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REDUCED MEASURES FOR OBSTACLE PROBLEMS

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a smooth bounded domain. In this paper, we study the problem

$$\begin{cases} -\Delta u + \beta(u) \ni \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where μ is a finite measure in Ω and β is a maximal monotone graph (m.m.g.) such that $(0,0) \in \operatorname{graph} \beta$.

Throughout most of the Introduction, we assume that

$$\operatorname{dom} \beta = (-\infty, a] \quad \text{for some } 0 \leq a < \infty \tag{1.2}$$

and

$$\beta(t) = \{0\} \quad \forall t \leq 0. \tag{1.3}$$

(However, the case where $\operatorname{dom} \beta = [-b, a]$, $b \geq 0$, is also of interest and will be discussed at the end of the Introduction.) A typical example of a m.m.g. β satisfying (1.2)–(1.3) is the following

$$\beta(t) = \begin{cases} \{g(t)\} & \text{if } t < a, \\ [g(a), \infty) & \text{if } t = a, \\ \emptyset & \text{if } t > a, \end{cases}$$

where $g : (-\infty, a] \to [0, \infty)$ is any continuous nondecreasing function such that $g(t) = 0$, for all $t \leq 0$.

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Let $\mathcal{M}(\Omega)$ be the space of finite measures μ on $\overline{\Omega}$ such that $\mu(\partial\Omega) = 0$; $\mathcal{M}(\Omega)$ can be also identified with $\big[C_0(\overline{\Omega})\big]^*$. Given $\mu \in \mathcal{M}(\Omega)$, we say that u is a solution of the obstacle problem (1.1) if the following holds:

(i) $u \in L^1(\Omega)$, $u \le a$ almost everywhere and $\Delta u \in \mathcal{M}(\Omega)$;

(ii) there exists $\nu \in \mathcal{M}(\Omega)$ such that $\nu \in \beta(u)$ (in a sense to be explained below) and

$$-\int_\Omega u\Delta\zeta + \int_\Omega \zeta\, d\nu = \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{1.4}$$

Here,

$$C_0^2(\overline{\Omega}) = \Big\{\zeta \in C^2(\overline{\Omega})\ ;\ \zeta = 0 \text{ on } \partial\Omega\Big\}.$$

Recall that if $u \in L^1(\Omega)$ is such that $\Delta u \in \mathcal{M}(\Omega)$, then u admits a representative which is quasicontinuous with respect to the Newtonian (H^1) capacity, denoted cap (see, e.g., [1]). We may thus assume that u is well defined q.e. (= quasi-everywhere = outside a set of zero capacity). In particular, u is ν-measurable for any finite measure ν which does not charge sets of zero capacity. Given any such measure ν, we then say that

$$\nu \in \beta(u) \tag{1.5}$$

if the following holds:

- $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere;
- ν_{s} is concentrated on $[u = a]$ and $\nu_{\mathrm{s}} \ge 0$.

Here, we denote by ν_{a} and ν_{s} the absolutely continuous and the singular parts of ν with respect to the Lebesgue measure in $\mathbb{R}^N$. In Section 2 below, we discuss other equivalent definitions of (1.5).

If the measure μ is an L^1 function, it is known (see Brezis-Strauss [11]) that (1.1) has a unique solution (and in this case $\nu \in L^1(\Omega)$). As we shall see below, problem (1.1) still admits a solution for every measure μ in $L^1 + H^{-1}$ (see Theorem 1 below). In this case, the measure ν need not belong to $L^1(\Omega)$ (it is easy to construct examples in dimension $N = 1$).

We say that $\mu \in \mathcal{M}(\Omega)$ is diffuse if $\mu(E) = 0$ for every Borel set $E \subset \Omega$ such that $\operatorname{cap}(E) = 0$. Equivalently, a measure μ is diffuse if and only if $\mu \in L^1 + H^{-1}$ (see Grun-Rehomme [19] and Boccardo-Gallouët-Orsina [5]). In dimension $N = 1$, every measure is diffuse (since $\operatorname{cap}(E) = 0$ if and only if $E = \emptyset$). However, when $N \ge 2$, there are measures which are not diffuse, for instance Dirac masses.

In this paper, we follow the same program as in Brezis-Marcus-Ponce [8]. The main difference is that here $\operatorname{dom}\beta \ne \mathbb{R}$ and $\operatorname{dom}\beta$ is closed (the case where $\operatorname{dom}\beta \neq \mathbb{R}$ and $\operatorname{dom}\beta$ is an open set has been studied by Dupaigne-Ponce-Porretta [16]). Our main concern in [8] was twofold:

(a) Identify the good measures for problem (1.1), i.e., those for which (1.1) has a solution.
(b) If μ is *not* a good measure, define some kind of "generalized" solution, i.e., a common limit of all natural approximation schemes.

Our first result gives the complete answer to question (a):

Theorem 1. *Let $\mu \in \mathcal{M}(\Omega)$. Then,* (1.1) *has a solution if and only if μ^+ is diffuse. Moreover, the solution is unique.*

In other words, μ is a good measure for (1.1) if and only if μ^+ is diffuse (equivalently, $\mu^+ \in L^1 + H^{-1}$).

Recall that any measure μ can be uniquely decomposed as (see, e.g., [17])

$$\mu = \mu_{\mathrm{d}} + \mu_{\mathrm{c}},$$

where μ_{d} is a diffuse measure and μ_{c} is a measure concentrated on some set of zero H^1-capacity. Note in particular that μ is diffuse if and only if $\mu_{\mathrm{c}} = 0$.

Given $\mu \in \mathcal{M}(\Omega)$, set

$$\mu^* = \mu - (\mu_{\mathrm{c}})^+. \tag{1.6}$$

An easy consequence of Theorem 1 is the

Corollary 1. *For every $\mu \in \mathcal{M}(\Omega)$, μ^* is the largest good measure $\leq \mu$.*

Indeed, let λ be a good measure $\leq \mu$. By Theorem 1, we know that $\lambda_{\mathrm{c}} \leq 0$. On the other hand, from $\lambda \leq \mu$ we deduce that $\lambda_{\mathrm{d}} \leq \mu_{\mathrm{d}}$ and $-(\lambda_{\mathrm{c}})^- \leq -(\mu_{\mathrm{c}})^-$. Thus,

$$\lambda = \lambda_{\mathrm{d}} + (\lambda_{\mathrm{c}})^+ - (\lambda_{\mathrm{c}})^- \leq \mu_{\mathrm{d}} - (\mu_{\mathrm{c}})^- = \mu^*.$$

Notation. Given μ, we denote by u^* the unique solution of

$$\begin{cases} -\Delta u^* + \beta(u^*) \ni \mu^* & \text{in } \Omega, \\ \qquad\qquad u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We say that v is a subsolution of (1.1) if $v \in L^1(\Omega)$, $v \leq a$ almost everywhere, $\Delta v \in \mathcal{M}(\Omega)$ and there exists diffuse $\nu \in \mathcal{M}(\Omega)$ such that $\nu \in \beta(v)$ and

$$-\int_\Omega u\Delta\zeta + \int_\Omega \zeta\, d\nu \leq \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{1.8}$$

Since $\nu \geq 0$, v is a subsolution of (1.1) if and only if one can find $f \in L^1(\Omega)$ such that $f \in \beta(v)$ almost everywhere and

$$-\int_\Omega v\Delta\zeta + \int_\Omega f\zeta \leq \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}),\ \zeta \geq 0 \text{ in } \Omega. \tag{1.9}$$

A "companion" to Corollary 1 is the following

Proposition 1. *For any $\mu \in \mathcal{M}(\Omega)$, u^* is the largest subsolution of* (1.1).

Several authors have studied the obstacle problem (1.1) associated with measures. Given $a \geq 0$, let $\beta = \beta_a$ be the m.m.g. defined as

$$\beta_a(t) = \begin{cases} \{0\} & \text{if } t < a, \\ \big[0, \infty\big) & \text{if } t = a, \\ \emptyset & \text{if } t > a. \end{cases} \tag{1.10}$$

When $a = 0$ and $\Omega = \mathbb{R}^N$, $N \geq 3$, Theorem 1 has been known for a long time to experts from potential theory (see Baxter [3] and the references therein). In [3], the following statement appears: Let $\lambda_1, \lambda_2 \in \mathcal{M}(\Omega)$ be two nonnegative measures and assume that

$$\operatorname{Pot}\lambda_1 \equiv (-\Delta)^{-1}\lambda_1 < +\infty \quad \lambda_1\text{-a.e.}; \tag{1.11}$$

then, (1.1) has a solution for $\mu = \lambda_1 - \lambda_2$. Note that (1.11) is equivalent to the condition

$$\lambda_1 \text{ is diffuse} \tag{1.12}$$

(see, e.g., [13, Lemma 3.1]). Hence, (1.11) implies that μ^+ $(\leq \lambda_1)$ is diffuse, so that μ satisfies the assumption of Theorem 1. When $a > 0$ and β is given by (1.10), Theorem 1 and Proposition 1 are due to Dall'Aglio-Leone [14] and Dall'Aglio-Dal Maso [13]. More precisely, given any measure μ, Dall'Aglio-Leone [14] have proved that there exists a largest element $\hat{u}$ in the class of functions w such that

$$\begin{cases} -\Delta w \leq \mu & \text{in } \Omega, \\ \quad w \leq a & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

Subsequently, Dall'Aglio-Dal Maso [13] have proved that $\hat{u}$ satisfies (1.1) relative to the measure μ^*. In other words, they have shown that $\hat{u} = u^*$.

We now return to a general m.m.g. β satisfying (1.2)–(1.3) and we investigate part (b) of the program. Let (β_n) be a sequence of continuous, nondecreasing functions $\beta_n : \mathbb{R} \to \mathbb{R}$, $\beta_n(0) = 0$, such that

$$\beta_n \to \beta \quad \text{in the sense of graphs}; \tag{1.13}$$

more precisely, for every $(t, s) \in G \equiv \operatorname{graph} \beta$, there exists a sequence $t_n \to t$ such that $\beta_n(t_n) \to s$. If $N \geq 2$, then we also assume that each β_n has subcritical growth; i.e.,

$$\big|\beta_n(t)\big| \leq C\big(|t|^p + 1\big) \quad \forall t \in \mathbb{R}, \tag{1.14}$$

for some constant $C > 0$ and some $p < \frac{N}{N-2}$ (both possibly depending on n). We suppose in addition that

$$\beta_n(t) = 0 \quad \forall t \leq 0. \tag{1.15}$$

Given any $\mu \in \mathcal{M}(\Omega)$, there exists a unique function u_n such that (see Bénilan-Brezis [4])

$$\begin{cases} -\Delta u_n + \beta_n(u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

The limiting behavior of the sequence (u_n) is given by the following

Theorem 2. *For every $\mu \in \mathcal{M}(\Omega)$, we have*

$$u_n \to u^* \quad \text{in } L^1(\Omega),$$

where u_n is the solution of (1.16) *and u^* is the solution of* (1.7)*.*

We emphasize that—as in [8]—the limit of (u_n) is *independent* of the approximating sequence (β_n).

Another approximation scheme consists of keeping β fixed, and approximating μ by convolution. More precisely, given a sequence of mollifiers (ρ_n), let v_n denote the solution of

$$\begin{cases} -\Delta v_n + \beta(v_n) \ni \rho_n * \mu & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Then, the sequence (v_n) converges to the same limit u^*. More precisely,

Theorem 3. *For every $\mu \in \mathcal{M}(\Omega)$, we have*

$$v_n \to u^* \quad \text{in } L^1(\Omega),$$

where v_n is the solution of (1.17) *and u^* is the solution of* (1.7)*.*

In Section 7, we consider similar questions for the case of *two* obstacles, i.e., $\operatorname{dom} \beta = [-b, a]$ for some $b \geq 0$. We define the notion of solution of

$$\begin{cases} -\Delta u + \beta(u) \ni \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.18}$$

by analogy with the case studied above. Our main results assert that

- Problem (1.18) has a solution if and only if μ is diffuse and, in this case, this solution coincides with the unique minimizer of the variational problem

$$\min_{\substack{v \in H_0^1(\Omega) \\ -b \leq v \leq a \text{ a.e.}}} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega j(v) - \int_\Omega v \, d\mu \right\},$$

where $\partial j = \beta$ and $j(0) = 0$.
- Given any measure μ, the solutions of the "natural" approximation problems (1.16) and (1.17) converge to the solution of (1.18) with data μ_{d} (= diffuse part of μ).

This last result is related to a theorem of Orsina-Prignet [20].

This paper is organized as follows. In Section 2, we examine the concept of $\nu \in \beta(u)$ for any m.m.g. β and we discuss other equivalent definitions. In Section 3, we study some properties related to equidiffuse sequences (see Definition 1 below). In Section 4, we prove that the sequences given by (1.16) and (1.17) both converge q.e. The proofs of Theorems 1–3 and Proposition 1 are then presented in Sections 5 and 6. In Section 7, we discuss problem (1.1) in the case of two obstacles; the proofs of these results are given in Section 8.

2. Various definitions of $\nu \in \beta(u)$

Let β be a m.m.g. with closed domain, let u be a quasicontinuous function in Ω such that $u \in \operatorname{dom} \beta$ q.e., and let $\nu \in \mathcal{M}(\Omega)$. Set

$$a = \sup \left\{ \operatorname{dom} \beta \right\} \quad \text{and} \quad b = -\inf \left\{ \operatorname{dom} \beta \right\}. \tag{2.1}$$

We say that

$$\nu \in \beta(u)$$

if the following conditions are satisfied:

(1) ν is diffuse; (2.1a)

(2) $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere; (2.1b)

(3) ν_{s} is concentrated on the set $[u = a] \cup [u = -b]$; (2.1c)

(4) $\nu_{\mathrm{s}} \geq 0$ on $[u = a]$ and $\nu_{\mathrm{s}} \leq 0$ on $[u = -b]$. (2.1d)

We first observe that this definition agrees with the cases we have already considered in the Introduction, namely $\operatorname{dom} \beta = (-\infty, a]$ and $\operatorname{dom} \beta = \mathbb{R}$.

In fact, recall that for every quasicontinuous function u, the set $[u = \pm\infty]$ has zero capacity. Thus, for every diffuse measure ν, we have

$$\nu_{\mathrm{s}} = 0 \quad \text{on } [u = \pm\infty].$$

In particular, if $\operatorname{dom}\beta = \mathbb{R}$, then

$$\nu \in \beta(u) \quad \text{if and only if} \quad \nu \in L^1(\Omega) \quad \text{and} \quad \nu \in \beta(u) \ \text{ a.e.}$$

We also observe that, since $\operatorname{dom}\beta$ is closed, then

$$u \in \operatorname{dom}\beta \ \text{ q.e.} \quad \text{if and only if} \quad u \in \operatorname{dom}\beta \ \text{ a.e.}$$

We present some equivalent forms of the assertion "$\nu \in \beta(u)$" in the next

Proposition 2. *Let u be a quasicontinuous function in Ω such that $u \in \operatorname{dom}\beta$ almost everywhere. Assume that $\nu \in \mathcal{M}(\Omega)$ is diffuse and $u \in L^1(\Omega;\nu)$. Then, the following assertions are equivalent:*

(a) $\nu \in \beta(u)$;
(b) $j(u) \in L^1(\Omega)$ *and* $j(t) - j(u) \geq \nu\,(t-u)$ *in* $\big[C_0(\overline{\Omega})\big]^*$, $\forall t \in \mathbb{R}$;
(c) $j^*(\nu) \in \mathcal{M}(\Omega)$ *and* $j^*(s) - j^*(\nu) \geq u\,(s-\nu)$ *in* $\big[C_0(\overline{\Omega})\big]^*$, $\forall s \in \mathbb{R}$;
(d) $j(u) \in L^1(\Omega)$ *and* $\int_\Omega j(v) - \int_\Omega j(u) \geq \int_\Omega (v-u)\,d\nu$ *for every* $v \in H_0^1(\Omega) \cap L^\infty$;
(e) $j^*(\nu) \in \mathcal{M}(\Omega)$ *and* $\int_\Omega j^*(\sigma) - \int_\Omega j^*(\nu) \geq \int_\Omega u\,d(\sigma - \nu)$ *for every* $\sigma \in \mathcal{M}(\Omega)$ *diffuse such that* $u \in L^1(\Omega;\sigma)$;
(f) $\int_\Omega j(u) + \int_\Omega j^*(\nu) = \int_\Omega u\,d\nu$.

Here, $j : \mathbb{R} \to [0,+\infty]$ is a convex function such that $\partial j = \beta$ and $j(0) = 0$; j^* denotes the convex conjugate of j. For any $\nu \in \mathcal{M}(\Omega)$, the measure $j^*(\nu)$ is a nonnegative Borel measure defined as

$$j^*(\nu) = j^*(\nu_{\mathrm{a}}) + a(\nu_{\mathrm{s}})^+ + b(\nu_{\mathrm{s}})^-, \tag{2.2}$$

where a and b are given by (2.1) (see, e.g., [6, 18, 21, 22]).

Proof. (a) $\Rightarrow$ (b), (c), (f). By assumption,

$$j(t) - j(u) \geq \nu_{\mathrm{a}}\,(t-u) \quad \text{a.e.} \quad \forall t \in \mathbb{R}. \tag{2.3}$$

Recall that $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere is equivalent to

$$u \in \beta^{-1}(\nu_{\mathrm{a}}) = \partial j^*(\nu_{\mathrm{a}}) \quad \text{a.e.}$$

Thus,

$$j^*(s) - j^*(\nu_{\mathrm{a}}) \geq u\,(s - \nu_{\mathrm{a}}) \quad \text{a.e.} \quad \forall s \in \mathbb{R}, \tag{2.4}$$

$$j(u) + j^*(\nu_{\mathrm{a}}) = u\nu_{\mathrm{a}} \quad \text{a.e.} \tag{2.5}$$

On the other hand, ν_{s} is ≥ 0 on $[u = a]$ and ≤ 0 on $[u = -b]$; thus,

$$\begin{aligned} u\nu_{\mathrm{s}} &= a\nu_{\mathrm{s}} = a(\nu_{\mathrm{s}})^{+} \quad \text{in } [u = a], \\ u\nu_{\mathrm{s}} &= -b\nu_{\mathrm{s}} = b(\nu_{\mathrm{s}})^{-} \quad \text{in } [u = -b]. \end{aligned}$$

Here we use the convention that $\infty \cdot 0 = 0$. Since ν_{s} is concentrated on the set $[u = a] \cup [u = -b]$, we deduce that

$$u\nu_{\mathrm{s}} = a(\nu_{\mathrm{s}})^{+} + b(\nu_{\mathrm{s}})^{-} \geq 0 \quad \text{in } \Omega. \tag{2.6}$$

Thus, in view of (2.2),

$$j^*(\nu) = j^*(\nu_{\mathrm{a}}) + u\nu_{\mathrm{s}} \quad \text{in } \Omega. \tag{2.7}$$

By (2.5)–(2.6), we have $j(u) \in L^1(\Omega)$ and $j^*(\nu) \in \mathcal{M}(\Omega)$. Therefore, (b) follows from (2.3) and (2.6); (c) follows from (2.4) and (2.7); (f) follows from (2.5) and (2.7).

(a) $\Rightarrow$ (d). Let $v \in H_0^1(\Omega) \cap L^\infty$. If $v > a$ or $v < -b$ on a set of positive measure, then $\int_\Omega j(v) = +\infty$ and (d) holds. We now assume that $-b \leq v \leq a$ almost everywhere; hence, this also holds q.e. By (2.3), we have

$$j(v) - j(u) \geq \nu_{\mathrm{a}}\,(v - u) \quad \text{a.e.}$$

Thus,

$$\int_\Omega j(v) - \int_\Omega j(u) \geq \int_\Omega \nu_{\mathrm{a}}\,(v - u). \tag{2.8}$$

Since, by assumption, $-b \leq v \leq a$ q.e., it follows from (2.6) that

$$(v - u)\,\nu_{\mathrm{s}} = (v - a)\,(\nu_{\mathrm{s}})^{+} - (v + b)\,(\nu_{\mathrm{s}})^{-} \leq 0 \quad \text{in } \Omega.$$

Thus,

$$\int_\Omega (v - u)\,d\nu_{\mathrm{s}} \leq 0. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain (d).

(a) $\Rightarrow$ (e). Let diffuse $\sigma \in \mathcal{M}(\Omega)$ be such that $u \in L^1(\Omega; \sigma)$. By (2.4), we have

$$j^*(\sigma_{\mathrm{a}}) - j^*(\nu_{\mathrm{a}}) \geq u(\sigma_{\mathrm{a}} - \nu_{\mathrm{a}}) \quad \text{a.e.} \tag{2.10}$$

On the other hand, $-b \leq u \leq a$ q.e. implies

$$j^*(\sigma) = j^*(\sigma_{\mathrm{a}}) + a(\sigma_{\mathrm{s}})^{+} + b(\sigma_{\mathrm{s}})^{-} \geq j^*(\sigma_{\mathrm{a}}) + u\sigma_{\mathrm{s}}. \tag{2.11}$$

Combining (2.7), (2.10) and (2.11), we deduce (e).

(b) $\Rightarrow$ (a). By assumption,

$$j(t) - j(u) \geq \nu\,(t - u) \quad \text{in } \Omega, \quad \forall t \in \mathbb{R}. \tag{2.12}$$

Taking the absolutely continuous parts of (2.12) with respect to Lebesgue measure, we obtain (2.3). Thus, $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere. Given $t \in (-b, a)$, it follows from the singular part of (2.12) that

$$\nu_{\mathrm{s}}\,(t-u) \leq 0 \quad \text{in } \Omega.$$

In other words,

$$\begin{aligned} \nu_{\mathrm{s}} &\leq 0 \quad \text{in } [u < t],\\ \nu_{\mathrm{s}} &\geq 0 \quad \text{in } [u > t]. \end{aligned}$$

Since $t \in (-b, a)$ was arbitrary, we deduce that

$$\begin{aligned} \nu_{\mathrm{s}} &= 0 \quad \text{in } [-b < u < a],\\ \nu_{\mathrm{s}} &\leq 0 \quad \text{in } [u = -b],\\ \nu_{\mathrm{s}} &\geq 0 \quad \text{in } [u = a]. \end{aligned}$$

This establishes (a).

(c) $\Rightarrow$ (a). Comparing the absolutely continuous parts of both sides of (c), we obtain (2.4). Thus, (2.3) holds and so $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere. From the singular parts of (c), we get

$$a(\nu_{\mathrm{s}})^+ + b(\nu_{\mathrm{s}})^- \leq u\nu_{\mathrm{s}} \quad \text{in } \Omega.$$

Since $-b \leq u \leq a$ q.e., we conclude that

$$a(\nu_{\mathrm{s}})^+ + b(\nu_{\mathrm{s}})^- = u\nu_{\mathrm{s}} \quad \text{in } \Omega. \tag{2.13}$$

Thus, ν_{s} is concentrated on $[u = a] \cup [u = -b]$, $\nu_{\mathrm{s}} \geq 0$ in $[u = a]$, and $\nu_{\mathrm{s}} \leq 0$ in $[u = -b]$. We conclude that (a) holds.

(f) $\Rightarrow$ (a). We first observe that, by (f), we have $j(u) \in L^1(\Omega)$ and

$$j(u) + j^*(\nu) = u\nu \quad \text{in } \big[C_0(\overline{\Omega})\big]^*. \tag{2.14}$$

In fact, note that we always have

$$j(u) + j^*(\nu_{\mathrm{a}}) \geq u\nu_{\mathrm{a}} \quad \text{a.e.}$$

and

$$a(\nu_{\mathrm{s}})^+ + b(\nu_{\mathrm{s}})^- \geq u\nu_{\mathrm{s}}.$$

Combining these two inequalities, we get

$$j(u) + j^*(\nu) \geq u\nu \quad \text{in } \big[C_0(\overline{\Omega})\big]^*.$$

Using (f), (2.14) follows. From the absolutely continuous part of (2.14), we deduce that (2.5) holds, and so $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere. Moreover, from the singular part of (2.14), we obtain (2.13). Therefore, (a) is satisfied.

(d) $\Rightarrow$ (b). Since the right-hand side of (d) is finite and $j \geq 0$, we have $j(u) \in L^1(\Omega)$. Let $t \in (-b, a)$. We claim that

$$\int_A j(t) - \int_A j(u) \geq \int_A (t-u)\, d\nu \quad \forall A \subset \Omega \text{ Borel.} \tag{2.15}$$

By outer regularity of Radon measures, it suffices to establish (2.15) when A is open and $A \subset\subset \Omega$. Given an open subset $\omega \subset\subset A$ and $k \geq |t|$, let $v \in H_0^1(\Omega)$, $|v| \leq k$ in Ω, be such that $v = t$ in ω and $v = T_k(u)$ in $\Omega \backslash A$, where T_k denotes the truncation at $\pm k$. Note that $j(v) \leq j(u)$ in $\Omega \backslash A$. By (d), we then have

$$\int_A j(v) - \int_A j(u) \geq \int_\Omega (v-u)\, d\nu.$$

As $\omega \uparrow A$, we get

$$\int_A j(t) - \int_A j(u) \geq \int_A (t-u)\, d\nu + \int_{\Omega \backslash A} \big(T_k(u) - u\big)\, d\nu. \tag{2.16}$$

Since $u \in L^1(\Omega; \nu)$, by monotone convergence we have

$$\int_{\Omega \backslash A} \big(T_k(u) - u\big)\, d\nu \to 0 \quad \text{as } k \to \infty.$$

Thus, as $k \to \infty$ in (2.16), we get (2.15). Assertion (b) immediately follows from (2.15).

(e) $\Rightarrow$ (c). Given $s \in \mathbb{R}$ and a Borel set $A \subset \Omega$, let

$$\sigma = s\mathcal{L}^N\lfloor_A + \nu\lfloor_{\Omega \backslash A},$$

where $\mathcal{L}^N$ denotes the Lebesgue measure in $\mathbb{R}^N$. Clearly, σ is a diffuse measure in Ω and $u \in L^1(\Omega; \sigma)$. By (e), we have

$$\int_A j^*(s) - \int_A j^*(\nu) \geq \int_A u\,(s\, d\mathcal{L}^N - d\nu).$$

Since $A \subset \Omega$ is arbitrary, we conclude that (c) holds. The proof of the proposition is complete.

Remark 1. Assume $\nu \in \beta(u)$. The assumption that $u \in L^1(\Omega; \nu)$ automatically holds in the following cases:

- $\operatorname{dom} \beta = (-\infty, a]$ and $\beta(t) = \{0\}$, $\forall t \leq 0$;
- $\operatorname{dom} \beta = [-b, a]$.

Note, however, that if $\nu \in \beta(u)$ and $\operatorname{dom} \beta = \mathbb{R}$, then it need not be true that $u \in L^1(\Omega; \nu)$ (even if β is a continuous function); see however Lemma 2 below.

As we mentioned in the Introduction, we are interested in the approximation of problem (1.1) with a sequence (β_n). Given a sequence of m.m.g. (β_n), then $\beta_n \to \beta$ in the sense of graphs if, for every $(t,s) \in \operatorname{graph}\beta$, there exists $(t_n,s_n) \in \operatorname{graph}\beta_n$ such that $t_n \to t$ and $s_n \to s$. Since the notion of $\nu \in \beta(u)$ can be stated in terms of j, where $\partial j = \beta$, it is useful to have a characterization of the convergence of (β_n) in terms of the convergence of the primitives (j_n). This is given by the next

Proposition 3. *Let (β_n) be a sequence of m.m.g. Let $j_n : \mathbb{R} \to [0,+\infty]$ be such that $j_n(0) = 0$ and $\partial j_n = \beta_n$. Then, the following statements are equivalent:*

(a) *$\beta_n \to \beta$ in the sense of graphs;*
(b) *for every $t \in \mathbb{R}$, if $t_n \to t$, then $j(t) \le \liminf_{n\to\infty} j_n(t_n)$; for every $t \in \mathbb{R}$, there exists $t_n \to t$ such that $j(t) = \lim_{n\to\infty} j_n(t_n)$.*

We refer the reader to Attouch [2, Theorem 3.66] for the proof of Proposition 3. In the literature, assertion (b) is called the Mosco-convergence of j_n to j.

We conclude this section with the standard

Proposition 4. *Let (β_n) be a sequence of m.m.g. such that $\beta_n \to \beta$ in the sense of graphs. Let $(t_n,s_n) \in \operatorname{graph}\beta_n$. If $t_{n_k} \to t$ and $s_{n_k} \to s$, then*

$$(t,s) \in \operatorname{graph}\beta.$$

Proof. Given $(x,y) \in \operatorname{graph}\beta$, let $(x_n,y_n) \in \operatorname{graph}\beta_n$ be such that $x_n \to x$ and $y_n \to y$. From the monotonicity of β_{n_k} we have

$$(s_{n_k} - y_{n_k})(t_{n_k} - x_{n_k}) \ge 0.$$

Thus, as $k \to \infty$, we get

$$(s-y)(t-x) \ge 0 \quad \forall (x,y) \in \operatorname{graph}\beta.$$

From the maximality of β, we conclude that $(t,s) \in \operatorname{graph}\beta$.

3. Equidiffuse sequences of measures

We begin this section with the following

Definition 1. *Let $(\lambda_n) \subset \mathcal{M}(\Omega)$. Given $\varepsilon > 0$, we say that (λ_n) is ε-*equidiffuse *if*

(*i*) *(λ_n) is bounded in $\mathcal{M}(\Omega)$;*

(*ii*) *there exists* $\delta > 0$ *such that*

$$\operatorname{cap}(A) < \delta \qquad \Longrightarrow \qquad |\lambda_n|(A) < \varepsilon \quad \forall n \geq 1,$$

for every Borel set $A \subset \Omega$.

The sequence (λ_n) *is* equidiffuse *if it is* ε*-equidiffuse for every* $\varepsilon > 0$.

Clearly, if (λ_n) is equidiffuse, then each measure λ_n is diffuse. We also observe that any *finite* set of diffuse measures is equidiffuse. Here is another example. Let $\mu \in \mathcal{M}(\Omega)$ and let (ρ_n) be a sequence of mollifiers. If $\|\mu_c\|_{\mathcal{M}} < \varepsilon$, then $(\rho_n * \mu)$ is ε-equidiffuse. It is then easy to see that $(\rho_n * \mu)$ is equidiffuse if and only if μ is diffuse.

The main result of this section is the

Theorem 4. *Let* (w_n) *be a sequence of quasicontinuous functions in* Ω. *Let* (β_n) *be a sequence of m.m.g. Let* (ν_n) *be a sequence of diffuse measures with* $\nu_n \in \beta_n(w_n)$ *in the sense of* (2.1a)–(2.1d). *Assume that*

(*i*) $w_n \to w$ *q.e.;*
(*ii*) $\beta_n \to \beta$ *in the sense of graphs;*
(*iii*) $\nu_n \overset{*}{\rightharpoonup} \nu$ *weak* in* $\mathcal{M}(\Omega)$*;*
(*iv*) *for every* $\varepsilon > 0$, *there exists a compact set* $K_\varepsilon \subset \Omega$ *such that* $\operatorname{cap}(K_\varepsilon) < \varepsilon$ *and* (ν_n) *is* ε*-equidiffuse in* $\Omega \setminus K_\varepsilon$.

Then,

$$w \in \operatorname{dom}\beta \ \text{ a.e.} \quad \text{and} \quad \nu_{\mathrm{d}} \in \beta(w). \tag{3.1}$$

Remark 2. Assumption (*iv*) is clearly weaker then the assumption "(ν_n) is equidiffuse"; moreover, the measure ν need not be diffuse. For example, for any measure μ (not necessarily diffuse), the sequence $(\rho_n * \mu)$ always satisfies (*iv*).

Before turning to the proof of Theorem 4, we present two lemmas. The first one is reminiscent of part of the Dunford-Pettis theorem:

Lemma 1. *Let* $(\lambda_n) \subset \mathcal{M}(\Omega)$ *be a sequence of diffuse measures such that*

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \quad \text{weak* in } \mathcal{M}(\Omega).$$

Let (w_n) *be a bounded sequence of quasicontinuous functions in* Ω *such that* $w_n \to w$ *q.e. as* $n \to \infty$. *If* (λ_n) *is* ε*-equidiffuse, then*

$$\limsup_{n\to\infty} \left| \int_\Omega w_n \zeta \, d\lambda_n - \int_\Omega w\zeta \, d\lambda_{\mathrm{d}} \right| \leq C\varepsilon \|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0(\overline{\Omega}), \tag{3.2}$$

for some constant $C > 0$ *depending on* $\sup \|\lambda_n\|_{\mathcal{M}}$ *and* $\sup \|w_n\|_{L^\infty}$.

Notice that since (w_n) is quasicontinuous in Ω and $w_n \to w$ q.e., then w is also quasicontinuous.

Proof. Let $\delta > 0$ be such that $\operatorname{cap}(A) < \delta$ implies $|\lambda_n|(A) < \varepsilon$, for all $n \geq 1$, and $|\lambda|(A) \leq \varepsilon$. In particular,

$$\|\lambda_{\mathrm{c}}\|_{\mathcal{M}} \leq \varepsilon. \tag{3.3}$$

Let $\zeta \in C_0(\overline{\Omega})$. Since $w\zeta$ is quasicontinuous on Ω and $\zeta = 0$ on $\partial\Omega$, there exists $\psi \in C_0(\overline{\Omega})$ such that

$$\operatorname{cap}\big(\big[|\psi - w\zeta| > \varepsilon\big]\big) < \delta. \tag{3.4}$$

Thus,

$$\int_\Omega |\psi - w\zeta|\,|\lambda_{\mathrm{d}}| \leq \varepsilon\|\lambda\|_{\mathcal{M}} + \int\limits_{[|\psi - w\zeta| > \varepsilon]} |\psi - w\zeta|\,|\lambda_{\mathrm{d}}| \leq C\varepsilon. \tag{3.5}$$

Moreover, since $w_n \to w$ q.e., by (3.4) we have

$$\limsup_{n\to\infty}\Big\{\operatorname{cap}\big(\big[|\psi - w_n\zeta| > \varepsilon\big]\big)\Big\} < \delta.$$

Proceeding as above, we get

$$\limsup_{n\to\infty}\int_\Omega |\psi - w_n\zeta|\,|\lambda_n| \leq C\varepsilon. \tag{3.6}$$

Finally, since $\psi \in C_0(\overline{\Omega})$ and $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $\mathcal{M}(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega \psi\, d\lambda_n = \int_\Omega \psi\, d\lambda. \tag{3.7}$$

Combining (3.3), (3.5)–(3.7), we get

$$\limsup_{n\to\infty}\left|\int_\Omega w_n\zeta\, d\lambda_n - \int_\Omega w\zeta\, d\lambda_{\mathrm{d}}\right| \leq 2C\varepsilon\|\zeta\|_{L^\infty}.$$

This establishes Lemma 1.

Our next lemma is a variant of Proposition 2, where the condition $u \in L^1(\Omega;\nu)$ is *not* assumed:

Lemma 2. *Let u be a quasicontinuous function in Ω such that $u \in \operatorname{dom}\beta$ almost everywhere, and assume that $\nu \in \mathcal{M}(\Omega)$ is diffuse. Then, $\nu \in \beta(u)$ if and only if*

$$\big[j(t) - j(u) - \nu\,(t-u)\big]\Phi_k(u) \geq 0 \quad \textit{in } \big[C_0(\overline{\Omega})\big]^*, \quad \forall t \in \mathbb{R}, \quad \forall k > 0, \tag{3.8}$$

where $\Phi_k \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is such that $\Phi_k \geq 0$ in $\mathbb{R}$ and $\Phi_k(t) = 1$ if $|t| \leq k$.

Proof. The implication "⇒" is established as in the proof of Proposition 2. We now prove the converse statement. For simplicity, we only consider the case $\operatorname{dom}\beta = \mathbb{R}$. By (3.8), we have

$$j(t) - j(u) \geq \nu\,(t-u) \quad \text{in } \big[|u| \leq k\big], \quad \forall t \in \mathbb{R}, \quad \forall k > 0.$$

Thus,

$$j(t) - j(u) \geq \nu_{\mathrm{a}}\,(t-u) \quad \text{and} \quad \nu_{\mathrm{s}}\,(t-u) \leq 0 \quad \forall t \in \mathbb{R}. \tag{3.9}$$

We conclude that $\nu_{\mathrm{a}} \in \beta(u)$ almost everywhere and $\nu_{\mathrm{s}} = 0$; in particular, $\nu = \nu_{\mathrm{a}} \in L^1(\Omega)$. Therefore, $\nu \in \beta(u)$.

We now present the

Proof of Theorem 4. We split the proof into two steps:

Step 1. $w \in \operatorname{dom}\beta$ almost everywhere.

It suffices to consider the case where $\operatorname{dom}\beta \neq \mathbb{R}$. Assume for simplicity that $\operatorname{dom}\beta = (-\infty, a]$ (the other cases are similar). Given $t_0 > a$, by Proposition 4 there exists a sequence $r_n \to +\infty$ such that if $(t_0, s) \in \operatorname{graph}\beta_n$, then $s \geq r_n$. Recall that (ν_n) is a bounded sequence in $\mathcal{M}(\Omega)$. Thus, in view of the assumption $(\nu_n)_{\mathrm{a}} \in \beta_n(w_n)$ almost everywhere, we have

$$\big|[w_n \geq t_0]\big| \leq \frac{1}{r_n}\int_\Omega (\nu_n)_{\mathrm{a}} \leq \frac{1}{r_n}\int_\Omega |\nu_n| \leq \frac{C}{r_n}.$$

We conclude that

$$\big|[w > t_0]\big| \leq \liminf_{n\to\infty} \big|[w_n \geq t_0]\big| = 0 \quad \forall t_0 > a.$$

This implies

$$\big|[w > a]\big| = 0.$$

In other words, $w \leq a$ almost everywhere and so $w \in \operatorname{dom}\beta$ almost everywhere.

Step 2. $\nu_{\mathrm{d}} \in \beta(w)$.

Given $\varepsilon > 0$, let K_ε be the compact set given by (iv). Let Φ_k be as in the statement of Lemma 2. Since

$$w_n\Phi_k(w_n) \to w\Phi_k(w) \quad \text{q.e.},$$

we can apply Lemma 1 to get

$$\limsup_{n\to\infty}\left|\int_\Omega w_n\Phi_k(w_n)\zeta\,d\nu_n - \int_\Omega w\Phi_k(w)\zeta\,d\nu_{\mathrm{d}}\right| \leq Ck\,\varepsilon\,\|\zeta\|_{L^\infty} \tag{3.10}$$

for every $\zeta \in C_0(\overline{\Omega \backslash K_\varepsilon})$. Let $t \in \mathbb{R}$ be such that $j(t) < \infty$. Since $\beta_n \to \beta$ in the sense of graphs, by Proposition 3 there exists a sequence $t_n \to t$ such that

$$j_n(t_n) \to j(t) \quad \text{as } n \to \infty. \tag{3.11}$$

Moreover, by Fatou,

$$\int_\Omega j(w)\zeta \le \liminf_{n\to\infty} \int_\Omega j_n(w_n)\zeta \quad \forall \zeta \in C_0(\overline{\Omega}),\ \zeta \ge 0 \text{ in } \Omega. \tag{3.12}$$

On the other hand, by Lemma 2 we have

$$\big[j_n(t_n) - j_n(w_n) - \nu_n\,(t_n - w_n)\big]\Phi_k(w_n) \ge 0 \quad \text{in } \big[C_0(\overline{\Omega})\big]^*. \tag{3.13}$$

Combining (3.10)–(3.13), we conclude that

$$\int_\Omega \big[j(t) - j(w) - \nu_{\mathrm{d}}\,(t - w)\big]\Phi_k(w)\zeta \ge -Ck\,\varepsilon\,\|\zeta\|_{L^\infty} \quad \forall \zeta \in C_0(\overline{\Omega \backslash K_\varepsilon}),\ \zeta \ge 0.$$

As $\varepsilon \to 0$, we obtain

$$\big[j(t) - j(w) - \nu_{\mathrm{d}}\,(t - w)\big]\Phi_k(w) \ge 0 \quad \text{in } \big[C_0(\overline{\Omega})\big]^*.$$

By Lemma 2, this implies $\nu_{\mathrm{d}} \in \beta(w)$.

4. A basic tool concerning q.e.-convergence

Given $\mu_n \in \mathcal{M}(\Omega)$, let u_n be such that

$$\begin{cases} -\Delta u_n = \mu_n & \text{in } \Omega, \\ \quad u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In this section, we study the pointwise convergence of (some subsequence of) (u_n) as $n \to \infty$. Assume that

$$\|\Delta u_n\|_{\mathcal{M}} \le C. \tag{4.2}$$

Using standard linear regularity theory, one can show that there exists a subsequence (u_{n_i}) such that $u_{n_i} \to u$ strongly in $W^{1,p}$, for $p < 2$, and thus

$$u_{n_i} \to u \ \ \mathrm{cap}_{W^{1,p}}\text{-q.e.} \quad \forall p < 2. \tag{4.3}$$

On the other hand, given a measure ν, diffuse with respect to the H^1-capacity, we can also extract a subsequence (u_{n_j}) (depending on ν) such that

$$u_{n_j} \to u \ \ \nu\text{-a.e.} \tag{4.4}$$

A natural question is whether

$$u_{n_k} \to u \ \ \text{q.e. (i.e., } \mathrm{cap}_{W^{1,2}}\text{-q.e.)}, \tag{4.5}$$

for some subsequence (u_{n_k}).

It turns out that the answer is negative, even if the sequence (μ_n) is equidiffuse (see Appendix below). Note however that (4.5) is *true* for some special sequences (μ_n), e.g., $\mu_n = \rho_n * \mu$ for some fixed measure μ.

A basic tool used in the proof of our main results is the following

Theorem 5. *Given $\mu \in \mathcal{M}(\Omega)$, let $u_n \in L^1(\Omega)$ be the solution of*

$$\begin{cases} -\Delta u_n + \beta_n(u_n) \ni \rho_n * \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where

(*i*) (β_n) *is a sequence of m.m.g. such that* $(0,0) \in \operatorname{graph} \beta_n$*;*
(*ii*) (ρ_n) *is a sequence of mollifiers.*

If $u_n \to u$ in $L^1(\Omega)$, then u is quasicontinuous and

$$\operatorname{cap}\Big(\big[|u_n - u| > \delta\big]\Big) \to 0 \quad \text{as } n \to \infty, \quad \forall \delta > 0. \tag{4.7}$$

In particular, there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \ \ \text{q.e.} \tag{4.8}$$

Warning. The reader might think that the conclusion (4.8) in Theorem 5 holds if $(\rho_n * \mu)$ is replaced by an equidiffuse sequence (μ_n) in (4.6). As we have already mentioned, the answer is negative even in the simple case $\beta_n \equiv 0$ (see Appendix).

In order to prove Theorem 5 we need the following

Lemma 3. *Assume $u_n \in L^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta u_n + \beta_n(u_n) \ni \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

where

(*i*) (β_n) *is a sequence of m.m.g. such that* $(0,0) \in \operatorname{graph} \beta_n$*;*
(*ii*) (μ_n) *is a bounded sequence in* $\mathcal{M}(\Omega)$.

Suppose (μ_n) is ε-equidiffuse in $\Omega \backslash F$ for some compact set $F \subset \Omega$ and some $\varepsilon > 0$. Then, for every open set $\omega \supset F$, there exists $n_0 \geq 1$ such that

$$\big(\mu_n + \Delta u_n\big)_{n \geq n_0} \quad \text{is } 2\varepsilon\text{-equidiffuse in } \Omega \setminus \overline{\omega}. \tag{4.10}$$

Proof. Let $\nu_n = \mu_n + \Delta u_n$. By assumption, ν_n is diffuse and $\nu_n \in \beta_n(u_n)$. Thus,

$$\big(\operatorname{sign} u_n\big)\, \nu_n = |\nu_n|.$$

By Kato's inequality (see [9]), we have

$$\big(\Delta|u_n|\big)_{\mathrm{d}} \geq \operatorname{sign} u_n\,(\Delta u_n)_{\mathrm{d}} = \operatorname{sign} u_n\,\big(\nu_n - (\mu_n)_{\mathrm{d}}\big) \geq |\nu_n| - |\mu_n|_{\mathrm{d}}$$

and

$$\big(\Delta|u_n|\big)_{\mathrm{c}} = -|\Delta u_n|_{\mathrm{c}} = -|\mu_n|_{\mathrm{c}}.$$

Thus,

$$-\Delta|u_n| + |\nu_n| \leq |\mu_n| \quad \text{in } \Omega.$$

Since $|u_n| \in W_0^{1,1}(\Omega)$, then for every superharmonic function $\zeta \in C_0^2(\overline{\Omega})$ we have

$$\int_\Omega \zeta\,|\nu_n| \leq \int_\Omega \zeta\,|\mu_n|. \tag{4.11}$$

Given $K \subset \Omega$, let ζ_K denote the capacitary potential of K. By density, (4.11) also holds with $\zeta = \zeta_K$ and then we have

$$|\nu_n|(K) \leq \int_\Omega \zeta_K\,|\nu_n| \leq \int_\Omega \zeta_K\,|\mu_n|. \tag{4.12}$$

Assume by contradiction that (ν_n) is not 2ε-equidiffuse in $\Omega \setminus \overline{\omega}$ for n sufficiently large. By inner regularity, there exists (ν_{n_j}) and a sequence of compact sets (K_j) in $\Omega \setminus \overline{\omega}$ such that

$$|\nu_{n_j}|(K_j) \geq \frac{3\varepsilon}{2} \quad \text{and} \quad \operatorname{cap}(K_j) \to 0. \tag{4.13}$$

Thus, $\zeta_{K_j} \to 0$ in $H_0^1(\Omega)$ and uniformly on F. Passing to a subsequence, we may assume that $\zeta_{K_j} \to 0$ q.e. By Lemma 1, we then have

$$\limsup_{n\to\infty} \int_\Omega \zeta_{K_n}\,|\mu_n| \leq \varepsilon. \tag{4.14}$$

Combining (4.12)–(4.14), we get a contradiction. Therefore, there exists $n_0 \geq 1$ such that (4.10) holds. The proof of the lemma is complete.

Proof of Theorem 5. By standard estimates, we have

$$\|\Delta u_n\|_{\mathcal{M}} \leq 2\|\rho_n * \mu\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}.$$

Thus, $\Delta u \in \mathcal{M}(\Omega)$; hence u is quasicontinuous. Moreover, note that (4.8) follows from (4.7) by using a standard diagonalization argument. We now split the proof of (4.7) into two steps:

Step 1. Proof of (4.7) when μ_{c} is concentrated on a compact set $K \subset \Omega$ of zero capacity.

Let $f_n = (\rho_n * \mu) + \Delta u_n$, so that $f_n \in L^1(\Omega)$ and $f_n \in \beta_n(u_n)$ almost everywhere, for all $n \geq 1$. We first establish the following

Claim 1. For every $\delta > 0$ and for every open set $\omega \supset K$, we have

$$\int_{(A_n \cup B_n)\setminus\overline{\omega}} |f_n| \to 0 \quad \text{as } n \to \infty, \tag{4.15}$$

where

$$A_n = \big[0 \le u_n \le u - \delta\big] \quad \text{and} \quad B_n = \big[u + \delta \le u_n \le 0\big]. \tag{4.16}$$

We show (4.15) for $|f_n|$ integrated over $A_n \setminus \overline{\omega}$; the term coming from $B_n \setminus \overline{\omega}$ can be estimated in a similar way. We consider two cases, depending on whether u^+ is bounded or not:

Case 1. $u^+ \in L^\infty(\Omega)$.

Let $M = \|u^+\|_{L^\infty}$. We claim that there exists $C_\delta > 0$ such that if $(t_n, s_n) \in \operatorname{graph} \beta_n$ and $t_n \le M - \delta$, then $s_n \le C_\delta$. Assume by contradiction that there exists a sequence $(t_{n_k}, s_{n_k}) \in \operatorname{graph} \beta_{n_k}$ such that

$$t_{n_k} \le M - \delta \quad \text{and} \quad s_{n_k} \to \infty.$$

Since $\|f_n\|_{L^1} \le \|\mu\|_{\mathcal{M}}$ and $f_n \in \beta(u_n)$ almost everywhere, we have

$$\big|[u_{n_k} > M - \delta]\big|\, s_{n_k} \le \int_{[u_{n_k} \ge t_{n_k}]} f_{n_k} \le C.$$

Thus,

$$\big|[u > M - \delta]\big| \le \liminf_{k \to \infty} \big|[u_{n_k} > M - \delta]\big| = 0.$$

We then deduce that $u \le M - \delta$ almost everywhere, a contradiction. Therefore, we have

$$0 \le f_n \le C_\delta \quad \text{on } A_n \setminus \overline{\omega}.$$

Since $\chi_{A_n} \to 0$ in $L^1(\Omega)$, we conclude that

$$0 \le \int_{A_n \setminus \overline{\omega}} |f_n| \le C_\delta |A_n| \to 0.$$

Case 2. $u^+ \notin L^\infty(\Omega)$.

Recall the standard estimate

$$\int_\Omega \big|\nabla T_k(u)\big|^2 \le k\|\Delta u\|_{\mathcal{M}} \quad \forall k > 0.$$

Thus,

$$\operatorname{cap}\big([u > k]\big) \le \frac{1}{k^2} \int_\Omega \big|\nabla T_k(u)\big|^2 \to 0 \quad \text{as } k \to \infty.$$

Given $\eta > 0$, let $k_0 \geq 1$ be such that

$$\operatorname{cap}\big([u > k]\big) < \eta \quad \forall k \geq k_0.$$

Clearly, $(\rho_n * \mu)$ is equidiffuse in $\Omega \setminus \overline{\omega}_0$ for every open set $\omega_0 \supset K$. Thus, by Lemma 3, the sequence (f_n) is equidiffuse in $\Omega \setminus \overline{\omega}$. Therefore, given $\varepsilon > 0$, one can take $k_0 \geq 1$ large enough such that

$$\int\limits_{[u>k]\setminus\overline{\omega}} |f_n| < \varepsilon \quad \forall k \geq k_0. \tag{4.17}$$

On the other hand, as in the proof of Case 1 above, there exists $C_{k_0} > 0$ such that if $(t_n, s_n) \in \operatorname{graph}\beta_n$ and $t_n \leq k_0$, then $s_n \leq C_{k_0}$. From this and (4.17), we then have

$$\int\limits_{A_n\setminus\overline{\omega}} |f_n| \;\leq \int\limits_{A_n\cap[u\leq k_0]} |f_n| \;+ \int\limits_{[u>k_0]\setminus\overline{\omega}} |f_n| \;\leq\; C_{k_0}|A_n| + \varepsilon.$$

Thus,

$$\limsup_{n\to\infty} \int\limits_{A_n\setminus\overline{\omega}} |f_n| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{n\to\infty} \int\limits_{A_n\setminus\overline{\omega}} |f_n| = 0.$$

The same argument shows that the conclusion holds when integrating $|f_n|$ on $B_n \setminus \overline{\omega}$. The proof of (4.15) is complete.

Claim 2. For every $\delta > 0$ and for every open set $\omega \supset K$, we have

$$\int\limits_{Z_n\setminus\overline{\omega}} \big|\nabla(u_n - u)\big|^2 \to 0 \quad \text{as } n \to \infty, \tag{4.18}$$

where $Z_n = \big[\delta < |u_n - u| < 2\delta\big]$.

Let $h_\delta : \mathbb{R} \to \mathbb{R}$ be the function given by

$$h_\delta(t) = \begin{cases} 0 & \text{if } -\delta \leq t \leq \delta,\\ 1 & \text{if } t \geq 2\delta,\\ -1 & \text{if } t \leq -2\delta,\\ \frac{1}{\delta}(t-\delta) & \text{if } \delta < t < 2\delta,\\ \frac{1}{\delta}(t+\delta) & \text{if } -2\delta < t < -\delta. \end{cases}$$

Note that given $\psi \in C^\infty(\overline{\Omega})$, $\psi \geq 0$ in $\overline{\Omega}$, and $v \in W_0^{1,1}(\Omega)$ such that $\Delta v \in \mathcal{M}(\Omega)$, we have

$$\delta \int_\Omega \big|\nabla h_\delta(v)\big|^2 \psi \leq -\int_\Omega (\Delta v)_{\mathrm{d}}\, h_\delta(v)\psi + \int_\Omega |\Delta v|_{\mathrm{c}}\, \psi - \int_\Omega \nabla v \cdot \nabla \psi\, h_\delta(v). \quad (4.19)$$

(This estimate clearly holds if v is smooth; the case of a general $v \in W_0^{1,1}$ then follows by approximation.)
Given open sets $\omega_1 \supset\supset \omega_0 \supset K$, we take $\psi \in C^\infty(\overline{\Omega})$ such that $\psi = 1$ on $\overline{\Omega} \setminus \omega_1$ and $\psi = 0$ on $\overline{\omega}_0$. It easily follows from Lemma 3 that $(\Delta u)_{\mathrm{c}}$ is supported in K. Thus,

$$\big|\Delta(u_n - u)\big|_{\mathrm{c}}\, \psi = |\Delta u|_{\mathrm{c}}\, \psi = 0. \quad (4.20)$$

We now apply (4.19) to $v = u_n - u$. By (4.20), we get

$$\begin{aligned} \delta \int_\Omega \big|\nabla h_\delta(v)\big|^2 \psi \\ \leq -\int_\Omega f_n\, h_\delta(u_n - u)\, \psi + \int_\Omega \lambda_n\, h_\delta(u_n - u) + C \int_\Omega \big|\nabla(u_n - u)\big|, \end{aligned} \quad (4.21)$$

where $\lambda_n = \big\{\rho_n * \mu - (\Delta u)_{\mathrm{d}}\big\}\psi$. The last integral converges to 0 as $n \to \infty$ because $u_n \to u$ in $W_0^{1,1}(\Omega)$. Since

$$h_\delta(u_n - u) \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega) \quad \text{as } n \to \infty,$$

and $\lambda_n = 0$ on ω_0, it is not difficult to see that

$$\lim_{n\to\infty} \int_\Omega \lambda_n\, h_\delta(u_n - u) = 0. \quad (4.22)$$

(To prove (4.22) one can proceed as in the proof of Proposition 2.1 in [9].)
We now show that

$$\liminf_{n\to\infty} \int_\Omega f_n\, h_\delta(u_n - u)\, \psi \geq 0. \quad (4.23)$$

In fact, note that the integrand is ≥ 0 on $\Omega \backslash (A_n \cup B_n)$, where A_n, B_n are given by (4.16). Thus,

$$\int_\Omega f_n\, h_\delta(u_n - u)\, \psi \geq \int_{A_n \cup B_n} f_n\, h_\delta(u_n - u)\, \psi \geq -\int_{(A_n \cup B_n)\backslash \overline{\omega}_0} |f_n|.$$

By Claim 1 and this estimate, we conclude that (4.23) holds. Combining (4.21)–(4.23), we obtain (4.18) with $\omega = \omega_1$.

We now conclude the proof of (4.7). Given $\varepsilon > 0$, we fix an open set $\omega_0 \supset K$ satisfying $\operatorname{cap}(\overline{\omega}_0) < \varepsilon$. Let $\psi \in C^\infty(\overline{\Omega})$ be such that $\psi = 1$ in

$\overline{\Omega} \setminus \omega_0$ and $\psi = 0$ in some neighborhood of K. Recall that, by standard estimates (see, e.g., [23]), we have

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad \forall p < \tfrac{N}{N-1}. \tag{4.24}$$

It then follows from (4.18) and (4.24) that

$$\lim_{n\to\infty} \int_\Omega \Big|\nabla\big[h_\delta(u_n - u)\,\psi\big]\Big|^2 = 0. \tag{4.25}$$

Let $W_n = \big[|u_n - u| > 2\delta\big]$. Since

$$\big|h_\delta(u_n - u)\big|\psi \geq 1 \quad \text{on } W_n \setminus \overline{\omega}_0,$$

we have

$$\text{cap}\,(W_n \setminus \overline{\omega}_0) \leq \int_\Omega \Big|\nabla\big|h_\delta(u_n - u)\,\psi\big|\Big|^2.$$

On the other hand, by subadditivity of cap,

$$\text{cap}\,(W_n) \leq \text{cap}\,(W_n \setminus \overline{\omega}_0) + \text{cap}\,(\overline{\omega}_0) \leq \text{cap}\,(W_n \setminus \overline{\omega}_0) + \varepsilon.$$

Therefore,

$$\limsup_{n\to\infty} \Big\{ \text{cap}\,(W_n) \Big\} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (4.7) follows (with δ replaced by 2δ).

Step 2. Proof of the theorem completed.

Clearly, it suffices to establish (4.7) for a subsequence (u_{n_j}). Let (μ_k) be a sequence in $\mathcal{M}(\Omega)$ so that $\mu_k \to \mu$ strongly in $\mathcal{M}(\Omega)$ and so that each measure $(\mu_k)_{\text{c}}$ is supported in some compact set of zero capacity. Let $v_{n,k} \in L^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta v_{n,k} + \beta_n(v_{n,k}) \ni \rho_n * \mu_k & \text{in } \Omega, \\ \qquad\qquad\qquad v_{n,k} = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, we have $\|\Delta v_{n,k}\|_{\mathcal{M}} \leq C$. Thus, there exist an increasing sequence of integers (n_j) and a sequence of functions (v_k) in $L^1(\Omega)$ such that

$$v_{n_j,k} \to v_k \quad \text{in } L^1(\Omega) \quad \text{as } j \to \infty, \quad \forall k \geq 1.$$

By standard estimates, we have

$$\int_\Omega \big|\Delta(v_{n_j,k} - u_{n_j})\big| \leq 2 \int_\Omega \rho_{n_j} * |\mu_k - \mu| \leq 2\|\mu_k - \mu\|_{\mathcal{M}}.$$

Thus,

$$\int_\Omega \big|\nabla T_\delta(v_{n_j,k} - u_{n_j})\big|^2 \leq 2\delta\|\mu_k - \mu\|_{\mathcal{M}}. \tag{4.26}$$

As $j \to \infty$, we also have

$$\int_\Omega \left|\nabla T_\delta(v_k - u)\right|^2 \leq 2\delta \|\mu_k - \mu\|_{\mathcal{M}}. \tag{4.27}$$

Combining (4.26) and (4.27), we get

$$\operatorname{cap}\left(\left[|v_{n_j,k} - u_{n_j}| > \delta\right]\right) + \operatorname{cap}\left(\left[|v_k - u| > \delta\right]\right) \leq \frac{4}{\delta}\|\mu_k - \mu\|_{\mathcal{M}}.$$

Thus,

$$\operatorname{cap}\left(\left[|u_{n_j} - u| > 3\delta\right]\right) \leq \operatorname{cap}\left(\left[|v_{n_j,k} - v_k| > \delta\right]\right) + \frac{4}{\delta}\|\mu_k - \mu\|_{\mathcal{M}}.$$

For $k \geq 1$ fixed, we apply the previous step to $(v_{n_j,k})$. We deduce that

$$\limsup_{j\to\infty}\left\{\operatorname{cap}\left(\left[|u_{n_j} - u| > 3\delta\right]\right)\right\} \leq \frac{4}{\delta}\|\mu_k - \mu\|_{\mathcal{M}}.$$

As $k \to \infty$, we obtain (4.7) for the subsequence (u_{n_j}). The proof of the theorem is complete.

Remark 3. An easy inspection of the proof of Theorem 5 shows that (4.7)–(4.8) are still valid if the sequence $(\rho_n * \mu)$ is replaced be any sequence of measures (μ_n) of the form

$$\mu_n = f_n + T_n + \rho_n * \nu,$$

where $f_n \in L^1(\Omega)$, $T_n \in H^{-1}(\Omega)$ satisfy

$$f_n \to f \text{ in } L^1(\Omega) \quad \text{and} \quad T_n \to T \text{ in } H^{-1}(\Omega),$$

and ν is a measure concentrated on a set of zero H^1-capacity.

We will also need the following variant of Theorem 5:

Theorem 6. *Given $\mu \in \mathcal{M}(\Omega)$, let $u_n \in L^1(\Omega)$ be the solution of*

$$\begin{cases} -\Delta u_n + \beta_n(u_n) = \mu & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.28}$$

where $\beta_n : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function with subcritical growth and such that $\beta_n(0) = 0$. If $u_n \to u$ in $L^1(\Omega)$, then u is quasicontinuous and

$$\operatorname{cap}\left(\left[|u_n - u| > \delta\right]\right) \to 0 \quad \text{as } n \to \infty, \quad \forall \delta > 0. \tag{4.29}$$

In particular, there exists a subsequence (u_{n_k}) such that

$$u_{n_k} \to u \ \text{ q.e.} \tag{4.30}$$

Proof. Let (ρ_k) be a sequence of mollifiers. Let $(u_{n,k})$ be the solution of

$$\begin{cases} -\Delta u_{n,k} + \beta_n(u_{n,k}) = \rho_k * \mu & \text{in } \Omega, \\ u_{n,k} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since β_n has subcritical growth, $u_{n,k} \to u_n$ in $L^1(\Omega)$ as $k \to \infty$. By the previous theorem, we have

$$\operatorname{cap}\Big(\big[|u_{n,k} - u_n| > \delta\big]\Big) \to 0 \quad \text{as } k \to \infty.$$

Thus, one can find an increasing sequence of integers (k_n) such that

$$\operatorname{cap}\Big(\big[|u_{n,k_n} - u_n| > \delta\big]\Big) \leq \frac{1}{2^n} \quad \text{and} \quad u_{n,k_n} \to u \ \text{ in } L^1(\Omega).$$

Therefore,

$$\operatorname{cap}\Big(\big[|u_n - u| > 2\delta\big]\Big) \leq \operatorname{cap}\Big(\big[|u_{n,k_n} - u| > \delta\big]\Big) + \frac{1}{2^n}.$$

Applying Theorem 5 to the sequence (u_{n,k_n}), the result follows.

5. Proof of Theorem 1

We first establish the following

Lemma 4. *Given any* $\mu \in \mathcal{M}(\Omega)$, *problem* (1.1) *has at most one solution.*

Proof. Assume u_1, u_2 both satisfy (1.1). We claim that

$$\Delta(u_1 - u_2)^+ \geq 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

In fact, let

$$\nu_i = \mu + \Delta u_i, \quad \text{for } i = 1, 2.$$

Recall that each ν_i is diffuse. Thus, $\Delta(u_1 - u_2)$ is also diffuse. Applying Kato's inequality (see [9, Theorem 1.1 and Remark 1]), we deduce that $\Delta(u_1 - u_2)^+$ is a diffuse measure and

$$\Delta(u_1 - u_2)^+ \geq \chi_{[u_1 > u_2]} \Delta(u_1 - u_2) = \chi_{[u_1 > u_2]}(\nu_1 - \nu_2) \quad \text{in } \Omega. \tag{5.2}$$

We now observe that

$$\chi_{[u_1 > u_2]}(\nu_1 - \nu_2) \geq 0 \quad \text{in } \Omega. \tag{5.3}$$

Indeed, recall that

$$(\nu_i)_{\mathrm{a}} \in \beta(u_i) \quad \text{a.e.}$$

Thus,

$$\chi_{[u_1 > u_2]}(\nu_1 - \nu_2)_{\mathrm{a}} \geq 0 \quad \text{a.e.} \tag{5.4}$$

On the other hand, since $(\nu_2)_s$ is concentrated on the set $[u_2 = a]$ and

$$[u_1 > u_2] \cap [u_2 = a] = \emptyset,$$

we have

$$(\nu_2)_s = 0 \quad \text{on } [u_1 > u_2].$$

Thus,

$$\chi_{[u_1>u_2]}(\nu_1 - \nu_2)_s = \chi_{[u_1>u_2]}(\nu_1)_s \geq 0 \quad \text{in } \Omega. \tag{5.5}$$

Combining (5.4)–(5.5), we obtain (5.3). It then follows from (5.2)–(5.3) that (5.1) holds. Since $u_1, u_2 \in W_0^{1,1}(\Omega)$, we get (see, e.g., [8, Proposition B.1])

$$(u_1 - u_2)^+ = 0 \quad \text{a.e.}$$

In other words, $u_1 \leq u_2$ almost everywhere. Reversing the roles of u_1 and u_2, we conclude that $u_1 = u_2$ almost everywhere. Therefore, problem (1.1) has at most one solution.

Lemma 5. *Let $\mu \in \mathcal{M}(\Omega)$ be such that μ^+ is diffuse. Given a sequence (β_n) satisfying* (1.13)–(1.15)*, let u_n be the solution of* (1.16)*. Then,*

$$u_n \to u \quad \textit{in } L^1(\Omega), \tag{5.6}$$

where u satisfies (1.1)*. In particular,*

$$\beta_n(u_n) \overset{*}{\rightharpoonup} \mu + \Delta u \quad \textit{weak}^* \textit{ in } \mathcal{M}(\Omega). \tag{5.7}$$

Proof. By standard estimates (see, e.g., [8]), we have

$$\int_\Omega |\beta_n(u_n)| \leq \|\mu\|_{\mathcal{M}} \quad \text{and} \quad \|\Delta u_n\|_{\mathcal{M}} \leq 2\|\mu\|_{\mathcal{M}}. \tag{5.8}$$

Passing to a subsequence if necessary, one can find $u \in L^1(\Omega)$ and $\nu \in \mathcal{M}(\Omega)$ such that

$$u_n \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad \beta_n(u_n) \overset{*}{\rightharpoonup} \nu \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega).$$

In particular,

$$-\int_\Omega u\Delta\zeta + \int_\Omega \zeta \, d\nu = \int_\Omega \zeta \, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{5.9}$$

Clearly, the sequence (u_n) satisfies the assumptions of Theorem 6. Thus, passing to a further subsequence if necessary, we may also assume that

$$u_n \to u \ \text{ q.e.}$$

Our goal is to show that u satisfies (1.1). By Lemma 4, this will imply that the limit u is actually independent of the subsequence, so that the entire sequence converges to the same limit u. We first establish the following

Claim. The sequence $\big(\beta_n(u_n)\big)$ is equidiffuse.

Let v_n be the solution of

$$\begin{cases} -\Delta v_n + \beta_n(v_n) = \mu^+ & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since μ^+ is diffuse, it follows from Lemma 3 that $\big(\beta_n(v_n)\big)$ is equidiffuse (note that in this case $K = \emptyset$). On the other hand, since $u_n \leq v_n$ almost everywhere and $\beta_n(t) = 0$, for all $t \leq 0$, we have

$$0 \leq \beta_n(u_n) \leq \beta_n(v_n) \quad \text{a.e.}$$

We conclude that $\big(\beta_n(u_n)\big)$ is also equidiffuse.

We can now apply Theorem 4 to conclude that $u \leq a$ almost everywhere and $\nu \in \beta(u)$. In other words, u is the (unique) solution of (1.1). The proof of the lemma is complete.

Proof of Theorem 1. ($\Leftarrow$) This follows from Lemma 5 above. By Lemma 4, the solution of (1.1) is unique.

($\Rightarrow$) Assume (1.1) has a solution u. In particular, there exists a diffuse measure ν such that

$$-\Delta u = \mu - \nu \quad \text{in } \mathcal{D}'(\Omega).$$

Since $u \leq a$ almost everywhere, it follows from the "Inverse" maximum principle that $(-\Delta u)_{\mathrm{c}} \leq 0$ (see [15]). Thus,

$$\mu_{\mathrm{c}} = (\mu - \nu)_{\mathrm{c}} \leq 0.$$

In other words, μ^+ is diffuse.

Corollary 2. *Let $\mu_i \in \mathcal{M}(\Omega)$, $i = 1, 2$, be such that μ_i^+ is diffuse. Let (u_i, ν_i) be the solution of* (1.1) *associated to μ_i. Then,*

$$\int_\Omega (\nu_1 - \nu_2)^+ \leq \int_\Omega (\mu_1 - \mu_2)^+. \tag{5.10}$$

Proof. Let $u_{i,n}$ be the solution of

$$\begin{cases} -\Delta u_{i,n} + \beta_n(u_{i,n}) = \mu_i & \text{in } \Omega, \\ u_{i,n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where (β_n) satisfies (1.13)–(1.15). Then, by Lemma 5, we have

$$\beta_n(u_{i,n}) \overset{*}{\rightharpoonup} \nu_i \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega),$$

so that

$$\beta_n(u_{1,n}) - \beta_n(u_{2,n}) \overset{*}{\rightharpoonup} \nu_1 - \nu_2 \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega). \tag{5.11}$$

On the other hand, by standard estimates (see, e.g., [8, Corollary B.1]) we have

$$\int_\Omega \big[\beta_n(u_{1,n}) - \beta_n(u_{2,n})\big]^+ \le \int_\Omega (\mu_1 - \mu_2)^+.$$

As $n \to \infty$, we conclude that (5.10) holds.

6. Proofs of Proposition 1 and Theorems 2, 3

Proof of Proposition 1. Let v be a subsolution of (1.1). In particular, there exists $f \in L^1(\Omega)$ such that $f \in \beta(u)$ almost everywhere and

$$-\Delta v + f \le \mu \quad \text{in } (C_0^2)^*.$$

Since $v \le a$ almost everywhere, it follows from the "Inverse" maximum principle that $(-\Delta v)_{\mathrm{c}} \le 0$. Thus,

$$-\Delta v + f \le \mu^* \quad \text{in } (C_0^2)^*.$$

Let (β_n) be a sequence satisfying (1.13)–(1.15) and such that $\beta_n \le \beta$, for all $n \ge 1$. In particular,

$$-\Delta v + \beta_n(v) \le -\Delta v + f \le \mu^* \quad \text{in } (C_0^2)^*.$$

In other words, v is a subsolution of

$$\begin{cases} -\Delta w_n + \beta_n(w_n) = \mu^* & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

By comparison (see [8, Corollary B.2]), we then have

$$v \le w_n \ \text{ a.e.} \quad \forall n \ge 1,$$

where w_n is the solution of (6.1). Since, by Lemma 5,

$$w_n \to u^* \quad \text{in } L^1(\Omega),$$

we conclude that $v \le u^*$ almost everywhere. Therefore, u^* is the largest subsolution of (1.1).

Proof of Theorem 2. Since

$$\|\Delta u_n\|_{\mathcal{M}} \le 2\|\mu\|_{\mathcal{M}},$$

one can find a subsequence (u_{n_k}) such that

$$u_{n_k} \to v \quad \text{in } L^1(\Omega),$$

for some $v \in L^1(\Omega)$. By Theorem 6, we also have

$$u_{n_k} \to v \ \text{ q.e.}$$

Let $\lambda \in \mathcal{M}(\Omega)$ be such that $\beta_{n_k}(u_{n_k}) \overset{*}{\rightharpoonup} \lambda$ weak* in $\mathcal{M}(\Omega)$. We now show that the sequence $\big(\beta_{n_k}(u_{n_k})\big)$ satisfies assumption (iv) of Theorem 4. In fact, given $\varepsilon > 0$, let $F \subset \Omega$ be a compact set such that $\operatorname{cap}(F) = 0$ and $|\mu_{\rm c}|(\Omega \setminus F) < \varepsilon$. Let $\omega \supset F$ be an open set such that $\operatorname{cap}(\overline{\omega}) < \varepsilon$. By Lemma 3, the sequence $\big(\beta_{n_k}(u_{n_k})\big)$ is 2ε-equidiffuse in $\Omega \setminus \overline{\omega}$, as claimed. Applying Theorem 4, we deduce that $v \le a$ almost everywhere and

$$\lambda_{\rm d} \in \beta(v).$$

We now show that $\lambda_{\rm c} = (\mu^+)_{\rm c}$. In fact, note that

$$\lambda \ge 0. \tag{6.2}$$

Let w denote the solution of

$$\begin{cases} -\Delta w = -\mu^- & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

By comparison, we have $w \le u_{n_k}$, for all $k \ge 1$. Thus, $w \le v$ almost everywhere. By the "Inverse" maximum principle, we then have

$$-(\mu^-)_{\rm c} = (-\Delta w)_{\rm c} \le (-\Delta v)_{\rm c} = \mu_{\rm c} - \lambda_{\rm c}. \tag{6.3}$$

Similarly, $v \le a$ almost everywhere implies

$$\mu_{\rm c} - \lambda_{\rm c} = (-\Delta v)_{\rm c} \le 0. \tag{6.4}$$

Combining (6.2)–(6.4), we conclude that

$$\lambda_{\rm c} = (\mu^+)_{\rm c}.$$

In other words, v satisfies

$$-\int_\Omega v\Delta\zeta + \int_\Omega \zeta\, d\nu = \int_\Omega \zeta\,\big(\mu - (\mu^+)_{\rm c}\big) = \int_\Omega \zeta\, d\mu^* \quad \forall \zeta \in C_0^2(\overline{\Omega}),$$

where $\nu = \lambda_{\rm d} \in \beta(v)$. Therefore, $v = u^*$. Since the limit v is independent of the subsequence (u_{n_k}), we must have

$$u_n \to u^* \quad \text{in } L^1(\Omega).$$

This establishes Theorem 2.

The proof of Theorem 3 is similar and will be omitted. Note that in this case one should apply Theorem 5 instead of Theorem 6.

7. The case of two obstacles

We shall assume throughout this section that β is any m.m.g. such that

$$\operatorname{dom}\beta = [-b, a] \quad \text{and} \quad (0,0) \in \operatorname{graph}\beta. \tag{7.1}$$

For simplicity, we restrict ourselves to the case where $a = b = 1$.

Given $\mu \in \mathcal{M}(\Omega)$, we say that u is a solution of (1.18) if $u \in L^1(\Omega)$, $|u| \le 1$ almost everywhere, $\Delta u \in \mathcal{M}(\Omega)$ and there exists a diffuse measure $\nu \in \beta(u)$ such that

$$-\int_\Omega u\Delta\zeta + \int_\Omega \zeta\, d\nu = \int_\Omega \zeta\, d\mu \quad \forall \zeta \in C_0^2(\overline{\Omega}). \tag{7.2}$$

We refer the reader to Section 2 for the definition of $\nu \in \beta(u)$ in the case of two obstacles. By Proposition 2, $\nu \in \beta(u)$ if and only if

$$\int_\Omega j(v) - \int_\Omega j(u) \ge \int_\Omega (v-u)\, d\nu \quad \forall v \in H_0^1(\Omega) \cap L^\infty, \tag{7.3}$$

where $\partial j = \beta$ and $j(0) = 0$.

The counterpart of Theorem 1 is given by the following

Theorem 7. *Let $\mu \in \mathcal{M}(\Omega)$. Then,* (1.18) *has a solution if and only if μ is diffuse. Moreover, this solution is unique.*

Solutions of (1.18) can be also obtained via minimization. More precisely, we have

Theorem 8. *Given a diffuse measure μ, let u be the solution of* (1.18). *Then, $u \in H_0^1(\Omega)$ and u coincides with the solution of the minimization problem*

$$\min_{\substack{v \in H_0^1(\Omega)\\ |v| \le 1 \ a.e.}} \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 + \int_\Omega j(v) - \int_\Omega v\, d\mu \right\}. \tag{7.4}$$

Given any measure μ in Ω, let $\mu^* = \mu_{\mathrm{d}}$ denote its diffuse part and let u^* be the unique solution of

$$\begin{cases} -\Delta u^* + \beta(u^*) \ni \mu^* & \text{in } \Omega, \\ \qquad\qquad u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

The analogs of Theorems 2 and 3 are given by the

Theorem 9. *Let (β_n) be a sequence satisfying* (1.13)–(1.14). *Given $\mu \in \mathcal{M}(\Omega)$, let w_n denote the solution of* (1.16) *or* (1.17). *Then,*

$$w_n \to u^* \quad \text{in } L^1(\Omega),$$

where u^* is the solution of (7.5).

Finally, we make a connection with another concept of generalized solution which was proposed in Brezis-Serfaty [10]. This is based on a duality principle mentioned in [7].

Given $f \in L^2(\Omega)$, consider the two minimization problems:

$$I_1(f) = \min_{\substack{v \in H_0^1(\Omega) \\ |v| \le 1 \text{ a.e.}}} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 + \int_\Omega j(v) - \int_\Omega vf \right\}, \tag{P}$$

$$I_2(f) = \min_{\substack{w \in H_0^1(\Omega) \\ \Delta w \in \mathcal{M}(\Omega)}} \left\{ \frac{1}{2} \int_\Omega |\nabla w|^2 + \int_\Omega j^*(f + \Delta w) \right\}. \tag{P*}$$

Here, j^* denotes the convex conjugate of j. It is not difficult to prove (see [7]) that (P) and (P*) admit the same minimizers. Moreover,

$$I_1(f) + I_2(f) = 0 \quad \forall f \in L^2(\Omega).$$

These two assertions remain valid when $f = \mu$ is a diffuse measure. Note that for general measures, problem (P) is not well posed; however, problem (P*) is well defined and admits a unique minimizer $U(\mu)$. The function $U(\mu)$ was thus regarded in [10] as a kind of generalized solution for the original problem (P) (or (1.18)).

The following result establishes the connection between the concept introduced in [10] and the notion of generalized solution in this paper:

Theorem 10. *For every $\mu \in \mathcal{M}(\Omega)$, we have*

$$I_2(\mu) = -I_1(\mu_{\mathrm{d}}) + \|\mu_{\mathrm{c}}\|_{\mathcal{M}}. \tag{7.6}$$

Moreover, the minimizers of $I_1(\mu_{\mathrm{d}})$ and $I_2(\mu)$ coincide; i.e., $U(\mu) = u^*$.

8. Proofs of Theorems 7–10

Proofs of Theorems 7 and 9.

Step 1. Equation (1.18) has, at most, one solution.

The argument is exactly the same as in the proof of Lemma 4. We leave the details to the reader.

Step 2. If (1.18) has a solution, then μ is diffuse.

Since u is bounded and $\Delta u \in \mathcal{M}(\Omega)$, we have $u \in H_0^1(\Omega)$. This easily implies that Δu is diffuse. Since ν is also diffuse, we deduce that $\mu = -\Delta u + \nu$ is diffuse as well.

Step 3. Proof of Theorem 9 completed.

Given $\mu \in \mathcal{M}(\Omega)$, let w_n denote the solution of (1.17) (respectively, (1.16)). Thus, for some subsequence (n_k) we have

$$u_{n_k} \to v \quad \text{in } L^1(\Omega) \quad \text{and} \quad \nu_{n_k} \overset{*}{\rightharpoonup} \lambda \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega).$$

By Theorem 5 (respectively, Theorem 6), we then have $u_{n_k} \to v$ q.e. By Lemma 3, the sequence (ν_{n_k}) satisfies the assumptions of Theorem 4. We conclude that

$$\lambda_{\mathrm{d}} \in \beta(v).$$

We claim that $\lambda_{\mathrm{c}} = \mu_{\mathrm{c}}$. In fact, note that $v \in H_0^1(\Omega)$. Therefore, Δv is diffuse. Since $-\Delta v + \lambda = \mu$, the claim follows by comparing the concentrated parts of both sides with respect to cap. We conclude that v satisfies (7.5), with $\nu = \lambda_{\mathrm{d}}$. By Step 1, the solution of (7.5) is unique. Hence,

$$u_n \to u^* \quad \text{in } L^1(\Omega).$$

Step 4. Proof of Theorem 7 completed.

By Step 2, if (1.18) has a solution, then μ is diffuse. Conversely, if μ is diffuse, then the existence of a solution of (1.18) follows from Step 3. By Step 1, the solution is unique.

Proof of Theorem 8. Since both problems (1.18) and (7.4) have unique solutions, it suffices to show that if u satisfies (1.18), then u minimizes (7.4). By Proposition 2, we have

$$\int_\Omega j(v) - \int_\Omega j(u) \geq \int_\Omega (v-u)\, d\nu \quad \forall v \in H_0^1(\Omega),\ |v| \leq 1, \tag{8.1}$$

where $\nu = \mu + \Delta u$. On the other hand, we have

$$\int_\Omega (v-u)\,\Delta u = -\int_\Omega \nabla(v-u)\cdot\nabla u \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2. \tag{8.2}$$

Combining (8.1) and (8.2), the result follows.

Proof of Theorem 10. Note that for every $w \in H_0^1(\Omega)$ such that $\Delta w \in \mathcal{M}(\Omega)$, it follows that Δw is a diffuse measure. Thus,

$$j^*(\mu + \Delta w) = j^*(\mu_{\mathrm{d}} + \Delta w) + |\mu_{\mathrm{c}}|.$$

This immediately implies that

$$I_2(\mu) = I_2(\mu_{\mathrm{d}}) + \|\mu_{\mathrm{c}}\|_{\mathcal{M}}.$$

In particular, the minimizers of $I_2(\mu)$ and $I_2(\mu_{\mathrm{d}})$ are equal; i.e., $U(\mu) = U(\mu_{\mathrm{d}})$. Therefore, it suffices to establish the theorem for μ diffuse. We now split the proof into two steps:

Step 1. The minimizers of $I_1(\mu)$ and $I_2(\mu)$ coincide.

In view of Theorem 8, it suffices to show that if u is the solution of (1.18), then u minimizes $I_2(\mu)$. By the equivalence (a) $\Leftrightarrow$ (e) in Proposition 2, we have

$$\int_\Omega j^*(\sigma) - \int_\Omega j^*(\mu + \Delta u) \geq \int_\Omega u\,(\sigma - \mu - \Delta u) \quad \forall \sigma \in \mathcal{M}(\Omega) \text{ diffuse.} \tag{8.3}$$

Let $w \in H_0^1(\Omega)$ with $\Delta w \in \mathcal{M}(\Omega)$. In particular, Δw is diffuse. Applying (8.3) to $\sigma = \mu + \Delta w$, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega j^*(\mu + \Delta w) \\ &\geq \frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega u(\Delta w - \Delta u) + \int_\Omega j^*(\mu + \Delta u) \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega j^*(\mu + \Delta u). \end{aligned}$$

Thus, u is the minimizer of $I_2(\mu)$.

Step 2. Proof of the theorem completed.

It remains to establish (7.6). Since we are assuming that μ is diffuse, and the solution u of (1.18) minimizes both $I_1(\mu)$ and $I_2(\mu)$, then we only need to show that

$$\int_\Omega |\nabla u|^2 + \int_\Omega \big\{ j(u) + j^*(\mu + \Delta u) \big\} - \int_\Omega u\, d\mu = 0,$$

which is equivalent to assertion (f) in Proposition 2. The proof of Theorem 10 is complete.

9. Appendix

Our goal in this Appendix is to show that there are sequences such that (u_n) is uniformly bounded and (Δu_n) is equidiffuse, but (u_n) need not have any subsequence converging q.e. For simplicity, we will assume throughout this Appendix that $N \geq 3$. We first establish the following

Proposition 5. *Let $(\mu_n) \subset \mathcal{M}(\mathbb{R}^N)$ be a sequence of nonnegative measures. Let*

$$v_n(x) = \frac{1}{(N-2)|B_1|} \int_{\mathbb{R}^N} \frac{d\mu_n(y)}{|x-y|^{N-2}} \quad \forall x \in \mathbb{R}^N. \tag{9.1}$$

If (v_n) is uniformly bounded, then (μ_n) is equidiffuse.

Proof. Clearly, it suffices to show that for every Borel set $A \subset \mathbb{R}^N$ we have

$$\mu_n(A) \leq C \operatorname{cap}(A) \quad \forall n \geq 1. \tag{9.2}$$

We only need to establish (9.2) when $A = K$ is a compact set. Let ζ_K denote the capacitary potential of K. Since v_n is bounded, μ_n is a diffuse measure. Thus,

$$\begin{aligned}\mu_n(K) &\le \int_{\mathbb{R}^N} \zeta_K \, d\mu_n \\ &= -\int_{\mathbb{R}^N} \zeta_K \Delta v_n = -\int_{\mathbb{R}^N} v_n \Delta\zeta_K \le \|v_n\|_{L^\infty} \int_{\mathbb{R}^N} |\Delta\zeta_K| \le C \operatorname{cap}(K).\end{aligned}$$

This establishes (9.2) when A is compact; the general case easily follows by approximation.

In conjunction with Proposition 5, we recall that there is an example of functions (w_n) constructed by Cioranescu-Murat [12, Example 2.1] such that

(i) $0 \le w_n \le 1$ a.e. in Ω;
(ii) (Δw_n) is equidiffuse;
(iii) $w_n \rightharpoonup 1$ weakly in $H^1(\Omega)$;
(iv) for every $\omega \subset\subset \Omega$, we have

$$\liminf_{n\to\infty} \Big\{ \operatorname{cap}\big([w_n = 0] \cap \omega\big) \Big\} > 0.$$

Properties (i), (iii) and (iv) are clear from their construction. We only need to check that (ii) holds. By construction, (Δw_n) is bounded in $\mathcal{M}(\Omega)$. Write

$$-\Delta w_n = \mu_n - \nu_n,$$

where $\mu_n, \nu_n \ge 0$ and μ_n, ν_n are mutually singular. We then extend μ_n, ν_n to be identically zero outside Ω. Let v_n be defined as in (9.1). An easy inspection from [12] shows that (v_n) is uniformly bounded in $\mathbb{R}^N$. By Proposition 5, (μ_n) is equidiffuse. Similarly, (ν_n) is also equidiffuse. We deduce that (ii) holds.

It now follows from (iii) and (iv) that $(w_n\varphi)$ has no subsequence converging q.e., for any $\varphi \in C_c^\infty(\Omega)$, $\varphi \not\equiv 0$. Note however that $\big(\Delta(w_n\varphi)\big)$ is equidiffuse in view of (ii).

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