Advances in Differential Equations Volume 10, Number 11 (2005), 1301–1320

ANALYTICITY OF SEMIGROUPS GENERATED BY OPERATORS WITH GENERALIZED WENTZELL BOUNDARY CONDITIONS

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(Submitted by: J.A. Goldstein)

Abstract. In this note we introduce a general framework which allows us to prove in a unified and systematic way that operators with Wentzell-type boundary conditions generate analytic semigroups on function spaces with bounded trace operator. The abstract generation result is illustrated in three concrete examples.

1. Introduction

Recently various authors have studied the generator property of second-order differential operators with generalized Wentzell boundary conditions defined on spaces of continuous functions; see, e.g., [2, Section 3], [4], [8], [9], [18], [21], [22], [28] and the references therein. In this context it is of particular interest if the generated semigroup is analytic. To this purpose we present in this note a general framework which allows us to prove, in a unified and systematic way, the analyticity of semigroups generated by operators with generalized Wentzell boundary conditions on function spaces with bounded trace operators. To do so we use techniques developed in the theory of one-sided coupled operator matrices (see [11], [12], [13]) which are mainly based on similarity transformations and perturbation arguments. As it turns out, our approach allows us to completely decouple Wentzell-type boundary conditions of an operator A obtaining in this way two much simpler operators: an operator A_0 defined on the same state space as A but

* Supported by INDAM.
Accepted for publication: June 2005.
AMS Subject Classifications: 47D06, 34G10, 47E05, 47F05.

with Dirichlet boundary conditions and the Dirichlet–Neumann operator N living on some associated boundary space.

This paper is organized as follows. In Section 2 we introduce our framework and show, as a motivation, how Wentzell-type boundary conditions are connected to dynamic boundary conditions. In Section 3 we present the main abstract result, the generation Theorem 3.1. Moreover, in Lemma 3.3 we analyze the relation between three operators derived from some "maximal" operator, which in many applications simplifies, to verify condition (ii) in Theorem 3.1. These results are then applied in Section 4 to three concrete operators: a degenerate second-order differential operator on $\mathrm{C}[0,1]$, the second derivative on $\mathrm{W}^{1,1}(0,1)$ and a second-order uniformly elliptic operator on an open bounded domain $\Omega \subset \mathbb{R}^n$. Finally, in the Appendix A we collect some definitions and auxiliary results which are needed in the main part of this note.

We remark that in the papers [5] and [6] we show how our method can be modified in order to study the generation of cosine families by operators with Wentzell-type boundary conditions.

2. The Abstract Framework

The Abstract Framework 2.1. We consider

(i) two Banach spaces X and ∂X, called *state* and *boundary space*, respectively;
(ii) a *boundary operator* $L \in \mathcal{L}(X, \partial X)$;
(iii) a densely defined *maximal operator* $A_m : D(A_m) \subseteq X \to X$;
(iv) a *feedback operator* $B : D(B) \subseteq X \to \partial X$;
(v) a *boundary dynamic operator* $C \in \mathcal{L}(\partial X)$.

Using these spaces and operators we define on X the operator $A \subseteq A_m$ with abstract *generalized Wentzell boundary conditions* by

$$Af := A_m f, \quad D(A) := \big\{ f \in D(A_m) \cap D(B) : LA_m f = Bf + CLf \big\}. \tag{2.1}$$

Differential operators with Wentzell boundary conditions have been first studied by Feller and Wentzell in the context of diffusion processes; see [23], [29]. Here we give a different motivation showing the connection to Cauchy problems with *dynamic* boundary conditions.

Motivation 2.2. For an operator A with generalized Wentzell boundary condition consider the abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t > 0, \\ u(0) = f. \end{cases} \tag{ACP}$$

If we assume that A generates an analytic semigroup $(T(t))_{t\geq 0}$ on X, then the Cauchy problem (ACP) is well posed for every $f \in X$ and its solution is given by $u(t) = T(t)f$. Since $u(t) \in D(A)$ for all $t > 0$ we obtain by the definition of $D(A)$ for $x(t) := Lu(t)$

$$\begin{array}{rcl} L\dot{u}(t) & = & LAu(t) \\ \| & & \| \\ (Lu)^{\cdot}(t) & = & Bu(t) + CLu(t) \\ \| & & \| \\ \dot{x}(t) & = & Bu(t) + Cx(t) \end{array}$$

for all $t > 0$. Hence (ACP) can be rewritten as a system

$$\begin{cases} \dot{u}(t) = A_m u(t), & t > 0, \\ x(t) = Lu(t), & t > 0, \\ \dot{x}(t) = Bu(t) + Cx(t), & t > 0, \\ u(0) = f,\ x(0) = Lf \end{cases} \tag{SdBC}$$

with *dynamic boundary conditions* which, in a framework adapted to an L^p setting, has recently been studied in [10].

If in these considerations we choose $B = 0$ and $C = 0$, we obtain $\dot{x}(t) = 0$ for all $t > 0$. Since $\mathbb{R}_+ \ni x(t) = Lu(t) = LT(t)f$ is continuous in $t = 0$, this implies that $x(t) = x(0) = Lf$ for all $t \geq 0$. In other words, if for $B = 0$ and $C = 0$ the operator A in (2.1) generates an analytic semigroup $(T(t))_{t\geq 0}$, then this semigroup preserves the boundary values; i.e., $LT(t)f = Lf$ for all $f \in X$. Clearly, by the identity theorem for analytic functions, this also implies $LT(z)f = Lf$ for all z in the sector of analyticity of $(T(t))_{t\geq 0}$.

3. The Main Abstract Result

In the sequel we will frequently use the abstract *Dirichlet operator* $L_0 := (L|_{\ker A_m})^{-1} : \partial X \to \ker A_m \subseteq X$ which is characterized by

$$L_0 x = f \quad \iff \quad \begin{cases} A_m f = 0, \\ Lf = x. \end{cases}$$

Our main abstract result is the following

Theorem 3.1. *Assume that*

(i) *the* abstract Dirichlet operator $L_0 := (L|_{\ker A_m})^{-1} : \partial X \to \ker A_m \subseteq X$ *exists and is bounded from* ∂X *to* X*;*
(ii) *the restriction of* A_m *to* $X_0 := \ker L$*; i.e.,* $A_0 := A_m|_{\ker L}$ *is sectorial of angle* α *on* X*;*

(iii) *the operator* $B : D(B) \subseteq X \to \partial X$ *is relatively* A_0*-bounded with* A_0*-bound zero;*

(iv) *the abstract* Dirichlet–Neumann operator $N := BL_0$, $D(N) := \big\{x \in \partial X : L_0 x \in D(B)\big\}$ *generates an analytic semigroup of angle* α *on* ∂X.

Then the operator A *defined in* (2.1) *generates an analytic semigroup of angle* α *on* X. *This semigroup is compact, if* A_0 *and* N *have compact resolvents.*

The proof of Theorem 3.1 is divided into several steps.

Step 1. *The space* X *is isomorphic to* $\mathfrak{X} := \{\binom{f}{x} \in X \times \partial X : Lf = x\}$, *via the transformations*

$$S : X \to \mathfrak{X}, \qquad f \mapsto \tbinom{f}{Lf},$$
$$S^{-1} : \mathfrak{X} \to X, \qquad \tbinom{f}{x} \mapsto f.$$

Step 2. *The operator* A *on* X *is similar to* $\mathcal{A} := SAS^{-1}$ *on* $\mathfrak{X}$ *given by*

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \quad D(\mathcal{A}) = \Big\{ \tbinom{f}{x} \in \mathfrak{X} : f \in D(A) \Big\}. \tag{3.1}$$

Proof. We first observe that

$$\tbinom{f}{x} \in D(\mathcal{A}) \iff S^{-1}\tbinom{f}{x} = f \in D(A),$$

proving the second part of (3.1). Now take $\binom{f}{x} \in D(\mathcal{A})$; i.e. $Lf = x$ and $f \in D(A)$. Then, using the fact that $LAf = Bf + Cx$, we obtain

$$\mathcal{A}\tbinom{f}{x} = SAS^{-1}\tbinom{f}{x} = SAf = \begin{pmatrix} Af \\ LAf \end{pmatrix} = \begin{pmatrix} Af \\ Bf + Cx \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} f \\ x \end{pmatrix}$$

as claimed. □

Step 3. *The operator* $\mathcal{A}$ *can be represented as* $\mathcal{A} = \tilde{\mathcal{A}}|_{\mathfrak{X}}$, *where* $\tilde{\mathcal{A}}$ *on*

$$\tilde{\mathfrak{X}} := X \times \partial X$$

is defined by

$$\tilde{\mathcal{A}} := \begin{pmatrix} A_0 & 0 \\ B & N + C \end{pmatrix} \begin{pmatrix} I & -L_0 \\ 0 & I \end{pmatrix}, \tag{3.2}$$

$$D(\tilde{\mathcal{A}}) := \Big\{ \tbinom{f}{x} \in X \times D(N) : f - L_0 x \in D(A_0) \Big\}. \tag{3.3}$$

Proof. We have

$$\begin{aligned} \tbinom{f}{x} \in D(\tilde{\mathcal{A}}|_{\mathfrak{X}}) &\iff \tbinom{f}{x} \in D(\tilde{\mathcal{A}}) \cap \mathfrak{X} \quad \text{and} \quad \tilde{\mathcal{A}}\tbinom{f}{x} \in \mathfrak{X} \\ &\iff f - L_0 x \in D(A_0),\ x \in D(N);\ \text{i.e. } L_0 x \in D(B),\ \text{and} \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{A}}\tbinom{f}{x} = \begin{pmatrix} A_0 & 0 \\ B & N+C \end{pmatrix} \begin{pmatrix} f - L_0 x \\ x \end{pmatrix} = \begin{pmatrix} A_m f \\ Bf + Cx \end{pmatrix} \in \mathfrak{X} \\ \iff \quad \tbinom{f}{x} \in \mathfrak{X},\ f \in D(A_m) \cap D(B) \text{ and } LA_m f = Bf + Cx \\ \iff \quad \tbinom{f}{x} \in \mathfrak{X} \text{ and } f \in D(A),\ \iff \ \tbinom{f}{x} \in D(\mathcal{A}), \end{aligned}$$

where the second equivalence also shows that $\tilde{\mathcal{A}}|_{\mathfrak{X}}\binom{f}{x} = \mathcal{A}\binom{f}{x}$. □

Step 4. *Let* $\tilde{\mathfrak{T}} := \left(\begin{smallmatrix} I & -L_0 \\ 0 & I \end{smallmatrix}\right) \in \mathcal{L}(\tilde{\mathfrak{X}})$. *Then* $\tilde{\mathfrak{T}}$ *is invertible with inverse* $\tilde{\mathfrak{T}}^{-1} = \left(\begin{smallmatrix} I & L_0 \\ 0 & I \end{smallmatrix}\right) \in \mathcal{L}(\tilde{\mathfrak{X}})$. *Moreover,* $\tilde{\mathfrak{T}}$ *maps* $\mathfrak{X}$ *onto* $\mathfrak{X}_0 := X_0 \times \partial X$ *and hence its restriction* $\mathfrak{T} := \tilde{\mathfrak{T}}|_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}_0$ *is invertible as well with inverse* $\mathfrak{T}^{-1} = \tilde{\mathfrak{T}}^{-1}|_{\mathfrak{X}_0} : \mathfrak{X}_0 \to \mathfrak{X}$. *For these operators and spaces we have*

$$\mathfrak{T}\mathcal{A}\mathfrak{T}^{-1} = \big(\tilde{\mathfrak{T}}\tilde{\mathcal{A}}\tilde{\mathfrak{T}}^{-1}\big)\big|_{\mathfrak{X}_0}. \tag{3.4}$$

Proof. Since $D(\tilde{\mathcal{A}}) \subset \mathfrak{X}$ equation (3.4) follows from Lemma A.1. □

Step 5. *The operator* $\hat{\mathcal{A}} := \tilde{\mathfrak{T}}\tilde{\mathcal{A}}\tilde{\mathfrak{T}}^{-1}$ *on* $\tilde{\mathfrak{X}}$ *is given by*

$$\hat{\mathcal{A}} = \begin{pmatrix} A_0 - L_0 B & -L_0(N+C) \\ B & N+C \end{pmatrix} \tag{3.5}$$

$$D(\hat{\mathcal{A}}) = D(A_0) \times D(N). \tag{3.6}$$

Moreover, it is sectorial on $\tilde{\mathfrak{X}}$ *(cf. Definition A.2).*

Proof. The representation of $\hat{\mathcal{A}}$ in (3.5) and (3.6) follows from the definition of $\tilde{\mathcal{A}}$ in (3.2) and (3.3). In fact, we have

$$\hat{\mathcal{A}} = \tilde{\mathfrak{T}}\tilde{\mathcal{A}}\tilde{\mathfrak{T}}^{-1} = \begin{pmatrix} I & -L_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ B & N+C \end{pmatrix} = \begin{pmatrix} A_0 - L_0 B & -L_0(N+C) \\ B & N+C \end{pmatrix},$$

$$D(\hat{\mathcal{A}}) = D(A_0) \times D(N).$$

To prove that $\hat{\mathcal{A}}$ is sectorial we decompose it as

$$\hat{\mathcal{A}} = \begin{pmatrix} A_0 & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} -L_0 B & -L_0(N+C) \\ B & C \end{pmatrix} =: \hat{\mathcal{A}}_0 + \hat{\mathfrak{B}}.$$

Here, by assumption, B is relatively A_0-bounded with A_0-bound zero, $L_0 \in \mathcal{L}(\partial X, X)$, A_0 is sectorial on X of angle α and N generates an analytic semigroup of angle α on ∂X. Hence the entries of $\hat{\mathcal{A}}_0$ and $\hat{\mathfrak{B}}$ satisfy the assertions of Lemma A.4 and we conclude that $\hat{\mathcal{A}}$ is sectorial of angle α. □

Step 6. Proof of Theorem 3.1. By Step 5 the operator $\hat{\mathcal{A}}$ is sectorial of angle α. Next we show that

$$\overline{D(A_0)} = X_0. \tag{3.7}$$

To this end we define the projections

$$P := L_0 L \in \mathcal{L}(X), \qquad Q := (I - P) \in \mathcal{L}(X) \tag{3.8}$$

with $\ker P = \operatorname{rg} Q = \ker L = X_0$ and $\operatorname{rg} P = \ker Q = \ker A_m$. Since by assumption A_m is densely defined on X, for every $f \in X_0$ there exists $(f_n)_{n\in\mathbb{N}} \subset D(A_m)$ such that $f_n \to f$ as $n \to \infty$. Then $Qf_n = f_n - Pf_n \to Qf = f$ as $n \to \infty$, where

$$Qf_n \in \ker L \cap \big(D(A_m) + \ker A_m\big) \subseteq \ker L \cap D(A_m) = D(A_0).$$

This implies (3.7) and hence $\overline{D(\hat{\mathcal{A}})} = \overline{D(A_0) \times D(N)} = X_0 \times \partial X = \mathfrak{X}_0$. By Lemma A.3 we then conclude that $\hat{\mathcal{A}}|_{\mathfrak{X}_0}$ generates an analytic semigroup on $\mathfrak{X}_0$. From the similarity of $\hat{\mathcal{A}}|_{\mathfrak{X}_0}$ and $\mathcal{A}$ proved in Step 4, see (3.4), we deduce that $\mathcal{A}$ generates an analytic semigroup of angle α on $\mathfrak{X}$. Again by the similarity of A and $\mathcal{A}$ shown in Step 2 this implies that A generates an analytic semigroup of angle α on X.

Now assume that A_0 and N have compact resolvents. Then by Lemma A.4 we obtain that $\hat{\mathcal{A}}$ has compact resolvent. Clearly this implies that also A has compact resolvent and from [15, Theorem II.4.29] it follows that the semigroup generated by A is compact. This completes the proof of Theorem 3.1. $\square$

We remark that the assumptions (i) and (ii) of Theorem 3.1 already imply that the maximal operator A_m is closed. More precisely, we have the following result.

Lemma 3.2. *If*

(i) *the abstract Dirichlet operator* $L_0 = (L|_{\ker A_m})^{-1} : \partial X \to \ker A_m \subseteq X$ *exists and is bounded from* ∂X *to* X, *and*

(ii) $A_0 := A_m|_{\ker L}$ *is closed* (*e.g.* $\rho(A_0) \neq \emptyset$),

then A_m *is closed as well.*

Proof. Let $P, Q \in \mathcal{L}(X)$ be the two projections defined in (3.8). Then we can decompose X into the topological direct sum

$$X = QX \oplus PX = \ker L \oplus \ker A_m, \tag{3.9}$$

where both components, $\ker L$ and $\ker A_m = \ker Q$, are complete. Moreover, $D(A_m) = QD(A_m) \oplus PD(A_m) = D(A_0) \oplus \ker A_m$ and hence A_m, relative to the decomposition in (3.9), takes the form $A_m = A_0 \oplus 0$. Since by assumption A_0 is closed the assertion follows. $\square$

From the previous result and the considerations in our Motivation 2.2 we can deduce the following lemma which sometimes is quite useful in order to check condition (ii) in Theorem 3.1. Here we consider two different restrictions of A_m and their common part in $X_0 = \ker L$, namely

$$A_0 := A_m|_{\ker L} \subset A_m, \qquad D(A_0) := \{f \in D(A_m) : Lf = 0\} = D(A_m) \cap \ker L,$$

$$A_1 \subset A_m, \qquad D(A_1) := \{f \in D(A_m) : LA_m f = 0\},$$

and

$$A_{00} := A_0|_{\ker L} = A_1|_{\ker L}, \qquad D(A_{00}) := \{f \in D(A_0) : LA_m f = 0\} = \{f \in D(A_1) : Lf = 0\}.$$

Lemma 3.3. *Assume that the Dirichlet operator $L_0 \in \mathcal{L}(\partial X, X)$ exists. Moreover, suppose that there are $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A_0)$ and $\|\lambda R(\lambda, A_0)\| \leq M$ for all $\lambda > \omega$. Then the following assertions are equivalent.*

(a) *A_0 is sectorial of angle α on X.*
(b) *A_{00} generates an analytic semigroup $(T_{00}(t))_{t\geq 0}$ of angle α on $X_0 = \ker L$.*
(c) *A_1 generates an analytic semigroup $(T_1(t))_{t\geq 0}$ of angle α on X.*

Proof. (a)⇒(b) follows immediately from (3.7) and Lemma A.3.

(b)⇒(c). Define $T_1(t) := T_{00}(t)Q + P$, where P, Q are the projections defined in (3.8). Then

$$T_1(t)T_1(s) = \big[T_{00}(t)Q + P\big]\big[T_{00}(s)Q + P\big] = T_{00}(t+s)Q + P = T_1(t+s)$$

for all $t, s \geq 0$. Moreover, $\lim_{t\searrow 0} T_1(t)f = Qf + Pf = f$ for all $f \in X$ and hence $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on X. To calculate its generator $(\tilde{A}_1, D(\tilde{A}_1))$ we consider

$$\frac{T_1(t)f - f}{t} = \frac{T_{00}(t)(I-P)f + Pf - f}{t} = \frac{T_{00}(t)Qf - Qf}{t}$$

which converges as $t \searrow 0$, with limit $\tilde{A}_1 f = A_{00}Qf$, if and only if $Qf \in D(A_{00})$. Hence

$$\begin{aligned} f \in D(\tilde{A}_1) \quad &\iff \quad Qf = (I-P)f \in D(A_{00}) \\ &\iff \quad (I-P)f \in D(A_m) \quad \text{and} \quad LA_m(I-P)f = 0 \\ &\iff \quad f \in D(A_m) \quad \text{and} \quad LA_m f = 0 \\ &\iff \quad f \in D(A_1) \end{aligned}$$

and $\tilde{A}_1 f = A_{00}(I-P)f = A_m(I-P)f = A_m f$. This shows $(A_1, D(A_1)) = (\tilde{A}_1, D(\tilde{A}_1))$. By defining $T_1(z) := T_{00}(z)Q + P$ for z in the sector of analyticity

$$\Sigma_\alpha := \left\{0 \neq z \in \mathbb{C} : |\arg z| < \alpha\right\}$$

of $(T_{00}(t))_{t\geq 0}$ it follows that $(T_1(t))_{t\geq 0}$ can be extended to an analytic semigroup of angle α showing (c).

(c)⇒(a). Since A_1 coincides with A defined in (2.1) for $B = 0$, $C = 0$, the considerations at the end of Section 2 imply that $T_1(z)X_0 \subset X_0$ for all $z \in \Sigma_\alpha$. Therefore we obtain

$$T_1(z)D(A_0) \subseteq D(A_1) \cap X_0 = D(A_0) \cap D(A_1).$$

Assuming without loss of generality that A_0 is invertible (otherwise consider $A_0 - \mu$ for some $\mu \in \rho(A_0)$ instead of A_0), Lemma 3.2 and the closed graph theorem imply that the operators

$$T_0(z) := A_0 T_1(z) A_0^{-1} = A_1 T_1(z) A_0^{-1} = A_m T_1(z) A_0^{-1}, \quad z \in \Sigma_\alpha$$

are well defined and bounded on X. By defining $T_0(0) := I$ we obtain a (in general not strongly continuous) semigroup $(T_0(t))_{t\geq 0} \subset \mathcal{L}(X)$. To prove that $(T_0(t))_{t\geq 0}$ is analytic we choose some $z_0 \in \Sigma_\alpha$. Then we can write

$$T_0(z) = \left[A_m T_1(z_0)\right]\left[T_1(z - z_0)A_0^{-1}\right]$$

for z sufficiently close to z_0. Since $\operatorname{rg} T_1(z_0) \subset D(A_m)$ the operator $A_m T_1(z_0)$ is bounded which implies that $(T_0(t))_{t\geq 0}$ is analytic. Our next aim is to show that $T_0(z)$ is uniformly bounded for $z \in \Sigma_{\alpha'}$ for every $\alpha' \in (0, \alpha)$. To this end we first note that on X_0 the resolvents of A_0 and A_1 coincide. Indeed, if $g_0 = R(\lambda, A_0)f \in D(A_0)$ and $g_1 = R(\lambda, A_1)f \in D(A_1)$ for $f \in X_0$ and $\lambda \in \rho(A_0) \cap \rho(A_1)$, then

$$L(\lambda - A_m)g_0 = L(\lambda - A_m)g_1 = Lf = 0.$$

By the definition of $D(A_0)$ and $D(A_1)$ this implies that $LA_m g_0 = 0$ and $Lg_1 = 0$. Hence $g_0, g_1 \in D(A_0) \cap D(A_1)$ and we conclude

$$R(\lambda, A_0)|_{X_0} = R(\lambda, A_1)|_{X_0}. \tag{3.10}$$

as claimed. Using this fact together with [15, Proposition II.5.12] we obtain

$$\begin{aligned}
\|T_0(z)f\|_X = \left\|A_1 T_1(z) A_0^{-1} f\right\|_X &\leq \overline{\lim_{t\searrow 0}} \left\|\tfrac{T_1(t)-I}{t}\, T_1(z) A_0^{-1} f\right\|_X \\
&\leq \|T_1(z)\| \cdot \overline{\lim_{t\searrow 0}} \left\|\tfrac{T_1(t)A_0^{-1}f - A_0^{-1}f}{t}\right\|_X \\
&\leq C \cdot \|T_1(z)\| \cdot \overline{\lim_{\lambda\to\infty}} \left\|\lambda A_1 R(\lambda, A_1) A_0^{-1} f\right\|_X \\
&= C \cdot \|T_1(z)\| \cdot \overline{\lim_{\lambda\to\infty}} \left\|\lambda A_0 R(\lambda, A_0) A_0^{-1} f\right\|_X \leq CM \cdot \|T_1(z)\| \cdot \|f\|_X
\end{aligned}$$

for some $C > 0$. Consequently, $(T_0(t))_{t\geq 0}$ is an analytic semigroup on X. To finish the proof of (a) it only remains to prove that its generator is given by A_0. To this end we calculate the Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda t} T_1(t) A_0^{-1}\, dt = R(\lambda, A_1) A_0^{-1} = R(\lambda, A_0) A_0^{-1},$$

where the second equality follows from (3.10). By [3, Proposition 1.7.6] this implies

$$\int_0^\infty \mathrm{e}^{-\lambda t} T_0(t)\, dt = \int_0^\infty \mathrm{e}^{-\lambda t} A_0 T_1(t) A_0^{-1}\, dt = A_0 R(\lambda, A_0) A_0^{-1} = R(\lambda, A_0)$$

and the assertion follows. □

Remark 3.4. The spectral conditions imposed on A_0 in Lemma 3.3 are necessary to obtain the implication (c)⇒(a). In fact, if X is reflexive and A_1 and A_{00} are both generators of analytic semigroups, A_0 cannot be sectorial, since sectorial operators on reflexive spaces are always densely defined.

4. Examples

In order to show the power of our abstract generation result we present in this section three examples: a degenerate second-order differential operator on $\mathrm{C}[0,1]$, the second derivative on $\mathrm{W}^{1,1}(0,1)$ and a uniformly elliptic second-order differential operator on $\mathrm{C}(\overline{\Omega})$.

4.1. A degenerate second-order differential operator on $\mathbf{C}[0,1]$.

As a first application of our abstract generation Theorem 3.1 we study in this subsection a degenerate differential operator.

Corollary 4.1. *Let* $a(s) := m(s) \cdot s^{\alpha_0} \cdot (1-s)^{\alpha_1}$, $s \in [0,1]$, *for some strictly positive* $m \in \mathrm{C}[0,1]$ *and* $\alpha_0, \alpha_1 \in (0,1)$. *Moreover, define* A_m *on* $\mathrm{C}[0,1]$ *by*

$$A_m f := a \cdot f'', \quad D(A_m) := \big\{ f \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : a \cdot f'' \in \mathrm{C}[0,1] \big\}. \tag{4.1}$$

Then $D(A_m) \subseteq \mathrm{C}^1[0,1]$ *and for all* $B \in \mathcal{L}(\mathrm{C}^1[0,1], \mathbb{C}^2)$ *and* $C \in \mathrm{M}_2(\mathbb{C})$ *the operator*

$$Af := A_m f, \qquad D(A) := \Big\{ f \in D(A_m) : \binom{(a\cdot f'')(0)}{(a\cdot f'')(1)} = Bf + C\binom{f(0)}{f(1)} \Big\} \tag{4.2}$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ *on* $\mathrm{C}[0,1]$.

Proof. Choosing $X := \mathrm{C}[0,1]$, $\partial X := \mathbb{C}^2$ and $Lf := \binom{f(0)}{f(1)}$ for $f \in X$, one easily verifies that the operators defined in (2.1) and (4.2) coincide. Here A_m is densely defined since $\mathrm{C}^2[0,1] \subset D(A_m)$, the first space being dense in

X. Note that under the above assumptions $\frac{1}{a} \in \mathrm{L}^1[0,1]$. Using this fact we proceed by verifying the assumptions of Theorem 3.1.

(i) $L_0 := (L|_{\ker A_m})^{-1} \in \mathcal{L}(\partial X, X)$ *exists:* We have $\ker A_m = \operatorname{lin}\{\varepsilon_0, \varepsilon_1\}$, where

$$\varepsilon_0(s) := 1 - s, \quad \varepsilon_1(s) := s.$$

Then one easily verifies that $L_0 := (L_{|\ker A_m})^{-1} : \mathbb{C}^2 \to \ker A_m$ is given by

$$L_0 \tbinom{x_0}{x_1} = x_0 \cdot \varepsilon_0 + x_1 \cdot \varepsilon_1.$$

(ii) A_0 *is sectorial of angle* $\frac{\pi}{2}$: To verify this assertion we will use Lemma 3.2. By [8, Theorem 4.2] we know that

$$A_1 \subset A_m, \quad D(A_1) = \big\{ f \in D(A_m) : (A_m f)(j) = 0;\ j = 0,1 \big\}$$

generates an analytic semigroup of angle $\frac{\pi}{2}$. Moreover, A_0 is dissipative. In fact, if $0 \neq f \in D(A_0) \subset \mathrm{C}_0(0,1)$, then[1] $j(f) := \overline{f(s_0)} \cdot \delta_{s_0}$ is in the duality set of f if $\|f\| = |f(s_0)|$ (see [15, Example II.3.26]). Clearly, $s_0 \in (0,1)$ and since $s \mapsto \operatorname{Re}[\overline{f(s_0)} \cdot f(s)]$ takes its maximum at $s = s_0$ it follows that

$$\operatorname{Re} \langle A_0 f, j(f) \rangle = a(s_0) \operatorname{Re}\big[\overline{f(s_0)} \cdot f''(s_0)\big] = a(s_0) \operatorname{Re}\big[\overline{f(s_0)} \cdot f\big]''(s_0) \le 0.$$

Finally, one easily verifies that A_0 is injective and that for every $g \in X$ the function

$$f(s) := \int_0^s \int_0^r \tfrac{g(\tau)}{a(\tau)}\, d\tau\, ds - s \cdot \int_0^1 \int_0^r \tfrac{g(\tau)}{a(\tau)}\, d\tau\, ds, \quad s \in [0,1]$$

belongs to $D(A_0)$ and satisfies $A_0 f = g$. Hence, A_0 is invertible and by [15, Propositions II.3.14 and II.3.23] we obtain $[0,\infty) \subset \rho(A_0)$ and $\|\lambda R(\lambda, A_0)\| \le 1$ for all $\lambda > 0$. The assertion now follows from Lemma 3.3.

(iii) B *is relatively* A_0*-bounded with* A_0*-bound zero:* Since $D(B) = \mathrm{C}^1[0,1]$ and $A_0 \subset A_m$ it suffices to show that the first derivative (with domain $\mathrm{C}^1[0,1]$) is relatively A_m-bounded with A_0-bound zero.

We first observe that, for $f \in D(A_m)$, from $\frac{1}{a} \in \mathrm{L}^1[0,1]$ it follows that $f'' = \frac{1}{a} \cdot A_m f \in \mathrm{L}^1[0,1]$. Hence, $f \in \mathrm{W}^{2,1}(0,1) \subset \mathrm{C}^1[0,1]$. This shows that $D(A_m) \subset \mathrm{C}^1[0,1] = D(B)$. Now let $s \in [0,1]$ and choose $0 \neq \varepsilon \in (-1,1)$ such that $s + \varepsilon \in [0,1]$. Then by Taylor's formula

$$f(s+\varepsilon) = f(s) + f'(s) \cdot \varepsilon + \int_s^{s+\varepsilon} (s + \varepsilon - r) f''(r)\, dr$$

and therefore

$$\big|f'(s)\big| \le \Big|\frac{f(s+\varepsilon) - f(s)}{\varepsilon}\Big| + \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \frac{|s+\varepsilon-r|}{a(r)} \cdot \big|(A_0 f)(r)\big|\, dr$$

[1] Here $\delta_s \in X'$ denotes the point evaluation in s, i.e., $\delta_s f := f(s)$

$$\leq \frac{2}{|\varepsilon|} \cdot \|f\|_\infty + \Big| \int_s^{s+\varepsilon} \frac{dr}{a(r)} \Big| \cdot \|A_0 f\|_\infty$$

for all $f \in D(A_0)$. Since $\frac{1}{a} \in \mathrm{L}^1[0,1]$ it follows that $\int_s^{s+\varepsilon} \frac{dr}{|a(r)|} \to 0$ uniformly for $s \in [0,1]$ as $\varepsilon \to 0$, and hence there exists $C_\varepsilon > 0$ with $\lim_{\varepsilon \to 0} C_\varepsilon = 0$ such that

$$\|f'\|_\infty \leq \frac{2}{|\varepsilon|} \cdot \|f\|_\infty + C_\varepsilon \cdot \|A_0 f\|_\infty.$$

This proves that the first derivative on $\mathrm{C}[0,1]$ is relatively A_0-bounded with A_0-bound zero proving the claim.

(iv) $N = BL_0$ *generates an analytic semigroup of angle* $\frac{\pi}{2}$: Since $\mathrm{rg}(L_0) \subset D(B)$ the operator N is well defined and bounded. Hence condition (iv) is satisfied.

Finally, since in the proof of (iii) we showed that $D(A_0) \subset \mathrm{C}^1[0,1]$, we obtain from the closed graph and Arzelà–Ascoli theorems that

$$[D(A_0)] \hookrightarrow \mathrm{C}^1[0,1] \overset{c}{\hookrightarrow} \mathrm{C}[0,1],$$

where $[D(A_0)] := (D(A_0), \|\cdot\|_{A_0})$, and "$\hookrightarrow$" and "$\overset{c}{\hookrightarrow}$" denote continuous and compact injections, respectively. Hence $[D(A_0)] \overset{c}{\hookrightarrow} \mathrm{C}[0,1]$ and [15, Proposition II.4.25] implies that A_0 has compact resolvent. Since the resolvent of N is compact as well, we conclude from Theorem 3.1 that A generates a compact, analytic semigroup on X. □

Remark 4.2. In part (iii) of the previous proof we showed that the first derivative is relatively A_m and hence also relatively $A \subset A_m$ bounded with bound 0. By [15, Theorem III.2.10] we therefore obtain that for A defined by (4.2) and all $P \in \mathcal{L}(\mathrm{C}^1[0,1], \mathrm{C}[0,1])$ the operator

$$A_P := A + P, \quad D(A_P) := D(A)$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}[0,1]$. In particular, choosing $Pf := p \cdot f' + q \cdot f$ for $p, q \in \mathrm{C}[0,1]$, $Bf := \binom{\beta_0 f'(0)}{\beta_1 f'(1)}$ and $C := \mathrm{diag}(\gamma_0, \gamma_1)$ for constants $\beta_j, \gamma_j \in \mathbb{C}$, $j = 0, 1$, we conclude that the operator

$$\begin{aligned} A_P f &:= a \cdot f'' + p \cdot f' + q \cdot f, \\ D(A_P) &:= \left\{ f \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : \begin{array}{c} a \cdot f'' \in \mathrm{C}[0,1], \\ (a \cdot f'')(j) = \beta_j f'(j) + \gamma_j f(j);\ j = 0, 1 \end{array} \right\} \end{aligned}$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{C}[0,1]$. This generalizes the main result in [19], where, beside other restrictions, only the case $p = q = 0$ and $\gamma_0 = \gamma_1 = 0$ is considered.

4.2. **The second derivative on $\mathrm{W}^{1,1}(0,1)$.** Our approach can not only be used in spaces of continuous functions but also works in Sobolev spaces with continuous trace operator. As an example we prove the following result which seems to be new. Here it will be convenient to use in the sequel the notation

$$\mathrm{W}_0^{k,1}(0,1) := \Big\{ f \in \mathrm{W}^{k,1}(0,1) : f(0) = f(1) = 0 \Big\}.$$

Corollary 4.3. *Let $B \in \mathcal{L}(\mathrm{W}^{2,1}(0,1), \mathbb{C}^2)$, $C \in \mathrm{M}_2(\mathbb{C})$. Then the operator*

$$\begin{aligned} &Af := f'', \\ &D(A) := \Big\{ f \in \mathrm{W}^{3,1}(0,1) : \tbinom{f''(0)}{f''(1)} = Bf + C\tbinom{f(0)}{f(1)} \Big\} \end{aligned} \tag{4.3}$$

generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ on $\mathrm{W}^{1,1}(0,1)$.

Proof. Choosing $X := \mathrm{W}^{1,1}(0,1)$, $\partial X := \mathbb{C}^2$, $A_m f := f''$ for $f \in D(A_m) := \mathrm{W}^{3,1}(0,1)$ and $Lf := \binom{f(0)}{f(1)}$, one easily sees that the conditions in our Abstract Framework 2.1 are verified and that the operators defined in (2.1) and (4.3) coincide.

We proceed by verifying the assumptions of Theorem 3.1, where (i) and (iv) follow as in the proof of Corollary 4.1.

(ii) A_0 *is sectorial of angle* $\frac{\pi}{2}$: To verify this assertion we first show that A_0 generates a Lipschitz continuous sine family, cf. [3, Section 3.15]. This fact then allows to obtain the desired estimate for its resolvent.

By [20, Section 2] or [30, Section 2] the operator

$$\tilde{A}_0 f := f'', \quad D(\tilde{A}_0) := \mathrm{C}_0^2[0,1]$$

generates a sine family $(\tilde{S}_0(t))_{t\ge 0}$ on $\mathrm{C}[0,1]$ given by

$$\big[\tilde{S}_0(t)f\big](s) = \tfrac{1}{2}\int_{s-t}^{s+t} \tilde{f}(r)\,dr, \quad t \ge 0,\ s \in [0,1].$$

Here $\tilde{f}$ denotes the odd, 2-periodic extension of $f \in X$ to $\mathbb{R}$. Since $(\tilde{S}_0(t))_{t\ge 0}$ leaves $\mathrm{W}^{1,1}(0,1) \subset \mathrm{C}[0,1]$ invariant we can restrict it to X obtaining in this way an operator family $(S_0(t))_{t\ge 0}$ on X. We will now check that $(S_0(t))_{t\ge 0}$ is an exponentially bounded Lipschitz continuous sine family with generator A_0, cf. [3, Definition 3.15.1].

(α) Note that for all $t \ge 0$ the range $\mathrm{rg}(S_0(t)) \subseteq \mathrm{W}_0^{1,1}(0,1)$. Moreover, by Poincaré's inequality, on $\mathrm{W}_0^{1,1}(0,1)$ the norm induced by X and the restriction of the semi-norm $X \ni f \mapsto \|f\|_{\mathrm{W}_0^{1,1}} := \|f'\|_{\mathrm{L}^1}$ are equivalent. Hence it suffices to show Lipschitz continuity of $(S_0(t))_{t\ge 0}$ for (the operator semi-norm induced by) the semi-norm $\|\cdot\|_{\mathrm{W}_0^{1,1}}$. Let $0 \le s \le t$ and take $f \in X$, then

$$\big[S_0(t)f - S_0(s)f\big](\cdot) = \tfrac{1}{2}\Big[\int_{\cdot-t}^{\cdot-s} \tilde{f}(r)\,dr + \int_{\cdot+s}^{\cdot+t} \tilde{f}(r)\,dr\Big].$$

Hence, by Corollary A.6 and since $\tilde{f}$ is odd, $\tilde{f}'$ is even and both are 2-periodic, we have

$$\begin{aligned}\big\|S_0(t)f - S_0(s)f\big\|_{\mathrm{W}_0^{1,1}} &\le \tfrac{1}{2}\Big[\int_{-1}^{1}\big|\tilde{f}(r-s) - \tilde{f}(r-t)\big|\,dr \\ &\qquad + \int_{-1}^{1}\big|\tilde{f}(r+t) - \tilde{f}(r+s)\big|\,dr\Big] \\ &\le (t-s)\cdot\int_{-1}^{1}|\tilde{f}'(r)|\,dr = 2\cdot(t-s)\cdot\|f\|_{\mathrm{W}_0^{1,1}}\,.\end{aligned}$$

(β) Observe that $[D(\tilde{A}_0)] \hookrightarrow \mathrm{W}^{1,1}(0,1) \hookrightarrow \mathrm{C}[0,1]$ and $A_0 = \tilde{A}_0|_{\mathrm{W}^{1,1}(0,1)}$. Hence [15, Proposition IV.1.15 & 2.17] imply that $\rho(A_0) = \rho(\tilde{A}_0) \supset (0,\infty)$ and $R(\lambda, A_0) = R(\lambda, \tilde{A}_0)|_{\mathrm{W}^{1,1}(0,1)}$. Moreover, since $S(0) = 0$ it follows from (α) that $\|S_0(t)f\|_{\mathrm{W}_0^{1,1}} \le 2t\cdot\|f\|_{\mathrm{W}_0^{1,1}}$ and we conclude that for all $f \in \mathrm{W}^{1,1}(0,1)$ and $\lambda > 0$

$$R(\lambda^2, A_0)f = R(\lambda^2, \tilde{A}_0)f = \int_0^\infty \mathrm{e}^{-\lambda t}\tilde{S}_0(t)f\,dt = \int_0^\infty \mathrm{e}^{-\lambda t}S_0(t)f\,dt, \quad (4.4)$$

where the first integral is understood in the space $\mathrm{C}[0,1]$ and the second one in $\mathrm{W}^{1,1}(0,1)$.

By (α) and (β) we conclude that $(S_0(t))_{t\ge0}$ is an exponentially bounded Lipschitz continuous sine family with generator A_0 on X.

By holomorphic extension (see [3, Proposition B.5]) we infer from (4.4) that for $\operatorname{Re}\lambda > 0$ we have $\lambda^2 \in \rho(A)$ and

$$R(\lambda^2, A_0) = \int_0^\infty \mathrm{e}^{-\lambda t}S_0(t)\,dt.$$

Now take $\mu := r\,\mathrm{e}^{i\theta}$ for $r > 0$, $\theta \in (-\pi,\pi)$, and let $\lambda := \sqrt{r}\,\mathrm{e}^{i\frac{\theta}{2}}$. Then $\operatorname{Re}\lambda = \sqrt{r}\,\cos(\frac{\theta}{2}) > 0$ and since $\mu = \lambda^2$ we conclude

$$\big\|\mu R(\mu, A_0)f\big\|_{\mathrm{W}_0^{1,1}} \le 2r\cdot\|f\|_{\mathrm{W}_0^{1,1}}\cdot\int_0^\infty t\cdot\mathrm{e}^{-\operatorname{Re}\lambda t}\,dt = \frac{2}{\cos^2(\frac{\theta}{2})}\cdot\|f\|_{\mathrm{W}_0^{1,1}}$$

for all $f \in X$ and $\mu \in \mathbb{C}\setminus\mathbb{R}$. Since $\operatorname{rg}(R(\mu, A_0)) \subseteq \mathrm{W}_0^{1,1}(0,1)$, Poincaré's inequality implies that A_0 is sectorial of angle $\frac{\pi}{2}$ as claimed.

(iii) *B is relatively A_0-bounded with A_0-bound zero:* Since by [15, Example III.2.2] the first derivative on $\mathrm{L}^1[0,1]$ is relatively bounded with respect to the second derivative with bound zero, for every $\varepsilon > 0$ there exists $M_\varepsilon \geq 1$ such that

$$\|g'\|_{\mathrm{L}^1} \leq \varepsilon \cdot \|g''\|_{\mathrm{L}^1} + M_\varepsilon \cdot \|g\|_{\mathrm{L}^1} \quad \text{for all } g \in \mathrm{W}^{2,1}(0,1).$$

Using this for $g = f'$ and setting $\|B\| := \|B\|_{\mathcal{L}(\mathrm{W}^{2,1}(0,1),\mathbb{C}^2)}$ we obtain

$$\begin{aligned}\|Bf\|_{\mathbb{C}^2} &\leq \|B\| \cdot \|f\|_{\mathrm{W}^{2,1}} = \|B\| \cdot \big(\|f''\|_{\mathrm{L}^1} + \|f\|_{\mathrm{L}^1}\big) \\ &\leq \|B\| \cdot \big(\varepsilon\|f'''\|_{\mathrm{L}^1} + M_\varepsilon\|f'\|_{\mathrm{L}^1} + \|f\|_{\mathrm{L}^1}\big) \\ &\leq \varepsilon \cdot \|B\| \cdot \|f\|_{\mathrm{W}^{3,1}} + M_\varepsilon \cdot \|B\| \cdot \|f\|_{\mathrm{W}^{1,1}},\end{aligned}$$

showing the claim.

Finally, by [1, Chapter VI] we have $[D(A_0)] = \mathrm{W}_0^{3,1}(0,1) \overset{c}{\hookrightarrow} \mathrm{W}^{1,1}(0,1) = X$ and hence [15, Proposition II.4.25] implies that A_0 has compact resolvent. Since the resolvent of $N \in \mathrm{M}_2(\mathbb{C})$ is compact as well, we conclude from Theorem 3.1 that A generates a compact, analytic semigroup of angle $\frac{\pi}{2}$ on X. □

Remarks 4.4. (i). In [17] it was shown by completely different methods that the second derivative with general Wentzell boundary conditions

$$\begin{aligned}&Af := f'',\\ &D(A) := \big\{f \in \mathrm{W}^{3,p}(0,1) : f''(j) = \beta_j f'(j) + \gamma_j f(j),\ j = 0,1\big\}\end{aligned} \tag{4.5}$$

generates an analytic semigroup on $\mathrm{W}^{1,p}(0,1)$ for all $1 < p < \infty$. Hence Corollary 4.3 complements this result to the non-reflexive case $p = 1$, allowing also much more general boundary conditions as in (4.5).

(ii). We point out that by similarity transformations and perturbation arguments as performed in [5], Corollary 4.3 can be generalized to more general second-order differential operators on $\mathrm{W}^{1,1}(0,1)$. This will be studied in more detail in the upcoming paper [6].

4.3. **Elliptic operators on $\mathbf{C}(\overline{\boldsymbol{\Omega}})$.** In this subsection we consider uniformly elliptic second-order differential operators with generalized Wentzell boundary conditions on $\mathrm{C}(\overline{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\Gamma := \partial\Omega$. To this end, we first take real functions

$$a_{jk} = a_{kj},\ a_j,\ a_0,\ b_0 \in \mathrm{C}^\infty(\overline{\Omega}), \quad 1 \leq j,k \leq n$$

satisfying the uniform ellipticity condition

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c\|\xi\|_2, \quad \text{for all } x \in \overline{\Omega},\ \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$$

and some fixed $c > 0$. Then we define the maximal operator in divergence form

$$A_m f := \sum_{j=1}^{n} \partial_j \Big(\sum_{k=1}^{n} a_{jk} \partial_k f \Big) + \sum_{k=1}^{n} a_k \partial_k f + a_0 f,$$
$$D(A_m) := \Big\{ f \in \bigcap_{p \geq 1} \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega) : A_m f \in \mathrm{C}(\overline{\Omega}) \Big\},$$

and the feedback operator

$$Bf := - \sum_{j,k=1}^{n} a_{jk} \nu_j L \partial_k f + b_0 L f,$$
$$D(B) := \Big\{ f \in \bigcap_{p \geq 1} \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega) : Bf \in \mathrm{C}(\Gamma) \Big\}.$$

Here $L : \mathrm{C}(\overline{\Omega}) \to \mathrm{C}(\Gamma)$ is the trace operator; i.e. $Lf = f|_\Gamma$, $f \in \mathrm{C}(\overline{\Omega})$, and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outer normal on $\Gamma = \partial\Omega$.

Corollary 4.5. *The operator*

$$\begin{aligned} Af &:= A_m f, \\ D(A) &:= \big\{ f \in D(A_m) \cap D(B) : LA_m f = Bf \big\} \end{aligned} \tag{4.6}$$

generates a compact and analytic semigroup on $\mathrm{C}(\overline{\Omega})$.

Proof. Choosing $X := \mathrm{C}(\overline{\Omega})$, $\partial X := \mathrm{C}(\Gamma)$, $Lf := f|_\Gamma$ and $C := 0$, one easily verifies that the operators defined in (2.1) and (4.6) coincide. Here A_m is densely defined since $\mathrm{C}^2(\overline{\Omega}) \subset D(A_m)$, the first space being dense in X.

Before proceeding we note that since by assumption all coefficients a_{jk} of A_m are in $\mathrm{C}^\infty(\overline{\Omega})$ we can represent it also in non-divergence form; i.e.,

$$A_m f = \sum_{j,k=1}^{n} a_{jk}\, \partial_j \partial_k f + \sum_{k=1}^{n} \Big(a_k + \sum_{j=1}^{n} \partial_j a_{jk} \Big) \partial_k f + a_0 f, \tag{4.7}$$

where, by bounded perturbation, we may assume without loss of generality that $a_0 \leq 0$.

Next we verify the assumptions of Theorem 3.1.

(i) $L_0 := (L|_{\ker A_m})^{-1} \in \mathcal{L}(\partial X, X)$ *exists:* By [24, Theorem 9.18], for every $x \in \mathrm{C}(\Gamma)$ there exists a unique $f \in D(A_m)$ such that $A_m f = 0$ and $Lf = x$. This implies that the Dirichlet operator exists. Its boundedness then follows from the maximum principle, see e.g. [24, Theorem 9.1].

(ii) A_0 *is sectorial*: This follows from (4.7) and [25, Corollary 3.1.21.(ii)].

(iii) *B is relatively A_0-bounded with A_0-bound zero:* We first note that $f \in D(A_0)$ is a solution of the Dirichlet problem

$$\begin{cases} A_m f = g, \\ Lf = 0 \end{cases}$$

for $g := A_m f \in X$. Note that, since Ω is bounded, $g \in \mathrm{L}^p(\Omega)$ for all $p > 1$. Moreover, by [7, Theorem IX.17], for $f \in D(A_m)$ we have $Lf = 0$ if and only if $f \in \mathrm{W}_0^{1,p}(\Omega)$. Hence we can apply elliptic regularity theory as in [24, Theorem 9.15] to conclude that $f \in \mathrm{W}^{2,p}(\Omega)$; i.e., $D(A_0) \subset \mathrm{W}^{2,p}(\Omega)$ for all $p > 1$. Now by Rellich's imbedding theorem (cf. [1, Theorem 6.2, Part III]), $\mathrm{W}^{2,p}(\Omega) \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{\Omega})$ for $p > n$ and the closed graph theorem implies that

$$[D(A_0)] \hookrightarrow \mathrm{W}^{2,p}(\Omega) \overset{c}{\hookrightarrow} \mathrm{C}^1(\overline{\Omega}) \hookrightarrow \mathrm{C}(\overline{\Omega}).$$

Hence, we are in the position to apply Ehrling's lemma (see [26, Theorem 6.99]) and conclude that for every $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$\|f\|_{\mathrm{C}^1(\overline{\Omega})} \leq \varepsilon \cdot \|f\|_{A_0} + C_\varepsilon \cdot \|f\|_\infty$$

for every $f \in D(A_0)$. Since $B \in \mathcal{L}(\mathrm{C}^1(\overline{\Omega}), \mathrm{C}(\Gamma))$ this implies that B is indeed A_0-bounded with relative bound zero.

(iv) *The Dirichlet-Neumann operator N generates an analytic semigroup on ∂X:* This is the main result in [16].

Finally, by the proof of (iii) we know that $D(A_0) \overset{c}{\hookrightarrow} X$ and hence [15, Proposition II.4.25] implies that A_0 has compact resolvent. Since by [16] the resolvent of N is compact as well, we conclude by Theorem 3.1 that A generates a compact, analytic semigroup. $\square$

This result generalizes the main theorem in [14], which already generalizes the main result in [4], where the Laplacian instead of a general uniformly elliptic operator was considered.

Appendix A. Some Auxiliary Results

Lemma A.1. *Let $\tilde{X}$ be a Banach space and X, X_0 two closed subspaces of $\tilde{X}$. Moreover, let $\tilde{T} \in \mathcal{L}(\tilde{X})$ be an invertible operator with $\tilde{T}X = X_0$. Then $T := \tilde{T}|_X \in \mathcal{L}(X, X_0)$ is invertible and for every operator $\tilde{A}$ on $\tilde{X}$ satisfying $D(\tilde{A}) \subseteq X$ one has*

$$TAT^{-1} = \big(\tilde{T}\tilde{A}\tilde{T}^{-1}\big)|_{X_0}, \tag{A.1}$$

where $A := \tilde{A}|_X$.

Proof. Let A_l and A_r denote the operator on the left- and on the right-hand side, respectively, of equation (A.1). Then

$$\begin{aligned} f \in D(A_r) \quad &\iff \quad f \in X_0,\ \tilde{T}^{-1} f \in D(\tilde{A}) \quad \text{and} \quad \tilde{T}\tilde{A}\tilde{T}^{-1} f \in X_0 \\ &\iff \quad f \in X_0,\ \tilde{T}^{-1} f \in D(\tilde{A}) = D(\tilde{A}) \cap X \text{ and } \tilde{A}\tilde{T}^{-1} f \in X \\ &\iff \quad f \in X_0 \quad \text{and} \quad T^{-1} f \in D(\tilde{A}|_X) = D(A) \\ &\iff \quad f \in D(A_l). \end{aligned}$$

Since it is easy to verify that $A_l f = A_r f$ for all $f \in D(A_l) = D(A_r)$, equation (A.1) is proved. □

Definition A.2. A closed, not necessarily densely defined, linear operator A on a Banach space X is called *sectorial* of angle $\alpha \in (0, \frac{\pi}{2}]$ if there exists $r \geq 0$ such that

$$\tilde{\Sigma}_{\frac{\pi}{2}+\alpha,r} := \Big\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \alpha\Big\} \cap \big\{z \in \mathbb{C} : |z| > r\big\} \subseteq \rho(A)$$

and if for each $\varepsilon \in (0, \alpha)$ there exists $M_\varepsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|} \quad \text{for all } \lambda \in \tilde{\Sigma}_{\frac{\pi}{2}+\alpha-\varepsilon,r}.$$

Lemma A.3. *The part of a sectorial operator A of angle α in $X_0 := \overline{D(A)}$ generates a (strongly continuous) analytic semigroup of angle α on X_0.*

Proof. The assertion follows from [3, Remark 3.7.13]. □

Lemma A.4. *Let A and N be sectorial operators of angle $\alpha > 0$ on Banach spaces X and Y, respectively. Moreover assume that $P_1 : D(A) \subset X \to X$ and $P_2 : D(A) \subset X \to Y$ are relatively A-bounded with A-bound zero and that $P_3 : D(N) \subset Y \to Y$ is N-bounded with N-bound zero. Then, for every N-bounded operator $Q : D(N) \subset Y \to X$, the operator matrix*

$$\mathcal{A} := \begin{pmatrix} A + P_1 & Q \\ P_2 & N + P_3 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} P_1 & Q \\ P_2 & P_3 \end{pmatrix} =: \mathcal{A}_0 + \mathcal{B}$$

with domain $D(\mathcal{A}) := D(A) \times D(N)$ is sectorial of angle α on $\mathcal{X} := X \times Y$. Moreover, if A and N have compact resolvents then also the resolvent of $\mathcal{A}$ is compact.

Proof. Choose $\varepsilon > 0$ and define $\mathcal{S}_\varepsilon := \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{X})$. Then $\mathcal{S}_\varepsilon$ is invertible with inverse $\mathcal{S}_\varepsilon^{-1} = \mathcal{S}_{\varepsilon^{-1}}$ and we obtain

$$\mathcal{A}_\varepsilon := \mathcal{S}_\varepsilon \mathcal{A} \mathcal{S}_\varepsilon^{-1} = \mathcal{A}_0 + \begin{pmatrix} P_1 & \varepsilon Q \\ \varepsilon^{-1} P_2 & P_3 \end{pmatrix} =: \mathcal{A}_0 + \mathcal{B}_\varepsilon, \qquad D(\mathcal{A}_\varepsilon) = D(\mathcal{A}).$$

By assumption there exist constants $a, b, C_\varepsilon > 0$ such that

$$\|P_1 x\|_X \le \varepsilon \|Ax\|_X + C_\varepsilon \|x\|_X, \qquad \|Qy\|_X \le a\|Ny\|_Y + b\|y\|_Y,$$
$$\|\varepsilon^{-1} P_2 x\|_Y \le \varepsilon \|Ax\|_X + C_\varepsilon \|x\|_X, \qquad \|P_3 y\|_Y \le \varepsilon \|Ny\|_Y + C_\varepsilon \|y\|_Y$$

for all $x \in D(A)$ and $y \in D(N)$, where we used the fact that with P_2 also $\varepsilon^{-1}P_2$ is A-bounded with A-bound zero. Choosing, for convenience, the 1-norm $\|\binom{x}{y}\|_{\mathcal{X}} := \|x\|_X + \|y\|_Y$ on $\mathcal{X}$, we obtain from these estimates

$$\begin{aligned}\left\|\mathcal{B}_\varepsilon \tbinom{x}{y}\right\|_{\mathcal{X}} &= \|P_1 x + \varepsilon Q y\|_X + \|\varepsilon^{-1} P_2 x + P_3 y\|_Y \\ &\le \|P_1 x\|_X + \varepsilon \|Qy\|_X + \varepsilon^{-1}\|P_2 x\|_Y + \|P_3 y\|_Y \\ &\le 2\varepsilon \|Ax\|_X + (1+a)\varepsilon \|Ny\|_Y + 2C_\varepsilon \|x\|_X + (\varepsilon b + C_\varepsilon)\|y\|_Y \\ &\le \delta_\varepsilon \left\|\mathcal{A}_0 \tbinom{x}{y}\right\|_{\mathcal{X}} + M_\varepsilon \left\|\tbinom{x}{y}\right\|_{\mathcal{X}}\end{aligned}$$

for $\delta_\varepsilon := \max\{2\varepsilon, (1+a)\varepsilon\}$ and $M_\varepsilon := \max\{2C_\varepsilon, \varepsilon b + C_\varepsilon\}$. Hence $\mathcal{B}_\varepsilon$ is $\mathcal{A}_0$-bounded with $\mathcal{A}_0$-bound $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Since $\mathcal{A}_0$ is sectorial of angle α, [15, Lemma III.2.6] implies that $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \mathcal{B}_\varepsilon$ is sectorial of angle α for $\varepsilon > 0$ sufficiently small. Since $\mathcal{A}$ and $\mathcal{A}_\varepsilon$ are similar, this implies the assertion.

Now assume that A and N, hence also $\mathcal{A}_0$, have compact resolvents. In order to prove that this implies the compactness of the resolvent of $\mathcal{A}$, by similarity it suffices to show that $\mathcal{A}_\varepsilon$ has compact resolvent for some $\varepsilon > 0$. However, for the resolvent of $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \mathcal{B}_\varepsilon$ we have the Neumann series representation

$$R(\lambda, \mathcal{A}_\varepsilon) = R(\lambda, \mathcal{A}_0) \sum_{k=0}^{\infty} \big(\mathcal{B}_\varepsilon R(\lambda, \mathcal{A}_0)\big)^k$$

(cf. [15, Section III.2, (2.5)]), which converges in operator norm for $\varepsilon > 0$ sufficiently small and λ large, implying that $R(\lambda, \mathcal{A}_\varepsilon)$ is indeed compact. □

We close this appendix with the following simple results on operator semigroups.

Lemma A.5. *Let G be the generator of a strongly continuous semigroup $(T(t))_{t\in\mathbb{R}}$ on a Banach space X satisfying $\|T(t)\| \le M$ for all $t \ge 0$. Then*

$$\|T(t)g - T(s)g\| \le M \cdot |t-s| \cdot \|Gg\| \quad \text{for all } s, t \ge 0 \text{ and } g \in D(G).$$

Proof. Assuming without loss of generality that $t \ge s$ the claim follows immediately from

$$\|T(t)g - T(s)g\| \le \int_s^t \|T(r)Gg\|\, dr \le M \cdot (t-s) \cdot \|Gg\|.$$

□

If we apply the previous lemma to the left-shift group $(T(t))_{t\in\mathbb{R}}$ defined by $[T(t)f](r) := f(r+t)$ on the Banach space

$$\mathrm{L}^1_\pi(-1,1) := \left\{f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) : f \text{ is 2-periodic}\right\}, \quad \|f\|_{\mathrm{L}^1_\pi} := \int_{-1}^{1} |f(r)|\, dr$$

we obtain the following result. Here we use that $(T(t))_{t\in\mathbb{R}}$ is strongly continuous, isometric and that its generator is given by[2]

$$Gf = f', \quad D(G) = \mathrm{L}^1_\pi(-1,1) \cap \mathrm{W}^{1,1}(-1,1).$$

Corollary A.6. *If $g \in \mathrm{L}^1_\pi(-1,1) \cap \mathrm{W}^{1,1}(-1,1)$, then*

$$\int_{-1}^{1} \big|g(r+t) - g(r+s)\big|\, dr \le |t-s| \cdot \int_{-1}^{1} \big|g'(r)\big|\, dr.$$

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