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MULTIPLICITY OF SOLUTIONS TO A SCALAR-FIELD EQUATION INVOLVING THE SOBOLEV CRITICAL EXPONENT WITH THE ROBIN CONDITION

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Abstract. A scalar-field type equation with the Robin condition is discussed. For some range of the parameter, the multiplicity of solutions is obtained. Moreover, one of the obtained solutions has a blowup property and the blowup point is determined.

1. Introduction

The elliptic equations with the Sobolev critical exponent have been intensively studied from various points of view. We are interested in the radial solutions and consider the following equation

$$\begin{cases} \dfrac{1}{r^{n-1}}(r^{n-1}u_r)_r + \lambda u + u^{(n+2)/(n-2)} = 0, \quad r \in (0,1), \\ u > 0, \quad r \in [0,1], \\ \kappa u_r(1) + u(1) = 0 \end{cases} \tag{1.1}$$

with $n \geq 3$ and $\kappa \geq 0$.

Under the homogeneous Dirichlet condition ($\kappa = 0$), we see from Brezis and Nirenberg [1] combined with the uniqueness results by Kwong and Li [7], and Zhang [13], that the following theorem holds:

Theorem A. *Let* $\kappa = 0$.

(i) *If* $n = 3$: *Then* (1.1) *has a solution if and only if* $\pi^2/4 < \lambda < \pi^2$ *and the solution is unique.*

(ii) *If* $n \geq 4$: *Then* (1.1) *has a solution if and only if* $0 < \lambda < \mu_{0,n}^2$ *and the solution is unique, where* $\mu_{0,n}^2$ *is the first eigenvalue of* $-\Delta$ *under the homogeneous Dirichlet condition in the unit ball of* $\mathbf{R}^n$.

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Thus, the dimension $n = 3$ is exceptional and is called the critical dimension. Here we concentrate on the three-dimensional case and investigate solutions $u \in C([0,1]) \cap C^2((0,1))$ to the problem

$$\begin{cases} \dfrac{1}{r^2}(r^2 u_r)_r + \lambda u + u^5 = 0, \quad r \in (0,1), \\ u > 0, \quad r \in [0,1], \\ \kappa u_r(1) + u(1) = 0, \end{cases} \tag{1.2}$$

where $\kappa > 0$.

For the case $\kappa > 0$, the following results are known by Kabeya, Yanagida and Yotsutani [5].

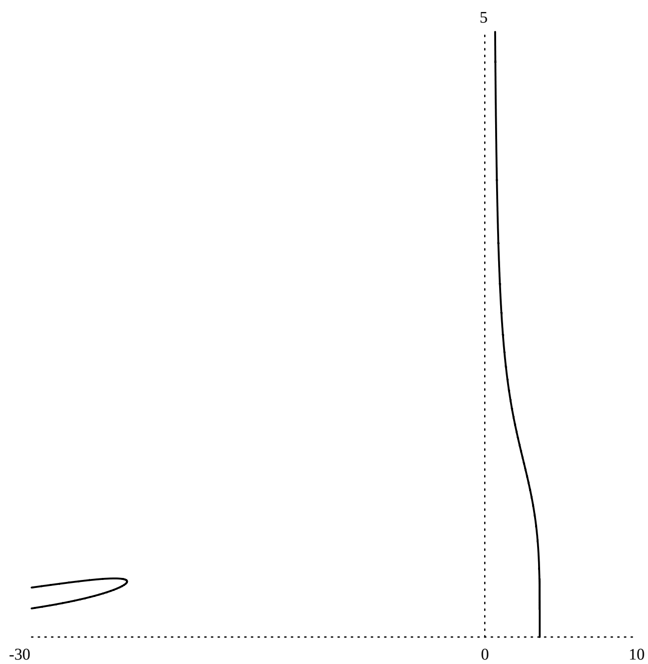

FIGURE 1. The bifurcation diagram of the equation (1.2) with $\kappa = 0.6$. The horizontal axis is λ and the vertical axis is $u(0;\lambda)$.

Theorem B (Theorem 1.3 of [5]). *Suppose that* $0 < \kappa \leq 1$. (1.2) *has a unique solution* $u(r;\lambda)$ *for* $\lambda \in (\mu_1^2, \mu_0^2)$, *and has no solution for* $\lambda \in$

$[-\zeta^2, \mu_1^2] \cup [\mu_0^2, \infty)$ *where* $\mu_0 = \mu_0(\kappa) \in (0, \pi)$, $\mu_1 = \mu_1(\kappa) \in [0, \pi/2)$ *and* $\zeta = \zeta(\kappa) \in [0, \infty)$ *are defined by*

$$\begin{aligned} &1 - \mu_0 \cot \mu_0 = 1/\kappa, \\ &1 + \mu_1 \tan \mu_1 = 1/\kappa, \\ &\zeta \coth \zeta = 1/\kappa \text{ for } \kappa \in (0,1), \ \zeta = 0 \text{ for } \ \kappa = 1, \end{aligned}$$

respectively. Moreover, $u(0; \lambda) \to \infty$ *as* $\lambda \downarrow \mu_1^2$ *and* $u(0; \lambda) \to 0$ *as* $\lambda \uparrow \mu_0^2$.

Note that the value μ_0^2 is the first eigenvalue of $-\Delta$ under the condition $\kappa \partial u/\partial \nu + u = 0$ in the unit ball of $\mathbf{R}^3$ and the linear problem

$$\begin{cases} u_{rr} + \dfrac{2}{r} u_r + \mu_1^2 u = 0, \quad r \in (0,1), \\ \kappa u_1(1) + u(1) = 0, \end{cases}$$

has the singular solution of the form

$$u(r) = \frac{\cos \mu_1 r}{\mu_1 r}.$$

In Figure 1, we show a numerically computed graph of $u(0; \lambda)$ for $\kappa = 0.6$. The rightmost branch is the branch rigorously shown in Theorem B. A curve on the left side is a part of a solution branch to the scalar field equation $(\lambda < 0)$ with one maximum point in $(0, 1)$.

On the other hand, for the case $\kappa > 1$, we have the following.

Theorem C ((ii) of Theorem 1.4 of [5]). *Let* $n = 3$. *Suppose that* $\kappa > 1$. (1.1) *has a unique solution for* $\lambda \in [-\mu_2^2, \mu_0^2)$ *and has no solution for* $\lambda \in [\mu_0^2, \infty)$, *where* μ_2 *is defined by*

$$\mu_2 \tanh \mu_2 = (\kappa - 1)/\kappa.$$

In Figure 2, we show a numerically computed graph of $u(0; \lambda)$ for $\kappa = 3$.

Two curves on the left side are parts of a solution branch to the scalar field equation $(\lambda < 0)$. The lower one has one maximum point in $(0, 1)$. The other has one local minimum and one local maximum in $(0, 1)$.

The rightmost curve is the solution branch bifurcating from $(\mu_0^2, 0)$. We note that each solution on this branch is strictly monotone decreasing. The branch with $\lambda \geq -\mu_2^2$ is rigorously assured by Theorem C. The multiplicity of solutions seems to hold for λ sufficiently close to $-\mu_2^2$ with $\lambda < -\mu_2^2$. We will show this fact.

Theorem 1.1. *Let* $n = 3$. *Suppose that* $\kappa > 1$. *There exists* $\varepsilon_0 > 0$ *such that* (1.1) *has at least two solutions* $u_1(r, \lambda)$, $u_2(r, \lambda)$ *for* $\lambda \in [-\mu_2^2 - \varepsilon_0, -\mu_2^2)$.

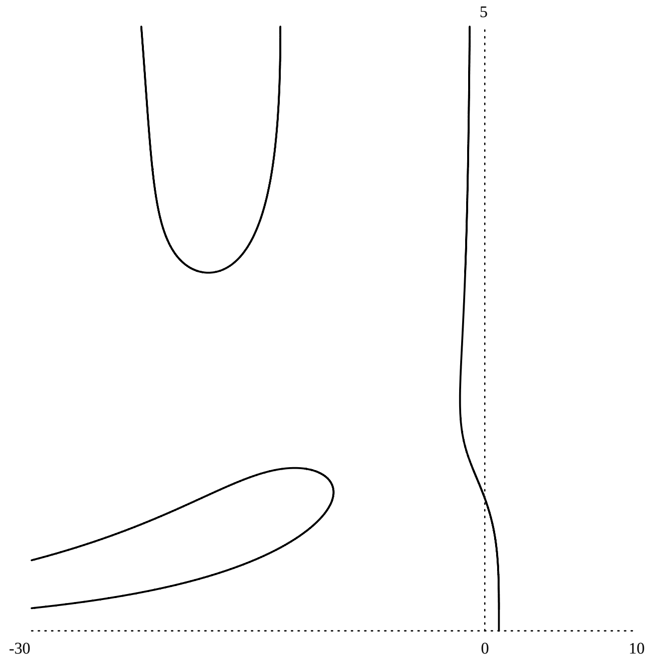

FIGURE 2. The bifurcation diagram of the equation (1.2) with $\kappa = 3$. The horizontal and the vertical axes are λ and $u(0;\lambda)$, respectively.

They are monotone decreasing in r and $u_1(0,\lambda)$ is uniformly bounded while $u_2(0,\lambda) \to \infty$ as $\lambda \uparrow -\mu_2^2$.

Here, we explain our strategy to prove the main result.

To show the sharpness of μ_2, we transform (1.2) to a Matukuma-type equation (3.1) in Section 3 under the exterior Neumann problem with rapid-decay condition at infinity (as introduced in Kabeya, Yanagida and Yotsutani [5]). This argument is done in Section 3. We will show that the structure of positive solutions to the transformed exterior Neumann problem (3.1) for $\lambda < -\mu_2^2$ differs from the case for $\lambda \geq -\mu_2^2$. These facts are proved by using Lemma 4.1 in Section 4 and the arguments based on the Pohozaev identity as in Kawano, Yanagida and Yotsutani [6], Yanagida and Yotsutani [11, 12] and [5].

The change of structure implies that the transformed problem has a solution whose maximum value is sufficiently small in the region $\lambda < -\mu_2^2$. Hence, for the corresponding original problem, the existence of a large solution to (1.2), whose supremum norm blows up at $\lambda = -\mu_2^2$ is ensured. Moreover, the large solution is approximated by a suitable function, whose properties come from the "self-similarity" of the solutions to $\Delta u + u^5 = 0$ in $\mathbf{R}^3$. The larger solution is indeed on the solution branch bifurcating from $(\mu_0^2, 0)$ and the branch never intersects with that under the Dirichlet or Neumann ones.

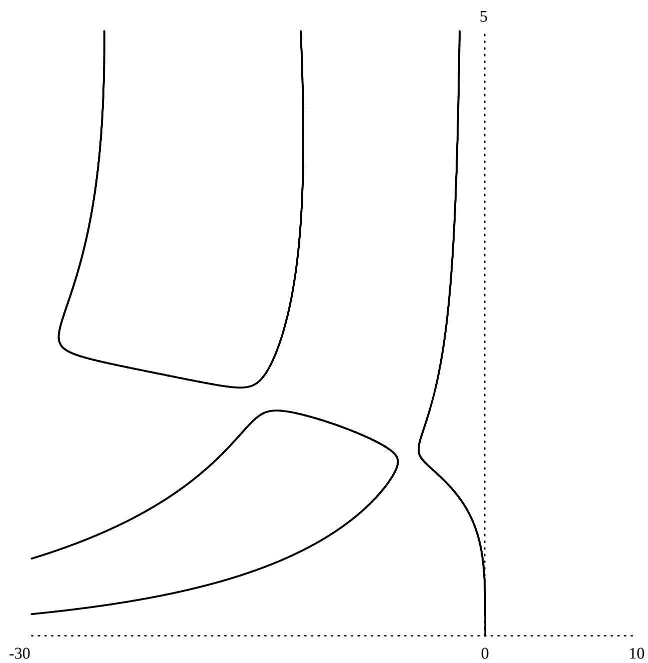

FIGURE 3. The (imperfect) bifurcation diagram of the equation (1.2) with $\kappa = 100$. The horizontal and the vertical axes are λ and $u(0;\lambda)$, respectively.

In addition, the smaller solution is also on the same branch. Thus, a priori, the solution branch with the Robin condition blows up between $\lambda = -\mu_2^2$ obtained by [5] and the blowup point of the Neumann problem $\lambda = -\mu_*^2$, which was investigated by [2]. See the rightmost blowing-up graph of the numerically computed graph in Figure 3 and compare this with Figure 4.

Letting $\kappa \to \infty$, we see that $\mu_2 \to \mu_*$ with $\mu_* \tanh \mu_* = 1$. The value μ_* is the same one as obtained in Budd, Knaap and Peletier [2] with $q = 1$ and $n = 3$. In Figures 3 and 4, we show the numerically computed graphs of $u(0;\lambda)$

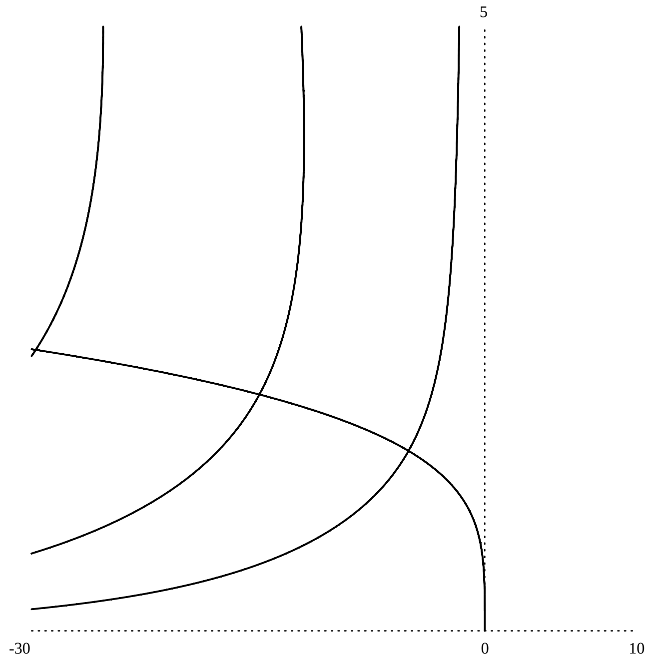

Figure 4. The bifurcation diagram of the equation (1.2) with homogeneous Neumann condition ($\kappa = \infty$).The horizontal and the vertical axes are λ and $u(0;\lambda)$, respectively.

for $\kappa = 100$ and $\kappa = \infty$ (Neumann boundary condition), respectively. Thus, the structure of positive solutions to (1.2) could be continuous with respect to κ even as $\kappa \to \infty$. For details on these branches, see Kabeya, Morishita and Ninomiya [4]. The blowup behavior will be discussed in Kabeya [3].

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by admitting a crucial proposition (Proposition 2.1). Section 3 has several technical lemmas and a proposition which are necessary to prove Proposition 2.1. Proposition 2.1 is proved in Section 4 together with an approximation lemma of a solution with a large initial value.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. In addition, we can prove not only the existence of two solutions, we can prove that the two solutions are on the same solution branch. Moreover, a part of the branch near the blowing-up point is indeed a continuous curve which bifurcates from $(\mu_0^2, 0)$ and connects to ∞ in $\mathbf{R} \times C([0,1])$ by Theorem 2.3 of Rabinowitz [9] (see also Theorem 2.12 of [9] and Section 2 of Rabinowitz [10]). We recall the statement of Theorem 2.3 of Rabinowitz [9], which gives us fundamental information on the solution branch. Our concern is also on the bifurcating branch from the trivial solution; his theorem is applicable to the Robin problem.

Theorem D (Theorem 2.3 of [9]). *Consider the problem*

$$\begin{cases} -(pu')' + qu = \lambda a(x)u + H(x,u,u',\lambda) \ \text{in } (0,\pi), \\ a_0 u(0) + b_0 u'(0) = 0, \ a_1 u(\pi) + b_1 u'(\pi) = 0 \end{cases} \tag{2.1}$$

where $p \in C^1([0,1])$ with $p > 0$ on $[0,\pi]$, $q \in C([0,\pi])$, $a \in C([0,\pi])$ with $a > 0$ on $[0,\pi]$, $H(x,y,z,\lambda) = O(\sqrt{y^2+z^2})$ near $(y,z) = (0,0)$ uniformly in bounded λ intervals and $H \in C([0,\pi] \times \mathbf{R}^3)$, $a_i^2 + b_i^2 \neq 0$ $(i = 0,1)$. Then (2.1) has a closed and connected positive solution branch $\mathcal{C} \subset \mathbf{R} \times C^{1+\beta}([0,\pi])$ $(0 < \beta < 1)$ which meets $(\lambda_1, 0)$ and ∞ with λ_1 being the first eigenvalue of the linearized equation of (2.1) around $u \equiv 0$.

In our case, $p = r^2$, $q \equiv 0$ and $a(x) \equiv r^2$, $a_0 = 0$, $a_1 = 1$, $b_0 = 1$, $b_1 = \kappa$ and suitable changes of variables reduce (2.1) to the problem on $[0,1]$. Seemingly, $p = r^2$ does not satisfy the assumption. However, this case is applicable since the essential point of the proof of Theorem 2.3 of [9] is that the eigenvalues of the linear problem $-(pu')' + qu = \lambda au$ correspond to those of the integral equation of the form $v = \mu Lv$ in $C[(0,\pi])$, where L is the integral operator so that $-(pu')' + qu = \lambda au$ is equivalent to $v = \mu Lv$. This correspondence holds for the first eigenvalue (which is simple) of $-\Delta$ with the Robin condition on the unit ball.

In fact, following the proof of Theorem 2.3 in [9], we discuss the applicability of Theorem D. We take $U_1(r) = 1$ and $U_2(r) = r^{-1} + \kappa^{-1} - 1$. We note that U_1 is a solution to $(r^2 U_r)_r = 0$ with $U_r(0) = 0$ and that U_2 is a solution to $(r^2 U_r)_r = 0$ with $U_r(1) + \kappa U(1) = 0$. We define the Green's function $g(r,s)$ by

$$g(r,s) = \begin{cases} U_1(s)U_2(r), & r \geq s, \\ U_1(r)U_2(s), & r < s. \end{cases}$$

Hence we see the exact form as

$$g(r,s)=\begin{cases}\dfrac{1}{r}+\dfrac{1}{\kappa}-1, & r\ge s,\\ \dfrac{1}{s}+\dfrac{1}{\kappa}-1, & r<s.\end{cases}$$

We define a map $G(\lambda,u)$ from $\mathbf{R}\times C([0,1])$ into $C([0,1])$ by

$$G(\lambda,u)=\int_0^1 g(r,s)s^2(\lambda u+u^5)\,ds. \tag{2.2}$$

$G(\lambda,u)$ is defined by applying the Green's function of $-\Delta$ with the Robin condition (divided by 4π) to $\lambda u+u^5$. We can see that (1.2) is equivalent to the integral equation $u=G(\lambda,u)$ and that G is a continuous and compact map from $\mathbf{R}\times C([0,1])$ into $C([0,1])$. Let us define a linear map L from $C([0,1])$ into itself by

$$Lu=\int_0^1 g(r,s)s^2u\,ds.$$

Then the largest eigenvalue of L is the reciprocal of the first eigenvalue of $-\Delta$ with $u_r(1)+\kappa u(1)=0$ and is simple. Hence, the assumptions in Theorem 1.3 in [9], which is used to prove Theorem 2.3 in [9], are assured and the applicability of Theorem D follows.

Invoking Theorem D, we have the following lemma.

Lemma 2.1. *For any $\alpha>0$, there exists $\lambda(\alpha)\in\mathbf{R}$ so that*

$$\begin{cases}\dfrac{1}{r^2}(r^2u_r)_r+\lambda(\alpha)u+u^5=0,\ r\in(0,1)\\ u(0)=\alpha,\quad u_r<0,\ r\in(0,1)\\ \kappa u_r(1)+u(1)=0\end{cases} \tag{2.3}$$

has a solution. Moreover, $\lambda(\alpha)$ is continuous for $\alpha\ge\alpha_0$ with some $\alpha_0>0$.

Proof. Let us consider the initial-value problem

$$\begin{cases}\dfrac{1}{r^2}(r^2u_r)_r+\lambda u+u^5=0,\ r\in(0,1)\\ u(0)=\alpha,\end{cases} \tag{2.4}$$

and define

$$\Xi(r;\lambda;\alpha):=\kappa r u_r(r;\lambda;\alpha)+u(r;\lambda;\alpha)$$

for the solution $u(r;\lambda;\alpha)$ to (2.4). At $\lambda = 0$, we see from the direct computations that

$$u(r;0;\alpha_*) = \frac{\alpha_*}{(1+\alpha_*^4 r^2/3)^{1/2}}$$

is the unique exact solution to (2.3) with $\alpha_* := \{3/(\kappa-1)\}^{1/4}$. Moreover, $\Xi(1;0;\alpha) < 0$ for $\alpha > \alpha_*$ and $\Xi(1;0;\alpha) > 0$ for $\alpha \in (0,\alpha_*)$ in view of

$$\Xi(1;0;\alpha) = \kappa u_r(1;0;\alpha) + u(1;0;\alpha) = (1+\frac{\alpha^4}{3})^{-3/2}(1-\frac{\kappa-1}{3}\alpha^4)\alpha.$$

For any $\alpha > \alpha_* = \{3/(\kappa-1)\}^{1/4}$, take $\lambda_{N,\alpha} < 0$ so that at $\lambda = \lambda_{N,\alpha}$, (2.4) has a positive solution with $u_r(r) < 0$ in $(0,1)$ and $u_r(1) = 0$. This can be possible due to the result by Ni [8]. Indeed, Lemma 2.2 in [2] shows that a solution to (2.4) is oscillatory, in other words, the solution has infinitely many critical points in $(0,\infty)$ for any α by using Theorem 2.1 of [8]. Moreover, by the result of [2], $\lambda_{N,\alpha} \to -\mu_*^2$ as $\alpha \to \infty$.

By suitable scaling as in (2.5) as below, we can take such $\lambda_{N,\alpha}$. Hence we see that $\Xi(1;\lambda_{N,\alpha};\alpha) > 0$ since $u_r(1;\lambda_{N,\alpha};\alpha) = 0$ and $u(1;\lambda_{N,\alpha};\alpha) > 0$ as stated above.

On the other hand, as we have seen above, $\Xi(1;0;\alpha) < 0$ for $\alpha > \alpha_*$. Moreover, $\Xi(1;0;\alpha) \to \Xi(1;0;\alpha_*) = 0$ as $\alpha \to \alpha_*$. From the continuity with respect to λ, there exists $\lambda > 0$ such that $\Xi(1;\lambda;\alpha) = 0$ for $\alpha > \alpha_*$. We take the largest such $\lambda < 0$ and denote it by $\lambda(\alpha)$. Note that $\lambda(\alpha) \to 0$ as $\alpha \downarrow \alpha_*$.

For $\alpha < \alpha_*$, $\Xi(1;0;\alpha) > 0$ and we see that $\Xi(1;\mu_0^2;\alpha) < 0$. Again, as above, we can take the smallest $\lambda > 0$ so that $\Xi(1;\lambda;\alpha) = 0$ and denote it also by $\lambda(\alpha)$. Due to the uniqueness result (Theorem C), we see that $\lambda(\alpha) < -\mu_2^2$ for any sufficiently large $\alpha > 0$.

Differentiating Ξ with respect to r, we have

$$\Xi_r(r;\lambda(\alpha);\alpha) = \kappa r u_{rr} + (\kappa+1)u_r = -(\kappa-1)u_r - \kappa r(\lambda(\alpha)+u^4)u.$$

Since $\Xi(1;\lambda(\alpha);\alpha) = 0$, we obtain $u_r(1;\lambda(\alpha);\alpha) = -u(1;\lambda(\alpha);\alpha)/\kappa$ and

$$\Xi_r(1;\lambda(\alpha);\alpha) = \Big\{\frac{\kappa-1}{\kappa^2} - (\lambda(\alpha)+u(1;\lambda(\alpha);\alpha)^4)\Big\}\kappa u(1;\lambda(\alpha);\alpha) \neq 0$$

if $u(1;\lambda(\alpha);\alpha)^4 \neq -\lambda(\alpha)+(\kappa-1)/\kappa^2$. We say that $\lambda(\alpha)$ is "non-degenerate" if $\Xi_r(1;\lambda(\alpha);\alpha) \neq 0$. Since $u(1;0;\alpha_*)^4 = 3(\kappa-1)/\kappa^2 \neq (\kappa-1)/\kappa^2$, $\lambda(\alpha)$ is "non-degenerate" for any $\alpha > \alpha_*$ which is close to α_*. For any "non-degenerate" $\lambda(\alpha_0)$, we have $\Xi_r(1;\lambda(\alpha_0);\alpha_0) \neq 0$. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\Xi(r;\lambda(\alpha_0);\alpha)$ changes its sign for some $r(\alpha) \in (1-\delta,1+\delta)$ for any $\alpha \in (\alpha_0-\varepsilon,\alpha_0+\varepsilon)$ in view of the continuity of w and w_r

with respect to α. $r(\alpha)$ is continuous with respect to α. By the change of variables

$$r = r(\alpha)\rho, \quad U(\rho) = r(\alpha)^{-1/2} u(r; \lambda(\alpha_0); \alpha) \tag{2.5}$$

$U(\rho)$ satisfies

$$\begin{cases} \dfrac{1}{\rho^2}(\rho^2 U_\rho)_\rho + r(\alpha)^2 \lambda(\alpha_0) U + U^5 = 0, \ \rho \in (0,1) \\ U(0) = \alpha r(\alpha)^{-1/2}, \\ \kappa U_\rho(1) + U(1) = 0. \end{cases} \tag{2.6}$$

The desired $\lambda(\alpha)$ is taken as $\lambda(\alpha) = r(\alpha)^2 \lambda(\alpha_0)$ and is continuous with respect to α.

From Lemma 4.1 in Section 4, we see that $u(1; \lambda; \alpha) \to 0$ as $\alpha \to \infty$ for any $\lambda \in [-\mu_*^2, -\mu_2^2]$. Thus, there exists $\alpha_0 > 0$ such that $u(1; \lambda(\alpha); \alpha) < -\lambda(\alpha) + (\kappa - 1)/\kappa^2$ and $\lambda(\alpha)$ is continuous for $\alpha > \alpha_0$. □

The following proposition is crucial for a proof of Theorem 1.1. We can determine the "blowup point" of $u(0; \lambda)$ in terms of λ.

Proposition 2.1. *Let $u(r; \lambda)$ be a monotone-decreasing solution to (1.2). If there exists $\lambda^* < 0$ such that $u(0; \lambda_j) \to \infty$ as $\lambda \uparrow \lambda^*$, then $\lambda^* = -\mu_2^2$.*

A proof of Proposition 2.1 is given in Section 4. Admitting Proposition 2.1, we prove Theorem 1.1.

As for the decaying property of the obtained two solutions, we use the fact that the number of critical points of a solution on a connected component of a solution branch is constant. A solution on the branch bifurcating from $(\mu_0^2, 0) \in \mathbf{R} \times C([0,1])$ has no critical point however large the initial value is.

Proof of Theorem 1.1. Let us define a set $\Gamma \subset \mathbf{R} \times C([0,1])$ by

$$\Gamma^* := \{ (\lambda, u(r; \lambda)) : u(r; \lambda) \text{ is a solution to (1.2) and decreasing} \}.$$

The set Γ^* is a bifurcating branch which bifurcates from $(\mu_0^2, 0)$. Again, essentially due to Theorem D, since μ_0^2 is a simple eigenvalue, and since Γ^* is a "zero-mode" branch, Γ^* is a connected and unbounded set in $\mathbf{R} \times C([0,1])$. Let us also define

$$\Gamma_1^* := \{ (\lambda, u(\cdot; \lambda)) \in \Gamma^* : \lambda \geq -\mu_2^2 \},$$
$$\Gamma_2^* := \{ (\lambda, u(\cdot; \lambda)) \in \Gamma^* : \lambda < -\mu_2^2 \},$$

and

$$\Gamma_1 := \{ (\lambda, u(0; \lambda)) \in \mathbf{R}^2 : (\lambda, u_\lambda) \in \Gamma_1^* \},$$
$$\Gamma_2 := \{ (\lambda, u(0; \lambda)) \in \mathbf{R}^2 : (\lambda, u_\lambda) \in \Gamma_2^* \}.$$

As stated in Theorem C in the introduction, a positive solution $u(r;\lambda)$ to (1.2) with $\lambda \geq -\mu_2^2$ is unique. Thus, Γ_1 coincides with the connected solution curve obtained in Theorem C. Since Γ_1 is a bounded set, Γ_2 must be an unbounded set. Moreover, since $u(r;-\mu_2^2)$ is monotone decreasing and since $(\lambda,(-\lambda)^{1/4}) \notin \Gamma_2$ for any $\lambda < 0$, any $(\lambda,u(r;\lambda)) \in \Gamma_2^*$ satisfies $u(0;\lambda) > (-\lambda)^{1/4}$. Since the map from $u(0;\lambda)$ to λ is continuous in view of Lemma 2.1 if $u(0;\lambda)$ is sufficiently large, Γ_2 is continuous if $u(0;\lambda)$ is large. Also we define

$$\Gamma := \{\, (\lambda, u(0;\lambda)) \in \mathbf{R}^2 : (\lambda, u(r;\lambda)) \in \Gamma^*\}.$$

To prove the sharpness of the value μ_2, we invoke the result by [2]. Let us consider the homogeneous Neumann problem

$$\begin{cases} \dfrac{1}{r^2}(r^2 v_r)_r + \lambda v + v^5 = 0, \ 0 < r < 1, \\ v > 0, \ 0 \leq r \leq 1, \\ v_r(1) = 0, \end{cases} \tag{2.7}$$

with $v \in C^2((0,1]) \cap C([0,1])$ and define three sets by

$$\Gamma_N^* := \{(\lambda, v(r;\lambda)) : v(r;\lambda) \text{ is a solution to (2.7) and monotone decreasing}\},$$

$$\Gamma_N := \{(\lambda, v(0;\lambda)) \in \mathbf{R}^2 : (\lambda, v_\lambda) \in \Gamma_N^*\},$$

and

$$\Gamma_C^* := \{(\lambda, (-\lambda)^{1/4}) \in \mathbf{R} \times C([0,1]) : 0 \geq \lambda \geq -\mu_N/4\},$$

where μ_N is the smallest positive radial eigenvalue of $-\Delta$ under the homogeneous Neumann condition. Note that Γ_C^* is the set of constant solution branches to (2.7) and can be regarded as a curve in $\mathbf{R}^2$. By this identification, we define

$$\Gamma_C := \{(\lambda, (-\lambda)^{1/4}) \in \mathbf{R}^2 : 0 \geq \lambda \geq -\mu_N/4\}.$$

Due to the result by [2], there exists $(\lambda, v(\cdot;\lambda)) \in \Gamma_N^*$ such that $v(0;\lambda) \to \infty$ as $\lambda \uparrow -\mu_*^2 (< -\mu_2^2)$, where μ_* is defined in Section 1.

By [2], there exists $\Lambda_* > 0$ such that $(\Gamma_N \cup \Gamma_C) \subset [-\Lambda_*, 0] \times [0,\infty)$. We also note that $(\Gamma_N \cup \Gamma_C) \cap \Gamma = \emptyset$. Indeed, if $u_\lambda \in (\Gamma_N \cup \Gamma_C) \cap \Gamma \neq \emptyset$, then u_λ satisfies both the Robin and the Neumann conditions. This is a contradiction since $u \equiv 0$ is excluded by the definition of Γ^*. For $(0, u(r;0)) \in \Gamma$, we have $u(0;0) > 0$. Hence, Γ is located entirely on the right of $\Gamma_N \cup \Gamma_C$. By Theorem C, Γ^* near the blowing-up point can only have a disjoint intersection with the region where $\lambda \geq -\mu_2^2$ because the uniqueness of solutions is ensured. Moreover, as commented above, since Γ cannot intersect Γ_N, Γ "blows up" between $-\mu_\infty^2 \leq \lambda \leq -\mu_2^2$ (see Figures 2 and 3 in Section 1). Thus, a monotone-decreasing solution with an arbitrarily large initial value exists for some $\lambda \in [-\mu_\infty^2, -\mu_2^2]$.

As λ approaches the blowup point, the initial value $u(0;\lambda)$ becomes large. Then, by Proposition 2.1, there exists a pair $(\lambda, u(r;\lambda))$ such that $u(0;\lambda) \to \infty$ as $\lambda \uparrow -\mu_2^2$. This solution corresponds to the larger solution, which is monotone decreasing and on Γ_2^*.

For a smaller solution, we can obtain the smaller one by continuing $u(r; -\mu_2^2)$ to the region where $-\lambda < -\mu_2^2$. Thus, there exists $\varepsilon_0 > 0$ such that (1.2) has at least two solutions for $(-\mu_2^2 - \varepsilon_0, -\mu_2^2)$. $\square$

3. Transformation to the exterior problem

In this section, we prove a crucial Proposition 3.1, which is stated below. This proposition is necessary to prove Proposition 2.1.

As mentioned in the Introduction, the key point is to transform our problem (1.2) to

$$\begin{cases} (\tau^2 w_\tau)_\tau + \tau^2 Q(\tau;\lambda)(\tau) w_+^5 = 0, \ \tau \in (\rho_\lambda, \infty) \\ w > 0, \ \tau \in (\rho_\lambda, \infty) \\ w_\tau(\rho_\lambda) = 0, \ \lim\limits_{\tau\to\infty} \tau w(\tau) < \infty \end{cases} \tag{3.1}$$

with changes of variables

$$\begin{aligned} \mu &:= \sqrt{-\lambda}, \\ \tau &:= \frac{h(r)}{g(r)} = \int_r^1 \frac{ds}{g(s)} + \rho_\lambda = \mu(\coth \mu r - \eta), \qquad (3.2) \\ w(\tau) &:= \frac{u(r)}{\tau} \\ g(r;\lambda) &= g(r) := \frac{\sinh^2(\mu r)}{\mu^2}, \\ \varphi(r;\lambda) &= \varphi(r) := \frac{\sinh(\mu r)}{\mu r}, \qquad (3.3) \\ \rho_\lambda &:= \frac{\kappa}{\varphi(1)(\varphi(1) + \kappa\varphi_r(1))}, \\ h(r) &:= g(r)\Big(\int_r^1 \frac{ds}{g(s)} + \rho_\lambda\Big) = \frac{\sinh 2\mu r + \eta(1 - \cosh 2\mu r)}{2\mu} \qquad (3.4) \end{aligned}$$

where

$$\eta := \frac{\kappa(\mu - \coth \mu) + \coth \mu}{\kappa(\mu \coth \mu - 1) + 1},$$

$w_+ := \max\{w, 0\}$ and

$$Q(\tau;\lambda) = \Big(\frac{\sinh 2\mu r + \eta(1-\cosh 2\mu r)}{2\mu r}\Big)^4 = (\frac{h(r)}{r})^4. \tag{3.5}$$

Note also that $\tau \to \infty$ as $r \downarrow 0$ that $d\tau/dr = -1/g(r)$. We see that $\eta < 0$ if $0 < \mu < \mu_2$, $\eta = 0$ if $\mu = \mu_2$, and $\eta > 0$ if $\mu > \mu_2$. ρ_λ is a continuous function with respect to λ.

Now we introduce the Pohozaev identity. Let

$$P(\tau; w) := \frac{1}{2}\tau^2 w_\tau(\tau w_\tau + w) + \frac{\tau^3}{6}Q(\tau;\lambda)w_+^6,$$

and

$$G(\tau;\lambda) := \frac{1}{6}\Big(\tau^3 Q(\tau;\lambda) - 3\int_{\rho_\lambda}^{\tau} s^2 Q(s;\lambda)\,ds\Big).$$

$G(\tau;\lambda)$ also satisfies the equality (see the equality below (4.2) of [5]):

$$G(\tau;\lambda) := \frac{1}{6}\int_{\rho_\lambda}^{\tau} s^3 Q_s(s;\lambda)\,ds = \frac{1}{6}\rho_\lambda^3 Q(\rho_\lambda;\lambda) + \frac{2}{3}\int_r^1 \frac{h^6(h(\varrho) - \varrho h_\varrho(\varrho))}{(g(\varrho))^3\varrho^5}\,d\varrho. \tag{3.6}$$

In order to determine the structure of solutions to (3.1), we consider the initial-value problem

$$\begin{cases} (\tau^2 w_\tau)_\tau + \tau^2 Q(\tau;\lambda)w_+^5 = 0, \ \tau \in (\rho_\lambda, \infty) \\ w(\rho_\lambda) = \zeta > 0, \ w_\tau(\rho_\lambda) = 0. \end{cases} \tag{3.7}$$

Note that (3.7) has a unique solution on $[\rho_\lambda, \infty)$ for any $\zeta > 0$. We denote a solution to (3.7) simply by w; however, in the case where the dependence on the parameters is necessary to be indicated, we denote w by $w(\tau;\zeta;\lambda)$.

Now we are in a position to state our crucial proposition. The following proposition is useful to show the sharpness of μ_2, and is a variant of Theorem 3 of [11].

Proposition 3.1. *Let us fix $\kappa > 1$. For arbitrarily chosen $\lambda_* < -\mu_2^2$, there exist $\varepsilon_1(\lambda_*) > 0$ and $\zeta_0(\lambda_*) > 0$ such that the unique solution $w(\tau;\zeta;\lambda)$ to* (3.7) *has a finite zero for any $\lambda \in [\lambda_* - \varepsilon_1(\lambda_*), \lambda_* + \varepsilon_1(\lambda_*)]$ and any $\zeta \in (0, \zeta_0(\lambda_*))$.*

To prove Proposition 3.1, we need several lemmas. These lemmas are for a solution w to (3.7). First, we note that the following identity holds as in Lemma 3.2 of [5].

Lemma 3.1. *Any solution w to (3.7) satisfies*

$$\frac{d}{d\tau}P(\tau;w) = G_\tau(\tau;\lambda)w_+^6 = \frac{\tau^3}{6}Q_\tau(\tau;\lambda)w_+^6 \tag{3.8}$$

and its integral form

$$P(\tau;w) = G(\tau;\lambda)w_+^6 - 6\int_{\rho_\lambda}^{\tau} G(s;\lambda)w_+^5 w_s\,ds. \tag{3.9}$$

Now we consider the behavior of w as $\zeta \downarrow 0$. The result is similar to Lemma 2.5 of Yanagida and Yotsutani [12].

Lemma 3.2. *For any given $L_1 > L_2 > 0$, take $T > 0$ such that $T > \tilde\rho(L_1, L_2) := \max_{\lambda\in[-L_1,-L_2]}\rho_\lambda$. Then a solution $w(\tau;\zeta)$ to (3.7) satisfies*

$$\lim_{\zeta\downarrow 0}\frac{w(\tau;\zeta)}{\zeta} = 1, \quad \lim_{\zeta\downarrow 0}\frac{1}{\zeta^6}P(\tau;w) = G(\tau;\lambda)$$

uniformly on $[\tilde\rho(L_1, L_2), T] \times [-L_1, -L_2]$.

Proof. The proof is essentially identical to that of Lemma 2.5 of [12]. So we give a sketchy proof. Let $W(\tau) := w(\tau;\zeta)/\zeta$. Then, it is easy to see that $W(\tau) \to 1$ uniformly on $[\tilde\rho(L_1, L_2), T]$ and on $\lambda \in [-L_1, -L_2]$ as $\zeta \downarrow 0$ by the integral form of (3.7)

$$W(\tau) = W(\rho_\lambda) - \zeta^4\int_{\rho_\lambda}^{\tau}(1 - \frac{s}{\tau})sQ(s;\lambda)W^5\,ds.$$

Moreover, since

$$W_\tau = -\zeta^4\int_{\rho_\lambda}^{\tau}(\frac{s}{\tau})^2 Q(s;\lambda)W^5\,ds,$$

we also have $W_\tau(\tau) \to 0$ as $\zeta \downarrow 0$ on $[\tilde\rho(L_1, L_2), R]$ and on $\lambda \in [-L_1, -L_2]$. Hence, by the integral expression of the Pohozaev identity as in Lemma 3.1, we get

$$\frac{1}{\zeta^6}P(\tau;w) = G(\tau;\lambda)W^6 - 6\int_{\rho_\lambda}^{\tau} G(s;\lambda)W^5 W_s\,ds$$

and the second term tends to 0 as $\zeta \to 0$ while $W \to 1$ as $\zeta \downarrow 0$. Thus we obtain the conclusion. □

As in Kawano, Yanagida and Yotsutani [6], τw (in general, $\tau^{n-2}w$ for the n-dimensional case) is nondecreasing as $\tau \to \infty$ if $w > 0$ on $[\rho_\lambda, \infty)$. Thus, we can classify a solution to (3.7) into three types.

(i) w is said to be a *slowly decaying solution* if $w > 0$ on $[\rho_\lambda, \infty)$ and $\lim_{\tau\to\infty}\tau w = \infty$.

(ii) w is said to be a *rapidly decaying solution* if $w > 0$ on $[\rho_\lambda, \infty)$ and $\lim_{\tau\to\infty} \tau w < \infty$.

(iii) w is said to be a *crossing solution* if w has a finite zero.

Similar to the whole space case as in [11, 12], a solution can be characterized by the Pohozaev identity.

Lemma 3.3. *Let w be a solution to* (3.7).

(i) *If w is a slowly decaying solution, then there exists a sequence $\{\tau_j\}$ such that $\tau_j \to \infty$ as $j \to \infty$ and $P(\tau_j; w(\tau_j)) < 0$ for any j.*

(ii) *If w is a rapidly decaying solution, then there exists a sequence $\{\hat{\tau}_j\}$ such that $\hat{\tau}_j \to \infty$ as $j \to \infty$ and $P(\hat{\tau}_j; w(\hat{\tau}_j)) \to 0$ as $j \to \infty$.*

(iii) *If w is a crossing solution, then $P(\tau; w) = P(\tau_0; w(\tau_0)) > 0$ on $[\tau_0, \infty)$, where τ_0 is the first zero of w.*

Proof. A proof is done by following the argument as in Lemmas 2.2 and 2.3 of [6], so we omit the details. □

The following lemma is a kind of converse proposition of (iii) of Lemma 3.3.

Lemma 3.4. *If $\liminf_{\tau\to\infty} P(\tau; w) > 0$, then w is a crossing solution.*

Proof. From (i) and (ii) of Lemma 3.3, w can be neither a slowly decaying solution nor a rapidly decaying solution. Thus, w must be a crossing solution. □

By Lemmas 3.3 and 3.4, if we can see that $G(\tau; \lambda) \geq c_* > 0$ with some $c_* > 0$ near $\tau = \infty$, we can prove that $w(\tau; \zeta)$ has a finite zero for sufficiently small $\zeta > 0$. We will prove that the structure of solutions changes in the case of $\lambda < -\mu_2^2$ from that of $\lambda \geq -\mu_2^2$. Now we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. From Lemmas 3.1 and 3.2, we first investigate the behavior of $G(\tau; \lambda)$ as $\tau \to \infty$. Due to (3.4), we have

$$(h - rh_r)_r = 2\mu r(-\sinh(2\mu r) + \eta \cosh(2\mu r)) \tag{3.10}$$

with $\lambda = \mu^2$. As commented in the beginning of Section 3, we recall that $\eta > 0$ if $\lambda < -\mu_2^2$ and $\eta = 0$ for $\lambda = -\mu_2^2$. As in the proof of Proposition 4.1 of [5], we have

$$\begin{aligned} h - rh_r &= \frac{1}{2\mu}\{\sinh 2\mu r - 2\mu r \cosh 2\mu r + 2\eta\mu r \sinh 2\mu r + \eta(1 - \cosh 2\mu r)\} \\ &= \eta\mu r^2 - \frac{4}{3}\mu^2 r^3 + \frac{1}{3}\mu^3(\theta_1^4 - 2\theta_2^4 + 4\eta\theta_3^3 - \eta\theta_4^4)r^4, \end{aligned} \tag{3.11}$$

where $\theta_i \in (0,1)$ $(i = 1,2,3,4)$ near $r = 0$ uniformly in μ in an interval including $-\mu_2^2$. Hence we see that $G_\tau(\tau;\lambda) > 0$ on $(T(\lambda),\infty)$ for $\lambda < -\mu_2^2$, where $T(\lambda)$ is defined by $T(\lambda) = \mu(\coth \mu R(\lambda) - \eta)$ in view of (3.2) with $R(\lambda)$ being the first zero of $h - rh_r$. By using the exact form of h and g, for fixed $\lambda_* < -\mu_2^2$, there exist $\varepsilon(\lambda_*) > 0$ and $0 < \tilde{R}(\lambda_*) < R(\lambda_*)$ such that

$$\frac{1}{2}r \leq h \leq 2r$$

holds on $[0,\tilde{R}(\lambda_*)]$ for any $\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]$ and

$$\frac{1}{2}r^2 \leq g \leq 2r^2$$

holds on $[0,\tilde{R}(\lambda_*)]$ for any $\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]$. Thus we see that

$$\frac{\eta\mu}{512r^3} \leq \frac{h^6(h - rh_r)}{g^3 r^5} \leq \frac{512\eta\mu}{r^3}$$

on $(0,\tilde{R}(\lambda_*)]$ for any $\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]$. Thus, $G(\tau;\lambda) \to \infty$ as $\tau \to \infty$ in view of (3.6) for any $\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]$. Hence we can take $T > \max_{\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]} \rho_\lambda$ such that $G(T;\lambda) > 0$ and $G(\tau) \geq G(T;\lambda)$ on $[T,\infty)$ for any $\lambda \in [\lambda_* - \varepsilon(\lambda_*), \lambda_* + \varepsilon(\lambda_*)]$.

In view of Lemma 3.2, there exists $\zeta_0(\lambda_*) > 0$ such that $w(T;\zeta;\lambda) > 0$ and $P(T; w(T;\zeta;\lambda)) > 0$ for any $\zeta \in (0,\zeta_0(\lambda_*))$ and for any $\lambda \in [\lambda_* - \varepsilon_1(\lambda_*), \lambda_* + \varepsilon_1(\lambda_*)]$. We see from (3.9) in Lemma 3.1 that

$$\begin{aligned} P(\tau;w) &= P(T;w) + G(\tau;\lambda)w_+(\tau)^6 - G(T;\lambda)w_+(T)^6 \\ &\quad - 6\int_T^\tau G(s;\lambda)w_+^5 w_s ds \\ &= P(T;w) + (G(\tau;\lambda) - G(T;\lambda))w_+(\tau)^6 \\ &\quad + 6\int_T^\tau (G(s;\lambda) - G(T;\lambda))w_+^5(-w_s)\, ds. \end{aligned}$$

As long as $w > 0$, w is monotone decreasing. Hence we see that $P(\tau;w) \geq P(T;w) > 0$ for (T,∞). In view of Lemma 3.4, $w(\tau;\zeta)$ is a crossing solution for any $\zeta \in (0,\zeta_0(\lambda_*))$ and for any $\lambda \in [\lambda_* - \varepsilon_1(\lambda_*), \lambda_* + \varepsilon_1(\lambda_*)]$. □

Remark 3.1. As $\lambda_* \uparrow -\mu_2^2$, $\varepsilon_1(\lambda_*) \to 0$ and $\zeta_0(\lambda_*) \to 0$. These constants are not globally uniform. We can also prove that $\lim_{\tau\to\infty} G(\tau;\lambda) = -\infty$ for $\lambda \geq -\mu_2^2$ and $w(\tau;\zeta)$ does not have a zero for any small $\zeta > 0$ (see Proposition 4.1 of [5]).

4. APPROXIMATION OF A SOLUTION AND A PROOF OF PROPOSITION 2.1

In this section, first, we show an approximation of a solution $u(r;\lambda)$ to (1.2) with a large initial value $u(0;\lambda)$. We define $u_+ = \max\{u, 0\}$ and denote the radial solution to

$$\frac{1}{r^2}(r^2 u_r)_r + \lambda u + u_+^5 = 0 \tag{4.1}$$

with $u(0;\lambda) = \alpha$ by $u(r;\alpha;\lambda)$. We set $U_\alpha(r) := \frac{\alpha}{(1+\alpha^4 r^2/3)^{1/2}}$. As we have seen in Section 2, $U_\alpha(r)$ is a positive solution to

$$\frac{1}{r^2}(r^2 U_r)_r + U^5 = 0 \quad r > 0, \tag{4.2}$$

with $U_\alpha(0) = \alpha(= u(0;\alpha;0))$. Then by Lemma 2.3 of Budd, Knaap and Peletier [2], we have

$$u(r;\alpha;\lambda) \geq U_\alpha(r) \tag{4.3}$$

on $r \in [0, 1]$ for any $\lambda < 0$. In this section, we do not require $u(r;\alpha;\lambda)$ to satisfy the boundary condition $\kappa u_r(1) + u(1) = 0$.

Lemma 4.1. *Let us arbitrarily fix $L_3 > L_4 > 0$. Then a monotone-decreasing solution $u(r;\alpha;\lambda)$ to (4.1) satisfies the following properties:*

(i) *$u(1;\alpha;\lambda) \to 0$ as $\alpha \to \infty$ uniformly on $\lambda \in [-L_3, -L_4]$;*
(ii) *$u(1;\alpha;\lambda) - \alpha U_1(\alpha^2) \leq C_* \alpha^{-1/3}$ uniformly on $\lambda \in [-L_3, -L_4]$ as $\alpha \to \infty$ with $C_* > 0$ dependent only on L_3 and L_4.*

Proof. Let us define

$$v(r) := \frac{u(r;\alpha;\lambda)}{\alpha}, \quad \Phi(\rho) := v(r)$$

with $\rho = \alpha^2 r$. Then Φ satisfies

$$\frac{1}{\rho^2}(\rho^2 \Phi_\rho)_\rho + \Phi^5 = -\frac{\lambda}{\alpha^4}\Phi. \tag{4.4}$$

Since $\max \Phi = 1$, we see that $\Phi(\rho) \to U_1(\rho)$ locally uniformly in $\mathbf{R}^3$ as $\alpha \to \infty$, where $U_1(\rho)$ is a solution to (4.2) with $U_1(0) = 1$. However, we need more accurate behavior of $u(r;\alpha;\lambda)$. For this purpose, subtracting (4.4) from (4.2), we have

$$\frac{1}{\rho^2}\left\{\rho^2(\Phi - U_1)_\rho\right\}_\rho + (\Phi^4 + \Phi^3 U_1 + \Phi^2 U_1^2 + \Phi U_1^3 + U_1^4)(\Phi - U_1) = -\frac{\lambda}{\alpha^4}\Phi. \tag{4.5}$$

Let us set $w(\rho) := \Phi - U_1$ and $c(\rho) := \Phi^4 + \Phi^3 U_1 + \Phi^2 U_1^2 + \Phi U_1^3 + U_1^4 \geq 0$. Note that $w(\rho)$ and $c(\rho)$ are radial functions and $w(\rho) \geq 0$ from (4.3). Since w is a solution to

$$(\rho^2 w_\rho)_\rho + \rho^2 c(\rho) w = -\frac{\lambda}{\alpha^4}\rho^2 \Phi,$$

we see that w satisfies

$$0 \leq w(\rho) = -\int_0^\rho (1 - \frac{s}{\rho}) s c(s) w(s)\, ds - \frac{\lambda}{\alpha^4}\int_0^\rho (1 - \frac{s}{\rho}) s \Phi(s)\, ds \tag{4.6}$$

in view of $w(0) = \Phi(0) - U_1(0) = 0$. Since $\lambda \in [-L_3, -L_4]$ and $\max \Phi = 1$, we have

$$w(\rho) \leq \frac{L_3}{\alpha^4}\int_0^\rho (1 - \frac{s}{\rho}) s\, ds \leq \frac{L_3}{6\alpha^4}\rho^2.$$

Thus, for fixed $R > 0$ and for any $\rho \in [0, \alpha^{4/3} R]$, we have

$$w(\rho) \leq \frac{bL_3 R^2}{6\alpha^{4/3}} \tag{4.7}$$

on $[0, \alpha^{4/3} R]$. Noting that $U_1(\alpha^{4/3} R) = (1 + \alpha^{8/3} R^2/3)^{-1/2}$, we see that there exists $\alpha_{**} > 0$ such that $\Phi(\alpha^{4/3} R) \leq 2R^{-1}\alpha^{-4/3}$ holds for any $\alpha > \alpha_{**}$ in view of (4.7). Since $u(r; \alpha; \lambda)$ is a monotone-decreasing solution to (4.1), we have

$$\begin{aligned} w(\alpha^2) &\leq \frac{L_3}{\alpha^4}\int_0^{\alpha^{4/3} R} (1 - \frac{s}{\alpha^2}) s\, ds + \frac{L_3}{\alpha^4}\frac{2R^{-1}}{\alpha^{4/3}}\int_{\alpha^{4/3} R}^{\alpha^2} (1 - \frac{s}{\alpha^2}) s\, ds \\ &\leq \frac{L_3 R^2}{2\alpha^{4/3}} + \frac{L_3 R^{-1}}{\alpha^{4/3}} + \frac{2L_3 R^2}{3\alpha^{10/3}} \end{aligned}$$

for any $\alpha > \alpha_{**}$. Taking $\alpha_{**} > 0$ larger, we see that $w(\alpha^2) \leq L_3 R^{-3}(R^3 + 2)\alpha^{-4/3}$ holds for any $\alpha > \alpha_{**}$. Thus, we obtain

$$u(1; \alpha; \lambda) - \alpha U_1(\alpha^2) \leq C_* \alpha^{-1/3}$$

for any $\alpha > \alpha_{**}$ and any $\lambda \in [-L_3, -L_4]$ with $C_* = L_3 R^{-1}(R^3 + 2)$. Since $\alpha U_1(\alpha^2) = \alpha(1 + \alpha^4/3)^{-1/2} \to 0$ as $\alpha \to \infty$, $u(1; \alpha; \lambda) \to 0$ uniformly in $[-L_3, -L_4]$. □

Finally, we prove Proposition 2.1 by using Lemma 4.1.

Proof of Proposition 2.1. By Theorem C, we may assume that $\lambda^* \leq -\mu_2^2$. Suppose that there exists a sequence $\{\tilde{\lambda}_j\}$ such that $u(0; \tilde{\lambda}_j) \to \infty$ and $\tilde{\lambda}_j \to -\mu_2^2 - \varepsilon_1$ with $\varepsilon_1 > 0$ as $j \to \infty$. Then from Lemma 4.1, $u(1; \tilde{\lambda}_j)$ is sufficiently small if j is sufficiently large. By taking $u(1; \tilde{\lambda}_j)/\rho_{\tilde{\lambda}_j}$ as an initial value to (3.1), the corresponding solution $w(\tau; \tilde{\lambda}_j)$ has a finite zero in $(0, 1)$

in view of Proposition 3.1 for any sufficiently large j. This is a contradiction since $u(r;\tilde{\lambda}_j) = \tau w(\tau;\tilde{\lambda}_j)$ is positive on $[0,1]$ by assumption. □

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