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CLASSICAL SOLUTIONS TO PARABOLIC SYSTEMS WITH FREE BOUNDARY OF STEFAN TYPE

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Abstract. Motivated by the classical model for the binary alloy solidification (crystallization) problem, we show the local in time existence and uniqueness of solutions to a parabolic system strongly coupled through free boundary conditions of Stefan type. Using a modification of the standard change of variables method and coercive estimates in a weighted Hölder space (the weight being a power of t) we obtain solutions with maximal global regularity (having at least equal regularity for $t > 0$ as at the initial moment).

1. Introduction

The classical Stefan problem is a simplified model for the solid-liquid phase transition in a pure material taking into account the heat diffusion in each phase with latent heat at the sharp transition interface, which is supposed kept at a given constant temperature [19, 16]. The melting or the crystallization of a two-component material, like a binary alloy, differs in an essential way from that free boundary model problem in several aspects. Firstly, the temperature of the mixture at the interface is not constant and depends on the relative concentration of each component, since each one having a different melting temperature determines the former through a thermodynamic phase diagram. Secondly, in a mixture of two components, one constituent is allowed to diffuse in the interior of each phase and its concentration exhibits a discontinuity across the solid-liquid interface.

Although qualitatively there is a fairly good understanding of these phenomena, there are several physical approaches to its modelling. For instance,

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using the theory of nonequilibrium thermodynamics a mathematical analysis has been proposed in terms of weak formulations (see [23], Chapter V and its references). As in the simpler case of the Stefan problem (see [16] and its bibliography) these generalized solutions admit the degeneracy of the interface into a mushy zone, giving place, even in a one-dimensional alloy problem, to non-uniqueness results as described in [11].

In this work we are interested in the mathematical analysis of the multi-dimensional case of the sharp interface model for the temperature-concentration system with a smooth free boundary arising from a phenomenological model suggested first in [19]. We shall show that, similarly to the classical two-phase Stefan problem for one equation (see [15, 16] or [22]), the corresponding classical free boundary problem for the coupled system, under certain non-degeneracy conditions, is well posed locally in time. As far as we are aware, this problem has been considered previously only from a numerical point of view by several authors (see [10, 4, 1, 20], for instance) without any rigorous mathematical analysis background, except a local existence result in the one-dimensional case in [17].

Nowadays there are several methods to obtain classical solutions to free boundary problems of parabolic type, but here we follow the approach of Solonnikov [22]. This method uses a suitable modification of a standard change of variables, considered by Hanzawa for the one-phase Stefan problem, that allows the transformation of the free boundary problem into an equivalent highly nonlinear parabolic problem in a fixed known domain. This method, and in particular that modification, is very useful for keeping the solution locally in time without loss of smoothness with respect to the regularity of the initial conditions. For the transformed problem in the fixed domain, a linearization procedure combined with sharp (coercive) estimates for solutions of the linear problem permits us to obtain local (in time) solutions by the contraction mapping principle. For parabolic equations with free boundaries of two-phase Stefan and Muskat–Verigin type this method has been developed in [9], where rigorous proofs of the classical solvability of the corresponding problems were given keeping the maximal regularity of the solutions.

An important tool, that was also used previously in [6, 8], with particular geometries of the domains, is the use of coercive estimates for the linear parabolic problems in weighted Hölder spaces, where the weight is a power of t. This allows us to reduce the initial compatibility conditions to a minimum. Although the second-order parabolic theory is now well understood in space-time weighted Hölder spaces (see [14], for instance), we use here a special class of weight introduced in [3] and studied in [21]. For our system, the transformed problem is the type of nonlinear system, of parabolic type with nonstandard transmission conditions at the initial interface, which has been treated with precise estimates for the solutions to the corresponding equations of second order in [5, 7]. We notice that this approach can also be extended to other problems like the ones considered in [18], yielding improvements in the assumptions on the initial data.

In this paper, after introducing the precise formulation of the free boundary problem for the coupled parabolic linear system of second order, in Section 2, we state the (local in time) existence and uniqueness results in weighted Hölder spaces, including, in particular, the local solvability in classical Hölder spaces under more restrictive assumptions on the initial data. In Section 3 we reduce the problem to a nonstandard transmission problem for a nonlinear parabolic system in the fixed known domain by means of a suitable transformation of variables, that reduces the free boundary to the initial given interface, together with a translation of unknown functions in order to work with functions with zero initial conditions. In Section 4 we show the existence and uniqueness of the solution of the linearized problem, for which we state precise estimates that are based in a model transmission problem that is solved in Appendix B. The application of the contraction mapping principle to the nonlinear problem is done in Section 5 and it is based on explicit estimates of the inverse Jacobian matrix of the domain transformation, which are shown in Appendix A.

2. Statement of the problem and results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$. Suppose there is a closed surface $\gamma(t)$ in Ω, $0 \leq t \leq T$, dividing Ω into two subdomains $\Omega_1(t)$ and $\Omega_2(t)$ such that $\partial\Omega_1(t) = \partial\Omega \cup \gamma(t)$, $\partial\Omega_2(t) = \gamma(t)$. At the initial time, $\gamma(0) = \Gamma$ and $\Omega_j(0) = \Omega_j$, $j = 1, 2$. We assume that the subdomains Ω_1 and Ω_2 are not degenerate, for instance, by assuming the smooth initial interface Γ satisfies a uniform ball property from both sides. Let

$$Q_{jT} = \Big\{(x,t)\colon\ x \in \Omega_j(t),\ t \in (0,T)\Big\}, \quad \Omega_{jT} = \Omega_j \times (0,T)\,, \qquad j = 1,2\,,$$
$$\Omega_T = \Omega \times (0,T)\,, \qquad \Sigma_T = \partial\Omega \times [0,T]\,, \qquad \Gamma_T = \Gamma \times [0,T]\,.$$

We consider a multidimensional two phases problem with unknown functions $u_j(x,t)$, $c_j(x,t)$, $j = 1, 2$, and a common free boundary $\gamma(t)$ satisfying the diffusion equations for $j = 1, 2$, ($\partial_t = \partial/\partial t$ and $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$)

$$\partial_t u_j - a_j \, \Delta u_j = 0 \quad \text{in} \;\; Q_{jT} \,, \tag{2.1}$$

$$\partial_t c_j - b_j \, \Delta c_j = 0 \quad \text{in} \;\; Q_{jT} \,, \tag{2.2}$$

with initial and boundary conditions

$$\gamma(t)\big|_{t=0} = \Gamma \,, \tag{2.3}$$

$$u_j\big|_{t=0} = u_{0j}(x), \quad c_j\big|_{t=0} = c_{0j}(x) \quad \text{in} \;\; \Omega_j, \quad j = 1, 2 \,, \tag{2.4}$$

$$u_1\big|_{\partial\Omega} = p(x,t), \quad c_1\big|_{\partial\Omega} = q(x,t) \,, \quad t \in (0,T) \,, \tag{2.5}$$

and the following conditions on the free boundary $\gamma(t)$, $t \in (0,T)$,

$$u_1 = u_2 \,, \quad c_j = \sigma_j(u_j), \quad j = 1, 2 \,, \tag{2.6}$$

$$\lambda_1 \, \partial_\nu u_1 - \lambda_2 \, \partial_\nu u_2 = -\kappa \, V_\nu \,, \tag{2.7}$$

$$k_1 \, \partial_\nu c_1 - k_2 \, \partial_\nu c_2 = -(c_1 - c_2) \, V_\nu \,. \tag{2.8}$$

Here κ, a_j, b_j, λ_j, k_j, $j = 1, 2$, are positive constants, $\nu(x,t)$ is the normal vector to $\gamma(t)$ directed to $\Omega_2(t)$, ∂_ν denotes the normal derivative, V_ν the normal velocity of $\gamma(t)$ and $\sigma_2 \geq \sigma_1$ are continuous functions given as in Figure 1.

This problem may describe the melting or the crystallization of a two-component system. The domain $\Omega_1(t)$ is occupied by the liquid phase with the temperature $u_1(x,t)$ and the concentration of the mixture $c_1(x,t)$; $u_2(x,t)$, $c_2(x,t)$ are the temperature and concentration of the mixture in the solid phase occupying $\Omega_2(t)$; $\gamma(t)$ is the free boundary separating the liquid and solid phases. The phase transition is described by the Stefan condition (2.7) with constant latent heat $\kappa > 0$.

On the free boundary $\gamma(t)$ the temperature u is continuous, but the concentration c is discontinuous and determined by a phase equilibrium diagram of the type shown in Figure 1, implying the jump $[\![c]\!]_{\gamma(t)} = (c_2 - c_1)\big|_{\gamma(t)} > 0$. The differential equation (2.8) represents the mass balance of the mixture and may be regarded as a Stefan-type condition with variable "latent heat".

Here the functions $\sigma_1(u)$, $\sigma_2(u)$ are defined on the interval $U = (u^*, u^{**})$, with $u^* < u^{**}$, and their graphs represent the *liquidus* and *solidus* lines, respectively, in the temperature-concentration plane.

In the classical Stefan problem for a monocomponent material, in addition to (2.7) we have the following condition on the free boundary $\gamma(t)$:

$$u_1 = u_2 = \theta \,,$$

where θ is the phase transition temperature, which is given by a known constant.

The Stefan problem describes physically the equilibrium process of melting or solidification of a pure substance and in each phase it holds always that

$$u_1 \geq \theta \quad \text{in } \Omega_1(t)\,, \qquad u_2 \leq \theta \quad \text{in } \Omega_2(t)\,. \tag{2.9}$$

The phase transition process described by problem (2.1)–(2.8) is not in equilibrium, as in the Stefan problem, and the melting (crystallization) temperature θ is a priori unknown and condition (2.9) does not hold in general. The presence of the unknown concentration c and unknown phase transition temperature on $\gamma(t)$ makes the problem a much more complex and difficult one.

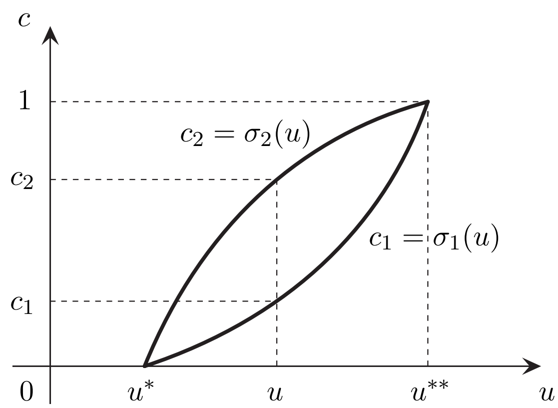

Figure 1

Let $\Gamma \in C^{2+\alpha}$, $N(\xi) = (N_1, ..., N_n)$ be a unit vector field on Γ such that $N(\xi) \in C^{2+\alpha}(\Gamma)$ and

$$\nu_0(\xi) \cdot N(\xi) = \nu_0(\xi)\, N^T(\xi) \geq d_1 > 0 \qquad \forall\, \xi \in \Gamma\,, \tag{2.10}$$

where $\nu_0(\xi)$ is the unit normal to Γ directed to Ω_2, N^T is a column vector. Here and in the future by d_k we denote positive constants.

For small $t \leq t_0$ we can represent the free boundary $\gamma(t)$ in the form ([9])

$$x = \xi + \rho(\xi, t)\, N(\xi)\,, \qquad \xi \in \Gamma, \quad t \in [0, t_0]\,, \tag{2.11}$$

where $\rho\big|_{t=0} = 0$ and $x = \xi$ at the initial moment. In particular, if $N(\xi) = \nu_0(\xi)$ we obtain the following equation to the free boundary:

$$x = \xi + \rho(\xi, t)\, \nu_0(\xi)\,, \qquad \xi \in \Gamma, \quad t \in [0, t_0]\,,$$

which is the representation of $\gamma(t)$ used earlier by A.M. Meirmanov [15] and E.I. Hanzawa [12].

Let the following assumptions hold:

A) $\sigma_j(u) \in C^5(\overline{U})$, $j=1,2$, σ_j are strictly increasing functions in $U=(u^*,u^{**})$, $\sigma_j(u^*)=0$, $\sigma_j(u^{**})=1$, $j=1,2$, $\sigma_1(u)<\sigma_2(u)$ for all $u\in U$ (see Figure 1);

B)

$$\Big(c_{02}(x)-c_{01}(x)\Big)\Big|_{\Gamma}\geq d_2>0\,, \tag{2.12}$$

$$\Big(\partial_{\nu_0}c_{0j}(x)-\sigma_j'(u_{0j}(x))\,\partial_{\nu_0}u_{0j}(x)\Big)\Big|_{\Gamma}\geq d_2\,,\qquad j=1,2\,, \tag{2.13}$$

$$0\leq\kappa\leq\delta_0\,,\qquad\Big|\nabla u_{01}(x)-\nabla u_{02}(x)\Big|\Big|_{\Gamma}\leq\delta_0\,,\qquad|\lambda_1-\lambda_2|\leq\delta_0\,, \tag{2.14}$$

where δ_0 is a small positive number;

C) $$\delta_j\leq c_{0j}(x)\leq 1-\delta_j\quad\text{ in }\ \Omega_j,\ \ 0<\delta_j<\frac{1}{2},\ \ j=1,2\,.$$

Now we formulate the local existence results for problem (2.1)–(2.8) in the weighted Hölder spaces $C_s^\ell(\Omega_T)$, $s\leq\ell$, ℓ a positive non-integer (with a power of t as the weight), of functions $u(x,t)$ with the norm ([3, 21]):

$$|u|_{s,\Omega_T}^{(\ell)}=\sup_{t\leq T}t^{\frac{\ell-s}{2}}[u]_{\Omega_t'}^{(\ell)}+\sum_{s<2k+|m|<\ell}\sup_{t\leq T}t^{\frac{2k+|m|-s}{2}}|\partial_t^k\partial_x^m u|_\Omega + \begin{cases}|u|_{\Omega_T}^{(s)}, & s\geq 0,\\ 0, & s<0\,,\end{cases} \tag{2.15}$$

where $\Omega_t'=\Omega\times(\frac{t}{2},t)$, $|v|_\Omega=\sup_{x\in\Omega}|v|$,

$$[u]_{\Omega_t}^{(\ell)}=\sum_{2k+|m|=[\ell]}[\partial_t^k\partial_x^m u]_{x,\Omega_t}^{(\ell-[\ell])}+\sum_{0<\ell-2k-|m|<2}[\partial_t^k\partial_x^m u]_{t,\Omega_t}^{\left(\frac{\ell-2k-|m|}{2}\right)}\,,$$

$$[v]_{x,\Omega_T}^{(\alpha)}=\sup_{(x,t),(z,t)\in\Omega_T}\Big|v(x,t)-v(z,t)\Big|\,|x-z|^{-\alpha}\,,$$

$$[v]_{t,\Omega_T}^{(\alpha)}=\sup_{(x,t),(x,t_1)\in\Omega_T}\Big|v(x,t)-v(x,t_1)\Big|\,|t-t_1|^{-\alpha}\,,\qquad\alpha\in(0,1)\,,$$

$|u|_{\Omega_T}^{(s)}$ denotes the norm of the classical Hölder space $C_x^{s,s/2}{}_t(\overline{\Omega}_T)$ ([13])

$$|u|_{\Omega_T}^{(s)}=\sum_{2k+|m|\leq[s]}|\partial_t^k\partial_x^m u|_{\Omega_T}+\begin{cases}0, & s\text{ an integer},\\ [u]_{\Omega_T}^{(s)}, & s\text{ not an integer}\,,\end{cases}$$

$$|v|_{\Omega_T} = \sup_{(x,t)\in\Omega_T} |v|;$$

and $C_s^\ell(\Omega_T) \equiv C_{x\ \ t}^{\ell,\ell/2}(\overline{\Omega}_T)$, if $s = \ell$. Similarly we define $C_s^\ell(\Gamma_T)$.

The solution to the problem in this space permits us to decrease the smoothness of the data and to reduce the order of their compatibility conditions up to $[\frac{s}{2}]$.

We write now the compatibility conditions that the data must satisfy at initial time.

The conditions of the zero order (for $0 < s < 1$) read

$$u_{01}\big|_{\partial\Omega} = p(x,0)\,, \qquad c_{01}\big|_{\partial\Omega} = q(x,0)\,, \tag{2.16}$$

$$u_{01}\big|_{\Gamma} = u_{02}\big|_{\Gamma}\,, \qquad c_{0j}\big|_{\Gamma} = \sigma_j(u_{0j})\big|_{\Gamma}, \quad j = 1,2\,, \tag{2.17}$$

and (for $1 \le s < 2$), with the notation $[\![\lambda\,\partial_{\nu_0} u_0]\!] = \lambda_2\,\partial_{\nu_0} u_{02} - \lambda_1\,\partial_{\nu_0} u_{01}$,

$$\frac{1}{\kappa}\,[\![\lambda\,\partial_{\nu_0} u_0]\!]\,\Big|_{\Gamma} = -\frac{1}{[\![c_0]\!]\big|_{\Gamma}}\,[\![k\,\partial_{\nu_0} c_0]\!]\,\Big|_{\Gamma}\,; \tag{2.18}$$

the conditions of the first order (for $2 \le s \le 2+\alpha$) are given by:

$$a_1\Delta u_{01}\big|_{\partial\Omega} = \partial_t p(x,0)\,, \qquad b_1\Delta c_{01}\big|_{\partial\Omega} = \partial_t q(x,0)\,, \tag{2.19}$$

$$\Big([\![a\,\Delta u_0]\!] - \frac{1}{[\![c_0]\!]}\,[\![\partial_{\nu_0} u_0]\!]\,[\![k\,\partial_{\nu_0} c_0]\!]\Big)\,\Big|_{\Gamma} = 0\,, \tag{2.20}$$

$$\Big(b_j\,\Delta c_{0j} - a_j\,\sigma_j'(u_{0j})\,\Delta u_{0j} - \frac{1}{[\![c_0]\!]}\,[\![k\,\partial_{\nu_0} c_0]\!]\,\big(\partial_{\nu_0} c_{0j} - \sigma_j'(u_{0j})\,\partial_{\nu_0} u_{0j}\big)\Big)\,\Big|_{\Gamma} = 0\,, \tag{2.21}$$

for $j = 1,2$.

Equations (2.16)–(2.18) are obtained from (2.5)–(2.8) with the initial conditions (2.4); differentiating the conditions (2.5),(2.6) with respect to t and applying (2.1), (2.2), (2.8), (2.4) leads to the compatibility conditions (2.19)–(2.21).

Theorem 2.1. *Let $\alpha \in (0,1)$, $1 < s \le 2+\alpha$. Let $\partial\Omega, \Gamma \in C^{2+\alpha}$ and the assumption* A) *hold. Then for each pair of functions $u_{0j}, c_{0j} \in C^s(\overline{\Omega}_j)$, $j = 1,2$, $p, q \in C_s^{2+\alpha}(\Sigma_T)$ satisfying conditions B), C) and the compatibility conditions of $[\frac{s}{2}]$-th order there exists $T_0 > 0$, such that problem* (2.1)–(2.8) *has a unique solution $u_j, c_j \in C_s^{2+\alpha}(Q_{jT_0})$, $j = 1,2$, $\rho \in C_s^{2+\alpha}(\Gamma_{T_0})$, $\partial_t\rho \in C_{s-1}^{1+\alpha}(\Gamma_{T_0})$ and the following estimate holds*

$$\sum_{j=1}^{2}\Big(|u_j|_{s,Q_{jt}}^{(2+\alpha)} + |c_j|_{s,Q_{jt}}^{(2+\alpha)}\Big) + |\rho|_{s,\Gamma_t}^{(2+\alpha)} + |\partial_t\rho|_{s-1,\Gamma_t}^{(1+\alpha)}$$

$$\leq \ C_1\Big(\sum_{j=1}^{2}\big(|u_{0j}|^{(s)}_{\Omega_j}+|c_{0j}|^{(s)}_{\Omega_j}\big)+|p|^{(2+\alpha)}_{s,\Sigma_t}+|q|^{(2+\alpha)}_{s,\Sigma_t}\Big), \qquad for\ \ 0<t\leq T_0 .$$

Putting in, in particular, $s=2+\alpha$ we obtain also the local solvability of the alloy free boundary problem in classical Hölder spaces [13] under more restrictive assumptions on the initial data.

Theorem 2.2. *Let* $\alpha\in(0,1)$, $\partial\Omega,\Gamma\in C^{2+\alpha}$, *and the assumption* A) *hold. Then for any functions* $u_{0j},c_{0j}\in C^{2+\alpha}(\overline{\Omega}_j)$, $j=1,2$, $p,q\in C^{2+\alpha,1+\alpha/2}_{\ x\qquad t}(\Sigma_T)$ *satisfying B), C) and the compatibility conditions* (2.16)–(2.21), *there exists* $T_0>0$ *such that problem* (2.1)–(2.8) *has a unique solution* $u_j,c_j\in C^{2+\alpha,1+\alpha/2}_{\ x\qquad t}(Q_{jT_0})$, $j=1,2$, $\rho\in C^{2+\alpha,1+\alpha/2}_{\ x\qquad t}(\Gamma_{T_0})$, $\partial_t\rho\in C^{1+\alpha,\frac{1+\alpha}{2}}_{\ x\quad\ t}(\Gamma_{T_0})$ *and the following estimate holds:*

$$\sum_{j=1}^{2}\Big(|u_j|^{(2+\alpha)}_{Q_{jt}}+|c_j|^{(2+\alpha)}_{Q_{jt}}\Big)+|\rho|^{(2+\alpha)}_{\Gamma_t}+|\partial_t\rho|^{(1+\alpha)}_{\Gamma_t}$$

$$\leq \ C_2\Big(\sum_{j=1}^{2}\big(|u_{0j}|^{(2+\alpha)}_{\Omega_j}+|c_{0j}|^{(2+\alpha)}_{\Omega_j}\big)+|p|^{(2+\alpha)}_{\Sigma_t}+|q|^{(2+\alpha)}_{\Sigma_t}\Big),$$

for $0<t\leq T_0$.

Remark 2.1. Here $c_j(s,t)$ represents the concentration of the mixture in the alloy in phase $j=1,2$. The condition C) guarantees that $c_j\in(0,1)$ for small $t\leq T_0$. Condition (2.12) means that the concentration of the mixture $c_0(x)$ at the initial time is a discontinuous and increasing function across Γ.

Remark 2.2. Condition (2.13) is physically compatible because $\sigma_j'(u)>0$, for all $u\in(u^*,u^{**})$ and $\partial_{\nu_0}u_{0j}(x)\big|_\Gamma<0$, $j=1,2$, due to the decrease of the temperature from the liquid phase into the solid one.

Remark 2.3. In (2.14) we suppose that the latent heat of melting κ and the jump of the initial temperature $|[\![\nabla u_0]\!]\big|_\Gamma|$, as well as the difference of heat conductivity coefficients $|\lambda_1-\lambda_2|$, are small values. These two last conditions lead to the condition $|[\![\lambda\,\partial_{\nu_0}u_0]\!]\big|_\Gamma|\leq C_3\,\delta_0$. We note also that the requirements (2.14) are adjusted in the compatibility condition (2.18).

3. Reduction to a problem in a fixed domain

Let $\lambda_0>0$ be a sufficiently small number, such that, every point $y\in\Omega$ situated in a $2\lambda_0$-neighbourhood $\mathcal{O}$ of Γ can be represented in the form

$$y=\xi+\lambda N(\xi)\,, \qquad \xi\in\Gamma,\quad |\lambda|<2\lambda_0\,, \tag{3.1}$$

$y = \xi$, if $\lambda = 0$. Let $\chi(\lambda)$ be a smooth cut-off function, such that $\chi = 1$, $|\lambda| < \lambda_0$, $\chi = 0$, $|\lambda| \geq 2\lambda_0$. We define the coordinates transformation $e_\rho\colon\, y \to x$ [9], which is the modification of Hanzawa mapping [12], by the formulas:

$$\begin{aligned} & x = y + \chi(\lambda)\,\rho(\xi,\tau)\,N(\xi)\,, \qquad y \in \mathcal{O}, \quad \xi \in \Gamma\,, \\ & x = y\,, \qquad y \in \Omega\backslash\mathcal{O}, \quad t = \tau\,, \end{aligned} \tag{3.2}$$

where $\rho\big|_{\tau=0} = 0$.

This mapping transforms the surface $\Gamma:\ \lambda = 0,\ y = \xi$ into the free boundary

$$\gamma(t)\colon\ \ x = \xi + \rho(\xi,t)\,N(\xi)\,, \qquad \xi \in \Gamma, \quad t \in [0,t_0]\,,$$

and the given fixed domains Ω_j into the unknown ones $\Omega_j(t)$, $j = 1, 2$.

To every point $y \in \mathcal{O}$ there correspond unique coordinates $\xi = \xi(y)$, $\lambda = \lambda(y)$ and inversely, every pair of coordinates (ξ,λ), $\xi \in \Gamma$, $|\lambda| < 2\lambda_0$, determine a unique point $y \in \mathcal{O}$. So, formula (3.1) sets up a one-to-one correspondence between the coordinates y and (ξ,λ) of every point in $\mathcal{O}$. Therefore, from the equation (3.1) we can express the coordinates (ξ,λ) via coordinates y: $\xi = \xi(y) = (\xi_1(y), ..., \xi_n(y))$, $\lambda = \lambda(y)$, where $\xi_k(y)$, $\lambda(y) \in C^{2+\alpha}(\overline{\mathcal{O}})$, $k = 1, ..., n$, because $\Gamma \in C^{2+\alpha}$. Thus, we can write the transformation (3.2) in the form

$$\begin{aligned} & x = y + \chi(\lambda(y))\,\rho(\xi(y),\tau)\,N(\xi(y))\,, \qquad y \in \mathcal{O}\,, \\ & x = y\,, \qquad y \in \Omega\backslash\mathcal{O}, \quad t = \tau. \end{aligned} \tag{3.3}$$

Remark 3.1. We note that the coordinate transformation (3.3) with $N \equiv \nu_0$ was used by Hanzawa [12]

$$x = y + \chi\,\rho\,\nu_0, \quad y \in \mathcal{O}\,, \qquad x = y, \quad y \in \Omega\backslash\mathcal{O} \tag{3.4}$$

but leads to the loss of smoothness of one unit, because the normal $\nu_0(\xi)$ to Γ in (3.4) is expressed via the first partial derivatives of a function setting the surface Γ. The mapping (3.3) is due to Solonnikov (see also [22] and [9]) and permits us to avoid this loss.

The Jacobian matrix of the transform (3.3) with respect to n variables $x_1, ..., x_n$ has the form

$$J = \left\{\frac{\partial x_i}{\partial y_j}\right\}_{1\leq i,\,j\leq n} = \begin{bmatrix} 1 + \partial_{y_1}(N_1\,\chi\,\rho) & \cdots & \partial_{y_n}(N_1\,\chi\,\rho) \\ \cdots & \cdots & \cdots \\ \partial_{y_1}(N_n\,\chi\,\rho) & \cdots & 1 + \partial_{y_n}(N_n\,\chi\,\rho) \end{bmatrix}$$

$$= \Big\{\delta_{ij} + \partial_{y_j}(N_i\,\chi\,\rho)\Big\}_{1\le i,\,j\le n} = I + \Big(\nabla^T(N\chi\,\rho)\Big)^T . \tag{3.5}$$

Here $\nabla = (\partial_{y_1}, \dots, \partial_{y_n})$, ∇^T is the column vector, $(a_1, \dots, a_n)^T(b_1, \dots, b_n)$ is the matrix $\{a_i b_j\}_{1\le i,\,j\le n}$, by A^T we denote the transposed matrix A, by δ_{ij} the Kronecker delta, by I the identity matrix $\{\delta_{ij}\}_{1\le i,\,j\le n}$.

In (3.5) $\rho\big|_{t=0} = 0$, $J\big|_{t=0} = I$; that is, at least, for small $t \le t_1$ the inverse matrix J^{-1} exists. In Appendix A we prove this.

In the new coordinates $\{y\}$ the differential operators, the normal ν to $\gamma(t)$ and the normal velocity of the free boundary $\gamma(t)$ are expressed, respectively, by the formulas:

$$\nabla_x{}^T\Big|_{x=y+\chi\rho N} = J^{-T}\,\nabla_y{}^T , \qquad \nabla_x\Big|_{x=y+\chi\rho N} = (J^{-T}\,\nabla_y{}^T)^T ,$$

$$\partial_t - a\,\Delta_x\Big|_{\substack{x=y+\chi\rho N\\ t=\tau}} = \partial_\tau - \chi\,\partial_\tau\rho N\,\nabla_x{}^T - a\,\nabla_x\nabla_x{}^T\Big|_{x=y+\chi\rho N}$$

$$= \partial_\tau - \chi\,\partial_\tau\rho N J^{-T}\,\nabla_y{}^T - a(J^{-T}\,\nabla_y{}^T)^T\,J^{-T}\,\nabla_y{}^T , \tag{3.6}$$

$$\nu\Big|_{\substack{x=\xi+\rho N\\ t=\tau}} = \nu_0\,J^{-1}\,|\nu_0\,J^{-1}|^{-1} ,$$

$$\partial_\nu\Big|_{\substack{x=\xi+\rho N\\ t=\tau}} = \nu_0\,J^{-1}J^{-T}\,|\nu_0\,J^{-1}|^{-1}\,\nabla_y{}^T , \tag{3.7}$$

$$V_\nu\Big|_{\substack{x=\xi+\rho N\\ t=\tau}} = \nu_0\,J^{-1}N^T\partial_\tau\rho\,|\nu_0\,J^{-1}|^{-1} . \tag{3.8}$$

Remark 3.2. For the sake of convenience we use again the variables t and ξ instead of τ and $\xi(y)$.

We apply the coordinate mapping (3.3) in problem (2.1)–(2.8) with the help of formulas (3.6), (3.7), (3.8). Then we obtain a nonlinear problem involving the unknown function $\rho(\xi, t)$,

$$u_j(y + \chi\,\rho N, t) = \widehat{u}_j(y, t) ,$$
$$c_j(y + \chi\,\rho N, t) = \widehat{c}_j(y, t) , \tag{3.9}$$

in the fixed given domains Ω_j, for $j = 1, 2$:

$$\partial_t\widehat{u}_j - \chi\,\partial_t\rho N J^{-T}\,\nabla^T\widehat{u}_j - a_j(J^{-T}\,\nabla^T)^T J^{-T}\,\nabla^T\widehat{u}_j = 0 \quad \text{in } \ \Omega_{jT} , \tag{3.10}$$

$$\partial_t\widehat{c}_j - \chi\,\partial_t\rho N J^{-T}\,\nabla^T\widehat{c}_j - b_j(J^{-T}\,\nabla^T)^T J^{-T}\,\nabla^T\widehat{c}_j = 0 \quad \text{in } \ \Omega_{jT} , \tag{3.11}$$

with initial and boundary conditions

$$\rho\big|_{t=0} = 0 \ \text{ on } \Gamma, \quad \widehat{u}_j\big|_{t=0} = u_{0j}(y), \quad \widehat{c}_j\big|_{t=0} = c_{0j}(y) \quad \text{in } \ \Omega_j , \tag{3.12}$$

$$\widehat{u}_1\big|_{\partial\Omega} = p(y,t), \quad \widehat{c}_1\big|_{\partial\Omega} = q(y,t)\,, \qquad t \in (0,T)\,, \tag{3.13}$$

and with the following transmission conditions on the initial interface

$$\widehat{u}_1 = \widehat{u}_2\,, \qquad \widehat{c}_j = \sigma_j(\widehat{u}_j) \quad \text{on } \Gamma_T, \quad j=1,2\,, \tag{3.14}$$

$$\nu_0\, J^{-1} J^{-T}(\lambda_1 \nabla^T \widehat{u}_1 - \lambda_2 \nabla^T \widehat{u}_2) = -\kappa\, \nu_0\, J^{-1} N^T\, \partial_t \rho \quad \text{on } \Gamma_T\,, \tag{3.15}$$

$$\nu_0\, J^{-1} J^{-T}(k_1 \nabla^T \widehat{c}_1 - k_2 \nabla^T \widehat{c}_2) = -(\widehat{c}_1 - \widehat{c}_2)\, \nu_0\, J^{-1} N^T\, \partial_t \rho \quad \text{on } \Gamma_T\,. \tag{3.16}$$

Although this highly nonlinear system has the advantage of being set in known domains, it will be treated in a more convenient form after a reduction to new unknowns with zero initial conditions.

We determine the auxiliary functions $\rho_0(\xi,t)$ on Γ_T under the conditions

$$\rho_0\big|_{t=0} = 0\,, \qquad \partial_t \rho_0\big|_{t=0} \equiv \partial_t \rho\big|_{t=0} = -\frac{[\![k\,\partial_{\nu_0} c_0]\!]\big|_\Gamma}{\nu_0 N^T [\![c_0]\!]\big|_\Gamma} \tag{3.17}$$

and $V_j(y,t)$, $Z_j(y,t)$, $j=1,2$, as the solutions of the Cauchy problems

$$\partial_t V_j - a_j\, \Delta V_j - \chi\, \partial_t \rho_0 N\, \nabla^T V_j = 0 \;\text{ in } \mathbb{R}^n_T\,, \qquad V_j\big|_{t=0} = \widetilde{u}_{0j}(y)\,, \tag{3.18}$$

$$\partial_t Z_j - b_j\, \Delta Z_j - \chi\, \partial_t \rho_0 N\, \nabla^T Z_j = 0 \;\text{ in } \mathbb{R}^n_T\,, \qquad Z_j\big|_{t=0} = \widetilde{c}_{0j}(y), \tag{3.19}$$

where $j=1,2$, the symbol "$\widetilde{\ }$" denotes the smooth extension of a function into the entire space $\mathbb{R}^n$ and $\mathbb{R}^n_T = \mathbb{R}^n \times (0,T)$.

Lemma 3.1 ([21, 9, 6, 8, 18])**.** *Let $1 < s \le 2+\alpha$, $\alpha \in (0,1)$. For arbitrary functions $u_{0j}, c_{0j} \in C^s(\overline{\Omega}_j)$, $j=1,2$, each one of the problems* (3.17)–(3.19) *has a unique solution $\rho_0 \in C^{3+\alpha}_{1+s}(\Gamma_T)$, $V_j, Z_j \in C^{2+\alpha}_s(\mathbb{R}^n_T)$, $j=1,2$, satisfying*

$$|\rho_0|^{(3+\alpha)}_{s+1,\Gamma_T} \le C_1 \sum_{k=1}^{2} |c_{0k}|^{(s)}_{\Omega_k}\,, \tag{3.20}$$

$$|V_j|^{(2+\alpha)}_{s,\mathbb{R}^n_T} \le C_2\Big(|u_{0j}|^{(s)}_{\Omega_j} + \sum_{k=1}^{2} |c_{0k}|^{(s)}_{\Omega_k}\Big)\,, \tag{3.21}$$

$$|Z_j|^{(2+\alpha)}_{s,\mathbb{R}^n_T} \le C_3 \sum_{k=1}^{2} |c_{0k}|^{(s)}_{\Omega_k}\,, \qquad j=1,2\,. \tag{3.22}$$

Now using the functions ρ_0, V_j, Z_j, $j=1,2$, we transform problem (3.10)–(3.16) into a more suitable form. We make use of the following substitutions:

$$\rho(\xi,t) = \rho_0(\xi,t) + \psi(\xi,t)\,, \tag{3.23}$$

$$\widehat{u}_j(y,t) \equiv u_j(y+\chi\,\rho N,t) = v_j(y,t) + V_j(y,t)\,, \tag{3.24}$$
$$\widehat{c}_j(y,t) \equiv c_j(y+\chi\,\rho N,t) = z_j(y,t) + Z_j(y,t)\,, \qquad j=1,2\,,$$

where ψ, v_j, z_j are new unknown functions satisfying zero initial conditions

$$\partial_t^k v_j\big|_{t=0} = 0, \quad \partial_t^k z_j\big|_{t=0} = 0, \quad \partial_t^{k_1}\psi\big|_{t=0} = 0\,, \qquad k = \begin{cases} 0, & 1 < s < 2, \\ 0,1, & 2 \le s \le 2+\alpha\,, \end{cases}$$

$k_1 = 0,1,\ j=1,2\,.$ We represent the composition of the functions $\sigma_j(v_j+V_j)$ in (3.14) as follows

$$\sigma_j(v_j+V_j) = \sigma_j(V_j) + \sigma_j'(V_j)\,v_j + v_j^2\int_0^1 (1-\lambda)\,\sigma_j''(V_j+\lambda v_j)\,d\lambda\,.$$

The expansion formulas (A.1), (A.2), (A.5) of Appendix A of the matrices $J^{-1},\ J_0^{-1} = J^{-1}\big|_{\rho=\rho_0}$ and the change (3.23), (3.24) of unknown functions permit us to extract linear principal terms with respect to the unknown functions, known functions and the remaining nonlinear terms. Thus, we obtain problem (3.10)–(3.16) in the form, for $j=1,2$, in Ω_{jT},

$$\partial_t v_j - a_j\,\Delta v_j - (\partial_t\psi - a_j\,\Delta\psi)\,\chi\,N J_0^{-T}\,\nabla^T V_j = f_j(y,t) + F_j(v_j,\psi)\,, \tag{3.25}$$
$$\partial_t z_j - b_j\,\Delta z_j - (\partial_t\psi - b_j\,\Delta\psi)\,\chi\,N J_0^{-T}\,\nabla^T Z_j = g_j(y,t) + G_j(z_j,\psi)\,, \tag{3.26}$$

with zero initial data, translated Dirichlet boundary conditions

$$v_1\big|_{\partial\Omega} = p_1(y,t), \quad z_1\big|_{\partial\Omega} = q_1(y,t)\,, \qquad t\in(0,T)\,, \tag{3.27}$$

and the corresponding transmission conditions

$$(v_1 - v_2)\Big|_\Gamma = \eta_0(y,t)\,, \tag{3.28}$$
$$\Big(z_j - \sigma_j'(V_j)\,v_j\Big)\Big|_\Gamma = \eta_j(y,t) + R_j(v_j)\Big|_\Gamma\,, \qquad j=1,2\,, \tag{3.29}$$
$$\Big(\lambda_1\,\partial_{\nu_0}v_1 - \lambda_2\,\partial_{\nu_0}v_2\Big)\Big|_\Gamma = \varphi_1(y,t) + \Phi_1(v_1,v_2,\psi)\Big|_\Gamma\,, \tag{3.30}$$
$$\Big(k_1\,\partial_{\nu_0}z_1 - k_2\,\partial_{\nu_0}z_2 - (Z_2 - Z_1)\,N J_0^{-T}\nu_0^T\,\partial_t\psi$$
$$-\,\nu_0\,N^T\Big[(k_1\,\nabla Z_1 - k_2\,\nabla Z_2)\,J_0^{-1}J_0^{-T} + (Z_1 - Z_2)\,N J_0^{-T}\,\partial_t\rho_0\Big]\,\nabla^T\psi\Big)\Big|_\Gamma$$
$$= \varphi_2(y,t) + \Phi_2(z_1,z_2,\psi)\Big|_\Gamma\,, \qquad t\in(0,T)\,. \tag{3.31}$$

We note that in condition (3.31) on Γ, we have used the relation

$$\nu_0(\xi)\nabla^T\psi(\xi,t) = 0, \qquad \xi\in\Gamma,$$

and in (3.30) we did not single out such a term.

Thus, we have reduced the free boundary problem (2.1)–(2.8) in the unknown domains $\Omega_1(t)$ and $\Omega_2(t)$ to the nonlinear problem (3.25)–(3.31) in given domains Ω_1, Ω_2 with unknown functions v_1, v_2, z_1, z_2, ψ satisfying zero initial data. In the left-hand sides of equations (3.25), (3.26) and in conditions on $\partial\Omega$ and Γ (3.27)–(3.31) there are linear terms. The functions $f_j, g_j, p_1, q_1, \eta_0, \eta_j, \varphi_j$, $j = 1, 2$, are known; F_j, G_j, R_j, Φ_j, $j = 1, 2$, are nonlinear functions — the rest of the expressions in (3.10), (3.11), (3.14)–(3.16) of problem (3.10)–(3.16) after separating linear terms and known functions, whose expressions (see Appendix A for notation of matrices J_0, J_{01}, J_1, J_{11}, J_{12}) are given by

$$f_j = \chi\,\partial_t\rho_0 N J_0^{-T}\,\nabla^T V_j - \partial_t V_j + a_j(J_0^{-T}\,\nabla^T)^T J_0^{-T}\,\nabla^T V_j\,, \tag{3.32}$$

$$g_j = \chi\,\partial_t\rho_0 N J_0^{-T}\,\nabla^T Z_j - \partial_t Z_j + b_j(J_0^{-T}\,\nabla^T)^T J_0^{-T}\,\nabla^T Z_j\,, \tag{3.33}$$

$$\begin{aligned} F_j &= \chi\,\partial_t(\rho_0+\psi)\,N J^{-T}(\nabla^T v_j - J_1^T J_0^{-T}\,\nabla^T V_j) \\ &\quad - a_j\Big[\nabla(J_{01}^T + J_1^T) + \Big((J_{01}^T + J_1^T)\,J^{-T}\,\nabla^T\Big)^T\Big] J^{-T}\,\nabla^T v_j \\ &\quad - a_j(\nabla\psi)\,\nabla^T(\chi\,N J_0^{-T}\,\nabla^T V_j) \\ &\quad + a_j\Big[\nabla(J_{01}^T + J_1^T) + \Big((J_{01}^T + J_1^T)\,J^{-T}\,\nabla^T\Big)^T\Big]\,J^{-T} J_{11}^T\,J_0^{-T}\,\nabla^T V_j \\ &\quad - a_j\Big[(J_0^{-T} J_1^T J^{-T}\,\nabla^T)^T + (J^{-T}\,\nabla^T)^T J^{-T} J_{12}^T\Big]\,J_0^{-T}\,\nabla^T V_j \\ &\equiv F(V_j, v_j, \psi; a_j)\,, \end{aligned} \tag{3.34}$$

$$G_j = F(Z_j, z_j, \psi; b_j)\,, \tag{3.35}$$

$$p_1 = p(y,t) - V_1(y,t)\Big|_{\partial\Omega}\,, \qquad q_1 = q(y,t) - Z_1(y,t)\Big|_{\partial\Omega}\,, \tag{3.36}$$

$$\eta_0 = \Big(V_2(y,t) - V_1(y,t)\Big)\Big|_\Gamma\,, \tag{3.37}$$

$$\eta_j = \Big(-Z_j + \sigma_j(V_j)\Big)\Big|_\Gamma\,, \tag{3.38}$$

$$R_j = v_j^2 \int_0^1 (1-\lambda)\,\sigma_j''(V_j + \lambda\,v_j)\,d\lambda\,, \tag{3.39}$$

$$\varphi_1 = -\nu_0\,J_0^{-1}\Big[J_0^{-T}(\lambda_1\nabla^T V_1 - \lambda_2\nabla^T V_2) + \kappa N^T\partial_t\rho_0\Big]\Big|_\Gamma\,, \tag{3.40}$$

$$\varphi_2 = -\nu_0\,J_0^{-1}\Big[J_0^{-T}(k_1\nabla^T Z_1 - k_2\nabla^T Z_2) + (Z_1 - Z_2)\,N^T\partial_t\rho_0\Big]\Big|_\Gamma\,, \tag{3.41}$$

$$\begin{aligned}\Phi_1 \;=\; &\nu_0\Big(J_{01}^T + J_1^T + J^{-1}(J_{01}+J_1)\Big)J^{-T}(\lambda_1\nabla^T v_1 - \lambda_2\nabla^T v_2)\\ &+\nu_0\, J_0^{-1}(J_0^{-T}J_1^T + J_1 J^{-1})\, J^{-T}(\lambda_1\nabla^T V_1 - \lambda_2\nabla^T V_2)\\ &-\kappa\,\nu_0(J^{-1}N^T\partial_t\psi - J_0^{-1}J_1J^{-1}N^T\partial_t\rho_0)\,, \qquad (3.42)\\ \Phi_2 \;=\; &-\nu_0\, J^{-1}(z_1-z_2)\, N^T\partial_t(\rho_0+\psi)\\ &+\nu_0\Big((J_{01}^T+J_1^T) + J^{-1}(J_{01}+J_1)\Big)\, J^{-T}(k_1\,\nabla^T z_1 - k_2\,\nabla^T z_2)\\ &-(Z_1-Z_2)\,\nu_0\, J^{-1}\Big[\Big((J_{01}+J_1)\, J_{11} - J_{12}\Big)\, J_0^{-1}N^T\partial_t\rho_0\\ &-J_1J_0^{-1}N^T\partial_t\psi\Big] - \nu_0\,\mathcal{M}(k_1\,\nabla^T Z_1 - k_2\,\nabla^T Z_2)\,, \qquad (3.43)\end{aligned}$$

$$\begin{aligned}\mathcal{M} \;=\; &J^{-1}\Big[(J_{01}+J_1)\, J_{11}^T + J_{01}^TJ_0^{-T}J_{11}^T - J_0^{-T}J_{12}^T\Big]\, J^{-T}\\ &+J^{-1}\Big((J_{01}+J_1)\, J_{11} - J_{12}\Big)\, J_0^{-1}J_0^{-T}\,.\end{aligned}$$

For the sake of convenience, we write problem (3.25)–(3.31) conventionally in operator form

$$\mathcal{A}[w] = h + \mathcal{N}[w]\,, \qquad (3.44)$$

where $w = (v_1, v_2, z_1, z_2, \psi)$, $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$, $\mathcal{A}[w]$ is a linear operator determined by the expressions in the left-hand sides of the four equations (3.25), (3.26) and seven boundary conditions (3.27)–(3.31), and $\mathcal{N}[w] = (F_1, F_2, G_1, G_2, 0, 0, 0, R_1\big|_\Gamma, R_2\big|_\Gamma, \Phi_1\big|_\Gamma, \Phi_2\big|_\Gamma)$ is a nonlinear operator. Moreover, $\mathcal{A} : \mathcal{B}(\Omega_T) \to \mathcal{H}(\Omega_T)$, $\mathcal{N} : \mathcal{B}(\Omega_T) \to \mathcal{H}(\Omega_T)$, where $\mathcal{B}(\Omega_T) = \overset{\circ}{C}{}_s^{2+\alpha}(\Omega_{1T})\times \overset{\circ}{C}{}_s^{2+\alpha}(\Omega_{2T})\times \overset{\circ}{C}{}_s^{2+\alpha}(\Omega_{1T})\times \overset{\circ}{C}{}_s^{2+\alpha}(\Omega_{2T})\times \overset{\circ}{\mathcal{D}}{}_s^{2+\alpha}(\Gamma_T)$, $\alpha \in (0,1)$, $s \le 2+\alpha$, represents the space of functions $w = (v_1, v_2, z_1, z_2, \rho)$ with the norm:

$$\|w\|_{\mathcal{B}(\Omega_T)} = \sum_{j=1}^{2}\Big(|v_j|_{s,\Omega_{jT}}^{(2+\alpha)} + |z_j|_{s,\Omega_{jT}}^{(2+\alpha)}\Big) + |\rho|_{s,\Gamma_T}^{(2+\alpha)} + |\partial_t\rho|_{s-1,\Gamma_T}^{(1+\alpha)}\ ; \qquad (3.45)$$

$\mathcal{H}(\Omega_T) = \overset{\circ}{C}{}_{s-2}^{\alpha}(\Omega_{1T})\times \overset{\circ}{C}{}_{s-2}^{\alpha}(\Omega_{2T})\times \overset{\circ}{C}{}_{s-2}^{\alpha}(\Omega_{1T})\times \overset{\circ}{C}{}_{s-2}^{\alpha}(\Omega_{2T})\times \overset{\circ}{C}{}_s^{2+\alpha}(\Sigma_T)$ $\times \overset{\circ}{C}{}_s^{2+\alpha}(\Sigma_T)\times \overset{\circ}{C}{}_s^{2+\alpha}(\Gamma_T)\times \overset{\circ}{C}{}_s^{2+\alpha}(\Gamma_T)\times \overset{\circ}{C}{}_s^{2+\alpha}(\Gamma_T)\times \overset{\circ}{C}{}_{s-1}^{1+\alpha}(\Gamma_T)\times \overset{\circ}{C}{}_{s-1}^{1+\alpha}(\Gamma_T)$ denotes the space of functions $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$ with the norm:

$$\|h\|_{\mathcal{H}(\Omega_T)} = \sum_{j=1}^{2}\Big(|f_j|_{s-2,\Omega_{jT}}^{(\alpha)} + |g_j|_{s-2,\Omega_{jT}}^{(\alpha)}\Big) + |p_1|_{s,\Sigma_T}^{(2+\alpha)} + |q_1|_{s,\Sigma_T}^{(2+\alpha)}$$

$$+ |\eta_0|^{(2+\alpha)}_{s,\Gamma_T} + \sum_{j=1}^{2} \Big(|\eta_j|^{(2+\alpha)}_{s,\Gamma_T} + |\varphi_j|^{(1+\alpha)}_{s-1,\Gamma_T} \Big) . \tag{3.46}$$

Here, $\overset{\circ}{C}{}^{\ell}_s(\Omega_T)$ is the subset of $C^{\ell}_s(\Omega_T)$ consisting of functions $u(x,t)$, such that, $\partial_t^k u\big|_{t=0} = 0$, $2k \le s$, if $s \ge 0$, $\overset{\circ}{C}{}^{\ell}_s(\Omega_T) = C^{\ell}_s(\Omega_T)$, if $s < 0$; and $\overset{\circ}{\mathcal{D}}{}^{2+\alpha}_s(\Gamma_T)$ is the space of functions $\rho(\xi,t) \in \overset{\circ}{C}{}^{2+\alpha}_s(\Gamma_T)$, $\partial_t\rho \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$.

We consider first the vector function h. We see from formulas (3.32), (3.33), (3.36)–(3.38), (3.40), (3.41) that the components are expressed via the auxiliary functions V_j, Z_j, $j = 1,2$, ρ_0 and the inverse matrix J_0^{-1}. In Theorem A.2 of the Appendix we prove the existence of J_0^{-1} for $t \le t_2$. Now we show that $h \in \mathcal{H}(\Omega_{t_2})$.

Lemma 3.2. *Let the conditions of Theorem* 2.1 *be fulfilled. Then the vector function* $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$ *belongs to the space* $\mathcal{H}(\Omega_{t_2})$ *and satisfies the estimate:*

$$\|h\|_{\mathcal{H}(\Omega_t)} \le C_4 \Big(\sum_{j=1}^{2} \Big(|u_{0j}|^{(s)}_{\Omega_j} + |c_{0j}|^{(s)}_{\Omega_j} \Big) + |p|^{(2+\alpha)}_{s,\Sigma_t} + |q|^{(2+\alpha)}_{s,\Sigma_t} \Big), \quad t \le t_2. \tag{3.47}$$

Proof. To show that $h \in \mathcal{H}(\Omega_{t_2})$ and to obtain an estimate (3.47) using the estimates (A.11), (A.53) and (3.20)–(3.22) of the product of functions in weighted Hölder spaces, of the matrix J_0^{-1} and of the functions ρ_0, V_j, Z_j, $j = 1,2$, respectively, we derive the required inequality (3.47).

We prove that the components — functions of the vector h — satisfy zero initial conditions. The functions f_j, g_j, $j = 1,2$, are equal to 0 at $t = 0$, if $s \ge 2$, by the condition $J_0^{-1}\big|_{t=0} = I$ and equations (3.18), (3.19).

On the basis of the initial data (3.17)–(3.19) and compatibility conditions (2.16)– (2.18) we have $p_1\big|_{t=0} = 0$, $q_1\big|_{t=0} = 0$, $\eta_0\big|_{t=0} = 0$, $\eta_j\big|_{t=0} = 0$, $\varphi_j\big|_{t=0} = 0$, $j = 1,2$.

Let $s \ge 2$. We show that the time derivatives $\partial_t p_1$, $\partial_t q_1$, $\partial_t \eta_0$, $\partial_t \eta_j$, $j = 1,2$, are also equal to zero at $t = 0$. We consider the functions

$$\partial_t p_1\big|_{t=0} = \Big(\partial_t p - \partial_t V_1\,\big|_{\partial\Omega}\Big)\Big|_{t=0}, \quad \partial_t q_1\big|_{t=0} = \Big(\partial_t q - \partial_t Z_1\,\big|_{\partial\Omega}\Big)\Big|_{t=0}. \tag{3.48}$$

From equations (3.18), (3.19) we obtain

$$\partial_t V_1\big|_{t=0} = a_1\,\Delta u_{01}\,, \quad \partial_t Z_1\big|_{t=0} = b_1\,\Delta c_{01}\,,\ y \in \partial\Omega,$$

by the assumption on the initial interface Γ and the definition of χ. Then, from (3.48) on the basis of the compatibility conditions (2.19) we find

$$\partial_t p_1\big|_{t=0} = 0, \quad \partial_t q_1\big|_{t=0} = 0.$$

We consider the time derivatives of the functions η_0, η_j $(j = 1, 2)$

$$\partial_t \eta_0 = \Big(\partial_t V_2 - \partial_t V_1\Big)\Big|_\Gamma, \qquad \partial_t \eta_j = \Big(-\partial_t Z_j + \sigma_j'(V_j)\, \partial_t V_j\Big)\Big|_\Gamma.$$

We substitute here $\partial_t V_j$, $\partial_t Z_j$, $\partial_t \rho_0$ found from formulas (3.17)–(3.19)

$$\partial_t \eta_0\big|_{t=0} = [\![a\, \Delta u_0]\!]\big|_\Gamma + (\partial_N u_{01} - \partial_N u_{02})\, \frac{[\![k\, \partial_{\nu_0} c_0]\!]}{\nu_0 N^T [\![c_0]\!]}\Bigg|_\Gamma, \tag{3.49}$$

$$\begin{aligned} \partial_t \eta_j\big|_{t=0} &= -b_j\, \Delta c_{0j}\,\big|_\Gamma + a_j\, \sigma_j'(u_{0j})\, \Delta u_{0j}\,\big|_\Gamma \\ &\quad + \Big(\partial_N c_{0j} - \sigma_j'(u_{0j})\, \partial_N u_{0j}\Big)\, \frac{[\![k\, \partial_{\nu_0} c_0]\!]}{\nu_0 N^T [\![c_0]\!]}\Bigg|_\Gamma, \quad j = 1, 2\,. \end{aligned} \tag{3.50}$$

We express the derivatives on the direction N via normal derivatives. Let $\xi \in \Gamma$ be an arbitrary point. The vector $N(\xi)$ may be represented in the form

$$N(\xi) = \alpha_1\, \nu_0(\xi) + \alpha_2\, \tau(\xi)\,, \tag{3.51}$$

where $\nu_0(\xi)$ and $\tau(\xi)$ are normal and tangential unit vectors at the point ξ and α_1, α_2 denote some numbers. Multiplying from the right both parts of the identity (3.51) by the vector $\nu_0^T(\xi)$ we obtain $\alpha_1 = N\nu_0^T \geq d_1$ (see (2.10)) and $N(\xi) = N\nu_0^T \nu_0 + \alpha_2\, \tau$.

Then we can write

$$\Big(\partial_N u_{01} - \partial_N u_{02}\Big)\Big|_\Gamma = \nu_0\, N^T \Big(\partial_{\nu_0} u_{01} - \partial_{\nu_0} u_{02}\Big)\Big|_\Gamma + \alpha_2 \Big(\partial_\tau u_{01} - \partial_\tau u_{02}\Big)\Big|_\Gamma, \tag{3.52}$$

$$\begin{aligned} \Big(\partial_N c_{0j} - \sigma_j'(u_{0j})\, \partial_N u_{0j}\Big)\Big|_\Gamma &= \nu_0\, N^T \Big(\partial_{\nu_0} c_{0j} - \sigma_j'(u_{0j})\, \partial_{\nu_0} u_{0j}\Big)\Big|_\Gamma \\ &\quad + \alpha_2 \Big(\partial_\tau c_{0j} - \sigma_j(u_{0j})\, \partial_\tau u_{0j}\Big)\Big|_\Gamma. \end{aligned} \tag{3.53}$$

We differentiate the compatibility conditions (2.17) with respect to the tangential direction τ

$$\Big(\partial_\tau u_{01} - \partial_\tau u_{02}\Big)\Big|_\Gamma = 0\,, \qquad \Big(\partial_\tau c_{0j} - \sigma_j'(u_{0j})\, \partial_\tau u_{0j}\Big)\Big|_\Gamma = 0\,, \qquad j = 1, 2\,.$$

On the basis of these identities from (3.52), (3.53) we find the formulas

$$\Big(\partial_N u_{01} - \partial_N u_{02}\Big)\Big|_\Gamma = -\nu_0\, N^T [\![\partial_{\nu_0} u_0]\!]\big|_\Gamma\,,$$

$$\Big(\partial_N c_{0j} - \sigma'_N(u_{0j})\,\partial_N u_{0j}\Big)\Big|_\Gamma = \nu_0\, N^T\Big(\partial_{\nu_0} c_{0j} - \sigma'_j(u_{0j})\,\partial_{\nu_0} u_{0j}\Big)\Big|_\Gamma,\ j=1,2.$$

We substitute these expressions into (3.49), (3.50) and, by the compatibility conditions (2.20), (2.21), we finally have $\partial_t\eta_0\big|_{t=0} = 0$, $\partial_t\eta_j\big|_{t=0} = 0$, $j = 1,2$. □

4. The linear problem

In order to solve the nonlinear problem (3.44) we consider first the linear operator $\mathcal{A}$: $\mathcal{B}(\Omega_T) \to \mathcal{H}(\Omega_T)$ and the equation

$$\mathcal{A}[w] = h$$

with unknown functions $w = (v_1(x,t), v_2(x,t), z_1(x,t), z_2(x,t), \psi(\xi,t))$ satisfying zero initial data, so that for $j = 1,2$:

$$\partial_t v_j - a_j\,\Delta v_j - \chi(\lambda(x))\,\alpha_j(x,t)\,(\partial_t\psi - a_j\,\Delta\psi) = f_j(x,t) \quad \text{in } \Omega_{jT}\,, \tag{4.1}$$

$$\partial_t z_j - b_j\,\Delta z_j - \chi(\lambda(x))\,\beta_j(x,t)\,(\partial_t\psi - b_j\,\Delta\psi) = g_j(x,t) \quad \text{in } \Omega_{jT}\,, \tag{4.2}$$

with Dirichlet and transmission boundary conditions for $t \in (0,T)$:

$$v_1\big|_{\partial\Omega} = p_1(x,t)\,, \qquad z_1\big|_{\partial\Omega} = q_1(x,t)\,, \tag{4.3}$$

$$(v_1 - v_2)\big|_\Gamma = \eta_0(x,t)\,, \tag{4.4}$$

$$(z_j - \gamma_j(x,t)\,v_j)\big|_\Gamma = \eta_j(x,t)\,, \qquad j = 1,2\,, \tag{4.5}$$

$$(\lambda_1\,\partial_{\nu_0} v_1 - \lambda_2\,\partial_{\nu_0} v_2)\big|_\Gamma = \varphi_1(x,t)\,, \tag{4.6}$$

$$\Big(k_1\,\partial_{\nu_0} z_1 - k_2\,\partial_{\nu_0} z_2 + d(x,t)\,\nabla^T\psi - \kappa_1(x,t)\,\partial_t\psi\Big)\Big|_\Gamma = \varphi_2(x,t)\,, \tag{4.7}$$

where $\chi(\lambda(x))$ is a cut-off function; a_j, b_j, λ_j, k_j, $j = 1,2$, are positive constants; and $d = (d_1, ..., d_n)$ is a given vector function.

Let the following assumptions hold:

a) $\partial\Omega, \Gamma \in C^{2+\alpha}$, $\partial\Omega \cap \Gamma = \emptyset$;

b) α_j, $\beta_j \in C^{1+\alpha}_{s-1}(\Omega_T)$, $\gamma_j \in C^{2+\alpha}_{s}(\Gamma_T)$, $\kappa_1, d_i \in C^{1+\alpha}_{s-1}(\Gamma_T)$, $j = 1,2$, $i = 1, ..., n$;

c)
$$\kappa_1(x,0)\big|_\Gamma \geq d_3 > 0\,, \tag{4.8}$$

$$\beta_j(x,0) - \gamma_j(x,0)\,\alpha_j(x,0)\,\big|_\Gamma \geq d_3\,, \qquad j = 1,2\,, \tag{4.9}$$

$$\Big|\alpha_1(x,0) - \alpha_2(x,0)\,\big|_\Gamma\Big| \leq \delta_0\,, \tag{4.10}$$

where δ_0 is a sufficiently small value.

The conditions in c) correspond, respectively, to the conditions (2.12), (2.13) and to the second condition (2.14).

Theorem 4.1. *Let* $\alpha \in (0,1)$, $s \in (1, 2+\alpha]$. *We assume that conditions* a)–c) *hold. Then for every functions* $f_j, g_j \in \overset{\circ}{C}{}^{\alpha}_{s-2}(\Omega_{jT})$, $p_1, q_1 \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\Sigma_T)$, $\eta_0, \eta_j \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\Gamma_T)$, $\varphi_j \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$, $j = 1, 2$, *problem* (4.1)–(4.7) *has a unique solution* $v_j, z_j \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\Omega_{jT})$, $j = 1, 2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\Gamma_T)$, $\partial_t \psi \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$, *which satisfies the estimate*

$$\begin{aligned} \|w\|_{\mathcal{B}(\Omega_T)} \equiv {} & \sum_{j=1}^{2} \Big(|v_j|^{(2+\alpha)}_{s,\Omega_{jT}} + |z_j|^{(2+\alpha)}_{s,\Omega_{jT}} \Big) + |\psi|^{(2+\alpha)}_{s,\Gamma_T} + |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_T} \\ & \le C_1 \Big(\sum_{j=1}^{2} \Big(|f_j|^{(\alpha)}_{s-2,\Omega_{jT}} + |g_j|^{(\alpha)}_{s-2,\Omega_{jT}} + |\eta_j|^{(2+\alpha)}_{s,\Gamma_T} + |\varphi_j|^{(1+\alpha)}_{s-1,\Gamma_T} \Big) \\ & \quad + |p_1|^{(2+\alpha)}_{s,\Sigma_T} + |q_1|^{(2+\alpha)}_{s,\Sigma_T} + |\eta_0|^{(2+\alpha)}_{s,\Gamma_T} \Big) \equiv C_1 \, \|h\|_{\mathcal{H}(\Omega_T)} \, . \end{aligned} \tag{4.11}$$

Proof. We rewrite the equations $(4.1)_2$, $(4.2)_2$ (equations (4.1), (4.2) for $j = 2$) in the form

$$\begin{aligned} & \partial_t v_2 - a_2 \, \Delta v_2 - \chi \, \alpha_1(x,t) \, (\partial_t \psi - a_2 \, \Delta \psi) \\ & \quad = f_2 - \chi \Big(\alpha_1(x,t) - \alpha_2(x,t) \Big) \, (\partial_t \psi - a_2 \, \Delta \psi) \quad \text{ in } \ \Omega_{2T} \, , \end{aligned} \tag{4.12}$$

$$\begin{aligned} & \partial_t z_2 - b_2 \, \Delta z_2 - \chi \big[\beta_2(x,t) + \gamma_2(x,t) \, \big(\alpha_1(x,t) - \alpha_2(x,t) \big) \big] \, (\partial_t \psi - b_2 \, \Delta \psi) \\ & \quad = g_2 - \chi \, \gamma_2(x,t) \, \big(\alpha_1(x,t) - \alpha_2(x,t) \big) \, (\partial_t \psi - b_2 \, \Delta \psi) \quad \text{ in } \ \Omega_{2T} \, . \end{aligned} \tag{4.13}$$

By condition (4.10) the terms containing the difference $\alpha_1(x,t) - \alpha_2(x,t)$ are small. We have written equations $(4.1)_2$,$(4.2)_2$ in the form (4.12),(4.13) to obtain the model problem (B.1)–(B.7) for convenience.

We construct a regularizer to prove the solvability of problem (4.1)–(4.7) and apply Schauder's method to find the estimate (4.11). Since this is a standard procedure (see [13], for instance) we give only the sketch of a proof.

We cover the domain Ω with balls $K_{i,\delta}$ and $K_{i,2\delta}$ of radii δ and 2δ, respectively, and with a common center $x^{(i)}$. Let $\{\zeta_i(x)\}$, $\{\mu_i(x)\}$ be the sets of smooth functions subordinated to this overlapping by the balls, such that, $\zeta_i = 1$, if $|x - x^{(i)}| \le \delta$ and $\zeta_i = 0$, if $|x - x^{(i)}| \ge 2\delta$, $\operatorname{supp} \mu_i = \overline{K}_{i,2\delta}$, $\sum_i \zeta_i \mu_i = 1$ and $|\partial^m \zeta_i|, |\partial^m \mu_i| \le C_{m,i} \, \delta^{-|m|}$, $m = (m_1, ..., m_n)$. Let $\mathcal{O} = \{x \in \Omega \colon \, x = \xi + \lambda N(\xi), \, \xi \in \Gamma, \, |\lambda| < 2\lambda_0\}$, be the $2\lambda_0$-neighbourhood of Γ.

We define a regularizer $\mathcal{R}$ by the formula

$$\mathcal{R}h = \Big\{\mathcal{R}_1 h, \mathcal{R}_2 h, \mathcal{R}_3 h, \mathcal{R}_4 h, \mathcal{R}_5 h\Big\} = \Big\{ \sum_i \mu_i \, v_{1,i}\,,\ \sum_i \mu_i \, v_{2,i}\,,\ \sum_i \mu_i \, z_{1,i}\,,\ \sum_i \mu_i \, z_{2,i}\,,\ \sum_i \mu_i \, \psi_i \Big\}, \quad (4.14)$$

where the functions $v_{j,i}$, $z_{j,i}$, ψ_i, $j = 1,2$, are found as follows.

Let a ball $K_{i,\delta}$ intersect the surfaces Γ $(i \in \mathcal{I}_1)$ or $\partial\Omega$ $(i \in \mathcal{I}_2)$; the point $\xi^{(i)}$ belongs to $\Gamma \cap K_{i,\delta}$ or $\partial\Omega \cap K_{i,\delta}$. We pass to the local coordinates $\{\overline{y}\}$ with the center in $\xi^{(i)}$ and the axis $\overline{y}_n$ directed on the normal $\nu_0(\xi^{(i)})$ to the surface Γ into Ω_2 or $\partial\Omega$ into Ω_1. We choose a radius δ sufficiently small so that the boundary $\Gamma \cap K_{i,2\delta}$ or $\partial\Omega \cap K_{i,2\delta}$ can be expressed by the equation $\overline{y}_n = q_i(\overline{y}')$, where $q_i \in C^{2+\alpha}$ and $q_i(0) = 0$, $\nabla' q_i(0) = 0$. Here we set $\overline{y} = (\overline{y}_1, ..., \overline{y}_n)$ and $\overline{y}' = (\overline{y}_1, ..., \overline{y}_{n-1})$.

We denote by $\widetilde{q}_i(\overline{y}')$ the extension of $q_i(\overline{y}')$ into $\mathbb{R}^{n-1}$ which preserves regularity. Then we "straighten" the boundary $\overline{y}_n = \widetilde{q}_i$ by the formulas $y' = \overline{y}'$, $y_n = \overline{y}_n - \widetilde{q}_i$. Let $y = Y_i(x)$ be the transformation of the coordinates $\{x\}$ to the coordinate $\{y\}$ consisting of the parallel carry, rotation of a coordinate system $\{\overline{y}\}$ around the point $\xi^{(i)}$ to turn an axis $\overline{y}_n$ on a normal $\nu_0(\xi^{(i)})$ and "straightening" of a boundary $\overline{y}_n = \widetilde{q}_i$. We put $\zeta_i f_j, \zeta_i g_j\big|_{x=Y_i^{-1}(y)} = f_{j,i}(y,t), g_{j,i}(y,t)$; $\zeta_i \eta_k$, $\zeta_i \varphi_j, \zeta_i p_1, \zeta_i q_1\big|_{\substack{x=Y_i^{-1}(y)\\ y_n=0}} = \eta_{k,i}(y',t), \varphi_{j,i}(y',t)$, $p_{1,i}(y',t)$, $q_{1,i}(y',t)$, $j = 1,2$, $k = 0,1,2$, and extend by zero the functions $f_{j,i}, g_{j,i}$ into $\mathcal{S}_j$ $(i \in \mathcal{I}_1)$ and $f_{1,i}, g_{1,i}$ into $\mathcal{S}_2$ $(i \in \mathcal{I}_2)$ and $\eta_{k,i}, \varphi_{j,i}, p_{1,i}, q_{1,i}$ into $\mathbb{R}^{n-1}$ retaining the preceding notation and we set

$$\mathcal{S}_1 = \mathbb{R}^n_-\,, \quad \mathcal{S}_2 = \mathbb{R}^n_+ \quad \text{and} \quad \mathcal{S}_{jT} = \mathcal{S}_j \times (0,T), \quad j = 1,2\,.$$

1. We define the functions $v'_{j,i}(y,t)$, $z'_{j,i}(y,t)$, $j = 1,2$, $\psi'_i(y',t)$, $i \in \mathcal{I}_1$, satisfying zero initial data, as the solution to the following problem:

$$\partial_t v'_{j,i} - a_j \, \Delta_y v'_{j,i} - \chi(\lambda(\xi^{(i)}))\, \alpha_1(x^{(i)},0)\, (\partial_t \psi'_i - a_j \, \Delta' \psi'_i) = f_{j,i}(y,t) \quad \text{in } \mathcal{S}_{jT}, \quad j = 1,2, \quad (4.15)$$

$$\partial_t z'_{1,i} - b_1 \, \Delta z'_{1,i} - \chi(\lambda(\xi^{(i)}))\, \beta_1(x^{(i)},0)\, (\partial_t \psi'_i - b_1 \, \Delta' \psi'_i) = g_{1,i}(y,t) \quad \text{in } \mathcal{S}_{1T}, \quad (4.16)$$

$$\partial_t z'_{2,i} - b_2 \Delta z'_{2,i} - \chi(\lambda(\xi^{(i)}))\big[\beta_2(x^{(i)},0) + \gamma_2(x^{(i)},0)\big(\alpha_1(x^{(i)},0) - \alpha_2(x^{(i)},0)\big)\big] \times (\partial_t \psi'_i - b_2 \, \Delta' \psi'_i) = g_{2,i}(y,t) \quad \text{in } \mathcal{S}_{2T}\,, \quad (4.17)$$

$$\left(v'_{1,i} - v'_{2,i}\right)\Big|_{y_n=0} = \eta_{0,i}(y',t)\,, \tag{4.18}$$

$$\left(z'_{j,i} - \gamma_j(\xi^{(i)},0)\, v'_{j,i}\right)\Big|_{y_n=0} = \eta_{j,i}(y',t)\,, \quad j=1,2\,, \tag{4.19}$$

$$\left(\lambda_1\,\partial_{y_n} v'_{1,i} - \lambda_2\,\partial_{y_n} v'_{2,i}\right)\Big|_{y_n=0} = \varphi_{1,i}(y',t)\,, \tag{4.20}$$

$$\left(k_1 \partial_{y_n} z'_{1,i} - k_2 \partial_{y_n} z'_{2,i}\right)\Big|_{y_n=0} + d'_i \nabla'^T \psi'_i - \kappa_1(\xi^{(i)},0)\partial_t \psi'_i = \varphi_{2,i}(y',t), \tag{4.21}$$

where $\chi(\lambda(\xi^{(i)}))=1$, $d'_i=(d_{1,i},...,d_{n-1,i})$, and

$$d'_i \nabla'^T_y \psi'_i = d(\xi^{(i)},0)\nabla^T_x \psi_i(x,t)\Big|_{\substack{x=Y_i^{-1}(y)\\ x=\xi\in\Gamma\cap K_{i,2\delta}}}.$$

Here we consider the equations $(4.15)_2$, and (4.17) in accordance with the representation of the original equations $(4.1)_2$,$(4.2)_2$ in the form (4.12),(4.13) respectively. We have also used the standard notation $\Delta' = \partial^2_{y_1} + \cdots + \partial^2_{y_{n-1}}$ and $\nabla' = (\partial_{y_1},...,\partial_{y_{n-1}})$.

2. If $i\in\mathcal{I}_2$, we find the functions $v'_{1,i}(y,t)$, $z'_{1,i}(y,t)$ as the solutions to the first boundary value problems

$$\begin{aligned} &\partial_t v'_{1,i} - a_1\,\Delta v'_{1,i} = f_{1,i}(y,t) \quad \text{in } \mathcal{S}_{2T}, \\ &v'_{1,i}\big|_{t=0}=0, \quad v'_{1,i}\big|_{y_n=0} = p_{1,i}(y',t) \end{aligned} \tag{4.22}$$

$$\begin{aligned} &\partial_t z'_{1,i} - b_1\,\Delta z'_{1,i} = g_{1,i}(y,t) \quad \text{in } \mathcal{S}_{2T}, \\ &z'_{1,i}\big|_{t=0}=0, \quad z'_{1,i}\big|_{y_n=0} = q_{1,i}(y',t). \end{aligned} \tag{4.23}$$

We note that (4.15)–(4.21) is the model problem (B.1)–(B.7) studied in Appendix B. So, by Theorem B.1, it has a unique solution $v'_{j,i}, z'_{j,i} \in \overset{\circ}{C}{}^{2+\alpha}_s(\mathcal{S}_{jT})$, $j=1,2$, $\psi'_i \in \overset{\circ}{C}{}^{2+\alpha}_s(R_T)$, $\partial_t\psi_i \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$, $i\in\mathcal{I}_1$, and it satisfies the estimate (B.9) (here R is the plane $y_n=0$, $R_T = R\times[0,T]$). The solutions $v'_{1,i}, z'_{1,i}$, $i\in\mathcal{I}_2$, to the problems (4.22),(4.23) exist and belong to the space $\overset{\circ}{C}{}^{2+\alpha}_s(\mathcal{S}_{2T})$ [21].

The estimates for the solutions to problem (4.15)–(4.21) and to the first boundary-value problems (4.22), (4.23) in coordinates $\{x\}$ take the form

$$\sum_{j=1}^{2}\left(|v_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}} + |z_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}}\right) + |\psi_i|^{(2+\alpha)}_{s,\Gamma_{i,T}} + |\partial_t\psi_i|^{(1+\alpha)}_{s-1,\Gamma_{i,T}} \tag{4.24}$$

$$\le C_2\Big[\sum_{j=1}^{2}\big(|\zeta_i f_j|^{(\alpha)}_{s-2,K^{(j)}_{i,T}} + |\zeta_i g_j|^{(\alpha)}_{s-2,K^{(i)}_{i,T}}$$
$$+ |\zeta_i\eta_j|^{(2+\alpha)}_{s,\Gamma_{i,T}} + |\zeta_i\varphi_j|^{(1+\alpha)}_{s-1,\Gamma_{i,T}}\big) + |\zeta_i\eta_0|^{(2+\alpha)}_{s,\Gamma_{i,T}}\Big], \quad i\in\mathcal{I}_1,$$
$$|v_{1,i}|^{(2+\alpha)}_{s,K^{(1)}_{i,T}} \le C_3\Big(|\zeta_i f_1|^{(\alpha)}_{s-2,K^{(1)}_{i,T}} + |\zeta_i p_1|^{(2+\alpha)}_{s,\Sigma_{i,T}}\Big), \qquad i\in\mathcal{I}_2, \tag{4.25}$$
$$|z_{1,i}|^{(2+\alpha)}_{s,K^{(1)}_{i,T}} \le C_4\Big(|\zeta_i g_1|^{(\alpha)}_{s-2,K^{(1)}_{i,T}} + |\zeta_i q_1|^{(2+\alpha)}_{s,\Sigma_{i,T}}\Big), \qquad i\in\mathcal{I}_2,$$

where $v'_{j,i},\ z'_{j,i}\big|_{y=Y_i(x)} = v_{j,i}(x,t),\ z_{j,i}(x,t),\ i\in\mathcal{I}_1\cup\mathcal{I}_2,\ \psi'_i\big|_{\substack{y=Y_i(x)\\ y_n=0}} = \psi_i(\xi,t)$, $\xi\in\Gamma\cap K_{i,2\delta},\ i\in\mathcal{I}_1,\ K^{(j)}_{i,T} = (K_{i,2\delta}\cap\Omega_j)\times(0,T),\ j=1,2$, and $\Gamma_{i,T} = (\Gamma\cap K_{i,2\delta})\times[0,T]$, $\Sigma_{i,T} = (\partial\Omega\cap K_{i,2\delta})\times[0,T]$.

3. Let the ball $K_{i,\delta}$, $i\in\mathcal{I}_3$, be entirely inside Ω_1 or Ω_2, but $K_{i,\delta}\cap\mathcal{O}\ne\emptyset$ (here $\mathcal{O}$ is $2\lambda_0$-neighbourhood of Γ). We determine the functions $v_{j,i}(x,t)$, $z_{j,i}(x,t)$, $i\in\mathcal{I}_3$, as the solutions of a Cauchy problem

$$\partial_t v_{j,i} - a_j\,\Delta v_{j,i} = \zeta_i f_j(x,t) + \chi(\lambda(\overline{x}^{(i)}))\,\alpha_1(x^{(i)},0)\,(\partial_t\psi_i - a_j\,\Delta\psi_i) \tag{4.26}$$

in $\mathbb{R}^n_T$, $v_{j,i}\big|_{t=0}=0$, where $j=1,2$, $\overline{x}^{(i)}\in K_{i,\delta}\cap\mathcal{O}$, $\mathbb{R}^n_T = \mathbb{R}^n\times(0,T)$;

$$\partial_t z_{1,i} - b_1\,\Delta z_{1,i} = \zeta_i g_1(x,t) + \chi(\lambda(\overline{x}^{(i)}))\,\beta_1(x^{(i)},0)\,(\partial_t\psi_i - b_1\,\Delta\psi_i)$$
$$\text{in}\ \ \mathbb{R}^n_T,\quad z_{1,i}\big|_{t=0}=0, \tag{4.27}$$
$$\partial_t z_{2,i} - b_2\,\Delta z_{2,i} =$$
$$= \zeta_i g_2(x,t) + \chi(\lambda(\overline{x}^{(i)}))\big[\beta_2(x^{(i)},0) + \gamma_2(x^{(i)},0)\big(\alpha_1(x^{(i)},0) - \alpha_2(x^{(i)},0)\big)\big]$$
$$\times\ (\partial_t\psi_i - b_2\,\Delta\psi_i)\quad\text{in}\ \ \mathbb{R}^n_T,\qquad z_{2,i}\big|_{t=0}=0; \tag{4.28}$$

here the functions f_j, g_j defined in $\Omega_j\cap K_{i,2\delta}$ are extended to $K_{i,2\delta}$ by preserving its regularity $\overset{\circ}{C}{}^{\alpha}_{s-2}$ and then the products $\zeta_i f_j$, $\zeta_i g_j$ are extended by zero into $\mathbb{R}^n$ (we keep the same notation for them).

In the right-hand sides of the equations of the problems (4.26)–(4.28) there are functions $\psi_i(\xi,t)$, $i\in\mathcal{I}_3$. We choose them as follows. Let x^0 be an arbitrary point in $K_{i,\delta}\cap\mathcal{O}$, $i\in\mathcal{I}_3$, and $\xi^0\in\Gamma$ be the origin of the vector $N(\xi^0)$, on which the point x^0 is situated. In the point ξ^0 there is determined, at least, one function $\psi_k(\xi)$, $k\in\mathcal{I}_1$, because each point $\xi^0\in\Gamma$ is contained, at least, in one ball $K_{k,\delta}$, $k\in\mathcal{I}_1$. For all $x\in K_{i,\delta}\cap\mathcal{O}$, $i\in\mathcal{I}_3$, we have the corresponding functions $\psi_k(\xi,t)$, $k\in n(i)\subset\mathcal{I}_1$, $n(i)\neq\emptyset$. For $i\in\mathcal{I}_3$ we set

$$\psi_i(\xi,t)=\sum_{k\in n(i)}\psi_k(\xi,t)\,\zeta_k(x)\Big(\sum_{k\in n(i)}\zeta_k(x)\Big)^{-1}\Big|_\Gamma$$

and put it in the right-hand sides of the equations of problems (4.26)–(4.28).

4. At last, if $K_{i,\delta}$, $i\in\mathcal{I}_4$, is entirely inside Ω_1 or Ω_2 and $K_{i,\delta}\cap\mathcal{O}=\emptyset$ we find the functions $v_{j,i}(x,t)$, $z_{j,i}(x,t)$, $i\in\mathcal{I}_4$, $j=1,2$, as the solutions of the Cauchy problems

$$\partial_t v_{j,i}-a_j\,\Delta v_{j,i}=\zeta_i f_j(x,t)\quad\text{in }\ \mathbb{R}^n_T\,,\qquad v_{j,i}\big|_{t=0}=0\,,\tag{4.29}$$

$$\partial_t z_{j,i}-b_j\,\Delta z_{j,i}=\zeta_i g_j(x,t)\quad\text{in }\ \mathbb{R}^n_T\,,\qquad z_{j,i}\big|_{t=0}=0;\tag{4.30}$$

for $j=1,2$, the functions $\zeta_i f_j$, $\zeta_i g_j$ are extended into $\mathbb{R}^n$ as in problems (4.26)–(4.28).

The Cauchy problems (4.26)–(4.30) have unique solutions

$$v_{j,i},z_{j,i}\in\overset{\circ}{C}{}^{2+\alpha}_s(\mathbb{R}^n_T),\qquad i\in\mathcal{I}_3\cup\mathcal{I}_4,\quad j=1,2,$$

and the following estimates hold

$$\begin{aligned}
&|v_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}}\le C_5\Big(|\zeta_i f_j|^{(\alpha)}_{s-2,K^{(j)}_{i,T}}+|\psi_i|^{(2+\alpha)}_{s,\Gamma_{i,T}}\Big),\\
&|z_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}}\le C_6\Big(|\zeta_i g_j|^{(\alpha)}_{s-2,K^{(j)}_{i,T}}+|\psi_i|^{(2+\alpha)}_{s,\Gamma_{i,T}}\Big),\qquad i\in\mathcal{I}_3,\quad j=1,2\,,
\end{aligned}\tag{4.31}$$

$$\begin{aligned}
&|v_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}}\le C_7\,|\zeta_i f_j|^{(\alpha)}_{s-2,K^{(j)}_{i,T}}\,,\\
&|z_{j,i}|^{(2+\alpha)}_{s,K^{(j)}_{i,T}}\le C_8\,|\zeta_i g_j|^{(\alpha)}_{s-2,K^{(j)}_{i,T}}\,,\qquad i\in\mathcal{I}_4,\quad j=1,2\,.
\end{aligned}\tag{4.32}$$

Thus, we have constructed the regularizer $\mathcal{R}h$ in (4.14).

We introduce the norms [13]

$$\{w\}_{\mathcal{B}(\Omega_T)}=\sup_i\|w\|_{\mathcal{B}(K_{i,T})}\,,\tag{4.33}$$

$$\{h\}_{\mathcal{H}(\Omega_T)}=\sup_i\|h\|_{\mathcal{H}(K_{i,T})}\,,\tag{4.34}$$

where $K_{i,T}=(K_{i,2\delta}\cap\Omega)\times(0,T)$.

The norms (4.33), (4.34) are equivalent to the norms $\|w\|_{\mathcal{B}(\Omega_T)}$, $\|h\|_{\mathcal{H}(\Omega_T)}$ determined in formulas (3.45), (3.46) and (4.11).

Lemma 4.1. *The operator* $\mathcal{R}\colon\mathcal{H}(\Omega_T)\to\mathcal{B}(\Omega_T)$ *is bounded:*

$$\{\mathcal{R}h\}_{\mathcal{B}(\Omega_T)}\le C_9\,\{h\}_{\mathcal{H}(\Omega_T)}\,.$$

This estimate holds on the basis of estimates (4.24), (4.25), (4.31), (4.32).

Lemma 4.2. *For any vector $h \in \mathcal{H}(\Omega_T)$ the following identity is fulfilled:*

$$\mathcal{A}\mathcal{R}h = h + Ph \equiv (E+P)h\,, \tag{4.35}$$

where $Ph = \{P_1h, P_2h, P_3h, P_4h, 0, 0, 0, P_5h, P_6h, P_7h, P_8h\}$ is a fully defined vector and E the identity operator.

Proof. We substitute $\mathcal{R}_1h - \mathcal{R}_5h$ (see (4.14)) into equations and conditions of the problem $\mathcal{A}[w] = h$ instead of the functions v_1, v_2, z_1, z_2, ψ respectively. After some computations taking into consideration that $v_{j,i}$, $z_{j,i}$, ψ_i are the solutions of the model problems (4.15)–(4.21), $i \in \mathcal{I}_1$, (4.22), (4.23), $i \in \mathcal{I}_2$, (4.26)–(4.28), $i \in \mathcal{I}_3$, (4.29), (4.30), $i \in \mathcal{I}_4$, we obtain formula (4.35), where the functions $P_1h - P_8h$ are the rest of $\mathcal{A}\mathcal{R}h$ after extraction of the vector h. We note also that a function P_2h contains the additional term

$$\chi(\lambda)\big(\alpha_1(x,t) - \alpha_2(x,t)\big)\sum_{i\in\mathcal{I}_1}\Big[\mu_i(\partial_t\psi_i - a_2\Delta\psi_i) - a_2\psi_i\Delta\mu_i - 2a_2\,\nabla\psi_i\nabla^T\mu_i\Big], \tag{4.36}$$

the term similar to (4.36) with $\gamma_2(x,t)\,(\alpha_1(x,t) - \alpha_2(x,t))$ and b_2 instead of $\alpha_1 - \alpha_2$ and a_2 is included in P_4h by the representations of the equations $(4.1)_2$,$(4.2)_2$ in the form (4.12),(4.13). □

Lemma 4.3. *Under the assumptions of Theorem* 4.1 *there exists $T_1 > 0$ such that the operator $\mathcal{A}$ has the right inverse bounded operator $\mathcal{A}_r^{-1} = \mathcal{R}(E + P)^{-1}\colon \mathcal{H}(\Omega_{T_1}) \to \mathcal{B}(\Omega_{T_1})$.*

Proof. With the help of the estimates (4.24), (4.25), (4.31), (4.32) for the functions $v_{j,i}$, $z_{j,i}$, $j = 1, 2$, ψ_i, we estimate the norm of Ph. Choosing the radius δ of the balls $K_{i,\delta}$ and $t \le T_1$ sufficiently small and making use of condition (4.10) we obtain the estimate

$$\{Ph\}_{\mathcal{H}(\Omega_t)} \le \varepsilon\{h\}_{\mathcal{H}(\Omega_t)} \quad \forall\, t \le T_1\,, \tag{4.37}$$

where $\varepsilon \in (0,1)$ (see (4.34)).

We consider an equation $h + Ph = h_1$, where $h_1 \in \mathcal{H}(\Omega_T)$. By estimate (4.37) it has a unique solution $h \in \mathcal{H}(\Omega_{T_1})$ and $\{h\}_{\mathcal{H}(\Omega_{T_1})} \le \frac{1}{1-\varepsilon}\,\{h_1\}_{\mathcal{H}(\Omega_{T_1})}$ for every vector $h_1 \in \mathcal{H}(\Omega_{T_1})$; that is, the inverse operator $(E+P)^{-1}$ exists and is bounded on the whole space $\mathcal{H}(\Omega_{T_1})$. Substituting $h = (E+P)^{-1}h_1$ into the left-hand side of the identity (4.35) and taking into account that $h + Ph = h_1$, we obtain the identity $\mathcal{A}\mathcal{R}(E+P)^{-1}h_1 = h_1$ for $h_1 \in \mathcal{H}(\Omega_{T_1})$ or $\mathcal{A}\mathcal{R}(E+P)^{-1} = E$.

From here it follows that an operator $\mathcal{A}$ has the inverse right bounded operator $\mathcal{A}_r^{-1} = \mathcal{R}(E+P)^{-1}$ determined in the whole space $\mathcal{H}(\Omega_{T_1})$. □

Therefore, problem $\mathcal{A}[w] = h$ has a solution $w = (v_1, v_2, z_1, z_2, \psi) \in \mathcal{B}(\Omega_{T_1})$ for every vector $h \in \mathcal{H}(\Omega_{T_1})$.

To find the estimate (4.11) to the solution we multiply both parts of the equations $(4.1)_1$, (4.12), $(4.2)_1$, (4.13) and conditions (4.3)–(4.7) by a cut-off function $\zeta_i(x)$. After some computations for the functions $\zeta_i v_j$, $\zeta_i z_j$, $j = 1, 2$, $\zeta_i \psi$, we obtain a model transmission problem of the type (B.1)–(B.7), or the first boundary-value problem, or Cauchy problem in connection with the position of the ball $K_{i,\delta}$, $i \in \mathcal{I} = \bigcup_{k=1}^{4} \mathcal{I}_k$. In the right-hand sides of the equations and transmission conditions of these model problems we shall have given functions multiplied by ζ_i and the operators similar to $P_1 h - P_8 h$, depending on w. We write the estimates for the solutions to these model problems. Evaluating the so-obtained operators, choosing a radius δ of balls and $t \le T_2$ sufficiently small, applying condition (4.10) and taking into account that $\zeta_i(x) = 1$ in $K_{i,\delta}$, we can achieve the estimate

$$\|w\|_{\mathcal{B}(\omega_{i,t})} \le C_{10} \, \|h\|_{\mathcal{H}(\omega_{i,t})} + \varepsilon \, \|w\|_{\mathcal{B}(\omega_{i,t})} \, , t \le T_2 \, ,$$

where $\varepsilon \in (0,1)$, $i \in \mathcal{I}$, $\omega_{i,t} = (\Omega \cap K_{i,\delta}) \times (0,t)$. From here we find

$$\begin{aligned} \{w\}_{\mathcal{B}(\Omega_t)} &= \sup_{i \in \mathcal{I}} \|w\|_{\mathcal{B}(\omega_{i,t})} \le C_{10}(1-\varepsilon)^{-1} \sup_{i \in \mathcal{I}} \|h\|_{\mathcal{H}(\omega_{i,t})} \\ &= C_{10}(1-\varepsilon)^{-1} \, \{h\}_{\mathcal{H}(\Omega_t)} \, , \qquad t \le T_2 \, . \end{aligned}$$

By the equivalence of the norms (4.33), (4.34) to the norms $\|w\|_{\mathcal{B}(\Omega_t)}$ and $\|h\|_{\mathcal{H}(\Omega_t)}$ we derive the estimate (4.11) for $t \le T_2$.

Thus, we have proved the theorem for $t \le \min(T_1, T_2)$. We extend the solution into the segment $[0,T]$ as, for example, in [9] and obtain Theorem 4.1 for $t \le T$.

5. Proof of Theorem 2.1

We have transformed the free boundary problem (2.1)–(2.8) into the nonlinear problem (3.25)–(3.31) in fixed given domains:

$$\mathcal{A}[w] = h + \mathcal{N}[w] \, , \tag{5.1}$$

where $w = (v_1, v_2, z_1, z_2, \psi)$ is unknown and h is a given vector function.

In Theorem 4.1 we have proved the existence and uniqueness of the solution to the linear problem $\mathcal{A}[w] = h$ defined by (4.1)–(4.7) in the space

$\mathcal{B}(\Omega_T)$ and established the estimate (4.11)

$$\|w\|_{\mathcal{B}(\Omega_T)} \le C_1 \|h\|_{\mathcal{H}(\Omega_T)} \quad \forall\, h \in \mathcal{H}(\Omega_T);$$

that is, the linear operator $\mathcal{A}$ has the inverse bounded operator $\mathcal{A}^{-1}$ in $\mathcal{H}(\Omega_T)$.

Under the assumptions of Theorem 2.1 all the conditions of Theorem 4.1 hold, so we can write problem (5.1) in the form

$$w = \mathcal{A}^{-1}[h + \mathcal{N}[w]]$$

and apply the estimate (4.11)

$$\|w\|_{\mathcal{B}(\Omega_t)} \le C_1 \Big(\|h\|_{\mathcal{H}(\Omega_t)} + \|\mathcal{N}[w]\|_{\mathcal{H}(\Omega_t)} \Big), \qquad t \le T\,.$$

Let $K(M)$ be the closed ball in the space $\mathcal{B}(\Omega_T)$:

$$K(M) = \Big\{ w \in \mathcal{B}(\Omega_{T_0})\colon\ \|w\|_{\mathcal{B}(\Omega_{T_0})} \le M \Big\}\,,$$

where $M = C_1 \|h\|_{\mathcal{H}(\Omega_{T_0})}(1-q)^{-1}$, $q \in (0,1)$.

We shall prove that the nonlinear operator $\mathcal{A}^{-1}[h+\mathcal{N}[w]]$ acts from $K(M)$ into $K(M)$ and is a contraction for small $t \le T_0$. For that we shall estimate the following norms:

$$I_1 = \Big\| \mathcal{A}^{-1}[h + \mathcal{N}[w]] \Big\|_{\mathcal{B}(\Omega_t)} \le C_1 \Big(\|h\|_{\mathcal{H}(\Omega_t)} + \|\mathcal{N}[w]\|_{\mathcal{H}(\Omega_t)} \Big)\,, \tag{5.2}$$

$$\begin{aligned} I_2 &= \Big\| \mathcal{A}^{-1}[h + \mathcal{N}[w]] - \mathcal{A}^{-1}[h + \mathcal{N}[\widetilde{w}]] \Big\|_{\mathcal{B}(\Omega_t)} \\ &\equiv \Big\| \mathcal{A}^{-1}[\mathcal{N}[w] - \mathcal{N}[\widetilde{w}]] \Big\|_{\mathcal{B}(\Omega_t)} \le C_1 \Big\| \mathcal{N}[w] - \mathcal{N}[\widetilde{w}] \Big\|_{\mathcal{H}(\Omega_t)} \qquad (5.3) \\ &\qquad\qquad \forall\, w, \widetilde{w} \in K(M), \quad t \le T\,. \end{aligned}$$

We rewrite in details the norms in the right-hand sides of the inequalities (5.2), (5.3) substituting there the corresponding functions from the right-hand sides of the equations and conditions of the problem (3.25)–(3.31)

$$I_1 \le C_1 \|h\|_{\mathcal{H}(\Omega_t)} + C_1 \sum_{j=1}^{2} \Big(|F_j|^{(\alpha)}_{s-2,\Omega_{jt}} + |G_j|^{(\alpha)}_{s-2,\Omega_{jt}} + |R_j|^{(2+\alpha)}_{s,\Gamma_t} + |\Phi_j|^{(1+\alpha)}_{s-1,\Gamma_t} \Big), \tag{5.4}$$

$$I_2 \le C_1 \Big(\sum_{j=1}^{2} \Big(\big| F_j(v_j,\psi) - F_j(\widetilde{v}_j,\widetilde{\psi}) \big|^{(\alpha)}_{s-2,\Omega_{jt}} + \big| G_j(z_j,\psi) - G_j(\widetilde{z}_j,\widetilde{\psi}) \big|^{(\alpha)}_{s-2,\Omega_{jt}}$$

$$+ \big|R_j(v_j) - R_j(\widetilde{v}_j)\big|^{(2+\alpha)}_{s,\Gamma_t}\Big) + \big|\Phi_1(v_1, v_2, \psi) - \Phi_1(\widetilde{v}_1, \widetilde{v}_2, \widetilde{\psi})\big|^{(1+\alpha)}_{s-1,\Gamma_t} \qquad (5.5)$$
$$+ \big|\Phi_2(z_1, z_2, \psi) - \Phi_2(\widetilde{z}_1, \widetilde{z}_2, \widetilde{\psi})\big|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)$$

for all $w, \widetilde{w} \in K(M)$, where the functions F_j, G_j, R_j, Φ_j, $j = 1, 2$, are defined by formulas (3.34), (3.35), (3.39), (3.42), (3.43).

To estimate the norms in (5.4), (5.5) we apply the estimates (A.6)–(A.11) for the norms of the functions and their products, the estimates (A.12)–(A.15), (A.18)–(A.28), (A.35)–(A.40), (A.41), (A.53) and (A.55),(A.56) for the matrices J_{01}, $J_1 = J_{11} + J_{12}$, $J_{01}J_1$, J^{-1}, J_0^{-1} and $J^{-1}(\psi) - J^{-1}(\widetilde{\psi})$ respectively, then we obtain

$$|F_j|^{(\alpha)}_{s-2,\Omega_{jt}} \le$$
$$\le C\Big[\Big(t^{\frac{1}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases} + t^{\frac{s}{2}}|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t} + t^{\frac{s-1}{2}}|\psi|^{(2+\alpha)}_{s,\Gamma_t}\Big)|v_j|^{(2+\alpha)}_{s,\Omega_t}$$
$$+ \Big(t^{\frac{s-1}{2}} + t^{\frac{s}{2}} + t^{\frac{1+s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases}\Big)|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big]; \qquad (5.6)$$

the estimate for the function G_j is like (5.6) with z_j instead of v_j;

$$|R_j|^{(2+\alpha)}_{s,\Gamma_t} \le C\, t^{\frac{s}{2}}\big(|v_j|^{(2+\alpha)}_{s,\Omega_t}\big)^2\Big(1 + |v_j|^{(2+\alpha)}_{s,\Omega_t} + \sum_{k=0}^{3} æ_k(t)\big(|v_j|^{(2+\alpha)}_{s,\Omega_t}\big)^k\Big),\ j = 1, 2, \qquad (5.7)$$

where $æ_k(t)$ are positive-definite functions for $t > 0$ consisting of powers of t and $æ_k(0) = 0$,

$$|\Phi_1|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\Big[\Big(\begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases} + t^{\frac{s-1}{2}}|\psi|^{(2+\alpha)}_{s,\Gamma_t}\Big)\sum_{k=1}^{2}|v_k|^{(2+\alpha)}_{s,\Omega_t}$$
$$+ \delta_0\big(|\psi|^{(2+\alpha)}_{s,\Gamma_t} + |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\big)\Big], \qquad (5.8)$$

$$|\Phi_2|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\Big[\Big(\Big(t^{\frac{s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases}\Big)|\psi|^{(2+\alpha)}_{s,\Gamma_t}$$
$$+ t^{\frac{s}{2}}|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)\sum_{k=1}^{2}|z_k|^{(2+\alpha)}_{s,\Omega_t} \qquad (5.9)$$

$$+\Big(\begin{cases} t^{\frac{s}{2}}, & 1<s<2\\ t, & 2\le s\le 2+\alpha\end{cases} + t + t^{\frac{s-1}{2}}\big(|\psi|^{(2+\alpha)}_{s,\Gamma_t} + |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\big)\Big)|\psi|^{(2+\alpha)}_{s,\Gamma_t}\Big];$$

the estimates for the norms of the differences of the functions F_j, G_j, R_j, Φ_j, $j=1,2$, in (5.5) are analogous to the estimates (5.6)–(5.9); they contain the norms $|v_j-\widetilde{v}_j|^{(2+\alpha)}_{s,\Omega_t}$, $|z_j-\widetilde{z}_j|^{(2+\alpha)}_{s,\Omega_t}$, $|\psi-\widetilde{\psi}|^{(2+\alpha)}_{s,\Gamma_t}$, $|\partial_t\psi-\partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_t}$. Here $C>0$ denotes some constant independent of $t\in(0,T)$.

We substitute those estimates into inequalities (5.4), (5.5), then we derive

$$I_1 \le C_1\,\|h\|_{\mathcal{H}(\Omega_t)} + r_1(t)\|w\|_{\mathcal{B}(\Omega_t)} \quad\text{and}\quad I_2 \le r_2(t)\|w-\widetilde{w}\|_{\mathcal{B}(\Omega_t)}, \tag{5.10}$$

where

$$r_j(t) = C_{1+j}\Big(\delta_0 + t^{\frac12} + t + t^{\frac{1+s}{2}} +$$

$$\Big(t^{\frac{s-1}{2}} + t^{\frac{s}{2}} + \begin{cases} t^{\frac{s}{2}}, & 1<s<2\\ t, & 2\le s\le 2+\alpha\end{cases}\Big)(1+M) + t^{\frac{s}{2}}M\big(M+\sum_{k=0}^{3}æ_k(t)M^k\big)\Big),$$

$j=1,2$, and

$$\|w\|_{\mathcal{B}(\Omega_t)},\ \|\widetilde{w}\|_{\mathcal{B}(\Omega_t)} \le M = C_1\,\|h\|_{\mathcal{H}(\Omega_{T_0})}\,(1-q)^{-1}, \qquad q\in(0,1),\quad \delta_0\ll 1\,.$$

We find $T_3>0$ from the inequalities

$$r_j(t)\le q\,, \qquad q\in(0,1),\quad j=1,2\,.$$

Then, from (5.10), we obtain

$$I_1 \le C_1\,\|h\|_{\mathcal{H}(\Omega_{T_0})} + qM = M\,, \qquad I_2 \le q\,\|w-\widetilde{w}\|_{\mathcal{B}(\Omega_t)} \qquad \forall\, t\le T_0\,,$$

where $T_0=\min(t_0,t_1,t_2,t_3,T_3)$. We recall that the parametrization of the free boundary by the equation (2.11) is valid for $t\le t_0$; in Theorems A.1–A.3 we prove the existence of the inverse matrices J^{-1} and J_0^{-1} for $t\le t_1$ and $t\le t_2$ respectively and the estimates of the difference $J^{-1}(\psi)-J^{-1}(\widetilde{\psi})$ for $t\le t_3$. Thus, we have

$$\big\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]\big\|_{\mathcal{B}(\Omega_t)} \,\le\, M\,,$$

$$\big\|\mathcal{A}^{-1}[h+\mathcal{N}[w]] - \mathcal{A}^{-1}[h+\mathcal{N}[\widetilde{w}]]\big\|_{\mathcal{B}(\Omega_t)} \,\le\, q\,\|w-\widetilde{w}\|_{\mathcal{B}(\Omega_t)}$$

for all $w,\widetilde{w}\in K(M)$, $t\le T_0$.

Whence it follows that the operator $\mathcal{A}^{-1}[h+\mathcal{N}[w]]$ is a contraction from the closed ball $K(M)$ into itself. Therefore problem (5.1) (i.e. (3.25)–(3.31)) has a unique solution $w=(v_1,v_2,z_1,z_2,\psi)\in\mathcal{B}(\Omega_{T_0})$ and it satisfies the estimate

$$\|w\|_{\mathcal{B}(\Omega_t)} = \big\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]\big\|_{\mathcal{B}(\Omega_t)} \le C_1(1-q)^{-1}\,\|h\|_{\mathcal{H}(\Omega_t)} \ \ \forall\, t\le T_0. \tag{5.11}$$

Now we return to the problem (2.1)–(2.8). From formulas (3.9), (3.23), (3.24) we have

$$\begin{aligned} &\rho = \rho_0 + \psi\,, \\ &u_j(x,t) = v_j(x - N\chi\rho, t) + V_j(x - N\chi\rho, t)\,, \\ &c_j(x,t) = z_j(x - N\chi\rho, t) + Z_j(x - N\chi\rho, t)\,, \qquad j = 1,2\,, \end{aligned} \tag{5.12}$$

where V_j, Z_j, and ρ_0 belong to $C_s^{2+\alpha}(\mathbb{R}_T^n)$ and $C_{1+s}^{3+\alpha}(\Gamma_T)$, respectively, and $v_j, z_j \in \overset{\circ}{C}{}_s^{2+\alpha}(\Omega_{jT_0})$, $j = 1,2$, $\psi \in \overset{\circ}{C}{}_s^{2+\alpha}(\Gamma_{T_0})$, $\partial_t\psi \in \overset{\circ}{C}{}_{s-1}^{1+\alpha}(\Gamma_{T_0})$. Then by direct evaluations of the norms we obtain that the compositions of the functions v_j, z_j, V_j, Z_j with ρ belong to $C_s^{2+\alpha}(Q_{jT_0})$, $j = 1,2$.

From formulas (5.12) we derive

$$\begin{aligned} &\sum_{j=1}^{2}\Big(|u_j|_{s,Q_{jt}}^{(2+\alpha)} + |c_j|_{s,Q_{jt}}^{(2+\alpha)}\Big) + |\rho|_{s,\Gamma_t}^{(2+\alpha)} + |\partial_t\rho|_{s-1,\Gamma_t}^{(1+\alpha)} \\ &\quad \le C\Big(\|w\|_{\mathcal{B}(\Omega_t)} + \sum_{j=1}^{2}\Big(|V_j|_{s,Q_{jt}}^{(2+\alpha)} + |Z_j|_{s,Q_{jt}}^{(2+\alpha)}\Big) + |\rho_0|_{1+s,\Gamma_t}^{(3+\alpha)}\Big)\,, \qquad t \le T_0, \end{aligned} \tag{5.13}$$

where $w = (v_1, v_2, z_1, z_2, \psi)$ satisfies the estimate (5.11) with $h = (f_1, f_2, g_1, g_2, p_1, q_1, \eta_0, \eta_1, \eta_2, \varphi_1, \varphi_2)$. For the functions V_j, Z_j, ρ_0 and this vector h we may use the estimates (3.20) – (3.22) and (3.47) of Lemmas 3.1 and 3.2 to complete the proof of Theorem 2.1. □

Appendix A. Expansions and estimates of the inverse Jacobian matrix J^{-1}

We consider a Jacobian matrix (3.5) J of the transformation of coordinates (3.3)

$$J = \Big\{\delta_{ij} + \partial_{y_j}\Big(N_i\,\chi(\rho_0 + \psi)\Big)\Big\}_{1\le i,\,j\le n} = I + J_{01} + J_1\,,$$

with identity matrix I, $J_{01} = \{\partial_{y_j}(N_i\,\chi\,\rho_0)\}_{1\le i,\,j\le n} = (\nabla^T N\chi\,\rho_0)^T$ and $J_1 = \{\partial_{y_j}(N_i\,\chi\,\psi)\}_{1\le i,\,j\le n}$, where $N(\xi) = (N_1,\dots,N_n)$, $\chi(\lambda(y)), N_i \in C^{2+\alpha}(\Gamma)$, $i = 1,\dots,n$, $\xi \in \Gamma$. We denote

$$\begin{aligned} &J_0 = I + J_{01}\,, \qquad J_1 = J_{11} + J_{12}\,, \\ &J_{11} = \{N_i\,\chi\,\partial_{y_j}\psi\}_{1\le i,\,j\le n} = N^T\chi\,\nabla\psi\,, \qquad J_{11}^T = (\nabla^T\psi)\,N\chi\,, \\ &J_{12} = \psi\{\partial_{y_j}(N_i\,\chi)\}_{1\le i,\,j\le n} = \psi(\nabla^T(N\chi))^T\,. \end{aligned}$$

We note that $J\big|_{t=0} = I$, $J_0\big|_{t=0} = I$, because $\rho_0\big|_{t=0} = 0$, $\psi\big|_{t=0} = 0$. That is, for small t the inverse matrices J^{-1}, J_0^{-1} exist (see Theorems A.1, A.2 below).

Now we write the expansion formulas of the matrices J^{-1}, J_0^{-1} by straightforward linear algebra computations.

Lemma A.1. *For $t \le t_1$ the matrix J^{-1} can be represented in the form*

$$J^{-1} = J_0^{-1} - J_0^{-1} J_1\, J^{-1} = J_0^{-1} - J^{-1} J_1\, J_0^{-1}\,, \tag{A.1}$$

$$J^{-1} = I - BJ^{-1}\,, \qquad J^{-1} = I - J^{-1}B\,, \tag{A.2}$$

$$J^{-1} = I - B + B^2 J^{-1} = I - B + J^{-1}B^2\,, \tag{A.3}$$

$$J^{-1} = (I - B)\,(I - B^2)^{-1} = (I - B^2)^{-1}\,(I - B)\,, \tag{A.4}$$

where $B = J_{01} + J_1$.

Lemma A.2. *For $t \le t_2$ the matrix J_0^{-1} can be represented in the form*

$$J_0^{-1} = I - J_{01} J_0^{-1}\,, \qquad J_0^{-1} = I - J_0^{-1} J_{01}\,. \tag{A.5}$$

To estimate the norms of the functions in the weighted Hölder spaces we shall use the following results. We denote by C different positive constants.

Lemma A.3 (Imbedding Theorem [3])**.** *If $q(x,t) \in C_s^{\ell}(\Omega_T)$, ℓ is a positive number, $s \le \ell$, and $k = 2\,i + |j| \le [\ell]$, then $\partial_t^i \partial_x^j q \in C_{s-k}^{\ell-k}(\Omega_T)$ and*

$$|\partial_t^i \partial_x^j q|_{s-k,\Omega_T}^{(\ell-k)} \le |q|_{s,\Omega_T}^{(\ell)}\,. \tag{A.6}$$

Lemma A.4 ([6])**.** *Let ℓ be a positive noninteger, r a nonnegative number, $s \le \ell$, $s + r \ge 0$, $f_1(x,t) \in \overset{\circ}{C}{}_{s+r}^{\ell+r}(\Omega_T)$, $f_2(x,t) \in \overset{\circ}{C}{}_s^{\ell}(\Omega_T)$, $q_1(x,t) \in C_{s+r}^{\ell+r}(\Omega_T)$, $q_2(x,t) \in C_s^{\ell}(\Omega_T)$, then for $t \le T$ the following estimates hold:*

$$|f_1|_{s,\Omega_t}^{(\ell)} \le C\, t^{\frac{r}{2}}\, |f_1|_{s+r,\Omega_t}^{(\ell+r)}\,, \tag{A.7}$$

$$|f_1 q_2|_{s,\Omega_t}^{(\ell)} \le C\, |f_1|_{s+r,\Omega_t}^{(\ell+r)} \Big[\sup_{\tau \le t} \tau^{\frac{r}{2}} |q_2|_{\Omega} + t^{\frac{s+r}{2}} |q_2|_{s,\Omega_t}^{(\ell)}\Big]\,, \tag{A.8}$$

$$|f_1 f_2|_{s,\Omega_t}^{(\ell)} \le C\, t^{\frac{s+r}{2}} |f_1|_{s+r,\Omega_t}^{(\ell+r)}\, |f_2|_{s,\Omega_t}^{(\ell)}\,, \tag{A.9}$$

$$|f_2\, q_1|_{s,\Omega_t}^{(\ell)} \le C\, |f_2|_{s,\Omega_t}^{(\ell)} \Big[t^{\frac{s+r}{2}} |q_1|_{s+r,\Omega_t}^{(\ell+r)} + |q_1|_{\Omega_t}\Big]\,, \tag{A.10}$$

$$|q_1 q_2|_{s,\Omega_t}^{(\ell)} \le C\, |q_1|_{s+r,\Omega_t}^{(\ell+r)}\, |q_2|_{s,\Omega_t}^{(\ell)}\,. \tag{A.11}$$

We estimate the norms of the matrices J_{01}, $J_1 = J_{11} + J_{12}$, then we prove the existence of the inverse matrices J^{-1}, J_0^{-1} and evaluate the difference $J^{-1}(\rho_0 + \psi) - J^{-1}(\rho_0 + \widetilde{\psi})$.

Lemma A.5. *Let* $\rho_0(\xi,t) \in C^{3+\alpha}_{1+s}(\Gamma_T)$, $\alpha \in (0,1)$, $1 < s \leq 2+\alpha$, $\rho_0\big|_{t=0} = 0$ *and* $|\rho_0|^{(3+\alpha)}_{1+s,\Gamma_T} \leq M_1$, $M_1 > 0$.

Then for the matrix J_{01} *the following estimates hold for* $t \leq T$*:*

$$\|J_{01}\|^{(\alpha)}_{s-2,\Gamma_t} = n \max_{i,j} |\partial_{y_j}(N_i\,\chi\,\rho_0)|^{(\alpha)}_{s-2,\Gamma_t} \leq C\,M_1 \begin{cases} t, & 1 < s < 2 \\ t^{\frac{4-s}{2}}, & 2 \leq s \leq 2+\alpha\,, \end{cases} \tag{A.12}$$

$$\|J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t} \leq C\,M_1 \begin{cases} t^{\frac{1}{2}}, & 1 < s < 2 \\ t^{\frac{3-s}{2}}, & 2 \leq s \leq 2+\alpha\,. \end{cases} \tag{A.13}$$

Proof. We write the norms of the element $\partial_{y_j}(N_i\,\chi\,\rho_0)$ in accordance with their definition (2.15); for example,

$$|\partial_{y_j}(N_i\,\chi\,\rho_0)|^{(\alpha)}_{s-2,\Gamma_t} = \sup_{\tau\leq t} \tau^{\frac{2+\alpha-s}{2}} [\partial_{y_j}(N_i\,\chi\,\rho_0)]^{(\alpha)}_{\Gamma'_\tau}$$
$$+ \sup_{\tau\leq t} |\partial_{y_j}(N_i\,\chi\,\rho_0)|_\Gamma \begin{cases} \tau^{\frac{2-s}{2}}, \\ 1, \end{cases} + \begin{cases} 0, & 1 < s \leq 2 \\ [\partial_{y_j}(N_i\,\chi\,\rho_0)]^{s-2}_{\Gamma_t}, & 2 < s \leq 2+\alpha \end{cases}$$

and evaluate each term taking into account that $\rho_0\big|_{t=0} = 0$ (here $\Gamma'_\tau = \Gamma\times[\frac{\tau}{2},\tau]$). □

Corollary A.1. *Let* $f(y,t) \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$, *then for* $t \leq T$

$$\|f J_{01}\|^{(\alpha)}_{s-2,\Gamma_t} \leq C\,M_1\,|f|^{(1+\alpha)}_{s-1,\Gamma_t} \begin{cases} t^{\frac{1+s}{2}}, & 1 < s < 2 \\ t^{\frac{3}{2}}, & 2 \leq s \leq 2+\alpha, \end{cases} \tag{A.14}$$

$$\|f J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t} \leq C\,M_1\,|f_1|^{(1+\alpha)}_{s-1,\Gamma_t} \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \leq s \leq 2+\alpha. \end{cases} \tag{A.15}$$

Proof. We apply estimate (A.8)

$$\|f J_{01}\|^{(\alpha)}_{s-2,\Gamma_t} \leq C|f|^{(1+\alpha)}_{s-1,\Gamma_t}\Big[\sup_{\tau\leq t} \tau^{\frac{1}{2}} n \max_{i,j} |\partial_{y_j}(N_i\chi\rho_0)|_\Gamma + t^{\frac{s-1}{2}} \|J_{01}\|^{(\alpha)}_{s-2,\Gamma_t}\Big],$$

$$\|f J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t} \leq C|f|^{(1+\alpha)}_{s-1,\Gamma_t}\Big[\sup_{\tau\leq t} n \max_{i,j} |\partial_{y_j}(N_i\chi\rho_0)|_\Gamma + t^{\frac{s-1}{2}} \|J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t}\Big].$$

On the basis of the estimates (A.12), (A.13) and

$$|\partial_{y_j}(N_i\,\chi\,\rho_0)| \leq \int_0^\tau |\partial_{\tau_1}\partial_{y_j}(N_i\,\chi\,\rho_0)|\,d\tau_1$$

$$\le \sup_{\tau_1 \le \tau} \Big(|\partial_{\tau_1} \partial_{y_j} (N_i\, \chi\, \rho_0)|_\Gamma \begin{cases} \tau_1^{\frac{2-s}{2}}, & 1 < s < 2 \\ 1, & 2 \le s \le 2+\alpha \end{cases} \Big) \begin{cases} \frac{2}{s}\, \tau^{\frac{s}{2}}, & 1 < s < 2 \\ \tau, & 2 \le s \le 2+\alpha \end{cases}$$

we obtain (A.14), (A.15).

We note that these estimates can be derived with the direct evaluations of the corresponding norms or with the help of an estimate (A.9), because $\partial_{y_j}\rho_0$ may be considered as the function belonging to $\overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$ by the condition $\rho_0\big|_{t=0} = 0$. □

Corollary A.2. *For the matrix J^2_{01} the following estimates hold for $t \le T$*

$$\|J^2_{01}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, M_1^2 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^{\frac{6-s}{2}}, & 2 \le s \le 2+\alpha\,, \end{cases} \tag{A.16}$$

$$\|J^2_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, M_1^2 \begin{cases} t^{\frac{1+s}{2}}, & 1 < s < 2 \\ t^{\frac{5-s}{2}}, & 2 \le s \le 2+\alpha\,. \end{cases} \tag{A.17}$$

Proof. These estimates are found with the help of the inequalities (A.9), (A.12), (A.13). □

Lemma A.6. *Let $\psi(\xi,t) \in \overset{\circ}{C}{}^{2+\alpha}_{s}(\Gamma_T)$, $\partial_t\psi \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(\Gamma_T)$, $\alpha \in (0,1)$, $1 < s \le 2+\alpha$, then for the matrix $J_1 = J_{11} + J_{12}$ the following estimates hold for $t \le T$:*

$$\|J_{11}\|^{(\alpha)}_{s-2,\Gamma_t} = n \max_{i,j} |N_i\, \chi\, \partial_{y_j}\psi|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\,, \tag{A.18}$$

$$\|J_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, |\psi|^{(2+\alpha)}_{s,\Gamma_t}\,, \tag{A.19}$$

$$\|J_{12}\|^{(\alpha)}_{s-2,\Gamma_t} = n \max_{i,j} |\psi\, \partial_{y_j}(N_i\, \chi)|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{3}{2}}\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\,, \tag{A.20}$$

$$\|J_{12}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\,. \tag{A.21}$$

Proof. The estimates (A.18), (A.20), (A.21) are derived with direct evaluation of the corresponding norms and by applying an inequality $|\partial^m_y\psi| \le \int_0^\tau |\partial_{\tau_1}\partial^m_y\psi|\, d\tau_1$, $|m| = 0, 1$; an estimate (A.19) follows from (A.6). □

Corollary A.3. *For the matrix $J_1 = J_{11} + J_{12}$ the following estimates are valid for $t \le T$*

$$\|J_1\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\,, \tag{A.22}$$

$$\|J_1\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\Big(|\psi|^{(2+\alpha)}_{s,\Gamma_t} + |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big). \tag{A.23}$$

Corollary A.4. *Let $f(y,t) \in \overset{\circ}{C}{}^{k+\alpha}_{s-2+k}(\Gamma_T)$, $k = 0,1,2$, then the following estimates are valid for $t \le T$*

$$\|fJ_{11}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{s-1}{2}}\, |\psi|^{(2+\alpha)}_{s,\Gamma_t}\, |f|^{(\alpha)}_{s-2,\Gamma_t}, \qquad k=0, \tag{A.24}$$

$$\|fJ_{11}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{s+k}{2}}\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\, |f|^{(k+\alpha)}_{s-2+k,\Gamma_t}, \qquad k=1,2, \tag{A.25}$$

$$\|fJ_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{s+k-2}{2}}\, |\psi|^{(2+\alpha)}_{s,\Gamma_t}\, |f|^{(k+\alpha)}_{s-2+k,\Gamma_t}, \qquad k=1,2, \tag{A.26}$$

$$\|fJ_{12}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{s+k+1}{2}}\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\, |f|^{(k+\alpha)}_{s-2+k,\Gamma_t}, \qquad k=0,1,2, \tag{A.27}$$

$$\|fJ_{12}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{s+k}{2}}\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\, |f|^{(k+\alpha)}_{s-2+k,\Gamma_t}, \qquad k=1,2. \tag{A.28}$$

Proof. These estimates are derived with the help of the inequalities (A.9) and (A.18)–(A.21). □

Corollary A.5. *For $t \le T$ the following estimates hold:*

$$\|J^2_{11}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{2+s}{2}} \Big(|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)^2,$$

$$\|J^2_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{s-1}{2}} \Big(|\psi|^{(2+\alpha)}_{s,\Gamma_t}\Big)^2, \tag{A.29}$$

$$\|J^2_{12}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{4+s}{2}} \Big(|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)^2, \tag{A.30}$$

$$\|J^2_{12}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{3+s}{2}} \Big(|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)^2,$$

$$\|J_{11}J_{12}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{3+s}{2}} \Big(|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)^2, \tag{A.31}$$

$$\|J_{11}J_{12}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{1+s}{2}}\, |\psi|^{(2+\alpha)}_{s,\Gamma_t}\, |\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_t}. \tag{A.32}$$

Proof. All inequalities can be derived with the help of direct evaluations of the corresponding norms. The estimate (A.29) follows from (A.9) and (A.19). We can find inequalities (A.30), (A.31), (A.32) applying (A.9) and estimates (A.18)–(A.21) of the matrices J_{11}, J_{12}. □

Corollary A.6. *For the matrix $J_1 = J_{11} + J_{12}$ the following estimates hold for $t \le T$*

$$\|J_1^2\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{2+s}{2}} \Big(|\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t} \Big)^2, \tag{A.33}$$

$$\|J_1^2\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{s-1}{2}} \Big(|\psi|^{(2+\alpha)}_{s,\Gamma_t} + t\, |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t} \Big)^2. \tag{A.34}$$

Corollary A.7. *Let $\rho_0 \in C^{3+\alpha}_{1+s}(\Gamma_T)$, $\rho_0\big|_{t=0} = 0$, $|\rho_0|^{(3+\alpha)}_{1+s,\Gamma_T} \le M_1$, $M_1 > 0$. Then for $t \le T$ the following estimates hold*

$$\|J_{01} J_{11}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, M_1 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \le s \le 2+\alpha \end{cases} |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t}, \tag{A.35}$$

$$\|J_{01} J_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, M_1 \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases} |\psi|^{(2+\alpha)}_{s,\Gamma_t}, \tag{A.36}$$

$$\|J_{01} J_{12}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, M_1 \begin{cases} t^{\frac{3+s}{2}}, & 1 < s < 2 \\ t^{\frac{5}{2}}, & 2 \le s \le 2+\alpha \end{cases} |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t}, \tag{A.37}$$

$$\|J_{01} J_{12}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, M_1 \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \le s \le 2+\alpha \end{cases} |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t}, \tag{A.38}$$

$$\|J_{01} J_1\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, M_1\, (1 + t^{\frac{1}{2}}) \begin{cases} t^{\frac{2+s}{2}}, & 1 < s < 2 \\ t^2, & 2 \le s \le 2+\alpha \end{cases} |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t}, \tag{A.39}$$

$$\|J_{01} J_1\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, M_1 \begin{cases} t^{\frac{s}{2}}, & 1 < s < 2 \\ t, & 2 \le s \le 2+\alpha \end{cases} \big(|\psi|^{(2+\alpha)}_{s,\Gamma_t} + |\partial_t \psi|^{(1+\alpha)}_{s-1,\Gamma_t} \big). \tag{A.40}$$

Proof. We consider the matrices J_{01} and J_{11}. With the help of an estimate (A.9) for the product of the functions we derive

$$\|J_{01} J_{11}\|^{(\alpha)}_{s-2,\Gamma_t} \le C\, t^{\frac{s-1}{2}}\, \|J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t}\, \|J_{11}\|^{(\alpha)}_{s-2,\Gamma_t}\,,$$

$$\|J_{01} J_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le C\, t^{\frac{s-1}{2}}\, \|J_{01}\|^{(1+\alpha)}_{s-1,\Gamma_t}\, \|J_{11}\|^{(1+\alpha)}_{s-1,\Gamma_t}\,.$$

On the basis of the estimates (A.13) for J_{01} and (A.18), (A.19) for J_{11} we obtain (A.35), (A.36). The inequalities (A.37),(A.38) are derived analogously by applying the estimates (A.9) and (A.13), (A.20), (A.21) for J_{01}, J_{12}. Formulas (A.39), (A.40) with the matrix $J_1 = J_{11} + J_{12}$ follow from (A.35)–(A.38). □

Remark A.1. All these lemmas and corollaries are applied in the proofs of Lemma 3.2, Theorem 2.1 (Section 5) and Theorems A.1–A.3.

Now we prove the existence of the inverse matrices J^{-1} and J_0^{-1}.

Theorem A.1. *Let* $\psi(\xi,t) \in \overset{\circ}{C}{}_s^{2+\alpha}(\Gamma_T)$, $\partial_t\psi \in \overset{\circ}{C}{}_{s-1}^{1+\alpha}(\Gamma_T)$, $\alpha \in (0,1)$, $s \in (1, 2+\alpha]$, $\rho_0(\xi,t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$, $\rho_0\big|_{t=0} = 0$ *and* $|\psi|_{s,\Gamma_T}^{(2+\alpha)} + |\partial_t\psi|_{s-1,\Gamma_T}^{(1+\alpha)} \le M$, $|\rho_0|_{1+s,\Gamma_T}^{(3+\alpha)} \le M_1$, $M > 0$, $M_1 > 0$.

Then there is $t_1 \in [0,T]$ *such that the inverse Jacobian matrix* J^{-1} *exists, is bounded,*

$$\|J^{-1}\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le C(1+M)\,, \qquad \nu = 0,1\,, \tag{A.41}$$

and can be represented in the form

$$J^{-1} = \sum_{k=0}^{\infty}(J_{01}+J_1)^{2k}\Big(I-(J_{01}+J_1)\Big)\,. \tag{A.42}$$

Proof. We consider the Jacobian matrix $J = I + J_{01} + J_1$. We have found formally the inverse Jacobian matrix J^{-1} (A.4) in Lemma A.1

$$J^{-1} = \Big(I-(J_{01}+J_1)^2\Big)^{-1}\Big(I-(J_{01}+J_1)\Big)\,. \tag{A.43}$$

To prove the existence of J^{-1} we show the existence of an inverse matrix $(I-(J_{01}+J_1)^2)^{-1}$. For that we estimate the norms of the matrix $(J_{01}+J_1)^2$. By the estimates (A.16), (A.17) for J_{01}^2, (A.33), (A.34) for $J_1^2 = (J_{11}+J_{12})^2$ and (A.39), (A.40) for $J_{01}J_1$, we obtain

$$\begin{aligned}\|(J_{01}+J_1)^2\|_{s-2,\Gamma_t}^{(\alpha)} &\le \|J_{01}^2\|_{s-2,\Gamma_t}^{(\alpha)} + \|J_{01}J_1\|_{s-2,\Gamma_t}^{(\alpha)} \\ &\quad + \|J_1J_{01}\|_{s-2,\Gamma_t}^{(\alpha)} + \|J_1^2\|_{s-2,\Gamma_t}^{(\alpha)} \le \mu_1(t),\end{aligned} \tag{A.44}$$

$$\|(J_{01}+J_1)^2\|_{s-1,\Gamma_t}^{(1+\alpha)} \le \mu_2(t)\,, \tag{A.45}$$

where

$$\mu_1(t) = C\,(M+M_1)^2\begin{cases} t^{\frac{2+s}{2}}, & 1<s<2 \\ t^{\frac{2+s}{2}} + t^{\frac{6-s}{2}}, & 2 \le s \le 2+\alpha\,, \end{cases}$$

$$\mu_2(t) = C\,(M+M_1)^2\Big(t^{\frac{s-1}{2}} + \begin{cases} t^{\frac{s}{2}}(1+t^{\frac12}), & 1<s<2 \\ t(1+t^{\frac{3-s}{2}}), & 2 \le s \le 2+\alpha\,, \end{cases}\Big).$$

Let $D = (J_{01}+J_1)^2$. Applying the estimate (A.9), we find

$$\|D^2\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le C\,t^{\frac{s-1}{2}}\,\|D\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)}\,\|D\|_{s-1,\Gamma_t}^{(1+\alpha)}\,, \qquad \nu = 0,1\,.$$

By mathematical induction, we have

$$\|D^k\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le \|D\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \Big(C\, t^{\frac{s-1}{2}}\, \|D\|^{(1+\alpha)}_{s-1,\Gamma_t} \Big)^{k-1}, \qquad \nu = 0,1\,.$$

Here we make use of the estimates (A.44), (A.45)

$$\|D^k\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le \mu_{1+\nu}(t) \Big(C\, t^{\frac{s-1}{2}}\, \mu_2(t) \Big)^{k-1}, \qquad \nu = 0,1\,.$$

We find $t_1 > 0$ from the inequalities

$$C\, t^{\frac{s-1}{2}}\, \mu_2(t) \le q, \quad \mu_{1+\nu}(t) \le q\,, \qquad \nu = 0,1, \quad q \in (0,1)\,,$$

then we obtain

$$\|(J_{01}+J_1)^{2k}\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le q^k\,, \qquad \nu = 0,1, \quad k = 1,2,..., \quad t \le t_1\,. \tag{A.46}$$

From here it follows

$$\sum_{k=0}^{\infty} \|(J_{01}+J_1)^{2k}\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}\,, \qquad \nu = 0,1, \quad t \le t_1\,. \tag{A.47}$$

On the basis of these estimates we shall have that the inverse matrix $(I - (J_{01}+J_1)^2)^{-1}$ exists, is expressed in the form

$$\Big(I - (J_{01}+J_1)^2 \Big)^{-1} = \sum_{k=0}^{\infty} (J_{01}+J_1)^{2k} \tag{A.48}$$

and is bounded

$$\Big\| \Big(I - (J_{01}+J_1)^2 \Big)^{-1} \Big\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le \frac{1}{1-q}\,, \qquad \nu = 0,1, \quad t \le t_1\,. \tag{A.49}$$

But then the matrix in the right-hand side of the formula (A.43) exists and is bounded by the estimates (A.12), (A.13) and (A.22), (A.23) of the matrices J_{01} and J_1, respectively, and (A.49)

$$\Big\| \big(I - (J_{01}+J_1)^2 \big)^{-1} \big(I - (J_{01}+J_1) \big) \Big\|^{(\alpha+\nu)}_{s-2+\nu,\Gamma_t} \le C(1+M)\,, \quad \nu = 0,1, \ \ t \le t_1. \tag{A.50}$$

Now we show that the matrix

$$\Big(I - (J_{01}+J_1)^2 \Big)^{-1} (I - J_{01} - J_1) \equiv \sum_{k=0}^{\infty} (J_{01}+J_1)^{2k}\, (I - J_{01} - J_1)$$

is equal to the inverse Jacobian matrix J^{-1}. For that we consider an identity

$$(I + J_{01} + J_1) \sum_{k=0}^{m} (J_{01}+J_1)^{2k}\, (I - J_{01} - J_1)$$

$$= \sum_{k=0}^{m} (J_{01}+J_1)^{2k} (I - J_{01} - J_1) (I + J_{01} + J_1) \qquad \text{(A.51)}$$
$$\equiv I - (J_{01}+J_1)^{2m+2} \quad \forall\, m,$$

where $I + J_{01} + J_1 = J$.

By the estimate (A.46) we have

$$\|(J_{01}+J_1)^{2m+2}\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le q^{m+1} \to 0, \ m \to \infty, \quad \nu = 0,1, \quad q \in (0,1). \qquad \text{(A.52)}$$

Moreover, the series $\sum_{k=0}^{m}(J_{01}+J_1)^{2k}$ converges to the bounded matrix satisfying the estimate (A.47). So we can pass to the limit as $m \to \infty$ in the identities (A.51) taking into account (A.52), then we obtain

$$(I + J_{01} + J_1) \sum_{k=0}^{\infty} (J_{01}+J_1)^{2k} (I - J_{01} - J_1)$$
$$= \sum_{k=0}^{\infty} (J_{01}+J_1)^{2k} (I - J_{01} - J_1) (I + J_{01} + J_1) = I\,.$$

These identities mean that the matrix $\sum_{k=0}^{\infty}(J_{01}+J_1)^{2k}\,(I - J_{01} - J_1)$ is the right and the left inverse matrix to the matrix $J = I + J_{01} + J_1$; i.e., it is the inverse matrix J^{-1}.

Thus we have proved that the inverse Jacobian matrix J^{-1} exists, satisfies the estimate (A.50), and is expressed in the form (A.42), which follows from the formulas (A.43) and (A.48). □

We consider now the matrix $J_0 = I + J_{01}$.

Theorem A.2. *Let* $\rho_0(\xi,t) \in C_{1+s}^{3+\alpha}(\Gamma_T)$, $\alpha \in (0,1)$, $s \in (1, 2+\alpha]$, $\rho_0\big|_{t=0} = 0$ *and* $|\rho_0|_{1+s,\Gamma_T}^{(3+\alpha)} \le M_1$.

Then there is $t_2 > 0$ *such that the inverse matrix* J_0^{-1} *exists, is bounded,*

$$\|J_{01}^{-1}\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le \frac{1}{1-q}\,, \qquad \nu = 0,1, \quad q \in (0,1), \quad t \le t_2\,, \qquad \text{(A.53)}$$

and can be represented in the form

$$J_0^{-1} = \sum_{k=0}^{\infty} (-1)^k\, J_{01}^k\,. \qquad \text{(A.54)}$$

Proof. To prove the existence of the inverse matrix J_0^{-1} it is sufficient to show that the series in (A.54) converges to a bounded matrix, i.e. satisfies the estimate (A.53). On the basis of the estimates (A.12), (A.13) for J_{01} we obtain

$$\|J_{01}^k\|_{s-2,\Gamma_t}^{(\alpha+\nu)} \le \mu_{3+\nu}(t)\,(t^{\frac{s-1}{2}}\,\mu_4(t))^{k-1}\,, \qquad k=1,2,\dots,$$

where

$$\mu_{3+\nu}(t) = C\,M_1 \begin{cases} t^{\frac{2-\nu}{2}}, & 1<s<2 \\ t^{\frac{4-s-\nu}{2}}, & 2\le s\le 2+\alpha \end{cases}, \qquad \nu=0,1\,.$$

We choose $t_2>0$ from the inequalities

$$\mu_3(t)\le q\,, \quad \mu_4(t)\le q\,, \quad t^{\frac{s-1}{2}}\,\mu_4(t)\le q\,, \quad q\in(0,1)\,,$$

then we shall have

$$\|J_{01}^k\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le q^k \quad \forall\, t\le t_2, \;\; \nu=0,1\,,$$

and

$$\sum_{k=0}^{\infty}\|J_{01}^k\|_{s-2+\nu,\Gamma_t}^{(\alpha+\nu)} \le \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}\,, \qquad t\le t_2\,.$$

That is, the matrix $\sum_{k=0}^{\infty}(-1)^k J_{01}^k$ exists, is bounded, and equals J_0^{-1}. □

In Section 5, we make use of the contraction principle. For that we have to estimate the difference of the matrices $J^{-1}-\widetilde{J}^{-1}$, where $J=I+J_{01}+J_1$, $\widetilde{J}=I+J_{01}+\widetilde{J}_1$, $J_1=\{\partial_{y_j}(N_i\,\chi\,\psi)\}_{1\le i,\,j\le n}$, $\widetilde{J}_1=\{\partial_{y_j}(N_i\,\chi\,\widetilde{\psi})\}_{1\le i,\,j\le n}$.

Theorem A.3. *Let* $\rho_0(\xi,t)\in C^{3+\alpha}_{1+s}(\Gamma_T)$, $\rho_0\big|_{t=0}=0$; $\psi(\xi,t),\widetilde{\psi}(\xi,t)\in \mathring{C}^{2+\alpha}_s(\Gamma_T)$, $\partial_t\psi,\partial_t\widetilde{\psi}\in\mathring{C}^{1+\alpha}_{s-1}(\Gamma_T)$, $\alpha\in(0,1)$, $s\in(1,2+\alpha]$, *and* $|\rho_0|^{(3+\alpha)}_{s+1,\Gamma_T}\le M_1$, $|\psi|^{(2+\alpha)}_{s,\Gamma_T}+|\partial_t\psi|^{(1+\alpha)}_{s-1,\Gamma_T}\le M$, $|\widetilde{\psi}|^{(2+\alpha)}_{s-\Gamma_T}+|\partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_T}\le M$, $M_1>0$, $M>0$.

Then there is $t_3>0$ *such that for* $t\le t_3$ *the following estimates hold:*

$$\|J^{-1}-\widetilde{J}^{-1}\|^{(\alpha)}_{s-2,\Gamma_t}\le C\,t\,|\partial_t\psi-\partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_t}\,, \tag{A.55}$$

$$\|J^{-1}-\widetilde{J}^{-1}\|^{(1+\alpha)}_{s-1,\Gamma_t}\le C\Big(|\psi-\widetilde{\psi}|^{(2+\alpha)}_{s,\Gamma_t}+|\partial_t\psi-\partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_t}\Big)\,. \tag{A.56}$$

Proof. We compare the difference $J^{-1}-\widetilde{J}^{-1}$ of the inverse Jacobian matrix J^{-1}, which we take in the form (A.3)

$$J^{-1}-\widetilde{J}^{-1} = -(J_1-\widetilde{J}_1)+(J_{01}+J_1)^2\,J^{-1}-(J_{01}+\widetilde{J}_1)^2\,\widetilde{J}^{-1}\,,$$

where $J_1-\widetilde{J}_1=\{\partial_{y_j}(N_i\,\chi(\psi-\widetilde{\psi}))\}_{1\le i,\,j\le n}$.

After some computations we shall have

$$J^{-1}-\widetilde{J}^{-1} \;=\; \Big(I-(J_{01}+\widetilde{J}_1)^2\Big)^{-1}\Big[-(J_1-\widetilde{J}_1)+(J_{01}+\widetilde{J}_1)\,(J_1-\widetilde{J}_1)\,J^{-1}$$

$$+ \,(J_1 - \widetilde{J}_1)\,(J_{01} + J_1)\,J^{-1}\Big]\,, \tag{A.57}$$

where the matrices $(I - (J_{01} + \widetilde{J}_1)^2)^{-1}$, J^{-1} exist (see Theorem A.1). We evaluate the norms of the matrices in the right-hand side of the identity (A.57) with the help of the estimates (A.11) for the product of the functions, (A.22), (A.23), (A.39), (A.40), (A.33), (A.34) for the matrices J_1; $J_{01}J_1$; J_1^2 respectively and (A.49), (A.41), then we find

$$\|J^{-1} - \widetilde{J}^{-1}\|^{(\alpha)}_{s-2,\Gamma_t} \le t\,(C + \mu_5(t))\,|\partial_t\psi - \partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_t}\,, \tag{A.58}$$
$$\|J^{-1} - \widetilde{J}^{-1}\|^{(1+\alpha)}_{s-1,\Gamma_t} \le (C + \mu_6(t))\Big(|\psi - \widetilde{\psi}|^{(2+\alpha)}_{s,\Gamma_t} + |\partial_t\psi - \partial_t\widetilde{\psi}|^{(1+\alpha)}_{s-1,\Gamma_t}\Big),$$

where

$$\mu_{5+\nu}(t) = C\,t^{\frac{s-1}{2}}\Big(M\,t^{\frac{1-\nu}{2}} + M_1\begin{cases} t^{\frac{1}{2}}, & 1 < s < 2 \\ t^{\frac{3-s}{2}}, & 2 \le s \le 2+\alpha \end{cases}\Big)(1+M)\,, \quad \nu = 0, 1\,.$$

We choose $t_4 > 0$ from the inequalities

$$\mu_5(t) \le 1\,, \qquad \mu_6(t) \le 1\,,$$

and then from (A.58) we obtain the estimates (A.55), (A.56) for $t \le t_3 = \min(t_1, t_4)$. □

Appendix B. The model transmission problem

Let $\mathcal{S}_1 = \mathbb{R}^n_-$ and $\mathcal{S}_2 = \mathbb{R}^n_+$ be half spaces $x_n < 0$ and $x_n > 0$ in $\mathbb{R}^n$, respectively, $\mathcal{S}_{jT} = \mathcal{S}_j \times (0, T)$; R a plane $x_n = 0$, $R_T = R \times [0, T]$.

In Section 4 we study the linear problem written in the form $(4.1)_1$, (4.12), $(4.2)_1$, (4.13), (4.3)–(4.7). The proof of the solvability of this problem is based on the following model transmission problem for the unknown functions $u_j(x,t)$, $c_j(x,t)$, $j = 1,2$, $\psi(x',t)$ satisfying zero initial data:

$$\partial_t u_j - a_j\,\Delta u_j - \alpha_1(\partial_t\psi - a_j\,\Delta'\psi) = f_j(x,t) \quad \text{in } \mathcal{S}_{jT}, \quad j = 1,2, \tag{B.1}$$
$$\partial_t c_1 - b_1\,\Delta c_1 - \beta_1(\partial_t\psi - b_1\,\Delta'\psi) = g_1(x,t) \quad \text{in } \mathcal{S}_{1T}, \tag{B.2}$$
$$\partial_t c_2 - b_2\,\Delta c_2 - \big[\beta_2 + \gamma_2(\alpha_1 - \alpha_2)\big](\partial_t\psi - b_2\,\Delta'\psi) = g_2(x,t) \quad \text{in } \mathcal{S}_{2T}, \tag{B.3}$$

$$(u_1 - u_2)\big|_{x_n=0} = \eta_0(x',t), \tag{B.4}$$
$$(c_j - \gamma_j u_j)\big|_{x_n=0} = \eta_j(x',t)\,, \quad j = 1,2, \tag{B.5}$$
$$(\lambda_1\,\partial_{x_n}u_1 - \lambda_2\,\partial_{x_n}u_2)\big|_{x_n=0} = \varphi_1(x',t), \tag{B.6}$$

$$(k_1\,\partial_{x_n}c_1 - k_2\,\partial_{x_n}c_2)\big|_{x_n=0} + d'\,\nabla'\psi - \kappa_1\,\partial_t\psi = \varphi_2(x',t)\,, \qquad t\in(0,T), \tag{B.7}$$

where all coefficients are constant, κ_1, a_j, b_j, γ_j, λ_j, k_j, $j=1,2$, positive and $d'=(d_1,...,d_{n-1})$.

Theorem B.1. *Let $\alpha\in(0,1)$, $s\in(1,2+\alpha]$. We assume*

$$\kappa_1>0\,, \qquad \mu_j=\beta_j-\alpha_j\,\gamma_j>0, \quad j=1,2\,. \tag{B.8}$$

Then for every pair of functions $f_j, g_j \in \overset{\circ}{C}{}^{\alpha}_{s-2}(\mathcal{S}_{jT})$, $\eta_0,\eta_j\in \overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$, $\varphi_j\in\overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$, $j=1,2$, the problem (B.1)–(B.7) *has a unique solution $u_j, c_j\in\overset{\circ}{C}{}^{2+\alpha}_{s}(\mathcal{S}_{jT})$, $\psi\in\overset{\circ}{C}{}^{2+\alpha}_{s}(R_T)$, $\partial_t\psi\in\overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$ and it satisfies the estimate*

$$\sum_{j=1}\big(|u_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}}+|c_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}}\big)+|\psi|^{(2+\alpha)}_{s,R_T}+|\partial_t\psi|^{(1+\alpha)}_{s-1,R_T} \tag{B.9}$$

$$\le C_1\Big(\sum_{j=1}^{2}\big(|f_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}}+|g_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}}+|\eta_j|^{(2+\alpha)}_{s,R_T}+|\varphi_j|^{(1+\alpha)}_{s-1,R_T}\big)+|\eta_0|^{(2+\alpha)}_{s,R_T}\Big)\,.$$

Proof. In the equations (B.1) and conditions (B.4), (B.6) we make the change

$$u_j=v_j+\alpha_1\,\psi\,, \qquad j=1,2\,, \tag{B.10}$$

and then we obtain the problem for the new unknown functions v_1, v_2

$$\begin{aligned}
&\partial_t v_1 - a_1\,\Delta v_1 = f_1 \quad \text{in } \ \mathcal{S}_{1T}\,,\\
&\partial_t v_2 - a_2\,\Delta v_2 = f_2 \quad \text{in } \ \mathcal{S}_{2T},\\
&(v_1-v_2)\big|_{x_n=0}=\eta_0\,,\\
&(\lambda_1\,\partial_{x_n}v_1-\lambda_2\,\partial_{x_n}v_2)\big|_{x_n=0}=\varphi_1\,.
\end{aligned} \tag{B.11}$$

We note that (B.11) is the transmission (or conjunction) problem [5], it's solution v_j belongs to $\overset{\circ}{C}{}^{2+\alpha}_{s}(\mathcal{S}_{jT})$, $j=1,2$, and satisfies the estimate

$$\sum_{j=1}^{2}|v_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}}\le C_2\Big(\sum_{j=1}^{2}|f_j|^{(2+\alpha)}_{s-2,\mathcal{S}_{jT}}+|\eta_0|^{(2+\alpha)}_{s,R_T}+|\varphi_1|^{(1+\alpha)}_{s-1,R_T}\Big)\,. \tag{B.12}$$

We have represented the equation $(4.1)_2$ in the form (4.12) to obtain problem (B.11) separated from other unknown functions.

We construct auxiliary functions Z_1, Z_2 as the solutions of the first boundary value problems in the half space $\mathcal{S}_j$

$$\begin{aligned} &\partial_t Z_j - b_j\,\Delta Z_j = g_j \quad \text{ in } \ \mathcal{S}_{jT}\,, \\ &Z_j\big|_{t=0} = 0\,, \qquad Z_j\big|_{x_n=0} = \eta_j + \gamma_j\, v_j\big|_{x_n=0}\,, \end{aligned} \tag{B.13}$$

where $j = 1, 2$ and v_j are known functions. Each one of the problems (B.13) has a unique solution $V_j \in \overset{\circ}{C}{}^{2+\alpha}_s(\mathcal{S}_{jT})$ [21, 13] and the following estimate holds for $j = 1, 2$

$$\begin{aligned} |Z_j|^{(2+\alpha)}_{s,\mathcal{S}_{jT}} \le C_3\Big(&|g_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}} + |\eta_j|^{(2+\alpha)}_{s,R_T} \\ &+ \sum_{k=1}^{2} |f_k|^{(\alpha)}_{s-2,\mathcal{S}_{kT}} + |\eta_0|^{(2+\alpha)}_{s,R_T} + |\varphi_1|^{(1+\alpha)}_{s-1,R_T}\Big)\,. \end{aligned} \tag{B.14}$$

After the substitution (B.10) and the following ones

$$\begin{aligned} &c_1 = z_1 + Z_1 + \beta_1\,\psi\,, \\ &c_2 = z_2 + Z_2 + \big(\beta_2 + \gamma_2(\alpha_1 - \alpha_2)\big)\psi \end{aligned} \tag{B.15}$$

in the equations (B.2), (B.3) and conditions (B.5), (B.7) we obtain the problem for the functions z_1, z_2, ψ with time derivative in the transmission condition

$$\partial_t z_1 - b_1\,\Delta z_1 = 0 \quad \text{ in } \ \mathcal{S}_{1T}\,, \tag{B.16}$$

$$\partial_t z_2 - b_2\,\Delta z_2 = 0 \quad \text{ in } \ \mathcal{S}_{2T}\,, \tag{B.17}$$

$$z_j\big|_{x_n=0} + \mu_j\,\psi = 0\,, \qquad j = 1, 2\,, \tag{B.18}$$

$$(k_1\,\partial_{x_n} z_1 - k_2\,\partial_{x_n} z_2)\big|_{x_n=0} + d'\,{\nabla'}^T\psi - \kappa_1\,\partial_t\psi = \varphi(x', t)\,, \tag{B.19}$$

where $\mu_j = \beta_j - \gamma_j\,\alpha_j > 0$, $j = 1, 2$, $\varphi = \varphi_2 - (k_1\,\partial_{x_n} Z_1 - k_2\,\partial_{x_n} Z_2)\big|_{x_n=0} \in \overset{\circ}{C}{}^{1+\alpha}_{s-1}(R_T)$,

$$\begin{aligned} |\varphi|^{(1+\alpha)}_{s-1,R_T} \le |\varphi_2|^{(1+\alpha)}_{s-1,R_T} + C_4\Big(&\sum_{j=1}^{2}\Big(|f_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}} + |g_j|^{(\alpha)}_{s-2,\mathcal{S}_{jT}} + |\eta_j|^{(2+\alpha)}_{s,R_T}\Big) \\ &+ |\eta_0|^{(2+\alpha)}_{s,R_T} + |\varphi_1|^{(1+\alpha)}_{s-1,R_T}\Big)\,. \end{aligned} \tag{B.20}$$

We apply the Fourier transform on x' and Laplace transforms on t to the problem (B.16)–(B.19)

$$\widetilde{v}(\xi', x_n, p) \;=\; \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_0^\infty e^{-pt}\, dt \int_{\mathbb{R}^{n-1}} v(x,t)\, e^{-ix'\xi'} dx' \,.$$

Then from the heat equations of the problem, we find the solution in the images of Fourier and Laplace transforms

$$\widetilde{z}_1 = B_1\, e^{r_1 x_n}, \quad x_n < 0\,, \qquad \widetilde{z}_2 = B_2\, e^{-r_2 x_n}, \quad x_n > 0\,, \tag{B.21}$$

where $r_j^2 = \frac{1}{b_j}(p + b_j \xi'^2)$, $B_j = B_j(\xi', p)$, $j = 1, 2$, are unknown functions. From the conditions on the plane $x_n = 0$

$$B_j + \mu_j\, \widetilde{\psi} = 0\,, \qquad j = 1, 2\,,$$

$$k_1\, r_1\, B_1 + k_2\, r_2\, B_2 - (\kappa_1\, p - i\, d'\xi')\widetilde{\psi} = \widetilde{\varphi}$$

we find B_j, $\widetilde{\psi}$

$$B_j = \frac{\mu_j}{\kappa_1}\, \frac{1}{\zeta}\, \widetilde{\varphi}, \quad j = 1, 2\,, \qquad \widetilde{\psi} = -\frac{1}{\kappa_1\, \zeta}\, \widetilde{\varphi}\,,$$

where

$$\zeta = p + \frac{k_1\, \mu_1}{\kappa_1}\, r_1 + \frac{k_2\, \mu_2}{\kappa_1}\, r_2 - \frac{i}{\kappa_1}\, d'\xi'\,,$$

$\mathrm{Re}\,\zeta > 0$ by conditions $k_j > 0$, $\mu_j > 0$, $\kappa_1 > 0$. So we can represent the fraction $\frac{1}{\zeta}$ in the form

$$\frac{1}{\zeta} = \int_0^\infty e^{-\zeta\sigma} d\sigma\,.$$

Substituting the functions B_j into formulas (B.21) and applying this expression of $\frac{1}{\zeta}$ we shall have

$$\widetilde{z}_j = \frac{\mu_j}{\kappa_1}\, \widetilde{\varphi} \int_0^\infty e^{-\zeta\sigma - r_j |x_n|}\, d\sigma\,, \qquad \widetilde{\psi} = -\frac{1}{\kappa_1}\, \widetilde{\varphi} \int_0^\infty e^{-\zeta\sigma}\, d\sigma\,.$$

With the help of the inverse Laplace and Fourier transforms [2] we find z_j and ψ in the closed form

$$z_j(x,t) = \frac{\mu_j}{\kappa_1} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \varphi(y', \tau)\, G_j(x' - y', x_n, t - \tau)\, dy'\,, \qquad j = 1, 2, \tag{B.22}$$

$$\psi(x', t) = -\frac{1}{\kappa_1} \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \varphi(y', \tau)\, G_j(x' - y', 0, t - \tau)\, dy', \tag{B.23}$$

where

$$
\begin{aligned}
G_1 \;=\;& 4\,b_1\,b_2 \int_0^t d\sigma \int_0^{t-\sigma} d\tau_1 \\
&\times \int_{\mathbb{R}^{n-1}} \partial_{x_n}\Gamma_1\Big(x'-\eta'-\frac{d'}{\kappa_1}\sigma,\; \frac{k_1\,\mu_1\,\sigma}{\kappa_1}-x_n,\; t-\sigma-\tau_1\Big) \\
&\times \partial_{\eta_n}\Gamma_2\Big(\eta', \frac{k_2\,\mu_2\,\sigma}{\kappa_1}-\eta_n,\; \tau_1\Big)\Big|_{\eta_n=0} d\eta'\,, \qquad x_n<0\,,
\end{aligned}
$$

$$
\begin{aligned}
G_2 \;=\;& 4\,b_1\,b_2 \int_0^t d\sigma \int_0^{t-\sigma} d\tau_1 \int_{\mathbb{R}^{n-1}} \partial_{\eta_n}\Gamma_1\Big(\eta'-\frac{d'}{\kappa_1}\sigma,\; \frac{k_1\,\mu_1\,\sigma}{\kappa_1}+\eta_n,\; \tau_1\Big)\Big|_{\eta_n=0} \\
&\times \partial_{x_n}\Gamma_2\Big(x'-\eta',\; \frac{k_2\,\mu_2\,\sigma}{\kappa_1}+x_n,\; t-\sigma-\tau_1\Big)d\eta'\,, \qquad x_n>0\,.
\end{aligned}
$$

Here $\Gamma_j(x,t)$ is the fundamental solution of the heat equation $\partial_t v - b_j\,\Delta v = 0$:

$$
\Gamma_j(x,t) = \frac{1}{\big(2\sqrt{\pi\, b_j\, t}\big)^n}\; e^{-\frac{x^2}{4\,b_j\,t}}\,.
$$

As was proved in [7] the potentials (B.22) with the kernels $G_j(x,t)$ belong to the space $\overset{\circ}{C}{}_s^{2+\alpha}(\mathcal{S}_{jT})$ and satisfy the estimate

$$
|z_j|_{s,\mathcal{S}_{jT}}^{(2+\alpha)} \le C_5\,|\varphi|_{s-1,R_T}^{(1+\alpha)}\,. \tag{B.24}
$$

The function $\psi(x',t)$ ((B.23)) as a trace of potential z_j on the plane $x_n=0$ also belongs to $\overset{\circ}{C}{}_s^{2+\alpha}(R_T)$ and the following estimate for it holds:

$$
|\psi|_{s,R_T}^{(2+\alpha)} \le C_6\,|\varphi|_{s-1,R_T}^{(1+\alpha)}\,. \tag{B.25}
$$

From the boundary condition (B.19), by the above conclusions and estimates (B.24), (B.25), we have

$$
\partial_t\psi = \frac{1}{\kappa_1}\Big(k_1\,\partial_{x_n}z_1 - k_2\,\partial_{x_n}z_2 + d'\,\nabla'\psi - \varphi\Big) \in \overset{\circ}{C}{}_{s-1}^{1+\alpha}(R_T)\,,
$$

$$
|\partial_t\psi| \le C_7\Big(\sum_{j=1}^{2}|z_j|_{s,\mathcal{S}_{jT}}^{(2+\alpha)} + |\psi|_{s,R_T}^{(2+\alpha)} + |\varphi|_{s-1,R_T}^{(1+\alpha)}\Big) \le C_8\,|\varphi|_{s-1,R_T}^{(1+\alpha)}\,. \tag{B.26}
$$

Recalling the substitutions (B.10), (B.15) we obtain that the functions $u_j = v_j+\alpha_1\psi$, $j=1,2$, $c_1 = z_1+Z_1+\beta_1\psi$ $(j=1)$, $c_2 = z_1+Z_2+(\beta_2+\gamma_2(\alpha_1-\alpha_2))\psi$ $(j=2)$ belong to the space $\overset{\circ}{C}{}_s^{2+\alpha}(\mathcal{S}_{jT})$, $j=1,2$. Applying estimates (B.12), (B.24), (B.14), (B.25), (B.26) and (B.20) of the functions v_j, z_j, Z_j, ψ, $\partial_t\psi$ and φ respectively leads to the required estimate (B.9). □

Remark B.1. The conditions on the coefficients (B.8) correspond to the conditions (4.8), (4.9) of Theorem 4.1 respectively.

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