||\partial_t^j v(t)||^2_{\mu+m+M-j,\epsilon-w(t),\sigma} \leq e^{\tilde{c}_{r_1} t} \int_0^t ||Pv(\tau)||^2_{\mu+M+\eta+1,\epsilon-w(\tau),\sigma} \, d\tau, \tag{6.2}$$

for $t \in [0,T]$, if $t \leq T^*$. From (6.2) it follows that

$$||v||^2_{C^{m-1}([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \leq e^{\tilde{c}_{r_1} t} \cdot t||Pv||^2_{C^0([0,T];H^{\mu+M+\eta+1,\epsilon-w(t),\sigma})},$$

while from

$$D_t^m v + \sum_{j=0}^{m-1} \sum_{|\alpha| \leq m-j} a_\alpha^{(j)}(t,x,D^{m-1}u) D_x^\alpha D_t^j v = f(t,x),$$

and again by (4.7) and (6.1), we have

$$\begin{aligned} &||v||^2_{C^m([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \\ &\leq ||f||^2_{C^0([0,T];H^{\mu+M+\eta+1,\epsilon-w(t),\sigma})} + c_{r_0} ||v||^2_{C^{m-1}([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})}. \end{aligned}$$

Now, we only have to define r_0, r_1, T_0 to make $v \in E$ and complete the proof. So let's fix r_0 and r_1 such that

$$\begin{cases} r_0 > ||f||_{C^0([0,T];H^{\mu+M+\eta+1,\epsilon-w(t),\sigma})}, \\ r_1^2 > ||f||^2_{C^0([0,T];H^{\mu+M+\eta+1,\epsilon-w(t),\sigma})} + c_{r_0} r_0^2; \end{cases} \tag{6.3}$$

then for $t \to 0$ there exists a $T_0 \leq 1$ such that if $t \leq T_0$ we have

$$||v||_{C^{m-1}([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} < r_0,$$

and consequently

$$||v||^2_{C^m([0,T];H^{\mu+m+M,\epsilon-w(t),\sigma})} \leq ||f||^2_{C^0([0,T];H^{\mu+M+\eta+1,\epsilon,\sigma})} + c_{r_0} r_0^2 < r_1^2.$$

The choice of r_0 and r_1 in (6.3) makes the proof of Proposition 3.1 complete. □

Proof of Theorem 2.3. Finally, let us come back to the proof of Theorem 2.3. We know now that the mapping (3.5) is well defined. Consider the sequence:

$$\begin{cases} u^{(0)} = 0 \\ u^{(k+1)} = S(u^{(k)}) \quad k \in \mathbb{N}. \end{cases} \tag{6.4}$$

Since $f(t,x,0)$ is of compact support, the same is uniformly true for all the u_k's, because they are solutions of linear hyperbolic problems. So, (6.4) admits a subsequence converging in the space $C^m([0,T]; H^{\mu+m+M-1,\epsilon,\sigma})$ to a solution u of problem (3.1).

By the usual arguments in the energy method, see [11] for example, inequality (6.2) implies that the solution is unique. Theorem 2.3 is completely proved. □

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