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OPTIMAL BOUNDARY CONTROL AND RICCATI THEORY FOR ABSTRACT DYNAMICS MOTIVATED BY HYBRID SYSTEMS OF PDES

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Abstract. We study the quadratic optimal control problem over a finite time horizon for a class of abstract systems with *non analytic* underlying semigroup e^{tA} and unbounded control operator B. It is assumed that a suitable decomposition of the operator $B^*e^{tA^*}$ is valid, where only one component satisfies a 'singular estimate', whereas for the other component specific regularity properties hold. Under these conditions, we prove well posedness of the associated differential Riccati equation, and in particular that the gain operator is *bounded* on a dense set. In spite of the unifying abstract framework used, the prime motivation (and application) of the resulting theory of linear-quadratic problems comes from optimal boundary control of a thermoelastic system with clamped boundary conditions. The non-trivial trace regularity estimate showing that this PDE mixed problem fits into the distinct class of models under examination—for which we have developed the present, novel optimal control theory—is established, as well.

1. Introduction

In the development of the optimal control theory for (linear) Partial Differential Equations (PDEs) over the last four decades, some major steps can be singled out. A first extension of the finite-dimensional results to

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the infinite-dimensional context was carried out with the study of mixed (initial/boundary-value) problems for PDEs with *distributed* control, where the corresponding abstract dynamics $y' = Ay + Bu$ yields a *bounded* control operator B. Next, motivated by the modeling of parabolic equations with *boundary* control, systems with *unbounded* control operators B have arisen, making the analysis of the corresponding Riccati equations much more challenging. As is well known, the substantial new difficulties have been overcome by exploiting analyticity of the free dynamics operator e^{tA}. A drastic change of scenario occurs in the case of hyperbolic flows. Justified by relevant PDE illustrations of hyperbolic (and hyperbolic-like) problems, less restrictive hypotheses on the unboundedness of B are necessary, resulting in a general unboundedness of the gain operator B^*P which appears in the associated Algebraic Riccati equations, and in a lack of well posedness of the Differential Riccati equations, unless a certain degree of smoothing of the observation operator is assumed. It is beyond the scope of the present paper to give a comprehensive account of the literature on the linear quadratic problem in an infinite-dimensional context. The reader is referred to the advanced monographs [7] and [19], and their numerous references, for an exhaustive treatment of the aforementioned theories. Still, we are pleased to recall explicitly the pioneering book [5] and the treatise [22], and some of the most valuable contributions to the development of the Riccati theory for a single PDE with boundary control. Namely, (*i*) the former works [10], [12, 13], [17], [11] for *parabolic* dynamics (see also [2, 3] for non-autonomous systems); and (*ii*) [14]—a cornerstone of the theory pertaining to *hyperbolic-like* dynamics—along with the subsequent refinements of [9], [23, 24]).

In recent years, new PDE models stem from modern technological applications, such as *structural acoustic problems*[1], thermo-elastic systems, and more generally systems of coupled PDEs of different type. It is natural that the two basic classes of evolutions for which a theory of the corresponding optimal control problems was available in the literature—the classes of parabolic-like and hyperbolic-like dynamics—were no longer suitable to describe many of those PDE models. Thus, motivated by the analysis of a structural acoustic system, which comprised a hyperbolic equation strongly coupled with a parabolic-like one, with the work [4] a novel class of abstract models (of the same form $y' = Ay+Bu$) has been identified, characterized by a 'singular estimate' for the operator $e^{tA}B$ near $t = 0$. This kind of estimate

[1]These systems of interconnected PDEs arise in the modeling of noise reduction problems within mechanical structures.

reveals that although the overall dynamics is not analytic, the smoothing properties of the analytic component of the system have somewhat propagated into the entire system. Under this abstract condition assumed on $e^{tA}B$—which captures an intrinsic feature of the coupled dynamics—a satisfactory optimal control theory is valid, which constitutes an extension of the analytic one (see [15], [20], [21], [16]). To date, it has been shown that this theory covers many relevant systems of PDEs, including various structural acoustic models, thermoelastic problems, 'sandwich beams' problems, etc. (see, e.g., [9], [21]).

However, several important systems still do not fit exactly into the above class, though they are close. An example is given by an established thermoelastic plate model with clamped boundary conditions[2], subject to Dirichlet controls acting on the boundary ([8]). The precise mathematical description of the mixed PDE problem will be given in Section 3. It is in response to this demand that an extension of the 'singular estimates model' (briefly, the 'third' class) has attracted considerable attention. In this new framework, the singular estimate does not need to hold for $B^*e^{tA^*}$, but only for a component of it. A prime illustration (as well as motivation) for this new class of evolutions is the aforementioned thermoelastic problem. Then, as usual and as it should be, the theory developed—namely, the assumptions imposed on the abstract dynamics, and the corresponding results—meets the requirements of concrete physical applications.

The goal of this paper is to provide a complete Riccati theory for a new class of systems extending the third class (hence, also the analytic class). It is important to emphasize that while the new class is motivated by significant PDE models—one may say systems of PDEs which display a hyperbolic/parabolic coupling, yet *with an overall hyperbolic character*—the mathematical aspects of the theory and the corresponding results are new and different from the canonical, by now, singular case. Indeed, the most notable differences are that the gain operator is no longer bounded on the state space—rather, on a dense subspace—and the optimal control function is no longer continuous in time, unlike the case of parabolic-like dynamics or systems yielding singular estimates.

1.1. **Orientation.** The plan of the paper is the following. In the next Section 2 we introduce the abstract dynamics under consideration, along with

[2]From a physical point of view, clamped boundary conditions are the most natural ones.

the corresponding assumptions, and we give the precise statement of the paper's main result (Theorem 2.3). Its long proof will occupy several sections (namely, Sections 4 through 6, in addition to Appendix B). Hence, Section 2 includes as well a series of focused observations regarding the model and the key steps of the proof.

In Section 3 we describe the PDE mixed problem motivating and illustrating the abstract theory. The nontrivial boundary regularity estimate, which is crucial in order to prove that this thermoelastic problem fits into the abstract framework, is shown here.

The proof of Theorem 2.3 is started with the analysis of the regularity of the operator L which maps the boundary input to the interior solution (and of its adjoint operator L^*, as well). The corresponding results are collected in Appendix B.

Section 4 is focused on the optimal pair, and includes in particular preliminary information concerning the properties of the evolution map $\Phi(t,s)$.

Section 5 contains the fundamental result that the gain operator $B^*P(t)$ is well defined and bounded on a dense subset of the state space (Proposition 5.3).

In Section 6, seeking to derive that the operator $P(t)$—initially defined in terms of the optimal state—does satisfy the Differential Riccati equation on $\mathcal{D}(A)$, we ascertain the differentiability of the evolution map $\Phi(t,s)$ on $\mathcal{D}(A)$.

1.2. **Notation.** The notation used in the paper is largely standard within the literature: in particular, the one pertaining to function spaces. We just make a few remarks, for the reader's convenience. Let H and X be a Hilbert and a Banach space, respectively. The symbol $(\cdot,\cdot)_H$ denotes the inner product in H (the sub-script will be omitted, when H is clear), whereas $\langle\cdot,\cdot\rangle_X$ represents the pairing of X^* and X.

We employ the symbol $\mathcal{D}(A)$, and not simply $D(A)$, to denote the domain of a closed operator A: the reason is that the letter D naturally stands for the Dirichlet mapping.

By χ_E we mean the characteristic function of the set E. If q is a Sobolev exponent ($q \neq 1$), we shall write q' in lieu of $\frac{q}{q-1}$. Regarding the estimates, we often employ the letter c (c_T, C) to denote any constant, whose value may therefore change from line to line in a given computation.

2. Assumptions and main result

We consider the abstract control system

$$\begin{cases} y'(t) = Ay(t) + Bu(t)\,, \quad t \in [0,T]\,, \\ y(0) = y_0 \end{cases} \tag{2.1}$$

with associated cost functional

$$J(u,y) = \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\right] dt \tag{2.2}$$

to be minimized over all $u \in L^2(0,T;U)$, where y is the solution of (2.1) due to u. The following hypotheses on the model will be assumed throughout the paper.

Hypothesis 2.1. *T is a positive number, Y, U and Z are separable complex Hilbert spaces, R is a bounded linear operator from Y to Z; $A : \mathcal{D}(A) \subseteq Y \to Y$ is a closed linear operator, with $A^{-1} \in \mathcal{L}(Y)$, which generates a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$ on Y. Finally, $B : U \to [\mathcal{D}(A^*)]'$ is a bounded linear operator.*

Hypothesis 2.2. *For each $t \in [0,T]$, the operator $B^* e^{tA^*}$ can be represented as*

$$B^* e^{tA^*} x = F(t)x + G(t)x, \qquad t \ge 0, \quad x \in \mathcal{D}(A^*), \tag{2.3}$$

where $F(t) : Y \to U$, and $G(t) : \mathcal{D}(A^*) \to U$, $t > 0$ are bounded linear operators satisfying the following assumptions:

(i) *there is $\gamma \in (\frac{1}{2}, 1)$ such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)} \le c t^{-\gamma} \qquad \forall t \in (0,T]; \tag{2.4}$$

(ii) *the operator $G(\cdot)$ belongs to $\mathcal{L}(Y, L^p(0,T;U))$ for all $p \in [1,\infty[$, with*

$$\|G(\cdot)\|_{\mathcal{L}(Y,L^p(0,T;U))} \le c_p < \infty \qquad \forall p \in [1,\infty[; \tag{2.5}$$

(iii) *there is $\varepsilon > 0$ such that:*

(a) *the operator $G(\cdot)A^{*-\varepsilon}$ belongs to $\mathcal{L}(Y, C([0,T],U))$, and in particular*

$$\|A^{-\varepsilon} G(t)^*\|_{\mathcal{L}(U,Y)} \le c < \infty \qquad \forall t \in [0,T]; \tag{2.6}$$

(b) *the operator R^*R belongs to $\mathcal{L}(\mathcal{D}(A^\varepsilon), \mathcal{D}(A^{*\varepsilon}))$, i.e.,*

$$\|A^{*\varepsilon} R^* R A^{-\varepsilon}\|_{\mathcal{L}(Y)} \le c < \infty; \tag{2.7}$$

(c) *there is* $q \in (1,2)$ (*depending, in general, on* ε) *such that the operator* $B^* e^{\cdot A^*} R^* R A^\varepsilon$ *has an extension belonging to* $\mathcal{L}(Y, L^q(0,T;U))$.

It is a standard matter to verify that under Hypotheses 2.1 and 2.2, for fixed $s \in [0,T)$, the problem of minimizing the functional

$$J_s(u,y) = \int_s^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 \right] dt \tag{2.8}$$

among the functions $u \in L^2(s,T;U)$ and $y \in L^2(s,T;Y)$ satisfying the mild form of the state equation

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \quad t \in [s,T] \\ y(s) = x \in Y \end{cases} \tag{2.9}$$

has a unique solution $(\hat{y}(\cdot,s;x), \hat{u}(\cdot,t;x))$. The mild form of (2.9) reads as

$$y(t,s;x) = e^{(t-s)A}x + (L_s u)(t), \qquad t \in [s,T], \tag{2.10}$$

where the operator L_s is defined by

$$(L_s u)(t) = \int_s^t e^{(t-s)A} Bu(\sigma)\, d\sigma \qquad \forall u \in L^2(s,T;U). \tag{2.11}$$

We note that the adjoint of L_s is the operator

$$(L_s^* y)(t) = \int_t^T B^* e^{(\tau-t)A^*} y(\tau)\, d\tau \qquad \forall y \in L^2(s,T;Y); \tag{2.12}$$

in particular it does not depend on s, or, more precisely, $L_t^* y = (L_s^* y)|_{(t,T)}$ for $t \in (s,T]$. The regularity properties of L_s and L_s^* will be analyzed in Appendix B.

Let us state our main result.

Theorem 2.3. *With reference to the control problem* (2.1)–(2.2), *under Hypotheses* 2.1 *and* 2.2 *the following statements are valid for each* $s \in [0,T)$.

(i) *For each* $x \in Y$ *the optimal pair* $(\hat{u}(\cdot,s;x), \hat{y}(\cdot,s;x))$ *satisfies*

$$\hat{y}(\cdot,s;x) \in C([s,T],Y), \quad \hat{u}(\cdot,s;x) \in \bigcap_{1 \le p < \infty} L^p(s,T;U). \tag{2.13}$$

(ii) *The operator* $\Phi(t,s) \in \mathcal{L}(Y)$, *defined for* $x \in Y$ *by*

$$\Phi(t,s)x = \hat{y}(t,s;x) = e^{(t-s)A}x + [L_s \hat{u}(\cdot,s;x)](t), \quad s \le t \le T, \tag{2.14}$$

is an evolution operator; i.e.,

$$\Phi(t,t) = I_Y\,, \qquad \Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s) \quad \text{for } s \le \tau \le t \le T.$$

(iii) *For each $t \in [0,T]$ the operator $P(t) \in \mathcal{L}(Y)$, defined by*

$$P(t)x = \int_t^T e^{(\tau-t)A^*} R^* R \Phi(\tau,t)x \, d\tau, \qquad x \in Y, \tag{2.15}$$

is self-adjoint and positive; it belongs to $\mathcal{L}(Y, C([0,T],Y))$ and is such that

$$(P(s)x,x)_Y = J_s(\hat{u}(\cdot,s;x), \hat{y}(\cdot,s;x)) \qquad \forall s \in [0,T].$$

(iv) *The gain operator $B^*P(t)$ belongs to $\mathcal{L}(\mathcal{D}(A^\varepsilon), C([0,T],U))$ and the optimal pair satisfies for $s \le t \le T$*

$$\hat{u}(t,s;x) = -B^*P(t)\hat{y}(t,s;x) \qquad \forall x \in Y. \tag{2.16}$$

(v) *The operator $\Phi(t,s)$ defined in (2.14) satisfies for $s < t \le T$:*

$$\frac{\partial \Phi}{\partial s}(t,s)x = -\Phi(t,s)(A - BB^*P(t))x \in L^{\frac{1}{\gamma}}(s,T;[\mathcal{D}(A^{*\varepsilon})]') \tag{2.17}$$

for all $x \in \mathcal{D}(A)$, and

$$\frac{\partial \Phi}{\partial t}(t,s)x = (A - BB^*P(t))\Phi(t,s)x \in C([s,T],[\mathcal{D}(A^*)]') \tag{2.18}$$

for all $x \in \mathcal{D}(A^\varepsilon)$.

(vi) *The operator $P(t)$ satisfies the following differential Riccati equation in $[0,T)$:*

$$\begin{aligned}(P'(t)x,y)_Y &+ (P(t)x,Ay)_Y + (P(t)Ax,y)_Y \\ &+(R^*Rx,y)_Y - (B^*P(t)x, B^*P(t)y)_Y = 0 \quad \forall\, x,y \in \mathcal{D}(A).\end{aligned} \tag{2.19}$$

The proof of Theorem 2.3 will be split into several sections. This section provides for the reader's convenience a series of focused observations. We first sketch out a comparison with the previous general classes of (linear) evolutions considered in the literature, and next we give some outlining ideas for the proof of Theorem 2.3.

2.1. **A comparison with previous abstract models.** Let us briefly recall the major classes of abstract systems (of the form (2.1)) introduced so far for the representation of mixed problems for PDEs with boundary/point control, and the essential results pertaining to the corresponding Riccati theories. In each case, the basic assumptions in Hypothesis 2.1 will be taken for granted.

First class: (Parabolic-like dynamics). Under the assumptions

- e^{tA} is an *analytic* semigroup,
- there exists $r \in [0,1[$ such that $A^{-r}B$ is bounded,

one has that $(-A^*)^{1-\epsilon}P$ is bounded, and hence the key property that $B^*P(t)$ is bounded on the state space Y (besides further regularity properties for the Riccati operator $P(t)$). This ensures well posedness of the corresponding differential Riccati equations.

Second class: (Hyperbolic-like dynamics). This class is characterized by the following 'trace regularity assumption', also known as *admissibility condition*:

- $\displaystyle\int_0^T ||B^* e^{tA^*} x||_U^2 \, dt \le C_T \, ||x||_Y^2, \quad \forall x \in \mathcal{D}(A^*).$

In this case it is known that $B^*P(t)$ may not be densely defined, unless the observation operator R has some smoothing property. If $T = \infty$, B^*P is defined only as a suitable extension B_e^*P.

Third class: (Systems which yield singular estimates). The sole assumption

- there exists $\gamma \in [0,1[$ such that $\|e^{tA}B\|_{\mathcal{L}(U,Y)} \le C\, t^{-\gamma}$ near $t = 0$,

ensures that $B^*P(t)$ is bounded on Y. The motivation for introducing this model has come from coupled systems of hyperbolic/parabolic PDEs.

The present class: We make reference to systems characterized by the assumptions listed in Hypothesis 2.2. For these models Theorem 2.3 establishes the following novel, distinct result: $B^*P(t)$ is densely defined and bounded on the space $\mathcal{D}(A^\varepsilon)$. Consequently, the differential Riccati equation is well posed, as desired.

Notice that if $G = 0$ this model reduces to the preceding one, whereas if $F = 0$ one may find some reminiscence of the second (hyperbolic) class. Indeed, more precisely,

(1) Hypothesis 2.2(ii) is stronger than the admissibility condition of the hyperbolic case, since the latter requires only $p = 2$.

(2) As for Hypothesis 2.2(iii), item (a) always holds true with $\varepsilon = 1$. However, a “small” ε is worth having, in order that the most challenging condition (iii)(c) is satisfied.
(3) It is important to emphasize that the assumption (iii)(b) just requires that the operator R^*R ‘maintains regularity’; in particular, it allows R to be the identity.
(4) Finally, a distinctive feature of condition (iii)(c) is the correlation between the parameters ε and q. Notice that admissibility corresponds to $\varepsilon = 0$ and $q = 2$, which are limiting points for the parameters' range.

2.2. **Outlining ideas for the proof of Theorem 2.3.** The approach of our analysis is of variational type, and it follows the abstract operator treatment of the quadratic optimal control problem for PDEs—and of the related Differential/Algebraic Riccati equations—developed in the last twenty-five years by the authors of [19]. Thus, the general (by now classical) programme to be carried out consists of the following, somewhat standard, steps. (*i*) First, the optimal pair $\{\hat{u}, \hat{y}\}$ is characterized only in terms of the data of the problem (see Remark 4.2); (*ii*) next, an operator $P(t)$ is constructed in terms of the optimal state (see formula (5.1)), and hence of the original data of the problem; (*iii*) finally, the operator $P(t)$ is shown to satisfy the differential Riccati equation.

However, at a technical level, new difficulties arise due to the distinct assumptions on the abstract dynamics. This specifically pertains to the third step, by far the most complex.

We recall that a prime theoretical issue concerning the solving of optimal control problems with quadratic cost functional for PDE mixed problems is to investigate the regularity of the operator L which maps the boundary input to the interior solution (and of its adjoint operator L^*, as well). This preliminary analysis, reported in Appendix B, leads to the detailed conclusions of Proposition B.3 and of Proposition B.4, and it culminates in the statements of Proposition B.5—which pinpoints the regularity of the operator R^*RL—and its Corollary B.7. In turn, Proposition B.5 is critically used in order to show Proposition 5.3, which establishes that the gain operator $B^*P(t)$ is well defined and bounded on $\mathcal{D}(A^\varepsilon)$; that is the crux of the matter of the whole theory.

Finally, in order to derive that the operator $P(t)$ defined in (5.1) satisfies the Differential Riccati equation on $\mathcal{D}(A)$, one needs to differentiate strongly the evolution map $\Phi(t,s)$ on $\mathcal{D}(A)$ (where $\Phi(t,s)x = \hat{y}(t,s;x)$). This is a challenging point, as well. We recall that in the case of systems which yield singular estimates for the operator $e^{At}B$, one has, as a consequence, that the same degree of singularity is transferred to the operator $\Phi(t,s)B$, and this property is critical in deriving the Differential Riccati equation. Of course this does not hold in the present case. Section 6 is entirely devoted to showing—by using direct and rigorous proofs—differentiability of $\Phi(t,s)x$ with respect to both variables, when $x \in \mathcal{D}(A)$, resulting in Theorem 6.1 and Theorem 6.2, respectively (the latter being much more intricate and lengthy).

3. A physically motivated PDE illustration

In this section we introduce a thermoelastic problem which constitutes a significant PDE application of the abstract theory developed in the paper. By showing this illustration at the very outset, we aim to facilitate the PDE interpretation of the various assumptions listed in Hypothesis 2.2.

Let Ω be a bounded domain of $\mathbb{R}^2$ with smooth boundary Γ, and let $T > 0$. We consider the following model of a thermoelastic plate with clamped boundary conditions, in the variables $w(t,x)$ (vertical displacement) and $\theta(t,x)$ (temperature):

$$\begin{cases} w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } Q := (0,T] \times \Omega \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \Sigma := (0,T] \times \Gamma \\ \theta = u & \text{on } \Sigma \\ w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1,\ \theta(0,\cdot) = \theta^0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\rho > 0$, so that the elastic equation is of Kirchhoff type. The dynamics of the plate is influenced by a thermal control u acting on the temperature at the boundary.

We associate to (3.1) the following quadratic cost functional to be minimized over all $u \in L^2(0,T;L^2(\Gamma))$:

$$J(w,w_t,\theta;u) = \int_0^T \int_\Omega (|\Delta w|^2 + |\nabla w_t|^2 + |\theta|^2)\,dx\,dt + \int_0^T \int_\Gamma |u|^2\,ds\,dt\,. \tag{3.2}$$

We want to rewrite problem (3.1) in the abstract form (2.1). To this purpose we start by introducing the realization of the Laplacian with Dirichlet boundary conditions

$$A_D f = -\Delta f, \quad \mathcal{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega), \tag{3.3}$$

and the realization of the Bilaplacian with clamped boundary conditions

$$\mathcal{A}f = \Delta^2 f, \quad \mathcal{D}(\mathcal{A}) = H^4(\Omega) \cap H_0^2(\Omega). \tag{3.4}$$

Denoting by $D : L^2(\Gamma) \to L^2(\Omega)$ the Dirichlet map; i.e.

$$h = Dg \quad \iff \quad \begin{cases} \Delta h = 0 & \text{in } \Omega \\ h = g & \text{on } \Gamma, \end{cases} \tag{3.5}$$

and setting

$$\mathcal{M} = I + \rho A_D\,, \quad \mathcal{D}(\mathcal{M}) = \mathcal{D}(A_D), \tag{3.6}$$

we may rewrite problem (3.1) as

$$\begin{cases} \mathcal{M}w_{tt} + \mathcal{A}w - A_D\theta = -A_D D u, \\ \theta_t + A_D\theta + A_D w_t = A_D D u \\ w(0) = w^0,\ w_t(0) = w^1,\ \theta(0) = \theta^0. \end{cases} \tag{3.7}$$

Now let us rearrange the left-hand side of (3.7). Consider the operator A given by

$$A := \begin{pmatrix} 0 & I & 0 \\ -\mathcal{M}^{-1}\mathcal{A} & 0 & \mathcal{M}^{-1}A_D \\ 0 & -A_D & -A_D \end{pmatrix}, \tag{3.8}$$

with domain

$$\mathcal{D}(A) := \mathcal{D}(\mathcal{A}^{3/4}) \times [\mathcal{D}(\mathcal{A}^{1/2}) \cap \mathcal{D}(A_D)] \times \mathcal{D}(A_D) : \tag{3.9}$$

it is proved in [18] that it generates a strongly continuous contraction semigroup $\{e^{tA}\}_{t\geq 0}$ in the space

$$Y := \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}^{1/2}) \times L^2(\Omega) = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega). \tag{3.10}$$

We also have

$$A^{-1} = \begin{pmatrix} -\mathcal{A}^{-1}A_D & -\mathcal{A}^{-1}\mathcal{M} & -\mathcal{A}^{-1} \\ I & 0 & 0 \\ -I & 0 & -A_D^{-1} \end{pmatrix} \in \mathcal{L}(Y). \tag{3.11}$$

Moreover, if we set $U := L^2(\Gamma)$, and define the operator

$$B := \begin{pmatrix} 0 \\ -\mathcal{M}^{-1}A_D D \\ A_D D \end{pmatrix} \in \mathcal{L}(U, [\mathcal{D}(A^*)]'), \tag{3.12}$$

then it is easy to see that

$$A^{-1}B = \begin{pmatrix} 0 \\ 0 \\ -D \end{pmatrix} \in \mathcal{L}(U, Y). \tag{3.13}$$

Using the operators (3.8) and (3.12), problem (3.7) reduces to the form (2.1) with respect to the variable $y = (w, w_t, \theta)$, with initial datum $y_0 = (w^0, w^1, \theta^0)$. In this way we have verified Hypothesis 2.1.
In order to verify Hypothesis 2.2, we note that the adjoint A^* of A is given by

$$A^* = \begin{pmatrix} 0 & -I & 0 \\ \mathcal{M}^{-1}\mathcal{A} & 0 & -\mathcal{M}^{-1}A_D \\ 0 & A_D & -A_D \end{pmatrix}, \tag{3.14}$$

with domain

$$\mathcal{D}(A^*) \equiv \mathcal{D}(A) = [H^3(\Omega) \cap H_0^2(\Omega)] \times H_0^2(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]. \tag{3.15}$$

Next, recalling the well-known property

$$D^* A_D g = \frac{\partial g}{\partial \nu}\Big|_\Gamma \qquad \forall g \in H^{\frac{3}{2}+\delta}(\Omega) \cap H_0^1(\Omega), \quad \forall \delta > 0, \tag{3.16}$$

it follows that

$$B^* y = B^* \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{\partial}{\partial \nu}(y_3 - y_2)\Big|_\Gamma \qquad \forall y \in \mathcal{D}(A^*). \tag{3.17}$$

We evaluate now the operator $B^* e^{tA^*}$. Following [8], we have

$$e^{tA^*} z = e^{tA^*} \begin{pmatrix} w^0 \\ w^1 \\ \theta^0 \end{pmatrix} = \begin{pmatrix} w(t) \\ -w_t(t) \\ \theta(t) \end{pmatrix} \tag{3.18}$$

where $(w(t), w_t(t), \theta(t))$ is the solution of the uncontrolled version of system (3.7) with initial datum $(w^0, -w^1, \theta^0)$. Hence, by (3.17) and the boundary conditions in (3.1),

$$B^* e^{tA^*} z = \frac{\partial}{\partial \nu}(w_t(t) + \theta(t))\Big|_\Gamma = \frac{\partial \theta(t)}{\partial \nu}\Big|_\Gamma. \tag{3.19}$$

It is shown in [8] that

$$\frac{\partial \theta(t)}{\partial \nu}\Big|_\Gamma = F(t)z + G(t)z, \tag{3.20}$$

where $F(t)$ and $G(t)$ are suitable, explicit operators satisfying the following estimates for every $y \in Y$:

$$\|F(t)y\|_Y \le C_\varepsilon t^{-\frac{3}{4}-\varepsilon} \|y\|_Y \qquad \forall \varepsilon \in (0, \tfrac{1}{4}), \quad \forall t \in (0, T], \tag{3.21}$$

$$\|G(\cdot)y\|_{L^p(0,T;U)} \le C_p \|y\|_Y \qquad \forall p \in [1, \infty). \tag{3.22}$$

Namely, (i) and (ii) of Hypothesis 2.2 hold true.

Regarding Hypothesis 2.2(iii), the validity of the regularity assumption in item (a), for arbitrarily small ε, was proved in [8], as well. Next, notice that in the present case the observation operator R coincides with the identity operator I, while $\mathcal{D}(A^{*\varepsilon}) \equiv \mathcal{D}(A^{\varepsilon})$ (see [8, Remark 2.5]). Consequently, Hypothesis 2.2(iii)(b) is readily verified.
As for the most challenging Hypothesis 2.2(iii)(c), it will be satisfied provided that we show that there exist $q \in (1,2)$ and $K \geq 0$ such that

$$\|B^* e^{\cdot A^*} A^{*\varepsilon} y\|_{L^q(0,T;U)} \leq K\|y\|_Y \qquad \forall y \in \mathcal{D}(A^{*\varepsilon})\,. \tag{3.23}$$

Indeed, combining (3.23) with Hypothesis 2.2(iii)(b) gives, *a fortiori*,

$$\begin{aligned} \|B^* e^{\cdot A^*} R^* R A^{\varepsilon} y\|_{L^q(0,T;U)} &= \|B^* e^{\cdot A^*} A^{*\varepsilon} A^{*-\varepsilon} R^* R A^{\varepsilon} y\|_{L^q(0,T;U)} \\ &\leq K\|A^{*-\varepsilon} R^* R A^{\varepsilon} y\|_Y \leq K\|R^* R\|_{\mathcal{L}(\mathcal{D}(A^{\varepsilon}),\mathcal{D}(A^{*\varepsilon}))}\|y\|_Y \quad \forall y \in \mathcal{D}(A^{*\varepsilon})\,. \end{aligned} \tag{3.24}$$

Set now $z = A^{*\varepsilon-1} y$: as, by (3.19),

$$B^* e^{tA^*} A^{*\varepsilon} y = B^* \frac{d}{dt} e^{tA^*} z = \frac{\partial \theta_t(t)}{\partial \nu}\Big|_{\Gamma}\,; \tag{3.25}$$

in order to verify Hypothesis 2.2(iii)(c) it will be sufficient to prove that

$$\Big\|B^* \frac{d}{dt} e^{\cdot A^*} z\Big\|_{L^q(0,T;U)} \leq C\|z\|_{D(A^{*1-\varepsilon})} \qquad \forall z \in \mathcal{D}(A^*)\,. \tag{3.26}$$

In view of the second equality in (3.25), the above estimate is equivalent (in PDE terms) to a boundary regularity estimate, which has been established in [1, Theorem 1.1] where, more precisely, it is shown that

$$\begin{aligned} &\Big\|\frac{\partial \theta_t}{\partial \nu}\Big\|_{L^q(0,T;L^2(\Gamma))} \\ &\qquad \leq C_{q,\varepsilon}\Big[\|w^0\|_{H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)} + \|w^1\|_{H_0^{2-\varepsilon}(\Omega)} + \|\theta^0\|_{H_0^{2-2\varepsilon}(\Omega)\cap H_0^1(\Omega)}\Big] \end{aligned}$$

for all $\varepsilon \in (0, \frac{1}{4})$ and $q \in (1, \min\{\frac{8}{7}, \frac{4}{3+4\varepsilon}\})$. Thus, the control problem (3.1)–(3.2) fits into the abstract theory of the present paper.

4. The optimal pair

Let us consider the optimal control problem (2.9)–(2.8) over the interval $[s,T]$, i.e., with s as initial time. Throughout the paper, we will make critical use of several regularity results relative to the operator L_s and to its adjoint L_s^* (defined by (2.11) and (2.12), respectively). For the sake of a more focused exposition, this fundamental analysis—usually performed at the outset—is postponed to Appendix B. So our first result establishes the distinct regularity properties of the optimal pair of problem (2.9)–(2.8).

Proposition 4.1. *Under Hypotheses* 2.1 *and* 2.2, *for the optimal control problem* (2.8)–(2.9) *there is a unique optimal pair* $(\hat{u}(\cdot,s;x),\hat{y}(\cdot,s;x))$, *such that*

$$\hat{y}(\cdot,s;x)\in C([s,T],Y),\quad \hat{u}(\cdot,s;x)\in\bigcap_{p<\infty}L^p(s,T;U)\,;$$

moreover,

$$\|\hat{y}(\cdot,s;x)\|_{C([s,T],Y)}+\|\hat{u}(\cdot,s;x)\|_{L^p(s,T;Y)}\le c_{p,T}\|x\|_Y\quad \forall p\in[1,\infty). \tag{4.1}$$

Proof. As remarked in the Introduction, it is a standard matter to show existence and uniqueness of the optimal pair in $L^2(s,T;Y)\times L^2(s,T;U)$; moreover, since

$$\hat{y}(t,s;x)=e^{(t-s)A}x+(L_s\hat{u}(\cdot,s:x))(t),\qquad t\in[s,T], \tag{4.2}$$

by Proposition B.3(ii) we immediately get $\hat{y}(\cdot,s;x)\in L^p(s,T;Y)$ with $p=\frac{2}{2\gamma-1}$.

Next, writing down the Euler equation $J_s'(\hat{y}(\cdot,s;x),\hat{u}(\cdot,s;x))=0$, and taking into account (4.2), we find the equation

$$\Lambda_s\hat{u}(\cdot,s;x)+N_sx=0, \tag{4.3}$$

where

$$\Lambda_su=u+L_s^*R^*RL_su,\qquad u\in L^2(s,T;U), \tag{4.4}$$

$$N_sx=L_s^*R^*Re^{(\cdot-s)A}x,\qquad x\in Y. \tag{4.5}$$

By (4.2) we may rewrite (4.3) as

$$\hat{u}(\cdot,s;x)=-L_s^*R^*R\hat{y}(\cdot,s;x). \tag{4.6}$$

We apply now a classical bootstrap process: we set

$$p_0=2,\qquad p_{n+1}=\frac{p_n}{1-(1-\gamma)p_n},\quad 0\le n<N,$$

where $N\in\mathbb{N}$ is the first positive integer such that $p_N\ge\frac{1}{1-\gamma}$; such an integer exists because

$$p_{n+1}-p_n=p_n\,\frac{(1-\gamma)p_n}{1-(1-\gamma)p_n}>\frac{4(1-\gamma)}{2\gamma-1}>0.$$

Now we use repeatedly Proposition B.3 and Proposition B.4: starting from the fact that $\hat{u}(\cdot,s;x)\in L^{p_0}(s,T;U)$ and $\hat{y}(\cdot,s;x)\in L^{p_1}(s,T;Y)$, we deduce as a first step

$$N_sx,\ L_s^*R^*RL_s\hat{u}(\cdot,s;x),\ \Lambda_s\hat{u}(\cdot,s;x)\in L^{p_2}(s,T;U),$$

which in turn gives $\hat{u}(\cdot,s;x) \in L^{p_2}(s,T;U)$ as well as, by (4.2), $\hat{y}(\cdot,s;x) \in L^{p_3}(s,T;Y)$. Thus, after n steps, we get

$$\hat{u}(\cdot,s;x) \in L^{p_{2n}}(s,T;U), \quad \hat{y}(\cdot,s;x) \in L^{p_{2n+1}}(s,T;Y).$$

This process stops as soon as $2n$ or $2n+1$ equals N: indeed, if $N = 2n$ we get at the n-th step $\hat{u}(\cdot,s;x) \in L^{p_N}(s,T:U)$ and $\hat{y}(\cdot,s;x) \in C([s,T],Y)$, so that in the next step we obtain $\hat{u}(\cdot,s;x) \in L^p(s,T;U)$ for all $p < \infty$; if $N = 2n+1$ we find directly, at the $(n+1)$-th step, $\hat{u}(\cdot,s;x) \in \bigcap_{1\le p<\infty} L^p(s,T;U)$ and $\hat{y}(\cdot,s;x) \in C([s,T],Y)$. □

Remark 4.2. It is clear that the dependence on x of the optimal pair is linear. More precisely, we observe that the operator Λ_s given by (4.4) is invertible in $L^2(s,T;Y)$ with $\|\Lambda_s^{-1}\|_{\mathcal{L}(L^2(s,T;Y))} \le 1$, since it has the form $I+U$ with U bounded, self-adjoint and positive. Hence we can rewrite (4.3) as

$$\hat{u}(\cdot,s;x) = -\Lambda_s^{-1} N_s x, \tag{4.7}$$

and, by (4.2),

$$\hat{y}(\cdot,s;x) = e^{(\cdot-s)A}x - L_s\Lambda_s^{-1}N_s x. \tag{4.8}$$

The result of Remark 4.2 can be improved. In fact we have:

Proposition 4.3. *Under Hypotheses* 2.1 *and* 2.2, *let* Λ_s *be the operator introduced in* (4.4). *Then*

(i) $\Lambda_s \in \mathcal{L}(L^q(s,t;U))$,
(ii) Λ_s *is bijective,*

where q is the exponent appearing in Hypothesis 2.2(*iii*)(*c*).

Proof. By Proposition B.3 and Proposition B.4, the operator $\Lambda_s = I + L_s^* R^* R L_s$ is well defined in $L^q(s,T;U)$ and clearly its range is contained in $L^q(s,T;U)$. In addition, its restriction to $L^{q'}(s,T;U)$ coincides with its adjoint Λ_s^* and we have $\Lambda_s^*(L^{q'}(s,T;U)) \subseteq L^{q'}(s,T;U)$ too. Now we observe that the operator $\Lambda_s^* : L^{q'}(s,T;U) \to L^{q'}(s,T;U)$ is injective (since $\Lambda_s^* u = 0$ with $u \in L^{q'}(s,T;U) \subseteq L^2(s,T;U)$ implies $u = 0$) and surjective (the equation $\Lambda_s^* u = v \in L^{q'}(s,T;U) \subseteq L^2(s,T;U)$ has a solution $u \in L^2(s,T;U)$; by repeated application of Proposition B.3 and Proposition B.4 we get $u \in L^{q'}(s,T;U)$). Hence, by classical functional analysis, $\Lambda_s : L^q(s,T;U) \to L^q(s,T;U)$ is injective with range $R(\Lambda_s)$ closed in $L^q(s,T;U)$. But, since $\ker \Lambda_s^* = \{0\}$,

$$\overline{R(\Lambda_s)} = \{f \in L^q(s,T;U) : \langle f,g\rangle_{L^{q'}(s,T;U)} = 0 \ \forall g \in \ker\Lambda_s^*\} = L^q(s,T;U),$$

so that $R(\Lambda_s) = L^q(s,T;U)$; i.e., $\Lambda_s : L^q(s,T;U) \to L^q(s,T;U)$ is bijective. □

Define now for $0 \le s \le t \le T$, according to Remark 4.2,

$$\Phi(t,s)x = \hat{y}(t,s;x), \qquad \Psi(t,s)x = \hat{u}(t,s;x). \tag{4.9}$$

In order to prove that $(t,s) \mapsto \Phi(t,s)$ is an evolution operator, we will need the following lemma, also of independent use in the sequel. Since the proofs of the next two statements are pretty routine, they will be omitted.

Lemma 4.4. *Under Hypotheses* 2.1 *and* 2.2, *let L_s and Λ_s be the operators defined in* (2.11) *and* (4.4). *Then the operator $I - L_s\Lambda_s^{-1}L_s^*R^*R$ has an inverse in $\mathcal{L}(L^2(s,T;Y))$, given by $I + L_sL_s^*R^*R$.*

Proposition 4.5. *Under Hypotheses* 2.1 *and* 2.2, *let $\Phi(t,s)$ and $\Psi(t,s)$ be defined by* (4.9). *Then for $0 \le s < r < t < T$ we have*

$$\Phi(t,r)\Phi(r,s)x = \Phi(t,s)x, \quad \Psi(t,r)\Phi(r,s)x = \Psi(t,s)x \qquad \forall x \in Y.$$

We know from Proposition 4.1 that $t \mapsto \Phi(t,s)x$ belongs to $C([s,T],Y)$ for all $x \in Y$. In the next statement we describe the regularity of $s \mapsto \Phi(t,s)x$.

Proposition 4.6. *Under Hypotheses* 2.1 *and* 2.2, *let $\Phi(t,s)$ be defined by* (4.9). *Then $s \mapsto \Phi(t,s)x$ belongs to $C([0,t],Y)$ for all $x \in Y$.*

Proof. We remark firstly that by (4.1) we have

$$\|\Phi(t,s)\|_{\mathcal{L}(Y)} \le c_T \qquad \text{for } 0 \le s \le t \le T. \tag{4.10}$$

Hence, if $s < t$ and $h \in [0, t-s)$, we get

$$\begin{aligned}\|\Phi(t,s+h)x - \Phi(t,s)x\|_Y &\le \|\Phi(t,s+h)\|_{\mathcal{L}(Y)}\|x - \Phi(s+h,s)x\|_Y \\ &\le c_T\|x - \Phi(s+h,s)x\|_Y \to 0 \quad \text{as } h \to 0^+.\end{aligned}$$

Next, if $h \in [0,s)$ we can write, choosing $p > \frac{1}{1-\gamma}$ and applying Proposition B.3(iv) and Proposition B.4(ii),

$$\begin{aligned}&\|\Phi(t,s-h)x - \Phi(t,s)x\|_Y \le \|\Phi(t,s)\|_{\mathcal{L}(Y)}\|\Phi(s,s-h)x - x\|_Y \\ &\le c_T\big[\|e^{hA}x - x\|_Y + \|[L_{s-h}L_{s-h}^*R^*R\Phi(\cdot,s-h)x](s)\|_Y\big] \\ &\le c_T\big[\|e^{hA}x - x\|_Y + \|L_{s-h}^*R^*R\Phi(\cdot,s-h)x\|_{L^p(s-h,s;Y)}\big] \\ &\le c_T\big[\|e^{hA}x - x\|_Y + \|\Phi(\cdot,s-h)x\|_{L^r(s-h,s;Y)}\big] \\ &\le c_T\big[\|e^{hA}x - x\|_Y + \|x\|_Y h^{\frac{1}{r}}\big] \to 0 \quad \text{as } h \to 0^+,\end{aligned}$$

where $r = \frac{p}{1+(1-\gamma)p}$. □

We now aim to show that the closed operator $\Phi(t,s)B$ admits a bounded extension on the space $D(A^{*\varepsilon})$. In order to do this, the following preparatory result is needed.

Proposition 4.7. *Under Hypotheses* 2.1 *and* 2.2, *let* Λ_s *be the operator introduced in* (4.4). *Then the closed operator* $L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}B$ *has a bounded extension belonging to* $\mathcal{L}(U, L^{\frac{q}{1-(1-\gamma)q}}(s,T;Y))$, *where* q *and* γ *are the exponents introduced in Hypothesis* 2.2(*iii*).

Proof. Let $u \in U$. By Lemma A.2 we have

$$e^{(\cdot-s)A}Bu \in L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]') \qquad \forall p \in [1, \tfrac{1}{\gamma})\,.$$

Hence, by Corollary B.7,

$$L_s^*R^*Re^{(\cdot-s)A}Bu \in L^q(s,T;U)\,.$$

Thus, Proposition 4.3 gives

$$\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Bu \in L^q(s,T;U)$$

and finally Proposition B.3 yields the result. □

Corollary 4.8. *Under Hypotheses* 2.1 *and* 2.2, *let* $\Phi(t,s)$ *be defined by* (4.9). *Then*

$$\Phi(\cdot,s)B \in \mathcal{L}(U, L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]')) \qquad \forall p \in [1, \tfrac{1}{\gamma})\,.$$

Proof. By (4.8) and (4.5) we have

$$\Phi(\cdot,s)x = e^{(\cdot-s)A}x - L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}x \qquad \forall x \in Y. \tag{4.11}$$

Now choose $y = Bu$, with $u \in U$, so that $y \in [\mathcal{D}(A^*)]'$. Then for almost all $t \in [s,T]$ we have $\Phi(t,s)Bu \in [\mathcal{D}(A^*)]'$, since the right member belongs to that space: in fact, by Lemma A.2 and Proposition 4.7 we have

$$e^{(\cdot-s)A}Bu \in L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]')) \qquad \forall p \in \big[1, \tfrac{1}{\gamma}\big)\,,$$

$$L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Bu \in L^{\frac{q}{1-(1-\gamma)q}}(s,T;[\mathcal{D}(A^{*\varepsilon})]')\,,$$

and since $\frac{q}{1-(1-\gamma)q} > \frac{1}{\gamma}$, we immediately get the result. □

Corollary 4.9. *Under Hypotheses* 2.1 *and* 2.2, *let* $\Phi(t,s)$ *be defined by* (4.9). *Then*

$$L_s\,L_s^*\,R^*R\Phi(\cdot,s)B \in \mathcal{L}(U, L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]')) \qquad \forall p \in [1, \tfrac{1}{\gamma})\,.$$

Proof. It follows by equation (4.11) and Corollary 4.8. □

5. The Riccati operator

Let us define the Riccati operator of the control problem (2.2)-(2.1) as follows:

$$P(t)x = \int_t^T e^{(\sigma-t)A^*} R^* R\Phi(\sigma,t)x\, d\sigma, \quad t \in [0,T], \quad x \in Y. \tag{5.1}$$

By (4.10) it is clear that

$$\|P(t)\|_{\mathcal{L}(Y)} \le c_T(T-t) \qquad \forall t \in [0,T]. \tag{5.2}$$

Proposition 5.1. *Under Hypotheses* 2.1 *and* 2.2, *let* $P(t)$ *be defined by* (5.1). *Then* $P(t) = P(t)^* \ge 0$ *and*

$$(P(t)x,x)_Y = J_t(\hat{y}(\cdot,t;x),\hat{u}(\cdot,t;x)) \qquad \forall t \in [0,T], \quad \forall x \in Y,$$

where J_t *is defined by* (2.8).

Proof. We have for each $x \in Y$

$$\begin{aligned}(P(t)x,x)_Y &= \int_t^T (R\Phi(\sigma,t)x, Re^{(\sigma-t)A}x)_Z d\sigma \\ &= \int_t^T (R\Phi(\sigma,t)x, R\Phi(\sigma,t)x)_Z d\sigma \\ &\quad + \int_t^T (R\Phi(\sigma,t)x, R[L_tL_t^*R^*R\Phi(\cdot,t)x](\sigma))_Z d\sigma \\ &= \int_t^T \|R\hat{y}(\sigma,t;x)\|_Z^2 d\sigma + \int_t^T \|[L_t^*R^*R\Phi(\cdot,t)x](\sigma)\|_U^2 d\sigma \\ &= \int_t^T \|R\hat{y}(\sigma,t;x)\|_Z^2 d\sigma + \int_t^T \|\hat{u}(\sigma,t;x)\|_U^2 d\sigma \\ &= J_t(\hat{y}(\cdot,t;x),\hat{u}(\cdot,t;x)) \ge 0,\end{aligned}$$

so that $P(t) \ge 0$. More generally we get, proceeding as before,

$$\begin{aligned}(P(t)x,z)_Y &= \int_t^T (R\hat{y}(\sigma,t;x), R\hat{y}(\sigma,t;z))_Z d\sigma + \int_t^T (\hat{u}(\sigma,t;x),\hat{u}(\sigma,t;z))_U d\sigma \\ &= (x,P(t)z)_Y \qquad \forall x,z \in Y,\end{aligned}$$

so that $P(t)$ is selfadjoint. □

Remark 5.2. An easy application of the dominated convergence theorem shows that $P(\cdot)x \in C([0,T],Y)$ and $\sup_{t\in[0,T]} \|P(t)x\|_Y \le c_T\|x\|_Y$ for all $x \in Y$; i.e. $P \in \mathcal{L}(Y, C([0,T],Y))$.

Our next result states well posedness and regularity of the gain operator $B^*P(t)$.

Proposition 5.3. *Under Hypotheses* 2.1 *and* 2.2, *let* $P(t)$ *be defined by* (5.1). *Then* $B^*P(\cdot) \in \mathcal{L}(\mathcal{D}(A^\varepsilon), C([0,T],U))$.

Proof. For $x \in \mathcal{D}(A^\varepsilon)$, using (2.3) and (4.2), we give a meaning to the expression

$$B^*P(t)x = \int_t^T B^* e^{(\sigma-t)A^*} R^*R\Phi(\sigma,t)x\,d\sigma \tag{5.3}$$

by splitting it as follows:

$$\begin{aligned} B^*P(t)x &= \int_0^{T-t} F(\sigma)R^*R\Phi(\sigma+t,t)x\,d\sigma + \int_0^{T-t} G(\sigma)R^*Re^{\sigma A}x\,d\sigma \\ &\quad + \int_0^{T-t} G(\sigma)R^*R[L_t\hat{u}(\cdot,t;x)](\sigma+t)\,d\sigma = I + II + III. \end{aligned}$$

Now, by (2.4),

$$\|I\|_U \le c\int_0^{T-t} \sigma^{-\gamma}\|x\|_Y\,d\sigma \le c_T\|x\|_Y\,,$$

and by Lemmas A.3 and A.4

$$\|II\|_U \le c_T\|R^*Re^{\cdot A}x\|_{C([0,T],\mathcal{D}(A^{*\varepsilon}))} \le c_T\|x\|_{\mathcal{D}(A^\varepsilon)}\,;$$

finally, by Proposition B.5,

$$\begin{aligned} \|III\|_U &\le c_T\|R^*RL_t\hat{u}(\cdot,t;x)\|_{C([t,T],\mathcal{D}(A^{*\varepsilon}))} \\ &\le c_T\|\hat{u}(\cdot,t;x)\|_{L^{q'}(t,T;U)} \le c_T\|x\|_Y\,. \end{aligned}$$

This shows that

$$\|B^*P(\cdot)x\|_{L^\infty(0,T;U)} \le c_T\|x\|_{\mathcal{D}(A^\varepsilon)}\,. \tag{5.4}$$

Let us prove continuity of $B^*P(\cdot)x$ at $t_0 \in [0,T]$.
If $t \ge t_0$, we have by Proposition 4.5 and (2.12),

$$\begin{aligned} &B^*P(t)x - B^*P(t_0)x = \\ &\quad = \int_t^T B^*e^{(\sigma-t)A^*}R^*R\Phi(\sigma,t)x\,d\sigma - \int_{t_0}^T B^*e^{(\sigma-t_0)A^*}R^*R\Phi(\sigma,t_0)x\,d\sigma \\ &\quad = \int_t^T B^*e^{(\sigma-t)A^*}R^*R\Phi(\sigma,t)(x-\Phi(t,t_0)x)d\sigma \\ &\qquad + [L_0^*R^*R\Phi(\cdot,t_0)x](t) - [L_0^*R^*R\Phi(\cdot,t_0)x](t_0) \end{aligned}$$

$$
\begin{aligned}
&= \int_t^T [F(\sigma-t)+G(\sigma-t)]R^*Re^{(\sigma-t)A}(x-\Phi(t,t_0)x)d\sigma \\
&\quad + \int_t^T [F(\sigma-t)+G(\sigma-t)]R^*R[L_t\hat{u}(\cdot,t;x-\Phi(t,t_0)x)](\sigma)d\sigma \\
&\quad +[L_0^*R^*Re^{(\cdot-t_0)A}x](t)-[L_0^*R^*Re^{(\cdot-t_0)A}x](t_0) \\
&\quad +[L_0^*R^*R[L_{t_0}\hat{u}(\cdot,t_0;x)]](t)-[L_0^*R^*R[L_{t_0}\hat{u}(\cdot,t_0;x)]](t_0) \\
&= (I_1+I_2)+(I_3+I_4)+I_5+I_6\,.
\end{aligned}
$$

By (2.4) and Proposition B.3(iv)

$$
\|I_1\|_U+\|I_3\|_U \le c\int_t^T (\sigma-t)^{-\gamma}\|x-\Phi(t,t_0)x\|_Y\,d\sigma = \mathrm{o}(1) \quad \text{as } t\to t_0^+.
$$

Next,

$$
\begin{aligned}
\|I_2\|_U \le &\int_0^{T-t} \|G(\sigma)R^*Re^{\sigma A}(x-e^{(t-t_0)A}x)\|_U d\sigma + \\
&+\int_0^{T-t} \|G(\sigma)R^*Re^{\sigma A}[L_{t_0}\hat{u}(\cdot,t_0;x)](t)\|_U d\sigma.
\end{aligned}
$$

The first term is estimated by Lemmas A.3 and A.4:

$$
\begin{aligned}
&\int_0^{T-t} \|G(\sigma)R^*Re^{\sigma A}(x-e^{(t-t_0)A}x)\|_U d\sigma \\
&\le c\|x-e^{(t-t_0)A}x\|_{\mathcal{D}(A^\varepsilon)} = \mathrm{o}(1) \quad \text{as } t\to t_0^+;
\end{aligned}
$$

the second term is estimated by Lemma A.3, Remark B.6 and (4.1):

$$
\begin{aligned}
&\int_0^{T-t} \|G(\sigma)R^*Re^{\sigma A}[L_{t_0}\hat{u}(\cdot,t_0;x)](t)\|_U d\sigma \le \\
&\quad \le c(T-t)\|R^*Re^{\cdot A}[L_{t_0}\hat{u}(\cdot,t_0;x)](t)\|_{C([0,T-t],\mathcal{D}(A^{*\varepsilon}))} \le \\
&\quad \le c\|\hat{u}(\cdot,t_0;x)\|_{L^{q'}(t_0,t;U)} = \mathrm{o}(1) \quad \text{as } t\to t_0^+.
\end{aligned}
$$

Similarly, I_4 is estimated using Lemma A.3 and Proposition B.5:

$$
\begin{aligned}
\|I_4\|_U &\le \int_0^{T-t} \|G(\sigma)R^*R[L_t\hat{u}(\cdot,t;x-\Phi(t,t_0)x)](\sigma)\|_U d\sigma \\
&\le c\|R^*RL_t\hat{u}(\cdot,t;x-\Phi(t,t_0)x)\|_{C([t,T],\mathcal{D}(A^{*\varepsilon}))} \\
&\le c\|\hat{u}(\cdot,t;x-\Phi(t,t_0)x)\|_{L^{q'}(t,T;U)} \\
&\le c\|x-\Phi(t,t_0)x\|_Y = \mathrm{o}(1) \quad \text{as } t\to t_0^+.
\end{aligned}
$$

Finally, by Proposition B.4(iv) and Lemma A.4, $\|I_5\|_U = \mathrm{o}(1)$ as $t \to t_0^+$, whereas by Propositions B.4(iv) and B.5 we get $\|I_6\|_U = \mathrm{o}(1)$ as $t \to t_0^+$. Summing up, we have proved that

$$\|B^*P(t)x - B^*P(t_0)x\|_U \to 0 \quad \text{as } t \to t_0^+. \tag{5.5}$$

If, instead, $t < t_0$, we write

$$\begin{aligned} B^*P(t)x - B^*P(t_0)x &= \int_t^{t_0} B^* e^{(\sigma-t)A^*} R^*R\Phi(\sigma,t)x\, d\sigma \\ &+ \int_{t_0}^T B^* e^{(\sigma-t)A^*} R^*R\Phi(\sigma,t_0)x[\Phi(t_0,t)x - x]d\sigma \\ &+ \int_{t_0}^T B^* e^{(\sigma-t_0)A^*} [e^{(t_0-t)A^*} - 1] R^*R\Phi(\sigma,t_0)x\, d\sigma. \end{aligned}$$

We introduce now the additional term $\int_t^{t_0} B^* e^{(\sigma-t)A^*} R^*Rx\, d\sigma$, obtaining

$$\begin{aligned} B^*P(t)x - B^*P(t_0)x &= \int_t^{t_0} B^* e^{(\sigma-t)A^*} R^*R[\Phi(\sigma,t)x - x]d\sigma \\ &+ \int_{t_0}^T B^* e^{(\sigma-t)A^*} R^*R[\Phi(\sigma,t_0)x - x]d\sigma + [L_0^* y_{t_0}(t) - L_0^* y_{t_0}(t_0)] \\ &= I + II + III, \end{aligned}$$

where

$$y_{t_0}(r) = \begin{cases} R^*R\Phi(r,t_0)x & \text{if } r \in [t_0,T] \\ R^*Rx & \text{if } r \in [0,t_0]. \end{cases}$$

Note that $y_{t_0} \in C([0,T],Y) \cap L^\infty(0,T;\mathcal{D}(A^{*\varepsilon}))$: indeed

$$A^{*\varepsilon} y_{t_0}(r) = \begin{cases} A^{*\varepsilon} R^*R \left(e^{(r-t_0)A} x + [L_{t_0}\hat{u}(\cdot,t_0;x)](r) \right) & \text{if } r \in [t_0,T] \\ A^{*\varepsilon} R^*Rx & \text{if } r \in [0,t_0], \end{cases}$$

so that Lemma A.4, Proposition B.5 and (4.1) imply

$$\begin{aligned} \|A^{*\varepsilon} y_{t_0}(r)\|_Y &\le c\|x\|_{\mathcal{D}(A^{*\varepsilon})} + c\|R^*RL_{t_0}\hat{u}(\cdot,t_0;x)\|_{C([t_0,T],\mathcal{D}(A^{*\varepsilon}))} \\ &\le c\|x\|_{\mathcal{D}(A^{*\varepsilon})} + c\|\hat{u}(\cdot,t_0;x)\|_{L^{q'}(0,T;U)} \le c\|x\|_{\mathcal{D}(A^{*\varepsilon})}. \end{aligned}$$

Thus we get, by Proposition B.4(iv), $\|III\|_U = \mathrm{o}(1)$ as $t \to t_0^-$. On the other hand, by (2.4), Lemma A.3, Lemma A.4 and Proposition B.5, we have

$$\|I\|_U = \left\| \int_t^{t_0} [F(\sigma-t) + G(\sigma-t)] R^*R(\Phi(\sigma,t)x - x) d\sigma \right\|_U$$

$$\leq c\int_t^{t_0}(\sigma-t)^{-\gamma}\|\Phi(\sigma,t)x-x\|_Y d\sigma+\int_t^{t_0}\|G(\sigma-t)R^*R[e^{(\sigma-t)A}x-x]\|_U d\sigma$$
$$+\int_t^{t_0}\|G(\sigma-t)R^*R[L_t\hat{u}(\cdot,t;x)](\sigma)\|_U\,d\sigma$$
$$\leq c\|x\|_Y(t_0-t)^{1-\gamma}+\|x\|_{D(A^\varepsilon)}[\mathrm{o}(1)+c(t_0-t)]=\mathrm{o}(1)\quad\text{as } t\to t_0^-,$$

and, using also (4.1) and (A.3),

$$\begin{aligned}\|II\|_U&=\Big\|\int_{t_0}^T[F(\sigma-t)+G(\sigma-t)]R^*R\Phi(\sigma,t_0)(\Phi(t_0,t)x-x)d\sigma\Big\|_U\\ &\leq c\|\Phi(t_0,t)x-x\|_Y\\ &+\Big\|\int_{t_0}^T G(\sigma-t)R^*Re^{(\sigma-t_0)A}(\Phi(t_0,t)x-x)d\sigma\Big\|_U\\ &+\Big\|\int_{t_0}^T G(\sigma-t)R^*RL_{t_0}\hat{u}(\cdot,t_0;\Phi(t_0,t)x-x)]d\sigma\Big\|_U\\ &=\mathrm{o}(1)+c\int_{t_0}^T(\sigma-t_0)^{-\varepsilon}\|\Phi(t_0,t)x-x\|_Y\,d\sigma\\ &+c\|\hat{u}(\cdot,t_0;\Phi(t_0,t)x-x)\|_{L^{q'}(t_0,T;U)}=\mathrm{o}(1)\quad\text{as } t\to t_0^-.\end{aligned}$$

This proves that

$$\|B^*P(t)x-B^*P(t_0)x\|_U\to 0\quad\text{as } t\to t_0^-. \tag{5.6}$$

Summing up, we have obtained

$$\lim_{t\to t_0}\|B^*P(t)x-B^*P(t_0)x\|_U=0\qquad\forall x\in D(A^\varepsilon),$$

and the proof is complete. □

Corollary 5.4. *Under Hypotheses* 2.1 *and* 2.2*, let* $P(t)$ *and* $\Phi(t,s)$ *be defined by* (5.1) *and* (4.9)*. Then we have*

(i) *if* $x\in Y$*, then*

$$P(\cdot)\Phi(\cdot,s)x\in C([s,T],Y),\qquad B^*P(\cdot)\Phi(\cdot,s)x\in\bigcap_{1\leq p<\infty}L^p(s,T;U);$$

(ii) *if* $x\in\mathcal{D}(A^\varepsilon)$*, then* $B^*P(\cdot)\Phi(\cdot,s)x\in C([s,T],U)$.

Proof. (i) The first result follows trivially by Remark 5.2 and Proposition 4.1. The second one follows by remarking that

$$B^*P(\cdot)\Phi(\cdot,s)x=L_s^*R^*R\Phi(\cdot,s)x \tag{5.7}$$

and applying Proposition B.4 (iii).

(ii) Let $x \in \mathcal{D}(A^\varepsilon)$. We write

$$B^*P(\cdot)\Phi(\cdot,s)x = L_s^*R^*Re^{(\cdot-s)A}x + L_s^*R^*R\hat{u}(\cdot,s;x)\,.$$

The first term on the right-hand side belongs to $C([s,T],U)$, because we have $e^{(\cdot-s)A}x \in C([s,T],\mathcal{D}(A^\varepsilon))$ and we can use Hypothesis 2.2(iii)(b) as well as Proposition B.4(iv). The second term is in $C([s,T],U)$ in view of Propositions 4.1, B.5 and B.3(iv). The estimate

$$\|B^*P(\cdot)\Phi(\cdot,s)x\|_{C([s,T],U)} \le c_T\|x\|_{\mathcal{D}(A^\varepsilon)} \qquad \forall x \in \mathcal{D}(A^\varepsilon) \tag{5.8}$$

also follows. □

Corollary 5.5. *Under Hypotheses* 2.1 *and* 2.2, *let* $P(t)$ *be defined by* (5.1). *Then for each* $x \in Y$ *we have*

$$-B^*P(t)\Phi(t,s)x = \Psi(t,s)x = \hat{u}(t,s;x) \qquad \text{for } 0 \le s < t < T; \tag{5.9}$$

in particular, for each $t \in (0,T]$ *the function* $s \mapsto \hat{u}(t,s;x)$ *is in* $C([0,t[,U)$ *if* $x \in Y$, *and is in* $C([0,t],U)$ *if* $x \in \mathcal{D}(A^\varepsilon)$.

Proof. If $x \in Y$, using (5.7) and (4.6) we immediately get (5.9). The continuity properties of $s \mapsto \hat{u}(t,s;x)$ follow by the properties of $\Phi(t,s)x$ (Proposition 4.6). □

As a consequence of the previous results, we show that the operator $P(\cdot)$ solves the Riccati integral equation.

Corollary 5.6. *Under Hypotheses* 2.1 *and* 2.2, *let* $P(t)$ *be defined by* (5.1). *Then for all* $x,y \in Y$ *and* $t \in [0,T]$ *we have*

$$\begin{aligned}(P(t)x,y)_Y &= \int_t^T (R\Phi(\sigma,t)x, R\Phi(\sigma,t)y)_Z\,d\sigma \\ &\quad + \int_t^T (B^*P(\sigma)\Phi(\sigma,t)x, B^*P(\sigma)\Phi(\sigma,t)y)_U\,d\sigma.\end{aligned} \tag{5.10}$$

Proof. It follows by Proposition 5.1 and (5.9). □

6. Differentiability of the evolution map $\Phi(t,s)$

In this section we analyze the differentiability properties of the operator $\Phi(t,s)$ defined by (4.9). The regularity results established in Theorems 6.1 and 6.2 below—together with boundedness of $B^*P(t)$ on $\mathcal{D}(A^\varepsilon)$, already proved in Proposition 5.3—enable us to conclude that $P(t)$ satisfies the

Differential Riccati equation on $\mathcal{D}(A)$, thus completing the proof of Theorem 2.3.

Theorem 6.1. *Under Hypotheses* 2.1 *and* 2.2, *let* $\Phi(t,s)$ *be defined by* (4.9). *Then for each* $t \in (s,T]$

$$\exists \frac{\partial}{\partial t}\Phi(t,s)x = [A - BB^*P(t)]\Phi(t,s)x \in \bigcap_{1\le p<\infty} L^p(s,T;[\mathcal{D}(A^*)]') \quad \forall x \in Y;$$

moreover,

$$\frac{\partial}{\partial t}\Phi(t,s)x \in C([s,T],[\mathcal{D}(A^*)]') \qquad \forall x \in \mathcal{D}(A^\varepsilon)\,.$$

Proof. We start from the equation

$$\Phi(t,s)x = e^{(t-s)A}x - [L_sL_s^*R^*R\Phi(\cdot,s)x](t), \quad t\in(s,T], \quad x\in Y, \tag{6.1}$$

which is a consequence of (4.11) and Lemma 4.4. For $h>0$ we write

$$\begin{aligned}
&\frac{\Phi(t+h,s)x - \Phi(t,s)x}{h}\\
&= \frac{e^{(t+h-s)A}x - e^{(t-s)A}x}{h} - \frac{1}{h}\int_t^{t+h} e^{(t+h-\sigma)A}B[L_s^*R^*R\Phi(\cdot,s)x](\sigma)d\sigma\\
&- \int_s^t \frac{e^{(t+h-\sigma)A} - e^{(t-\sigma)A}}{h}B[L_s^*R^*R\Phi(\cdot,s)x](\sigma)d\sigma = I + II + III.
\end{aligned}$$

Now if $y \in \mathcal{D}(A^*)$ we have, as $h \to 0^+$,

$$(I,y)_Y = \Big(e^{(t-s)A}x, \frac{e^{hA^*}-I}{h}y\Big)_Y \to (e^{(t-s)A}x, A^*y)_Y\,, \tag{6.2}$$

whereas, by (5.7),

$$\begin{aligned}
(II,y)_Y &= -\frac{1}{h}\int_t^{t+h}(B^*P(\sigma)\Phi(\sigma,s)x, B^*e^{(t+h-\sigma)A^*}y)_U d\sigma\\
&\to -(B^*P(t)\Phi(t,s)x, B^*A^{*-1}A^*y)_U\\
&= ((A^{-1}B)B^*P(t)\Phi(t,s)x, A^*y)_Y;
\end{aligned}\tag{6.3}$$

finally, by (2.11) and (5.7),

$$\begin{aligned}
(III,y)_Y &= -\Big(L_s[B^*P(\cdot)\Phi(\cdot,s)x](t), \frac{e^{hA^*}-I}{h}y\Big)_Y\\
&\to -(L_s[B^*P(\cdot)\Phi(\cdot,s)x](t), A^*y)_Y.
\end{aligned}\tag{6.4}$$

By (6.2), (6.3) and (6.4) we get as $h \to 0^+$

$$\Big(\frac{\Phi(t+h,s)x - \Phi(t,s)x}{h}, y\Big)_Y \to (\Phi(t,s)x - (A^{-1}B)B^*P(t)\Phi(t,s)x, A^*y)_Y. \tag{6.5}$$

Now we turn to the limit as $h \to 0^-$. As

$$\begin{aligned}&\frac{\Phi(t+h,s)x - \Phi(t,s)x}{h}\\ &= \frac{e^{(t+h-s)A}x - e^{(t-s)A}x}{h} - \int_s^{t+h} \frac{e^{(t+h-\sigma)A} - e^{(t-\sigma)A}}{h} B[L_s^* R^* R\Phi(\cdot,s)x](\sigma)d\sigma\\ &+ \frac{1}{h}\int_{t+h}^t e^{(t-\sigma)A}B[L_s^*R^*R\Phi(\cdot,s)x](\sigma)d\sigma = I + II + III,\end{aligned}$$

we get as before

$$(I,y)_Y = \Big(e^{(t+h-s)A}x, \frac{I - e^{-hA^*}}{h}y\Big)_Y \to (e^{(t-s)A}x, A^*y)_Y\,,$$

$$\begin{aligned}(II,y)_Y &= -\Big([L_sB^*P(\cdot)\Phi(\cdot,s)x](t+h), \frac{I - e^{-hA^*}}{h}y\Big)_Y\\ &\to -([L_sB^*P(\cdot)\Phi(\cdot,s)x](t), A^*y)_Y\,,\end{aligned}$$

$$\begin{aligned}(III,y)_Y &= \frac{1}{h}\int_{t+h}^t (B^*P(\sigma)\Phi(\sigma,s)x, B^*e^{(t-\sigma)A^*}y)_Y d\sigma\\ &\to -(B^*P(t)\Phi(t,s)x, (A^{-1}B)^*A^*y)_Y\\ &= -((A^{-1}B)B^*P(t)\Phi(t,s)x, A^*y)_Y\,;\end{aligned}$$

hence when $h \to 0^-$ we obtain again (6.5). This proves that

$$\begin{aligned}\frac{\partial}{\partial t}(\Phi(t,s)x, y)_Y &= (\Phi(t,s)x, A^*y)_Y - ((A^{-1}B)B^*P(t)\Phi(t,s)x, y)_Y\\ &= \langle (A - BB^*P(t))\Phi(t,s)x, y\rangle_{\mathcal{D}(A^*)};\end{aligned} \tag{6.6}$$

i.e., the derivative $\frac{\partial}{\partial t}\Phi(t,s)x$ exists as an element of $[\mathcal{D}(A^*)]'$. But Proposition 4.1 implies $A\Phi(\cdot,s)x \in C([s,T],[\mathcal{D}(A^*)]')$, and Corollary 5.4(i) gives

$$B\,B^*P(\cdot)\Phi(t,s)x \in \bigcap_{1\le p<\infty} L^p(s,T;[\mathcal{D}(A^*)]')\,.$$

Moreover, if $x \in \mathcal{D}(A^\varepsilon)$ then $B^*P(\cdot)\Phi(\cdot,s)x \in C([s,T],[\mathcal{D}(A^*)]')$ by Corollary 5.4(ii). This proves the result. □

The proof of the differentiability of $\Phi(t,s)$ with respect to s is more involved and it requires a number of lemmas. The result is the following:

Theorem 6.2. *Under Hypotheses* 2.1 *and* 2.2, *let* $\Phi(t,s)$ *be defined by* (4.9). *Then for each* $s \in [0,t)$

$$\exists \frac{\partial}{\partial s}\Phi(t,s)x = -\Phi(t,s)[A - BB^*P(s)]x \in L^{\frac{1}{\gamma}}(s,T;[\mathcal{D}(A^{*\varepsilon})]') \quad \forall x \in \mathcal{D}(A)\,,$$

where γ *is the exponent appearing in* (2.4).

Proof. We split the proof into five steps.
Step 1. Splitting of the ratio $\frac{\Phi(t,s+h)x-\Phi(t,s)x}{h}$. We start from equation (6.1). For $h > 0$ we can write

$$\begin{aligned}&\frac{\Phi(t,s+h)x - \Phi(t,s)x}{h}\\ &= \frac{e^{(t-s-h)A}x - e^{(t-s)A}x}{h} - \Big[L_{s+h}L^*_{s+h}R^*R\frac{\Phi(\cdot,s+h)-\Phi(\cdot,s)}{h}x\Big](t)\\ &\quad - \frac{L_{s+h}L^*_{s+h} - L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x](t),\end{aligned}$$

whereas for $h < 0$ we have

$$\begin{aligned}&\frac{\Phi(t,s+h)x - \Phi(t,s)x}{h}\\ &= \frac{e^{(t-s-h)A}x - e^{(t-s)A}x}{h} - \frac{L_{s+h}L^*_{s+h} - L_sL^*_s}{h}[R^*R\Phi(\cdot,s+h)x](t)\\ &\quad - \Big[L_sL^*_sR^*R\frac{\Phi(\cdot,s+h)-\Phi(\cdot,s)}{h}x\Big](t).\end{aligned}$$

Hence, we obtain for $h > 0$, recalling Lemma 4.4,

$$\begin{aligned}\frac{\Phi(t,s+h)x - \Phi(t,s)x}{h} &= [I + L_{s+h}L^*_{s+h}R^*R]^{-1}\\ &\quad\times\Big[\frac{e^{(t-s-h)A}x - e^{(t-s)A}x}{h} - \frac{L_{s+h}L^*_{s+h} - L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x]\Big](t)\\ &= [I - L_{s+h}\Lambda^{-1}_{s+h}L^*_{s+h}R^*R]\\ &\quad\times\Big[\frac{e^{(t-s-h)A}x - e^{(t-s)A}x}{h} - \frac{L_{s+h}L^*_{s+h} - L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x]\Big](t);\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{\Phi(t,s+h)x-\Phi(t,s)x}{h}\\
&=\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}-\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x]\\
&-L_{s+h}\Lambda^{-1}_{s+h}L^*_{s+h}R^*R\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}\\
&+L_{s+h}\Lambda^{-1}_{s+h}L^*_{s+h}R^*R\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x],\quad h>0.
\end{aligned}\tag{6.7}$$

Similarly, for $h<0$, using Lemma 4.4,

$$\begin{aligned}
\frac{\Phi(t,s+h)x-\Phi(t,s)x}{h}&=[I+L_sL^*_sR^*R]^{-1}\\
&\times\Big[\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}-\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s+h)x]\Big](t)\\
&=[I-L_s\Lambda^{-1}_sL^*_sR^*R]\\
&\times\Big[\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}-\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}R^*R\Phi(\cdot,s+h)x]\Big](t);
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{\Phi(t,s+h)x-\Phi(t,s)x}{h}\\
&=\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}-\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}R^*R\Phi(\cdot,s+h)x(t)\\
&-L_s\Lambda^{-1}_sL^*_sR^*R\Big[\frac{e^{(t-s-h)A}x-e^{(t-s)A}x}{h}\Big]\\
&+L_s\Lambda^{-1}_sL^*_sR^*R\Big[\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}R^*R\Phi(\cdot,s+h)x\Big](t),\quad h<0.
\end{aligned}\tag{6.8}$$

Both expressions in (6.7) and (6.8) contain four terms, whose limits as $h\to 0^\pm$ must be evaluated.

Step 2. The limit of the first two terms in (6.7) and (6.8). First of all, for the first term we clearly have, for each $x\in\mathcal{D}(A)$ and $p\in[1,\infty)$:

$$\lim_{h\to 0^\pm}\frac{e^{(\cdot-s-h)A}x-e^{(\cdot-s)A}x}{h}=-Ae^{(\cdot-s)A}x\quad\text{in } L^p(s,T;Y).\tag{6.9}$$

The limit of the second term in (6.7) and (6.8) results from the following claim:

Lemma 6.3. *Under Hypotheses* 2.1 *and* 2.2, *for each* $x \in \mathcal{D}(A)$ *we have*

$$-\frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h}[R^*R\Phi(\cdot,s)x] \overset{*}{\rightharpoonup} e^{(\cdot - s)A}BB^*P(s)x \quad \text{as } h \to 0^+ \tag{6.10}$$

in the space $L^{\frac{1}{\gamma}}(s+\delta, T; [\mathcal{D}(A^{*\varepsilon})]') = [L^{\frac{1}{1-\gamma}}(s+\delta, T; \mathcal{D}(A^{*\varepsilon}))]'$, *for each* $\delta \in (0, T-s)$, *and*

$$-\frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h}[R^*R\Phi(\cdot,s+h)x] \overset{*}{\rightharpoonup} e^{(\cdot - s)A}BB^*P(s)x \text{ as } h \to 0^- \tag{6.11}$$

in the space $L^{\frac{1}{\gamma}}(s, T; [\mathcal{D}(A^{*\varepsilon})]') = [L^{\frac{1}{1-\gamma}}(s, T; \mathcal{D}(A^{*\varepsilon}))]'$.

Proof. For $h > 0$ and $t \in [s+h, T]$, we have

$$\begin{aligned}
&- \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h}[R^*R\Phi(\cdot,s)x](t) \\
&= \frac{1}{h}\int_s^{s+h} e^{(t-\sigma)A}B\int_\sigma^T B^*e^{(\tau-t)A^*}R^*R\Phi(\tau,s)x\,d\tau d\sigma \\
&= \frac{1}{h}\int_s^{s+h} e^{(t-\sigma)A}BB^*P(\sigma)\Phi(\sigma,s)x\,d\sigma \\
&= \frac{1}{h}L_s\left(B^*P(\cdot)\Phi(\cdot,s)x\cdot\chi_{[s,s+h]}(\cdot)\right)(t).
\end{aligned}$$

Similarly, for $h < 0$ and $t \in [s, T]$ we have:

$$\begin{aligned}
&- \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h}[R^*R\Phi(\cdot,s+h)x](t) \\
&= -\frac{1}{h}\int_{s+h}^{s} e^{(t-\sigma)A}B\int_\sigma^T B^*e^{(\tau-t)A^*}R^*R\Phi(\tau,s+h)x\,d\tau d\sigma \\
&= -\frac{1}{h}\int_{s+h}^{s} e^{(t-\sigma)A}BB^*P(\sigma)\Phi(\sigma,s+h)x\,d\sigma \\
&= -\frac{1}{h}L_{s+h}\left(B^*P(\cdot)\Phi(\cdot,s+h)x\cdot\chi_{[s+h,s]}(\cdot)\right)(t).
\end{aligned}$$

Now if $h > 0$ and $\delta > 0$ we get by Proposition B.3(i) and Corollary 5.4(ii):

$$\left\|\frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h}[R^*R\Phi(\cdot,s)x]\right\|_{L^{1/\gamma}(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]')}$$

$$
\begin{aligned}
&= \Big\| \frac{1}{h} L_s \left(B^* P(\cdot)\Phi(\cdot,s)x \cdot \chi_{[s,s+h]}(\cdot) \right) \Big\|_{L^{1/\gamma}(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]')} \\
&\le c_T \frac{1}{h} \| B^* P(\cdot)\Phi(\cdot,s)x \cdot \chi_{[s,s+h]}(\cdot) \|_{L^1(s,T;U)} \\
&= c_T \frac{1}{h} \| B^* P(\cdot)\Phi(\cdot,s)x \|_{L^1(s,s+h;U)} \\
&\le c_T \| B^* P(\cdot)\Phi(\cdot,s)x \|_{L^\infty(s,T;U)} \le c_T \|x\|_{\mathcal{D}(A^\varepsilon)} ;
\end{aligned}
$$

similarly, for $h<0$ we obtain

$$
\begin{aligned}
&\Big\| \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h} [R^* R \Phi(\cdot, s+h)x] \Big\|_{L^{1/\gamma}(s+h,T;[\mathcal{D}(A^{*\varepsilon})]')} \\
&= \Big\| \frac{1}{h} L_{s+h} \left(B^* P(\cdot)\Phi(\cdot,s+h)x \cdot \chi_{[s+h,s]}(\cdot) \right) \Big\|_{L^{1/\gamma}(s+h,T;[\mathcal{D}(A^{*\varepsilon})]')} \\
&\le c_T \frac{1}{|h|} \| B^* P(\cdot)\Phi(\cdot,s+h)x \cdot \chi_{[s+h,s]}(\cdot) \|_{L^1(s+h,T;U)} \\
&= c_T \frac{1}{|h|} \| B^* P(\cdot)\Phi(\cdot,s+h)x \|_{L^1(s+h,s;U)} \\
&\le c_T \| B^* P(\cdot)\Phi(\cdot,s+h)x \|_{L^\infty(s+h,T;U)} \le c_T \|x\|_{\mathcal{D}(A^\varepsilon)} .
\end{aligned}
$$

Hence,

$$
\Big\| \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h} [R^* R \Phi(\cdot, s)x] \Big\|_{L^{1/\gamma}(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]')} \le c_T \quad \text{for } h>0, \tag{6.12}
$$

$$
\Big\| \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h} [R^* R \Phi(\cdot, s+h)x] \Big\|_{L^{1/\gamma}(s,T;[\mathcal{D}(A^{*\varepsilon})]')} \le c_T \text{ for } h<0. \tag{6.13}
$$

Now fix $p \in (1, \frac{1}{\gamma})$, so that $p' > \frac{1}{1-\gamma}$, and let $\varphi \in L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))$. If $h>0$ we have

$$
\begin{aligned}
&\Big| \int_{s+\delta}^T \Big\langle - \frac{L_{s+h}L^*_{s+h} - L_s L^*_s}{h} [R^* R \Phi(\cdot, s)x](t) \\
&\qquad -e^{(t-s)A} B B^* P(s)x, \varphi(t) \Big\rangle_{\mathcal{D}(A^{*\varepsilon})} dt \Big| \\
&= \Big| \int_{s+\delta}^T \frac{1}{h} \int_s^{s+h} \Big[(B^* P(\sigma)\Phi(\sigma,s)x, B^* e^{(t-\sigma)A^*}\varphi(t))_U \\
&\qquad -(B^* P(s), B^* e^{(t-s)A^*}\varphi(t))_U \Big] d\sigma dt \Big|
\end{aligned}
$$

$$\leq \Big|\int_{s+\delta}^{T}\frac{1}{h}\int_s^{s+h}(B^*P(\sigma)\Phi(\sigma,s)x - B^*P(s)x, B^*e^{(t-\sigma)A^*}\varphi(t))_U\,d\sigma dt\Big|$$
$$+\Big|\int_{s+\delta}^{T}\frac{1}{h}\int_s^{s+h}(B^*P(s)x, B^*e^{(t-\sigma)A^*}[I-e^{(\sigma-s)A^*}]\varphi(t))_U\,d\sigma\,dt\Big|$$
$$= I_1 + I_2\,.$$

In order to estimate I_1 and I_2, we recall that $\sigma \mapsto B^*P(\sigma)\Phi(\sigma,s)x$ is continuous in $[s,T]$ by Corollary 5.4(ii). Hence by (A.3)

$$\begin{aligned} I_1 &= \mathrm{o}(1)\int_s^T \frac{1}{h}\int_s^{(s+h)\wedge t}\|B^*e^{(t-\sigma)A^*}\varphi(t)\|_U\,d\sigma dt \\ &\leq \mathrm{o}(1)\frac{1}{h}\int_s^{s+h}\int_\sigma^T \frac{c_T}{(t-\sigma)^\gamma}\|\varphi(t)\|_{\mathcal{D}(A^{*\varepsilon})}\,dtd\sigma \\ &\leq \mathrm{o}(1)\frac{c_T}{h}\int_s^{s+h}\Big(\int_\sigma^T\frac{dt}{(t-\sigma)^{\gamma p}}\Big)^{\frac{1}{p}}\|\varphi\|_{L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))}\,d\sigma \\ &\leq \mathrm{o}(1)c_T\|\varphi\|_{L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))} = \mathrm{o}(1) \qquad \text{as } h\to 0^+. \end{aligned}$$

In addition, using again (A.3),

$$I_2 \leq \frac{1}{h}\int_s^{s+h}\int_\sigma^T\frac{c_T}{(t-\sigma)^\gamma}\|[I-e^{(\sigma-s)A^*}]\varphi(t)\|_{\mathcal{D}(A^{*\varepsilon})}\,dt\,d\sigma;$$

now the integrand goes to 0 as $h\to 0^+$, and is dominated by the function

$$c_T(t-\sigma)^{-\gamma}\|\varphi(t)\|_{\mathcal{D}(A^{*\varepsilon})}, \tag{6.14}$$

whose integral with respect to the variables (t,σ) is finite. Thus we conclude that

$$I_1 + I_2 \to 0 \qquad \text{as } h\to 0^+. \tag{6.15}$$

Similarly, for $h<0$ and $\varphi\in L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))$ we have

$$\Big|\int_s^T\Big\langle -\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s+h)x](t)$$
$$-e^{(t-s)A}BB^*P(s)x,\varphi(t)\Big\rangle_{\mathcal{D}(A^{*\varepsilon})}\,dt\Big|$$
$$= \Big|\int_s^T\frac{1}{h}\int_{s+h}^s\Big[(B^*P(\sigma)\Phi(\sigma,s+h)x, B^*e^{(t-\sigma)A^*}\varphi(t))_U$$
$$-(B^*P(s),B^*e^{(t-s)A^*}\varphi(t))_U\Big]d\sigma dt\Big|$$

$$\leq \Big| \int_s^T \frac{1}{h}\int_{s+h}^s (B^*P(\sigma)\Phi(\sigma,s+h)x, B^*e^{(t-\sigma)A^*}(I-e^{\sigma-s)A^*})\varphi(t))_U \, d\sigma dt\Big|$$
$$+\Big| \int_s^T \frac{1}{h}\int_{s+h}^s (B^*P(\sigma)\Phi(\sigma,s+h)x - B^*P(s)x, B^*e^{(t-s)A^*}\varphi(t))_U \, d\sigma dt\Big|$$
$$= J_1 + J_2 \,.$$

We see that, as before,

$$\begin{aligned} J_2 &= \mathrm{o}(1)\int_{s+h}^s \frac{1}{h}\int_s^T \frac{c_T}{(t-s)^\gamma}\|\varphi(t)\|_{\mathcal{D}(A^{*\varepsilon})}\,dt\,d\sigma \\ &\leq \mathrm{o}(1)c_T\|\varphi\|_{L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))} = \mathrm{o}(1) \qquad \text{as } h\to 0^+, \end{aligned}$$

whereas

$$J_1 \leq \frac{1}{h}\int_{s+h}^s\int_s^T \frac{c_T}{(t-\sigma)^\gamma}\|[I-e^{(s-\sigma)A^*}]\varphi(t)\|_{\mathcal{D}(A^{*\varepsilon})}\,dt\,d\sigma;$$

the integrand goes to 0 as $h\to 0^-$, and is dominated by the function (6.14). Thus

$$J_1+J_2\to 0 \qquad \text{as } h\to 0^-. \tag{6.16}$$

Now we recall that, since both $\mathcal{D}(A^\varepsilon)$ and $\mathcal{D}(A^{*\varepsilon})$ are separable (Remark A.1(iv)), the space $L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]')$ is the dual of $L^{p'}(s,T;\mathcal{D}(A^{*\varepsilon}))$; hence by (6.15) and (6.16) we conclude that for each $p\in(1,\frac{1}{\gamma})$

$$\begin{gathered} -\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x] \overset{*}{\rightharpoonup} e^{(\cdot-s)A}BB^*P(s)x \\ \text{as } h\to 0^+ \text{ in } L^p(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]') \quad \forall\delta>0, \end{gathered} \tag{6.17}$$

and

$$\begin{gathered} -\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s+h)x] \overset{*}{\rightharpoonup} e^{(\cdot-s)A}BB^*P(s)x \\ \text{as } h\to 0^- \text{ in } L^p(s,T;[\mathcal{D}(A^{*\varepsilon})]'). \end{gathered} \tag{6.18}$$

Now we want to improve the above convergences up to $p=1/\gamma$. Set for simplicity

$$\varphi_h = -\frac{L_{s+h}L^*_{s+h}-L_sL^*_s}{h}[R^*R\Phi(\cdot,s)x].$$

Choose any sequence $\{h_n\}$ such that $h_n\to 0^+$: then, recalling (6.12), by weak* compactness there are $v\in L^{\frac{1}{\gamma}}(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]')$ and a subsequence $\{h_{n_k}\}\subseteq\{h_n\}$ such that

$$\varphi_{h_{n_k}} \overset{*}{\rightharpoonup} v \quad \text{in } L^{1/\gamma}(s+\delta,T;[\mathcal{D}(A^{*\varepsilon})]')\,.$$

Taking into account (6.17), we also have as $k \to \infty$

$$\varphi_{h_{n_k}} \overset{*}{\rightharpoonup} v \quad \text{in } L^p(s+\delta, T; [\mathcal{D}(A^{*\varepsilon})]'),$$

so that $v(t) = e^{(t-s)A}BB^*P(s)x$ a.e. in $[s+\delta, T]$ as elements of $[\mathcal{D}(A^{*\varepsilon})]'$, and finally

$$e^{(\cdot-s)A}BB^*P(s)x = v \in L^{\frac{1}{\gamma}}(s+\delta, T; [\mathcal{D}(A^{*\varepsilon})]').$$

Moreover, by the arbitrariness of the sequence $\{h_n\}$, we obtain as $h \to 0^+$

$$\varphi_h \overset{*}{\rightharpoonup} e^{(\cdot-s)A}BB^*P(s)x \quad \text{in } L^{\frac{1}{\gamma}}(s+\delta, T; [\mathcal{D}(A^{*\varepsilon})]'). \tag{6.19}$$

Proceeding quite similarly, setting

$$\psi_h = -\frac{L_{s+h}L^*_{s+h} - L_sL^*_s}{h}[R^*R\Phi(\cdot, s+h)x],$$

we find as $h \to 0^-$

$$\psi_h \overset{*}{\rightharpoonup} e^{(\cdot-s)A}BB^*P(s)x \quad \text{in } L^{\frac{1}{\gamma}}(s, T; [\mathcal{D}(A^{*\varepsilon})]'). \tag{6.20}$$

By (6.19) and (6.20) the proof of Lemma 6.3 is complete. □

Step 3. Two auxiliary results. In the evaluation of the limits of the third term in (6.7) and (6.8) we need two lemmas, the first of which improves Proposition 4.3.

Lemma 6.4. *Under Hypotheses* 2.1 *and* 2.2, *let* Λ_s *be the operator introduced in* (4.4). *Then for each* $p \in [2, \infty)$ *there is* $c_{p,T} > 0$ *such that*

$$\|\Lambda_s^{-1}\|_{\mathcal{L}(L^p(s,T;U))} \le c_{p,T}.$$

Proof. We already know, by Remark 4.2, that

$$\|\Lambda_s^{-1}\|_{\mathcal{L}(L^2(s,T;U))} \le 1.$$

Let $v \in L^p(s,T;U)$, $p > 2$; then there exists $u = \Lambda_s^{-1}v \in L^2(s,T;U)$, and moreover $u = v - L_s^*R^*RL_su$. Hence, applying Proposition B.4(ii) and Proposition B.3(ii), we get

$$\begin{aligned}\|u\|_{L^p(s,T;U)} &\le \|v\|_{L^p(s,T;U)} + \|L_s^*R^*RL_su\|_{L^p(s,T;U)}\\ &\le \|v\|_{L^p(s,T;U)} + c_T\|L_su\|_{L^{p_1}(s,T;U)} \le \|v\|_{L^p(s,T;U)} + c_T\|u\|_{L^{p_2}(s,T;U)},\end{aligned}$$

where

$$p_1 = \frac{p}{1+(1-\gamma)p}, \quad p_2 = \begin{cases} \frac{p}{1+2(1-\gamma)p} & \text{if } \frac{p}{1+2(1-\gamma)p} > 1 \\ 1+\delta, \ \delta > 0, & \text{if } \frac{p}{1+2(1-\gamma)p} \le 1. \end{cases} \tag{6.21}$$

If $p_2 \le 2$, we deduce

$$\begin{aligned}\|u\|_{L^p(s,T;U)} &\le \|v\|_{L^p(s,T;U)} + c_T\|u\|_{L^{p_2}(s,T;U)}\\ &\le \|v\|_{L^p(s,T;U)} + c_T\|u\|_{L^2(s,T;U)}\\ &\le c_T\left(\|v\|_{L^p(s,T;U)} + \|v\|_{L^2(s,T;U)}\right) \le c_T\|v\|_{L^p(s,T;U)}\,.\end{aligned}$$

If $p_2 > 2$, we iterate the above argument:

$$\begin{aligned}\|u\|_{L^p(s,T;U)} &\le \|v\|_{L^p(s,T;U)} + c_T\|u\|_{L^{p_2}(s,T;U)}\\ &\le \|v\|_{L^p(s,T;U)} + c_T\left(\|v\|_{L^{p_2}(s,T;U)} + \|L_s^*R^*RL_su\|_{L^{p_2}(s,T;U)}\right)\\ &\le c_T\|v\|_{L^p(s,T;U)} + c_T\|u\|_{L^{p_3}(s,T;U)}\end{aligned}$$

where

$$p_3 = \begin{cases}\frac{p}{1+4(1-\gamma)p} & \text{if } \frac{p}{1+4(1-\gamma)p} > 1\\ 1+\delta,\ \delta>0, & \text{if } \frac{p}{1+4(1-\gamma)p} \le 1.\end{cases}$$

After a finite number of steps, we fall into the first case, and we conclude that

$$\|u\|_{L^p(s,T;U)} \le c_{p,T}\|v\|_{L^p(s,T;U)}\,. \qquad \square$$

Lemma 6.5. *Under Hypotheses* 2.1 *and* 2.2, *let* Λ_s *be the operator introduced in* (4.4). *Then for all* $p \in [2,\infty[$ *and* $\varphi \in L^{r\vee 2}(s,T;U)$, *with* $r = \frac{p}{1+2(1-\gamma)p}$, *we have*

$$\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s+h,T;U)} \to 0 \quad \textit{as } h \to 0^+,$$
$$\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s,T;U)} \to 0 \quad \textit{as } h \to 0^-.$$

Proof. Let $h > 0$. With p_1 and p_2 given by (6.21) we have, using Lemma 6.4, Proposition B.4(ii) and Corollary B.3,

$$\begin{aligned}\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s+h,T;U)} &= \|\Lambda_{s+h}^{-1}[\Lambda_s - \Lambda_{s+h}]\Lambda_s^{-1}\varphi\|_{L^p(s+h,T;U)}\\ &\le c_T\|[\Lambda_s - \Lambda_{s+h}]\Lambda_s^{-1}\varphi\|_{L^p(s+h,T;U)}\\ &= c_T\|L_s^*R^*R(L_s - L_{s+h})\Lambda_s^{-1}\varphi\|_{L^p(s+h,T;U)}\\ &\le c_T\|[L_s - L_{s+h}]\Lambda_s^{-1}\varphi\|_{L^{p_1}(s+h,T;Y)}\\ &= c_T\|L_s[\Lambda_s^{-1}\varphi\cdot\chi_{[s,s+h]}]\|_{L^{p_1}(s+h,T;Y)}\\ &\le c_T\|\Lambda_s^{-1}\varphi\cdot\chi_{[s,s+h]}\|_{L^{p_2}(s+h,T;U)} = c_T\|\Lambda_s^{-1}\varphi\|_{L^{p_2}(s,s+h;U)}\,,\end{aligned}$$

so that

$$\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s+h,T;U)} \to 0 \quad \text{as } h \to 0^+.$$

Let now $h < 0$. Similarly we have

$$\begin{aligned}\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s,T;U)} &= \|\Lambda_s^{-1}[\Lambda_s - \Lambda_{s+h}]\Lambda_{s+h}^{-1}\varphi\|_{L^p(s,T;U)} \\ &\le c_T\|[\Lambda_s - \Lambda_{s+h}]\Lambda_{s+h}^{-1}\varphi\|_{L^p(s,T;U)} \\ &= c_T\|L_s^* R^* R(L_s - L_{s+h})\Lambda_{s+h}^{-1}\varphi\|_{L^p(s,T;U)} \\ &\le c_T\|[L_s - L_{s+h}]\Lambda_{s+h}^{-1}\varphi\|_{L^{p_1}(s,T;Y)} \\ &= c_T\|L_{s+h}[\Lambda_{s+h}^{-1}\varphi\cdot\chi_{[s+h,s]}]\|_{L^{p_1}(s,T;Y)} \\ &\le c_T\|\Lambda_{s+h}^{-1}\varphi\cdot\chi_{[s+h,s]}\|_{L^{p_2}(s+h,T;U)} = c_T\|\Lambda_{s+h}^{-1}\varphi\|_{L^{p_2}(s+h,s;U)}\,,\end{aligned}$$

with p_1 and p_2 as above. If $|h|$ is sufficiently small, so that

$$\|L_{s+h}^* R^* R L_{s+h}\|_{\mathcal{L}(L^{p_2}(s+h,s;U))} \le \frac{1}{2}$$

we get

$$\Lambda_{s+h}^{-1}\varphi = (I + L_{s+h}^* R^* R L_{s+h})^{-1}\varphi = \sum_{k=0}^{\infty}(-1)^k (L_{s+h} R^* R L_{s+h})^k\varphi;$$

hence

$$\|\Lambda_{s+h}^{-1}\varphi\|_{L^{p_2}(s+h,s;U)} \le 2\|\varphi\|_{L^{p_2}(s+h,s;U)}$$

and consequently

$$\|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]\varphi\|_{L^p(s,T;U)} \to 0 \quad \text{as } h \to 0^-.$$

This conclusion concludes the proof of Lemma 6.5. □

Step 4. The limit of the third term in (6.7) and (6.8). For the third term in (6.7) we can write

$$\begin{aligned}&-L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* R^* R\Big[\frac{e^{(\cdot-s-h)A} - e^{(\cdot-s)A}}{h}x\Big](t) - L_s\Lambda_s^{-1}L_s^* R^* R[e^{(\cdot-s)A}Ax](t) \\ &= L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* R^* R\Big[\frac{e^{(\cdot-s-h)A} - e^{(\cdot-s)A}}{-h}\,x - e^{(\cdot-s)A}Ax\Big](t) \\ &\quad + \big[L_{s+h}[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]L_s^* R^* R e^{(\cdot-s)A}Ax\big](t) \\ &\quad + [L_{s+h} - L_s][\Lambda_s^{-1}L_s^* R^* R e^{(\cdot-s)A}Ax](t);\end{aligned}$$

hence, using Proposition B.3(iv), Lemma 6.4, Proposition B.4(iii), and Lemma 6.5,

$$\Big\| - L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* R^* R\Big[\frac{e^{(\cdot-s-h)A} - e^{(\cdot-s)A}}{h}\,x\Big]$$

$$
\begin{aligned}
&\quad -L_s\Lambda_s^{-1}L_s^*R^*R[e^{(\cdot-s)A}Ax]\Big\|_{L^\infty(s+h,T;Y)}\\
&\le \Big\|L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^*R^*R\Big[\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{-h}\,x - e^{(\cdot-s)A}Ax\Big]\Big\|_{L^\infty(s+h,T;Y)}\\
&\quad +\Big\|\Big[L_{s+h}[\Lambda_{s+h}^{-1}-\Lambda_s^{-1}]L_s^*R^*Re^{(\cdot-s)A}Ax\Big]\Big\|_{L^\infty(s+h,T;Y)}\\
&\quad +\Big\|L_s\Big[\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Ax\cdot\chi_{[s,s+h]}\Big]\Big\|_{L^\infty(s+h,T;Y)}\\
&\le c_T\Big\|\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{-h}\,x - e^{(\cdot-s)A}Ax\Big\|_{L^{\frac{1}{1-\gamma}}(s+h,T;Y)}\\
&\quad +c_T\Big\|[\Lambda_{s+h}^{-1}-\Lambda_s^{-1}]L_s^*R^*Re^{(\cdot-s)A}Ax\Big\|_{L^p(s+h,T;U)}\\
&\quad +c_T\Big\|\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Ax\cdot\chi_{[s,s+h]}\Big\|_{L^p(s,T;U)}\to 0\quad\text{as } h\to 0^+,
\end{aligned}
$$

where $p > \frac{1}{1-\gamma}$; thus

$$
\begin{aligned}
&-L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^*R^*R\Big[\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{h}\,x\Big](t)\\
&\quad\to L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Ax\quad\text{in } L^\infty(s+\delta,T;Y)\quad\text{as } h\to 0^+
\end{aligned}
\tag{6.22}
$$

for every $\delta\in(0,T-s)$.

On the other hand, consider the third term in (6.8): for $h<0$ we have, similarly but more easily,

$$
\begin{aligned}
&\Big\|-L_s\Lambda_s^{-1}L_s^*R^*R\Big[\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{h}\,x\Big](t)\\
&\qquad -L_s\Lambda_s^{-1}L_s^*R^*R[e^{(\cdot-s)A}Ax]\Big\|_{L^\infty(s,T;Y)}\\
&=\Big\|L_s\Lambda_s^{-1}L_s^*R^*R\Big[\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{-h}\,x-e^{(\cdot-s)A}Ax\Big]\Big\|_{L^\infty(s,T;Y)}\\
&\le c_T\Big\|\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{-h}\,x-e^{(\cdot-s)A}Ax\Big\|_{L^{\frac{1}{1-\gamma}}(s,T;Y)}\to 0\quad\text{as } h\to 0^+;
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&-L_s\Lambda_s^{-1}L_s^*R^*R\Big[\frac{e^{(\cdot-s-h)A}-e^{(\cdot-s)A}}{h}\,x\Big]\\
&\quad\to L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}Ax\quad\text{in } L^\infty(s,T;Y)\quad\text{as } h\to 0^-.
\end{aligned}
\tag{6.23}
$$

Step 5. The limit of the fourth term in (6.7) and (6.8). Finally we consider the last term in (6.7) and (6.8). Fix $\delta \in (0, T-s)$ and $h \in (0,\delta]$. For each $z \in L^{\frac{1}{1-\gamma}}(s,T;Y)$ we have:

$$
\begin{aligned}
&\Big| \int_{s+\delta}^{T} \Big(L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* R^*R \Big[\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h} R^*R\Phi(\cdot,s)x\Big](t) \\
&\qquad + [L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}BB^*P(s)x](t), z(t)\Big)_Y dt \Big| \\
&\quad \le \Big| \int_{s+\delta}^{T} \Big(\Big[\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h} R^*R\Phi(\cdot,s)x\Big](t), \big[R^*R\big(L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* \\
&\qquad\quad - R^*RL_s\Lambda_s^{-1}L_s^*\big)z\big](t)\Big)_Y dt \Big| \\
&\qquad + \Big| \int_{s+\delta}^{T} \Big(\Big[\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h} R^*R\Phi(\cdot,s)x\Big](t) \\
&\qquad\quad + e^{(t-s)A}BB^*P(s)x, \big[R^*R\left(L_s\Lambda_s^{-1}L_s^*\right)z\big](t)\Big)_Y dt \Big| = T_1 + T_2.
\end{aligned}
$$

Concerning T_1, we note that its first factor is bounded in $L^{\frac{1}{\gamma}}(s,T;[\mathcal{D}(A^{*\varepsilon})]')$ by (6.12); its second factor can be rewritten as

$$
\begin{aligned}
&\big[R^*R\big(L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* - R^*RL_s\Lambda_s^{-1}L_s^*\big)z\big] \\
&\qquad = R^*RL_{s+h}[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]L_s^*z + R^*RL_s\Lambda_s^{-1}L_s^*(z\cdot\chi_{[s,s+h]}),
\end{aligned}
$$

and hence it is estimated, by Proposition B.5, Lemmas 6.5 and 6.4, and Proposition B.4(iii), as follows:

$$
\begin{aligned}
&\big\|\big[R^*R\big(L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^* - R^*RL_s\Lambda_s^{-1}L_s^*\big)z\big]\big\|_{L^{1/(1-\gamma)}(s+\delta,T;\mathcal{D}(A^{*\varepsilon}))} \\
&\quad \le c_T \|[\Lambda_{s+h}^{-1} - \Lambda_s^{-1}]L_s^*z\|_{L^{q'}(s+\delta,T;U)} + c_T\|L_s^*(z\cdot\chi_{[s,s+h]})\|_{L^{q'}(s,T;U)} \\
&\quad \le \mathrm{o}(1) + \|z\|_{L^{1/(1-\gamma)}(s,s+h;Y)} = \mathrm{o}(1) \quad \text{as } h \to 0^+.
\end{aligned}
$$

Thus, we get $T_1 \to 0$ as $h \to 0^+$.

For the term T_2 we simply remark that $R^*RL_s\Lambda_s^{-1}L_s^*z \in C([s+\delta,T],Y)$ by Propositions B.4(iii) and B.5; hence, by the weak* convergence of the first factor to 0 (Lemma 6.3), we get $T_2 \to 0$ as $h \to 0^+$.

Thus, we have obtained, for each $\delta \in (0, T-s)$,

$$
\begin{aligned}
&L_{s+h}\Lambda_{s+h}^{-1}L_{s+h}^*R^*R\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h}[R^*R\Phi(\cdot,s)x] \\
&\qquad \rightharpoonup -e^{(\cdot-s)A}BB^*P(s)x \qquad \text{in } L^{\frac{1}{\gamma}}(s+\delta,T;Y) \quad \text{as } h \to 0^+.
\end{aligned}
\tag{6.24}
$$

On the other hand, for $h < 0$, similarly but more easily we have for each $z \in L^{\frac{1}{1-\gamma}}(s,T;Y)$:

$$\begin{aligned}
&\Big| \int_s^T \Big(L_s\Lambda_s^{-1}L_s^*R^*R\Big[\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h}R^*R\Phi(\cdot,s+h)x\Big](t) \\
&\qquad + \Big[L_s\Lambda_s^{-1}L_s^*R^*Re^{(\cdot-s)A}BB^*P(s)x\Big](t), z(t)\Big)_Y dt\Big| \\
&= \Big| \int_s^T \Big(\Big[\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h}R^*R\Phi(\cdot,s+h)x \\
&\qquad + e^{(\cdot-s)A}BB^*P(s)x\Big](t), [R^*RL_s\Lambda_s^{-1}L_s^*z](t)\Big)_Y dt\Big|,
\end{aligned}$$

and as before, by Lemma 6.3, Propositions B.5 and B.4(iii), and Lemma 6.4, we obtain

$$\begin{aligned}
&L_s\Lambda_s^{-1}L_s^*R^*R\frac{L_{s+h}L_{s+h}^* - L_sL_s^*}{h}[R^*R\Phi(\cdot,s+h)x] \\
&\quad \rightharpoonup -e^{(\cdot-s)A}BB^*P(s)x \qquad \text{in } L^{\frac{1}{\gamma}}(s,T;Y) \quad \text{as } h \to 0^+.
\end{aligned} \tag{6.25}$$

This concludes the proof of Theorem 6.2. □

Appendix A. Basic facts

This Appendix contains some preliminary results which follow directly from Hypotheses 2.1 and 2.2.

Remarks A.1. (i) Since $A^{-1} \in \mathcal{L}(Y, \mathcal{D}(A))$, it is easily seen that $A = A^{**}$. Consequently, the operator $A^{-1}B$ belongs to $\mathcal{L}(U,Y)$, in view of the estimate

$$\begin{aligned}
|(A^{-1}Bu,y)_Y| &= |((A^{**})^{-1}Bu,y)_Y| \\
&= |\langle Bu, A^{*-1}y\rangle_{\mathcal{D}(A^*)}| \le \|B\|_{\mathcal{L}(U,[\mathcal{D}(A^*)]')}\|u\|_U\|y\|_Y
\end{aligned}$$

which holds for all $u \in U$ and $y \in Y$. In particular

$$B^*e^{tA^*} = (A^{-1}B)^*e^{tA^*}A^* \in \mathcal{L}(\mathcal{D}(A^*),U).$$

Hence, $G(t) = B^*e^{tA^*} - F(t)$ is an element of $\mathcal{L}(\mathcal{D}(A^*),U)$ for each $t \in (0,T]$, as claimed in Hypothesis 2.2.

(ii) For each $w \in Y$ define the element $A^\varepsilon w \in [\mathcal{D}(A^{*\varepsilon})]'$ according to the rule

$$\langle A^\varepsilon w, x\rangle_{\mathcal{D}(A^{*\varepsilon})} = (w, A^{*\varepsilon}x)_Y \qquad \forall x \in \mathcal{D}(A^{*\varepsilon});$$

then it is easy to check that the following characterizations hold true:

$$\begin{cases} [\mathcal{D}(A^{*\varepsilon})]' \simeq \{z \in [\mathcal{D}(A^*)]' : z = A^\varepsilon w,\, w \in Y\}, \\ \|z\|_{[\mathcal{D}(A^{*\varepsilon})]'} \simeq \|A^{-\varepsilon} z\|_Y \,, \end{cases} \tag{A.1}$$

$$\begin{cases} [\mathcal{D}(A^{\varepsilon})]' \simeq \{z \in [\mathcal{D}(A)]' : z = A^{*\varepsilon} v,\, v \in Y\}, \\ \|v\|_{[\mathcal{D}(A^{*\varepsilon})]'} \simeq \|A^{*-\varepsilon} z\|_Y \,. \end{cases} \tag{A.2}$$

(iii) Conditions (2.5) and (2.6) are regularity assumptions on $G(\cdot)$: (2.6) says in particular that the norm $\|G(t)\|_{\mathcal{L}(\mathcal{D}(A^{*\varepsilon}),U)}$ is bounded in $[0,T]$.

(iv) The separability of Y implies that both $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are separable spaces with their respective graph norms: indeed, if the sequence $\{x_n\} \subset Y$ is dense in Y, then it is easy to see that $\{n(n-A)^{-1}x_n\}$ is dense in $\mathcal{D}(A)$ and $\{n(n-A^*)^{-1}x_n\}$ is dense in $\mathcal{D}(A^*)$. The same property holds for the spaces $\mathcal{D}(A^\varepsilon)$ and $\mathcal{D}(A^{*\varepsilon})$.

The following lemmas illustrate further important consequences of the assumptions imposed on the model.

Lemma A.2. *Under assumptions* 2.1 *and* 2.2, *we have*

$$\|e^{tA}B\|_{\mathcal{L}(U,\mathcal{D}(A^{*\varepsilon})')} \le c\, t^{-\gamma} \qquad \forall t \in (0,T],$$

and hence

$$\|B^* e^{tA^*}\|_{\mathcal{L}(\mathcal{D}(A^{*\varepsilon}),U)} \le c\, t^{-\gamma} \qquad \forall t \in (0,T]. \tag{A.3}$$

Proof. If $u \in U$, by Remark A.1(i) one has for all $z \in \mathcal{D}(A^{*\varepsilon})$

$$\big|\langle e^{tA}Bu, z\rangle_{\mathcal{D}(A^{*\varepsilon})}\big| \le \|A^{-\varepsilon}e^{tA}Bu\|_Y \|z\|_{\mathcal{D}(A^{*\varepsilon})}\,,$$

and on the other hand, for all $y \in Y$,

$$\begin{aligned} \big|(A^{-\varepsilon}e^{tA}Bu, y)_Y\big| &= \big|(u, B^* e^{tA^*} A^{*-\varepsilon} y)_U\big| \\ &= \big|(u, F(t)A^{*-\varepsilon}y)_U + (u, G(t)A^{*-\varepsilon}y)_U\big| \\ &\le c\|u\|_U \left(t^{-\gamma}\|A^{*-\varepsilon}y\|_Y + \|y\|_Y\right) \le ct^{-\gamma}\|u\|_U\|y\|_Y\,. \end{aligned}$$

Hence,

$$\|A^{-\varepsilon}e^{tA}Bu\|_Y \le ct^{-\gamma}\|u\|_U \qquad \forall u \in U,$$

and the result follows. □

Lemma A.3. *Under assumptions* 2.1 *and* 2.2, *for each* $f \in C([0,T],\mathcal{D}(A^{*\varepsilon}))$ *we have* $G(\cdot)f(\cdot) \in C([0,T],U)$.

Proof. Fix $t_0 \in [0,T]$; then by Hypothesis 2.2(iii)(a) we have

$$G(\cdot)f(t_0) = G(\cdot)A^{*-\varepsilon}[A^{*\varepsilon}f(t_0)] \in C([0,T],U)\,,$$

with

$$\|G(\cdot)f(t_0)\|_{C([0,T],U)} \le c\|f(t_0)\|_{\mathcal{D}(A^{*\varepsilon})} \le c\|f\|_{c([0,T],\mathcal{D}(A^{*\varepsilon})}\,.$$

In particular, evaluating $G(\cdot)f(t_0)$ at $t = t_0$ we get

$$\|G(t_0)f(t_0)\|_U \le c\|f\|_{C([0,T],\mathcal{D}(A^{*\varepsilon}))}\,,$$

and since t_0 is arbitrary we obtain

$$f \in B(0,T;\mathcal{D}(A^{*\varepsilon})) \quad \Longrightarrow \quad G(\cdot)f(\cdot) \in B(0,T;U). \tag{A.4}$$

Let us now prove continuity of $t \mapsto G(t)f(t)$ at t_0. We have

$$\begin{aligned}
&\|G(t)f(t) - G(t_0)f(t_0)\|_U \\
&\quad \le \|G(t)[f(t) - f(t_0)]\|_U + \|[G(t) - G(t_0)]f(t_0)\|_U \\
&\quad \le \|G(\cdot)[f(t) - f(t_0)]\|_{C([0,T],U)} + \|G(t)f(t_0) - G(t_0)f(t_0)\|_U \\
&\quad \le c\|f(t) - f(t_0)\|_{\mathcal{D}(A^{*\varepsilon})} + \|G(t)f(t_0) - G(t_0)f(t_0)\|_U \\
&\quad = \mathrm{o}(1) \quad \text{as } t \to t_0\,. \qquad \square
\end{aligned}$$

Lemma A.4. *Under Hypotheses* 2.1 *and* 2.2, *we have*

$$R^*R \in \mathcal{L}([\mathcal{D}(A^{*\varepsilon})]', [\mathcal{D}(A^{\varepsilon})]')\,.$$

Proof. We start from Remark A.1(ii): if $z \in [\mathcal{D}(A^{*\varepsilon})]'$ and $w = A^{-\varepsilon}z \in Y$, we have

$$\begin{aligned}
\|R^*Rz\|_{[\mathcal{D}(A^{\varepsilon})]'} &= \|A^{*-\varepsilon}R^*Rz\|_Y = \|A^{*-\varepsilon}R^*RA^{\varepsilon}w\|_Y \\
&\le \|A^{*-\varepsilon}R^*RA^{\varepsilon}\|_{\mathcal{L}(Y)}\|w\|_Y = \|[A^{*-\varepsilon}R^*RA^{\varepsilon}]^*\|_{\mathcal{L}(Y)}\|z\|_{[\mathcal{D}(A^{*\varepsilon})]'} \\
&= \|R^*R\|_{\mathcal{L}(\mathcal{D}(A^{\varepsilon}),\mathcal{D}(A^{*\varepsilon}))}\|z\|_{[\mathcal{D}(A^{*\varepsilon})]'}\,. \qquad \square
\end{aligned}$$

Lemma A.5. *Under Hypotheses* 2.1 *and* 2.2, *we have*

$$t \mapsto B^*e^{tA^*}R^*R \in \mathcal{L}([\mathcal{D}(A^{*\varepsilon})]', L^q(0,T;U))\,.$$

Proof. If $z \in [\mathcal{D}(A^{*\varepsilon})]'$ and $w = A^{-\varepsilon}z \in Y$, we have by Hypothesis 2.2(iii)(c)

$$\begin{aligned}
\Big(\int_0^T \|B^*e^{tA^*}R^*Rz\|_U^q\,dt\Big)^{1/q} &= \Big(\int_0^T \|B^*e^{tA^*}R^*RA^{\varepsilon}w\|_U^q\,dt\Big)^{1/q} \\
&\le \|B^*e^{\cdot A^*}R^*RA^{\varepsilon}\|_{\mathcal{L}(Y,L^q(0,T;U))}\|w\|_Y \le c\|z\|_{[\mathcal{D}(A^{*\varepsilon})]'}\,. \qquad \square
\end{aligned}$$

Lemma A.6. *Under Hypotheses* 2.1 *and* 2.2, *we have*

$$G^* \in \mathcal{L}(L^1(0,T;U), [\mathcal{D}(A^{*\varepsilon})]').$$

Proof. If $x \in Y$ we have for all $p \in (1,\infty)$ and $\varphi \in L^{p'}(0,T;U)$:

$$(x, G^*\varphi)_Y = \int_0^T (G(t)x, \varphi(t))_U \, dt.$$

Now if $x \in \mathcal{D}(A^{*\varepsilon})$ and $y = A^{*\varepsilon}x$, by Hypothesis 2.2(iii)(a) we may write for all $\varphi \in L^1(0,T;U)$:

$$\begin{aligned} |(x, G^*\varphi)_Y| &= \Big| \int_0^T (G(t)x, \varphi(t))_U \, dt \Big| \\ &= \Big| \int_0^T (G(t)A^{*-\varepsilon}y, \varphi(t))_U \, dt \Big| = \Big| \int_0^T (y, A^{-\varepsilon}G(t)^*\varphi(t))_U \, dt \Big| \\ &\le c\|y\|_Y \sup_{t\in[0,T]} \|A^{-\varepsilon}G(t)^*\|_{\mathcal{L}(U,Y)} \|\varphi\|_{L^1(0,T;U)} \,. \end{aligned}$$

This shows that $\|G^*\varphi\|_{\mathcal{D}(A^{*\varepsilon})'} \le c\|\varphi\|_{L^1(0,T;U)}$.

APPENDIX B. REGULARITY ANALYSIS OF THE OPERATORS L_s AND L_s^*

This Appendix is focused on the regularity analysis of the operator L_s (and of its adjoint L_s^*) defined by (2.11) ((2.12), respectively). It is important to emphasize that while the basic regularity of L_s and L_s^* is a prerequisite of the whole theory and hence is needed throughout the paper, it will be specifically the conclusion of Proposition B.5 that will play a major role in the proof of boundedness of the gain operator $B^*P(t)$ on $\mathcal{D}(A^\varepsilon)$ (see Proposition 5.3).

In view of the decomposition (2.3), it is natural to rewrite L_s as the sum $L_{s1} + L_{s2}$, where

$$\begin{cases} L_{s1}u(t) := \displaystyle\int_s^t F(t-\sigma)^* u(\sigma)\, d\sigma \,, \\ L_{s2}u(t) := \displaystyle\int_s^t G(t-\sigma)^* u(\sigma)\, d\sigma \,, \end{cases} \qquad u \in L^p(s,T;U)\,. \tag{B.1}$$

Because of the distinct assumptions imposed on $F(t)$ and $G(t)$, a separate analysis of the operators L_{s1} and L_{s2} is necessary. The corresponding boundedness (continuity) results are stated in Proposition B.1 and Proposition B.2 below, respectively. For the sake of brevity, and since the component $F(t)$ of $B^*e^{A^*t}$ satisfies a singular estimate, the proof of Proposition B.1—which pertains to the operator L_{s1}—will be omitted (the reader is referred, e.g., to [21]).

Proposition B.1. *Under Hypotheses* 2.1 *and* 2.2, *let* L_{s1} *be the operator defined by* (B.1). *Then we have*

(i) *if* $1 \le p < \frac{1}{1-\gamma}$, *then* $L_{s1} \in \mathcal{L}(L^p(s,T;U), L^q(s,T;Y))$ *with* $q = \frac{p}{1-(1-\gamma)p}$;

(ii) *if* $p = \frac{1}{1-\gamma}$, *then* $L_{s1} \in \mathcal{L}(L^p(s,T;U), L^q(s,T;Y))$ *for all* $q \in [1,\infty[$;

(iii) *if* $p > \frac{1}{1-\gamma}$, *then* $L_{s1} \in \mathcal{L}(L^p(s,T;U), C([s,T],Y))$.

Moreover, in all cases the norm of L_{s1} *does not depend on* s.

Proposition B.2. *Under Hypotheses* 2.1 *and* 2.2, *let* L_{s2} *be the operator defined by* (B.1). *Then we have*

(i) *if* $p = 1$, *then* $L_{s2} \in \mathcal{L}(L^1(s,T;U), C([s,T],[\mathcal{D}(A^{*\varepsilon})]'))$;

(ii) *if* $p > 1$, *then* $L_{s2} \in \mathcal{L}(L^p(s,T;U), C([s,T],Y))$.

Moreover, in all cases the norm of L_s *does not depend on* s.

Proof. We note that, in principle, $L_{s2}u(t)$ just belongs to $[\mathcal{D}(A^*)]'$ when u is U-valued. Let us start by proving the second assertion. Indeed, for each $\varphi \in L^1(s,T;\mathcal{D}(A^*))$ we have, using Hypothesis 2.2(iii)(a):

$$\begin{aligned}
&\Big| \int_s^T \langle L_{s2}u(t), \varphi(t)\rangle_{\mathcal{D}(A^*)} \Big| \\
&\quad = \Big| \int_s^T \Big\langle \int_s^t G(t-\sigma)^* u(\sigma)d\sigma, \varphi(t) \Big\rangle_{\mathcal{D}(A^*)} dt \Big| \\
&\quad \le \int_s^T \int_s^t |(u(\sigma), G(t-\sigma)\varphi(t))_U| d\sigma dt \\
&\quad \le \int_s^T \|u\|_{L^p(s,t;U)} \|G(t-\cdot)\varphi(t)\|_{L^{p'}(s,t;U)} dt \\
&\quad \le c_{p',T} \int_s^T \|u\|_{L^p(s,t;U)} \|\varphi(t)\|_Y dt \le c_{p',T} \|u\|_{L^p(s,T;U)} \|\varphi\|_{L^1(s,T;Y)}.
\end{aligned}$$

As $L^1(s,T;\mathcal{D}(A^*))$ is dense in $L^1(s,T;Y)$, we deduce $L_{s2}u \in [L^1(s,T;Y)]' = L^\infty(s,T;Y)$, and

$$\|L_{s2}u\|_{L^\infty(s,T;Y)} \le c_{p',T} \|u\|_{L^p(s,T;U)}. \tag{B.2}$$

Next, we claim that $L_{s2}u$ is continuous when u is continuous. Indeed, fix $t_0 \in (s,T]$; if $t \in [s,t_0[$ we have for each $x \in \mathcal{D}(A^*)$:

$$\begin{aligned}
&|\langle L_{s2}u(t) - L_{s2}u(t_0), x\rangle_{\mathcal{D}(A^*)}| = \\
&\quad = \Big| \Big\langle \int_0^{t-s} G(q)^* u(t-q)dq, x \Big\rangle_{\mathcal{D}(A^*)} - \Big\langle \int_0^{t_0-s} G(q)^* u(t_0-q)dq, x \Big\rangle_{\mathcal{D}(A^*)} \Big|
\end{aligned}$$

$$= \Big| \int_0^{t-s} (u(t-q) - u(t_0 - q), G(q)x)_U \; dq \Big|$$
$$+ \Big| \int_{t-s}^{t_0-s} (u(t_0 - q), G(q)x)_U \, dq \Big|$$
$$\le \sup_{|\sigma-\tau|\le|t-t_0|} \|u(\sigma) - u(\tau)\|_U \int_0^{t-s} \|G(q)x\|_U \, dq$$
$$+ \|u\|_{C([s,T],U)} \int_{t-s}^{t_0-s} \|G(q)x\|_U \, dq$$
$$\le c_T \|G(\cdot)x\|_{L^2(s,T;U)} \Big(\sup_{|\sigma-\tau|\le|t-t_0|} \|u(\sigma) - u(\tau)\|_U + \sqrt{t_0 - t} \Big)$$
$$\le c_T \|x\|_Y \Big(\sup_{|\sigma-\tau|\le|t-t_0|} \|u(\sigma) - u(\tau)\|_U + \sqrt{t_0 - t} \Big)$$

and the last member tends to 0 as $t \to t_0^-$. Similarly one gets for $t_0 \in [s, T[$ and $t \in (t_0, T]$

$$|\langle L_{s2}u(t) - L_{s2}u(t_0), x\rangle_{D(A^*)}| \le c_T \|x\|_Y \; \omega(t - t_0)$$

with $\lim_{r\to 0^+} \omega(r) = 0$. By density we deduce

$$\|L_{s2}u(t) - L_{s2}u(t_0)\|_Y \to 0 \qquad \text{as } t \to t_0,$$

so that our claim is proved.

Finally, using (B.2), an easy density argument yields that

$$L_{s2}u \in C([s, T], Y)$$

whenever $u \in L^p(s, T; U)$ with $p > 1$. The second assertion is proved.

The first statement can be proved quite similarly: if $u \in L^1(s, T; U)$, then for each $w \in \mathcal{D}(A^{*\varepsilon})$ we have, using Remark A.1(ii)-(iii):

$$\langle L_{s2}u(t), w\rangle_{\mathcal{D}(A^{*\varepsilon})} = \Big\langle \int_s^t G(t - \sigma)^* u(\sigma) \, d\sigma, w \Big\rangle_{\mathcal{D}(A^{*\varepsilon})}$$
$$= \int_s^t (u(\sigma), (G(t - \sigma)A^{*-\varepsilon})A^{*\varepsilon}w)_U \, d\sigma$$
$$\le \int_s^T \|u(\sigma)\|_U \, d\sigma \sup_{[0,T]} \|G(\cdot)A^{*-\varepsilon}(A^{*\varepsilon}w)\|_U \le c\|u\|_{L^1(s,T;U)} \|w\|_{\mathcal{D}(A^{*\varepsilon})}.$$

The continuity of $L_{s2}u$ follows as before when $u \in C([s, T], U)$. □

Summing up, we obtain the following regularity statements about the operator L_s.

Proposition B.3. *Under Hypotheses* 2.1 *and* 2.2, *let* L_s *be the operator defined by* (2.11). *We have*

(i) *if* $p = 1$, *then* $L_s \in \mathcal{L}(L^1(s,T;U), L^{\frac{1}{\gamma}}(s,T;\mathcal{D}(A^{*\varepsilon})'))$;
(ii) *if* $1 < p < \frac{1}{1-\gamma}$, *then* $L_s \in \mathcal{L}(L^p(s,T;U), L^q(s,T;Y))$ *with* $q = \frac{p}{1-(1-\gamma)p}$;
(iii) *if* $p = \frac{1}{1-\gamma}$, *then* $L_s \in \mathcal{L}(L^p(s,T;U), L^q(s,T;Y))$ *for all* $q \in [1,\infty[$;
(iv) *if* $p > \frac{1}{1-\gamma}$, *then* $L_s \in \mathcal{L}(L^p(s,T;U), C([s,T],Y))$.

Moreover, in all cases the norm of L_s *does not depend on* s.

Proof. Every statement follows combining the results of Propositions B.1 and B.2. □

Regarding the adjoint operator L_s^*, the following boundedness results are valid.

Proposition B.4. *Under Hypotheses* 2.1 *and* 2.2, *let* L_s *be the operator defined by* (2.11). *For its adjoint* L_s^*, *defined by* (2.12), *we have*

(i) *if* $p = 1$, *then* $L_s^* \in \mathcal{L}(L^1(s,T;Y), L^r(s,T;U))$ *for all* $r \in [1, \frac{1}{\gamma})$;
(ii) *if* $p \in (1, \frac{1}{1-\gamma})$, *then* $L_s^* \in \mathcal{L}(L^p(s,T;Y), L^r(s,T;U))$ *with* $r = \frac{p}{1-(1-\gamma)p}$;
(iii) *if* $p \geq \frac{1}{1-\gamma}$, *then* $L_s^* \in \mathcal{L}(L^p(s,T;Y), L^r(s,T;U))$ *for all* $r \in [1,\infty)$;
(iv) *if* $p = \frac{1}{1-\gamma}$, *then* $L_s^* \in \mathcal{L}(L^{\frac{1}{1-\gamma}}(s,T;\mathcal{D}(A^{*\varepsilon})), C([s,T],U))$.

Moreover, in all cases the norm of L_s *does not depend on* s.

Proof. It follows by duality, using Proposition B.3. Namely:

(i) Fix $r \in (1, \frac{1}{\gamma})$ and let $y \in L^1(s,T;Y)$. As $r' > \frac{1}{1-\gamma}$, for every $v \in L^{r'}(s,T;U)$ we can write, by Corollary B.3(iii),

$$\Big|\int_s^T (L_s^* y(t), v(t))_U dt\Big| = \Big|\int_s^T (y(t), L_s v(t))_Y dt\Big| \leq \|y\|_{L^1(s,T;Y)} \|L_s v\|_{L^\infty(s,T;Y)} \leq c_{r,T} \|y\|_{L^1(s,T;Y)} \|v\|_{L^{r'}(s,T;Y)},$$

so that $L_s^* y \in L^r(s,T;U)$, and

$$\|L_s^* y\|_{L^r(s,T;U)} \leq c_{q,T} \|y\|_{L^1(s,T;Y)}.$$

(ii) Now let $p \in (1, \frac{1}{1-\gamma})$ and $y \in L^p(s,T;Y)$. Then $p' > \frac{1}{\gamma}$ and defining $r = (\frac{p}{1-(1-\gamma)p})' = \frac{p'}{1+(1-\gamma)p'}$ we have $r \in (1, \frac{1}{1-\gamma})$ and $p' = \frac{r}{1-(1-\gamma)r}$; thus, as before, for every $v \in L^r(s,T;U)$ we get by Corollary B.3(ii)

$$\Big| \int_s^T (L_s^* y(t), v(t))_U dt \Big| = \Big| \int_s^T (y(t), L_s v(t))_Y dt \Big| \\ \le \|y\|_{L^p(s,T;Y)} \|L_s v\|_{L^{p'}(s,T;Y)} \le c_{p,T} \|y\|_{L^p(s,T;Y)} \|v\|_{L^r(s,T;Y)},$$

and consequently $L_s^* y \in L^{r'}(s,T;Y)$ and

$$\|L_s^* y\|_{L^{r'}(s,T;U)} \le c_{p,T} \|y\|_{L^p(s,T;Y)}.$$

As $r' = \frac{p}{1-(1-\gamma)p}$, we obtain the desired result.

(iii) Fix $r \in (1,\infty)$. For each $v \in L^{r'}(s,T;U)$, since

$$L_s v \in L^{\frac{r'}{1-(1-\gamma)r'}}(s,T;Y) \quad \text{and} \quad \Big(\frac{r'}{1-(1-\gamma)r'}\Big)' = \frac{r}{1+(1-\gamma)r},$$

we have similarly:

$$\begin{aligned} \Big| \int_s^T (L_s^* y(t), v(t))_U dt \Big| &\le \|y\|_{L^{\frac{r}{1+(1-\gamma)r}}(s,T;Y)} \|L_s v\|_{L^{\frac{r'}{1-(1-\gamma)r'}}(s,T;Y)} \\ &\le c_{r,T} \|y\|_{L^{\frac{r}{1+(1-\gamma)r}}(s,T;Y)} \|v\|_{L^{r'}(s,T;U)}. \end{aligned}$$

Hence, $L_s^* v \in L^r(s,T;U)$.

(iv) Set $p_0 = \frac{1}{1-\gamma}$. Let $p \ge p_0$ and $y \in L^p(s,T;\mathcal{D}(A^{*\varepsilon}))$: noting that $p_0' = \frac{1}{\gamma} < \frac{1}{1-\gamma} = p_0$ (since $\gamma > \frac{1}{2}$), as before, we get, by Corollary B.3(i),

$$\Big| \int_s^T (L_s^* y(t), v(t))_U dt \Big| = \Big| \int_s^T (y(t), L_s v(t))_Y dt \Big| \\ \le \|y\|_{L^{p_0}(s,T;\mathcal{D}(A^{*\varepsilon}))} \|L_s v\|_{L^{p_0'}(s,T;\mathcal{D}(A^{*\varepsilon})')} \\ \le c_{p_0,T} \|y\|_{L^{p_0}(s,T;\mathcal{D}(A^{*\varepsilon}))} \|v\|_{L^1(s,T;Y)};$$

i.e., $L_s^* y \in L^\infty(s,T;Y)$ and

$$\|L_s^* y\|_{L^\infty(s,T;Y)} \le c_T \|y\|_{L^{p_0}(s,T;\mathcal{D}(A^{*\varepsilon}))} \le c_T \|y\|_{L^p(s,T;\mathcal{D}(A^{*\varepsilon}))}.$$

It remains to show that $L_s^* y$ is continuous when $y \in L^{p_0}(s,T;\mathcal{D}(A^{*\varepsilon}))$. This will be a consequence of the following properties:

(a) $L_s^* y \in C([s,T],Y)$ when y is a Y-valued step function;

(b) the Y-valued step functions are dense in $L^{p_0}(s,T;Y)$.

Property (b) is well known; we just have to prove (a). Let $s = t_0 < t_1 < \ldots < t_m = T$, set $I_k = [t_{k-1}, t_k[$ and define

$$y(t) = \sum_{k=1}^{m} v_k \chi_{I_k}(t), \qquad v_1, \ldots, v_m \in Y.$$

Then for $t \in [s, T]$ we have

$$\begin{aligned} L_s^* y(t) &= \sum_{k=1}^{m} \int_t^T B^* e^{(\tau - t)A^*} v_k \chi_{I_k}(\tau)\, d\tau \\ &= \sum_{k=1}^{m} \int_0^{T-t} (F(\sigma) + G(\sigma)) v_k \chi_{I_k}(\sigma + t)\, d\sigma \\ &= \sum_{k=1}^{m} \int_{0 \vee (t_{k-1} - t)}^{0 \vee (t_k - t)} (F(\sigma) + G(\sigma)) v_k \, d\sigma. \end{aligned}$$

By Hypothesis 2.2(i)-(ii), $\sigma \mapsto G(\sigma) v_k$ and $\sigma \mapsto F(\sigma) v_k$ are integrable functions in $[0, T]$; hence $L_s^* y$ is a finite sum of continuous functions. This proves (a) and completes the proof. □

Proposition B.5. *Under Hypotheses* 2.1 *and* 2.2, *let* L_s *be the operator defined by* (2.11). *Then we have* $R^* R L_s \in \mathcal{L}(L^{q'}(s, T; U), C([s, T], \mathcal{D}(A^{*\varepsilon})))$.

Proof. Let $u \in L^{q'}(s, T; U)$: we first show that $R^* R L_s u \in L^\infty(s, T; \mathcal{D}(A^{*\varepsilon}))$. Indeed, for all $z \in \mathcal{D}(A^\varepsilon)$ we have by Hypothesis 2.2(iii)(c)

$$\begin{aligned} &|(R^* R L_s u(t), A^\varepsilon z)_Y| = \left| \int_s^t (u(\sigma), B^* e^{(t-\sigma)A^*} R^* R A^\varepsilon z)_U \, d\sigma \right| \\ &\le \|u\|_{L^{q'}(s,T;U)} \|B^* e^{\cdot A^*} R^* R A^\varepsilon\|_{\mathcal{L}(Y, L^q(0,T;U))} \|z\|_Y \le c \|u\|_{L^{q'}(s,T;U)} \|z\|_Y \,, \end{aligned}$$

so that $A^{*\varepsilon} R^* R L_s u \in L^\infty(s, T; U)$ and

$$\|R^* R L_s u\|_{L^\infty(s,T;\mathcal{D}(A^{*\varepsilon}))} \le c \|u\|_{L^{q'}(s,T;U)} \,.$$

In order to prove that $R^* R\, L_s u \in C([s, T], \mathcal{D}(A^{*\varepsilon}))$ for all $u \in L^{q'}(s, T; U)$, it is enough to show that this result holds when u belongs to the space $Z = \{u \in H^{1,p}(s, T; U) : u(s) = 0\}$, $p > \frac{1}{1-\gamma}$, which is dense in $L^{q'}(s, T; U)$. Thus, fix $p > \frac{1}{1-\gamma}$ and let $u \in H^{1,p}(s, T; U)$ be such that $u(s) = 0$: then we can write, integrating by parts,

$$R^* R L_s u(t) = R^* R \int_s^t e^{(t-\sigma)A} B u(\sigma)\, d\sigma$$

$$= R^*R \int_s^t \Big(-\frac{d}{d\sigma} e^{(t-\sigma)A} A^{-1} \Big) Bu(\sigma)\, d\sigma \tag{B.3}$$
$$= -R^*RA^{-1}Bu(t) + R^*RA^{-1}L_s u'(t).$$

Now we have

$$R^*RA^{-1}L_s u' \in C([s,T], \mathcal{D}(A^{*\varepsilon})), \tag{B.4}$$

since $L_s u' \in C([s,T],Y)$ by Corollary B.3(iv) and $R^*RA^{-1} \in \mathcal{L}(Y, \mathcal{D}(A^{*\varepsilon}))$; hence we have only to verify that

$$R^*RA^{-1}Bu \in C([s,T], \mathcal{D}(A^{*\varepsilon})). \tag{B.5}$$

Indeed, we have for $s \le \tau < t \le T$ and for all $z \in \mathcal{D}(A^\varepsilon)$:

$$\begin{aligned}
&(A^{*\varepsilon}R^*RA^{-1}Bu(t) - A^{*\varepsilon}R^*RA^{-1}Bu(\tau), z)_Y \\
&\quad = (u(t)-u(\tau), B^*A^{*-1}R^*RA^\varepsilon z)_U \\
&\quad = \Big(u(t)-u(\tau), B^*e^{(t-\tau)A^*}A^{*-1}R^*A^\varepsilon z - \int_0^{t-\tau} B^*e^{\sigma A^*}R^*RA^\varepsilon z\, d\sigma\Big)_U;
\end{aligned}$$

hence, using Hypothesis 2.2(i)-(iii), and the fact that u is $\frac{1}{p'}$-Hölder continuous,

$$\begin{aligned}
&\Big|(A^{*\varepsilon}R^*RA^{-1}Bu(t) - A^{*\varepsilon}R^*RA^{-1}Bu(\tau), z)_Y\Big| \\
&\qquad \le \|u(t)-u(\tau)\|_U \|F(t-\tau)A^{*-1}R^*RA^\varepsilon z\|_U \\
&\qquad\quad + \|u(t)-u(\tau)\|_U \|G(t-\tau)A^{*-1}R^*RA^\varepsilon z\|_U \\
&\qquad\quad + \|u(t)-u(\tau)\|_U \int_0^{t-\tau} \|B^*e^{\sigma A^*}R^*RA^\varepsilon z\|_U\, d\sigma \\
&\qquad \le c(t-\tau)^{\frac{1}{p'}-\gamma}\|z\|_Y + c(t-\tau)^{\frac{1}{p'}}\|z\|_Y + c(t-\tau)^{\frac{1}{p'}+\frac{1}{q'}}\|z\|_Y\,.
\end{aligned}$$

This proves (B.5). By (B.3), (B.4) and the above remarks, the result follows. □

Remark B.6. The same proof shows that, for fixed s, t with $0 \le s < t \le T$, one has

$$\|R^*Re^{\cdot A}L_s u(t)\|_{C([0,T-t+s],\mathcal{D}(A^{*\varepsilon}))} \le c\|u\|_{L^{q'}(s,t;U)} \qquad \forall u \in L^{q'}(s,t;U).$$

Corollary B.7. *Under Hypotheses* 2.1 *and* 2.2, *let* L_s *be the operator defined by* (2.11). *Then for its adjoint* L_s^*, *defined by* (2.12), *we have*

$$L_s^*R^*R \in \mathcal{L}(L^1(s,T;[\mathcal{D}(A^{*\varepsilon})]'), L^q(s,T;U))\,.$$

Proof. It follows easily, by duality, from Proposition B.5. □

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