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EQUILIBRIUM SOLUTIONS OF THE BÉNARD EQUATIONS WITH AN EXTERIOR FORCE

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Abstract. We study the inhomogeneous Bénard equations on the infinite layer, $\Omega = \mathbb{R}^2 \times (-\frac{1}{2}, \frac{1}{2})$, provided with an exterior force $f = f(z)$, depending only on the bounded variable $z \in [-\frac{1}{2}, \frac{1}{2}]$. There is a unique equilibrium solution $v = v(z)$ depending only on z. We study the stability of small $v(z)$, once under L-periodic perturbations, and once under spatially localized perturbations, i.e., perturbations in $\mathcal{L}^2(\Omega)$. Loss of stability may occur in the neighbourhood of the critical Rayleigh numbers λ_L and λ_ω, where λ_L refers to the L-periodic setting, λ_ω to the $\mathcal{L}^2(\Omega)$ setting. Among others we give a characterization of λ_ω in terms of Orr-Sommerfeld theory. It is shown that if $\lambda_L \neq \lambda_\omega$ then $v(z)$ may be stable under L-periodic perturbations, but is necessarily unstable under $\mathcal{L}^2(\Omega)$ perturbations. The proofs are based on energy methods and on Bloch space theory.

1. Introduction

We consider the Bénard equations containing an exterior force:

$$\begin{aligned} &\partial_t v = \nu\Delta v + \lambda k\mathcal{V} - (v\nabla)v + \widetilde{f} - \nabla p, \quad \operatorname{div}(v) = 0 \\ &\partial_t \mathcal{V} = \kappa\Delta\mathcal{V} + \lambda v_3 - (v\nabla)\mathcal{V} + f_4 = 0, \quad v = \mathcal{V} = 0 \text{ at } \pm\tfrac{1}{2}, \end{aligned} \tag{1.1}$$

where $k = (0,0,1)^t$, $f = (\widetilde{f}, f_4)$, $\widetilde{f} = (f_1, f_2, f_3)$ for $(x,y,z) \in \mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}]$. The force f admits a physical interpretation. If e.g. the fluid is conducting and under the influence of an impressed magnetic field $H = (A,B,C)$ then $f = ((H\nabla)A, (H\nabla)B, (H\nabla)C, 0)$, so as to get an approximation to the magnetohydrodynamic equations which arise if H is small ([12], introduction). If $f = (0,0,0,f_4)$, then an exterior temperature source is present; the general case is a mixture of the two. For certain ranges of λ, (1.1) admits an equilibrium solution $v, \mathcal{V}, p$ (see [13]). Here we investigate the stability of

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some particular equilibria in two different settings. If $f = 0$ then $v = \mathcal{V} = 0$, $p =$ constant is an equilibrium of (1). Its stability depends on λ and on the class of admissible perturbations. In a standard setting, a period pair $L = (L_1, L_2)$, $(L_j > 0)$ is given; stability of $v = \mathcal{V} = 0$ with respect to L-periodic perturbations is studied. A critical Rayleigh number λ_L is found such that $\lambda < \lambda_L$ entails stability while $\lambda_L < \lambda$ implies instability; see [6], [7] and the cited literature. A different view is taken in [14] where the stability of $v = \mathcal{V} = 0$ under localized perturbations, i.e., perturbations in $\mathcal{L}^2(\Omega)$, $\Omega = R^2 \times (-\frac{1}{2}, \frac{1}{2})$ is investigated. Again one finds a critical λ_ω such that $\lambda < \lambda_\omega$ entails stability and $\lambda > \lambda_\omega$ instability. We will give an analytical characterization of λ_ω in terms of Orr-Sommerfeld theory, not given in [14]. Here we focus on a class of equilibria which emerges if $f = f(z)$ depends on $z \in [-\frac{1}{2}, \frac{1}{2}]$ only. There then exists precisely one equilibrium $v = v(z)$, $\mathcal{V} = \mathcal{V}(z)$, $p = p(z)$ depending on z only, i.e.,

$$\nu \partial_z^2 v_j = f_j, \quad j = 1, 2, \quad v_3 = 0, \quad \kappa \partial_z^2 \mathcal{V} = f_4, \quad \partial_z p + \lambda \mathcal{V} = f_3, \qquad (1.2)$$
$$v_j = \mathcal{V} = 0 \quad \text{at } z = \pm\tfrac{1}{2}.$$

Since $f(z)$ is assumed to be small we replace it by δf, $\delta \in \mathbb{R}$ small; the equilibrium $w_0 = (v(z), \mathcal{V}(z))$ is then replaced by δw_0. Since loss of stability is likely to occur in the neighbourhood of λ_L or λ_ω, we set $\lambda = \lambda^* + \varepsilon$, $\lambda^* \in \{\lambda_L, \lambda_\omega\}$. We now investigate the loss of stability of δw_0 for small ε, δ with respect to L-periodic and $\mathcal{L}^2(\Omega)$ perturbations. While a complete picture of the stability behaviour of δw_0 for ε, δ small cannot be obtained, a series of results of interest can be proved. Thus it is shown that if $\lambda^* = \lambda_L \neq \lambda_\omega$ then δw_0 is $\mathcal{L}^2(\Omega)$ unstable for $|\varepsilon| \leq \varepsilon_0$, $|\delta| \leq \delta_0$, some $\varepsilon_0, \delta_0 > 0$. On the other hand it is proved that for suitable forces $f(z)$ there are $\varepsilon \neq 0$, $\delta \neq 0$ arbitrarily small such that δw_0 is stable under L-periodic perturbations.

The methods of proof are based on an energy-bootstrap method, used in [10], [15], and on spectral methods based on Bloch space theory, described e.g. in [16], [18], [8]. A number of open questions, which remain, are discussed in the last section.

2. Notation

$\mathbb{R}$, $\mathbb{C}$ denote the real and complex numbers, $\overline{\zeta}$ is the complex conjugate of ζ. We set $\partial_j = \partial_{x_j}$, $x_1 = x$, $x_2 = y$, $x_3 = z$ and $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_p^{\alpha_p}$ for any multiindex $\alpha = (\alpha_1, \dots, \alpha_p)$. For $\Omega \subseteq \mathbb{R}^3$, $H^p(\Omega) = W^{p,2}(\Omega)$ are the usual Sobolev spaces, sometimes affixed with a label in order to specify a particular situation. The scalar product on $H^p(\Omega)$ is given by

$$(u,v)_p = \sum_{|\alpha|\le p} (D^\alpha u, D^\alpha v)_0, \quad (u,v)_0 = \int_\Omega u\overline{v}\,dx.$$

We have $H^0(\Omega) = \mathcal{L}^2(\Omega)$ and set $\|\ \|_{H^p} = \|\ \|_{H^p(\Omega)}$, $\|\ \|_2 = \|\ \| = \|\ \|_{\mathcal{L}^2(\Omega)}$ in a fixed context. For fields u, v in $H^p(\Omega)^m$ we set

$$\langle u,v\rangle_p = \sum_1^m (u_j,v_j)_p, \quad \langle\ ,\ \rangle_0 = \langle\ ,\ \rangle, \quad \|u\| = \|\ \|_2 = \|\ \|_{\mathcal{L}^2}, \quad \|u\|^2_{H^p} = \langle u,v\rangle_p.$$

Thus, $\|u\|_{\mathcal{L}^2(\Omega)}$, $\|u\|_{H^p}$ are interpreted according to the context. For $u = (u_1,u_2,u_3)$, $v = (v_1,v_2,\ldots,v_m)$ vector fields, and θ a scalar we set

$$(u\nabla)\theta = \sum_1^3 u_j\partial_j\theta, \quad (u\nabla)v = ((u\nabla)v_1,\ldots,(u\nabla)v_m). \tag{2.1}$$

For Δ the Laplacian, $f = (f_1,\ldots,f_m)$ we set $\Delta f = (\Delta f_1,\ldots,\Delta f_m)$; if $m = 4$ we set $\widetilde{f} = (f_1,f_2,f_3)$ and $f = (\widetilde{f}, f_4)$. For $u = (u_1,\ldots,u_4)$, $v = (v_1,\ldots,v_m)$ we stipulate

$$(u\nabla)v = (\widetilde{u}\nabla)v. \tag{2.2}$$

3. Functional setting

In order to handle the stability problems mentioned, we need some functional analytic preliminaries. We first consider the periodic case; we fix periods $L = (L_1,L_2)$, $L_j > 0$ and set

$$Q_L = (0,L_1)\times(0,L_2), \quad Q = Q_L\times I, \quad I = (-\tfrac{1}{2},\tfrac{1}{2}), \quad \Omega = \mathbb{R}^2\times I. \tag{3.1}$$

In order to apply Bloch space theory, we treat the L-periodic functions as a special case of the so-called θ-periodic functions; for details we refer to [16], chapters 2, 6, 7, to [18], Volume 4 or to [9]. Here we present the minimum needed for our arguments. We set

$$M_0 = [0,2\pi]^2, \quad \dot{M}_0 = M_0 - \{(0,0),(0,2\pi),(2\pi,0),(2\pi,2\pi)\}. \tag{3.2}$$

For $\theta = (\theta_1,\theta_2)\in M_0$ we define:

$$f \in C^p_\theta(Q) \text{ iff } f\in C^p(\overline{\Omega}), \text{ and for } j,k\in\mathbb{Z}, \quad (x,y,z)\in\overline{\Omega}: \tag{3.3}$$
$$f(x+jL_1, y+kL_2, z) = e^{i(j\theta_1+k\theta_2)}f(x,y,z).$$

Likewise $f\in C^1_{\theta,0}(Q)$ if and only if $f\in C^1_\theta(Q)$ and $f(x,y,\pm\frac{1}{2}) = 0$, $x,y\in\mathbb{R}$. θ-periodic Sobolev spaces are defined as follows:

(*) $f \in H^p_\theta(Q)$ if $f \in \mathcal{L}^2(Q)$ and if there is a sequence $f_n \in C^p_\theta(Q)$, $n \geq 1$ which is Cauchy in $H^p(Q)$ and such that $\|f - f_n\|_{\mathcal{L}^2(Q)} \to 0$ as $n \uparrow \infty$.

(**) We let $f \in H^1_{\theta,0}(Q)$ if and only if $f \in H^1_\theta(Q)$ and if the sequence f_n in (*) is in $C^1_{\theta,0}(Q)$.

Now fix $\theta \in M_0$ arbitrarily. We let $E_\theta \subseteq \mathcal{L}^2(Q)^3$ be the $\mathcal{L}^2$ closure of the set of $f \in H^1_{\theta,0}(Q)^3$ such that $\operatorname{div}(f) = 0$; we let P_θ be the orthogonal projection from $\mathcal{L}^2(Q)^3$ onto E_θ. We then stipulate

$$\mathcal{E}_\theta = E_\theta \times \mathcal{L}^2(Q); \text{ for } f = (\widetilde{f}, f_4), \quad \widetilde{f} = (f_1, f_2, f_3) \text{ we set } \mathcal{P}_\theta f = (P_\theta \widetilde{f}, f_4), \tag{3.4}$$

i.e., $\mathcal{P}_\theta$ is the orthogonal projection from $\mathcal{L}^2(Q)^4$ onto $\mathcal{E}_\theta$.

A θ-periodic operator A_θ is then defined as follows:

for $f = (f_1, f_2, f_3, f_4) = (\widetilde{f}, f_4)$ we stipulate:

$$\begin{aligned} &f \in \operatorname{dom}(A_\theta) \text{ iff } f \in (H^2_\theta(Q) \cap H^1_{\theta,0}(Q))^4 \text{ and } \operatorname{div}\widetilde{f} = 0, \\ &A_\theta f = -\mathcal{P}_\theta(\nu\Delta\widetilde{f}, \kappa\Delta f_4) = -(\nu P_\theta \Delta\widetilde{f}, \kappa\Delta f_4). \end{aligned} \tag{3.5}$$

Proposition 3.1. *A_θ is selfadjoint on $\mathcal{E}_\theta$, positive, i.e., $\geq k > 0$ for some θ-independent k, and has compact resolvents.*

Remark. For $\theta \in \dot{M}_0$ this is proved in [16], Chapter 6; for $\theta \in M_0 - \dot{M}_0$ this is the well-known periodic case, treated e.g. in [6]. Now let $M = (m_{jk})$ be the 4×4 matrix with entries $m_{34} = m_{43} = 1$ and $m_{jk} = 0$ otherwise. The operator

$$B_\theta(\lambda) = -A_\theta + \lambda \mathcal{P}_\theta M \tag{3.6}$$

is then also selfadjoint, bounded from above, and has compact resolvents. The L-periodic case is a special case which arises if

$$\theta \in \{(0,0), (0, 2\pi), (2\pi, 0), (2\pi, 2\pi)\} = M_0 - \dot{M}_0.$$

In order to simplify the notation we set

$$\begin{aligned} &E_{(0,0)} = E_L, \quad P_{(0,0)} = P_L, \quad \mathcal{E}_{(0,0)} = \mathcal{E}_L, \quad \mathcal{P}_{(0,0)} = \mathcal{P}_L \\ &A_{(0,0)} = A_L, \quad B_{(0,0)}(\lambda) = B_L(\lambda) = -A_L + \lambda \mathcal{P}_L M, \quad H^P_{(0,0)} = H^P_L; \end{aligned}$$

and in a context where L is constant we simplify further by dropping the index L, i.e., by setting $E_L = E$, $P_L = P$, $\mathcal{E}_L = \mathcal{E}$, $A_L = A$ etc. We stipulate:

$$\begin{aligned} &\sigma(\theta, \lambda) \text{ is the rightmost point in } \sigma(B_\theta(\lambda)); \quad \sigma(L, \lambda) \\ &\text{is the rightmost point in } \sigma(B_L(\lambda)). \end{aligned} \tag{3.7}$$

Proposition 3.2. (a) *For* $\theta \in M_0$ *there is* $\lambda^* = \lambda^*(\theta)$ *such that* $\sigma(\theta,\lambda) < 0$ *for* $\lambda \in [0,\lambda^*]$, (b) $\sigma(\theta,\lambda)$ *is strictly monotone increasing on* $[\lambda^*,\infty)$ *and continuous,* (c) $\lim \sigma(\theta,\lambda) = \infty$ *as* $\lambda \to \infty$.

Remark. Proposition 3.2 applies in particular to $\sigma((0,0),\lambda) = \sigma(L,\lambda)$; see Section 2 in [14]. One can prove $\lambda^* = 0$ but this was not proved in [14] and is irrelevant for what follows. By Proposition 3.2 there is a unique λ_L (a critical Rayleigh number) such that

$$\sigma(L,\lambda_L) = 0, \quad \sigma(L,\lambda) \lesseqgtr 0 \text{ if } \lambda \lesseqgtr \lambda_L. \tag{3.8}$$

As to the nonlinearity in (1.1) we set ((2.1), (2.2))

$$\mathcal{N}_\theta(w,v) = -\mathcal{P}_\theta(w\nabla)v, \quad \mathcal{N}_L(w\nabla)v = -\mathcal{P}_L(w\nabla)v \tag{3.9}$$

and in a context of fixed period L we even write $\mathcal{N}(w,v)$ for $\mathcal{N}_L(w,v)$. For $\mathcal{N}_\theta(\,,\,)$, standard estimates hold; here we need

$$\|\mathcal{N}_\theta(w,v)\| \le C\|w\|_6\|\nabla v\|_3 \text{ where } \|w\|_p^p = \sum_j \int_Q |w_j|^p\,dx \tag{3.10}$$

$$\text{and } \|\nabla v\|_p^p = \sum_{j,k} \int_Q |\partial_k v_j|^p\,dx.$$

Taking into account our notational conventions, we can rewrite (1.1) in terms of the L-periodic setting, thereby eliminating the pressure with the aid of $\mathcal{P} = \mathcal{P}_L$ so as to get

$$\partial_t w = (-A + \lambda\mathcal{P}M)w + \mathcal{N}(w,w) + \mathcal{P}f, \quad w \in \operatorname{dom}(A) \tag{3.11}$$

($A = A_L$, $\mathcal{N} = \mathcal{N}_L$). This is an evolution equation in the sense of [19], page 196, admitting classical theorems on local existence and uniqueness.

In order to consider (1.1) on the infinite layer $\Omega = \mathbb{R}^2 \times (-\frac{1}{2},\frac{1}{2})$ we recall the Sobolev spaces $H^p(\Omega)$, $H_0^1(\Omega)$. We let E_ω be the $\mathcal{L}^2$ closure of the $f \in H_0^1(\Omega)^3$ such that $\operatorname{div}(f) = 0$; P_ω is the orthogonal projection from $\mathcal{L}^2(\Omega)^3$ onto E_ω. We set

$$\mathcal{E}_\omega = E_\omega \times \mathcal{L}^2(\Omega); \text{ if } f = (\widetilde{f}, f_4), \quad \widetilde{f} = (f_1,f_2,f_3), \quad \text{then} \tag{3.12}$$
$$\mathcal{P}_\omega f = (P_\omega \widetilde{f}, f_4).$$

The Stokes operator A_ω is then given by

$$f \in \operatorname{dom}(A_\omega) \text{ iff } f = (\widetilde{f}, f_4) \in (H^2(\Omega) \cap H_0^1(\Omega))^4 \text{ and} \tag{3.13}$$
$$\operatorname{div}(\widetilde{f}) = 0; \quad A_\omega f = -\mathcal{P}_\omega(\nu\Delta\widetilde{f}, \kappa\Delta f_4) = -(\nu P_\omega \Delta\widetilde{f}, \kappa\Delta f_4).$$

A_ω is selfadjoint on $\mathcal{E}_\omega$ and > 0. With $M = (m_{jk})$ as above we have that the operators

$$B_\omega(\lambda) = -A_\omega + \lambda \mathcal{P}_\omega M, \quad \lambda \geq 0 \tag{3.14}$$

are also selfadjoint and bounded from above. We stipulate:

$$\sigma_{\mathcal{L}^2}(\lambda) \text{ is the rightmost point in } \sigma(-A_\omega + \lambda \mathcal{P}_\omega M). \tag{3.15}$$

Proposition 3.3. (a) *There is* $\lambda^* \geq 0$ *such that* $\sigma_{\mathcal{L}^2}(0) = \sigma_{\mathcal{L}^2}(\lambda) < 0$ *for* $\lambda \in [0, \lambda^*]$, (b) $\sigma_{\mathcal{L}^2}(\lambda)$ *is strictly monotone increasing and continuous on* $[\lambda^*, \infty)$, (c) $\lim \sigma_{\mathcal{L}^2}(\lambda) = \infty$ *as* $\lambda \to \infty$.

Remarks. For a proof see [14], section 2. The nonlinearity $\mathcal{N}_\omega(\, , \,)$ is defined via (3.9) but with $\mathcal{P}_\omega$ in place of $\mathcal{P}_\theta$; it satisfies (3.10) but with another C, and with $\int_\Omega$ in place of $\int_Q$. If we replace A, $\mathcal{P}$, $\mathcal{N}$ in (3.10) by A_ω, $\mathcal{P}_\omega$, $\mathcal{N}_\omega$, then we get an evolution equation in the sense of [19], page 196 which is the abstract version of (1.1) on $\mathcal{L}^2(\Omega)$.

By Proposition 3.3 there is a unique λ_ω such that

$$\sigma_{\mathcal{L}^2}(\lambda_\omega) = 0, \quad \sigma_{\mathcal{L}^2}(\lambda) \lesseqgtr 0 \text{ iff } \lambda \lesseqgtr \lambda_\omega. \tag{3.16}$$

λ_ω is the critical Rayleigh number for the $\mathcal{L}^2(\Omega)$ setting.

4. Stability behaviour near a critical Rayleigh number

In order to study the stability of the equilibrium δv given by δf via (1.2) we replace w in (3.1) by $w + \delta v$, and for convenience λ by $\lambda + \varepsilon$. By a rearrangement of terms we obtain

$$\partial_t w = (-A + \lambda \mathcal{P} M) w + \varepsilon \mathcal{P} M w + \delta \{\mathcal{N}(w, v) + \mathcal{N}(v, w)\} + \mathcal{N}(w, w), \tag{4.1}$$
$$w \in \operatorname{dom}(A).$$

(4.1) is to be understood in the L-periodic setting, i.e., $A = A_L$, $\mathcal{P} = \mathcal{P}_L$, $\mathcal{N} = \mathcal{N}_L$. Since the operator G, given by

$$Gw = G_L w = -\mathcal{N}(v, w) - \mathcal{N}(w, v) \tag{4.2}$$

is bounded relative to A with relative bound zero, the linear part of the right-hand side of (4.1) is a holomorphic semigroup generator with compact resolvents. To study the stability of δv thus amounts to study the stability of the equilibrium $w = 0$ of (4.1). A simple situation is described by:

Proposition 4.1. (a) *Let* $\sigma(L, \lambda) < 0$, *then there are* $\delta_0, \varepsilon_0 > 0$ *such that* $|\delta| \leq \delta_0$, $|\varepsilon| \leq \varepsilon_0$ *entails that* $w = 0$ *is an asymptotically stable equilibrium of* (4.1). (b) *Let* $\sigma(L, \lambda) > 0$; *then there are* ε_0, δ_0 *such that* $|\delta| \leq \delta_0$, $|\varepsilon| \leq \varepsilon_0$ *implies that* $w = 0$ *is a Lyapounov unstable solution of* (4.1).

Proof. If $\sigma(L,\lambda) < 0$, then $\sigma(-A+\lambda\mathcal{P}M)$ lies strictly to the left of $\sigma = 0$. By standard perturbation theory ([5], pages 212, 368) there are $\varepsilon_0, \delta_0 > 0$ such that $|\varepsilon| \le \varepsilon_0$, $|\delta| \le \delta_0$ entails

$$\sigma(-A+\lambda\mathcal{P}M+\varepsilon\mathcal{P}M-\delta G) \subseteq \{z : \mathrm{re}(z) \le -k\}, \text{ some } k > 0$$

whence asymptotic stability follows ([2], Chapter 5).

The proof of (b) is similar and thus omitted. □

Remark. By Proposition 4.1, $\lambda = \lambda_L$ is the interesting case. Some preliminary information is given by:

Proposition 4.2. *Let $\lambda = \lambda_L$ in* (4.1). (a) *if $\varepsilon < 0$ there is $\delta_\varepsilon > 0$ such that $|\delta| \le \delta_\varepsilon$ entails asymptotic stability of $w = 0$.* (b) *If $\varepsilon > 0$, then there is $\delta_\varepsilon > 0$ such that $|\delta| \le \delta_\varepsilon$ entails that $w = 0$ is Lyapounov unstable.*

Proof. This is by arguments similar to those in the proof of Proposition 4.1. □

In order to state our first stability result we need:

Definition 1. A real holomorphic function $\psi(z)$ on $I = [-\frac{1}{2}, \frac{1}{2}]$ has property (C) if (a) $\int_I \psi\, dz > 0$, (b) $\psi(\frac{1}{2} - h) = ah^P + O(h^{p+1})$ some $a \ne 0$, $p \ge 1$, in a neighbourhood of $z = \frac{1}{2}$.

Proposition 4.3. *Let ψ_j, $j \le N$ have property (C); the set of $v \in H_0^1(I)$ such that $\int_I \psi_j \partial_z v\, dz > 0$, $j \le N$ is nonempty and thus open in $H_0^1(I)$, $\| \;\|_{H^1}$.*

The proof is given in the Appendix.

Theorem 1. *Let $L = (L_1, L_2)$ and assume $\sigma(L,\lambda) = 0$; i.e., $\lambda = \lambda_L$. Then there are functions ψ_j, $j \le N$ having property (C) such that the following holds. If v_4 in* (1.2) *satisfies $\int_I \psi_j \partial_z v_4\, dz < 0$, $j \le N$, then there are $\delta_0, c > 0$ as follows: if $-\delta = |\delta| \in [0, \delta_0]$ and $|\varepsilon| \le c|\delta|$, then* (4.1) *is weakly energy stable; i.e.,*

$$\langle(-A+(\lambda+\varepsilon)\mathcal{P}M)w, w\rangle + \delta\langle\mathcal{N}(w,v), w\rangle \le 0 \text{ for } w \in \mathrm{dom}(A). \tag{4.3}$$

Likewise for $\int \psi_j \partial_z v\, dz > 0$, $j \le N$ and $\delta = |\delta| \in [0, \delta_0]$.

Remark. Theorem 1 seems restricted; however it gains in impact in conjunction with

Theorem 2. *Let $v(z)$ be as in* (1.2) *and assume $\sigma(L,\lambda) = 0$. There are $\varepsilon_1, \delta_1 > 0$ as follows: if $|\varepsilon| \le \varepsilon_1$, $|\delta| \le \delta_1$ and if* (4.1) *is weakly energy stable, i.e., satisfies* (4.3)*, then $w = 0$ is a Lyapounov stable equilibrium of* (4.1).

Corollary 1. *Under the assumption of Theorem* 1, *there is* $\delta_2 > 0$ *as follows: if* $-\delta = |\delta| \in [0, \delta_2]$ *and* $|\varepsilon| \le c|\delta|$, *then* $w = 0$ *is Lyapounov stable; likewise with* $\delta \in [0, \delta_2]$, $|\varepsilon| \le c\delta$.

The proof of Theorem 1 is based on three preparatory steps, the first of which is a summary of some parts of Orr-Sommerfeld theory. Thus let $\lambda_L = \lambda$ as assumed; i.e., $\sigma = 0$ is the rightmost eigenvalue of $\sigma(-A+\lambda\mathcal{P}M)$. We need some information about the eigenspace L_0 of $\sigma = 0$; see [3], [6], [7], see also [1] for the comparable case of plane Couette flow. Let $\alpha, \beta \in \mathbb{Z}$; we stipulate

$$\widehat{\alpha} = \frac{2\pi\alpha}{L_1}\,, \quad \widehat{\beta} = \frac{2\pi\beta}{L_2}\,, \quad e_{\alpha\beta} = e^{i(\widehat{\alpha}x + \widehat{\beta}y)}, \quad \gamma^2 = \widehat{\alpha}^2 + \widehat{\beta}^2. \tag{4.4}$$

The eigenspace L_0 is then spanned by finitely many elements of the form

$$e_{\alpha\beta}\Big(\frac{i\widehat{\alpha}\partial_z w}{\gamma^2}\,,\ \frac{i\widehat{\beta}\partial_z w}{\gamma^2}\,, w, \mathcal{V}\Big) = \varphi_{\alpha\beta}, \text{ with } \gamma^2 = \widehat{\alpha}^2 + \widehat{\beta}^2 > 0, \tag{4.5}$$

where $w, \mathcal{V}$ are real solutions of the Orr-Sommerfeld equations

$$\begin{aligned} &\nu(\partial_z^2 - \gamma^2)^2 w = \lambda\gamma^2\mathcal{V}, \quad \kappa(\partial_z^2 - \gamma^2)\mathcal{V} + \lambda w = 0, \quad (w\mathcal{V} \not\equiv 0) \\ &w(\pm\tfrac{1}{2}) = (\partial_z w)(\pm\tfrac{1}{2}) = \mathcal{V}(\pm\tfrac{1}{2}) = 0. \end{aligned} \tag{4.6}$$

With $\varphi_{\alpha\beta}$ also $\overline{\varphi}_{\alpha\beta} = \varphi_{-\alpha,-\beta}$ is an eigenvector of the above form. Real eigenvectors $\phi^j_{\alpha\beta}$, $j = 1, 2$ are given by

$$\phi^1_{\alpha\beta} = \varphi_{\alpha\beta} + \overline{\varphi}_{\alpha\beta}, \quad \phi^2_{\alpha\beta} = i(\overline{\varphi}_{\alpha\beta} - \varphi_{\alpha\beta}). \tag{4.7}$$

Like the $\varphi_{\alpha\beta}$, $\overline{\varphi}_{\alpha\beta}$ the $\phi^1_{\alpha\beta}$, $\phi^2_{\alpha\beta}$ are orthogonal; normalizing factors could be introduced but are not needed. The following statements are proved in the Appendix of [17]:

Proposition 4.4.

(a) $\langle\mathcal{N}(\phi^1_{\alpha\beta}, v), \phi^2_{\alpha\beta}\rangle = -\langle\mathcal{N}(\phi^2_{\alpha\beta}, v), \phi^1_{\alpha\beta}\rangle$.

(b) $\langle\mathcal{N}(\phi^j_{\alpha\beta}, v), \phi^j_{\alpha\beta}\rangle = -2L_1L_2\int_I w\mathcal{V}\partial_z v_4\,dz$, $j = 1, 2$.

(c) $\langle\mathcal{P}M\phi^1_{\alpha\beta}, \phi^1_{\alpha\beta}\rangle = \langle\mathcal{P}M\phi^2_{\alpha\beta}, \phi^2_{\alpha\beta}\rangle$, $\langle\mathcal{P}M\phi^1_{\alpha\beta}, \phi^2_{\alpha\beta}\rangle = 0$.

As a corollary we obtain:

Proposition 4.5.

$$\langle\mathcal{N}(\eta_1\phi^1_{\alpha\beta} + \eta_2\phi^2_{\alpha\beta}, v), \eta_1\phi^1_{\alpha\beta} + \eta_2\phi^2_{\alpha\beta}\rangle = -(\eta_1^2 + \eta_2^2)(2L_1L_2)\int_I w\mathcal{V}\partial_z v_4\,dz.$$

The eigenspace L_0 is now determined by a list $(\alpha_j, \beta_j) \in \mathbb{Z}^2$, $j \leq \mathbb{N}$ such that

$$\{(\alpha_j, \beta_j), (-\alpha_j, -\beta_j)\} \cap \{(\alpha_k, \beta_k), (-\alpha_k, -\beta_k)\} = \phi \text{ for } j \neq k, \qquad (4.8)$$
$$\gamma_j^2 = \widehat{\alpha}_j^2 + \widehat{\beta}_j^2 > 0.$$

We assume $\alpha_j \beta_j \neq 0$, $j \leq \mathbb{N}$; the case where $\alpha_j = 0$ or $\beta_j = 0$, for some j, is subsumed under the arguments below. With each pair α_j, β_j functions $w_j, \mathcal{V}_j \neq 0$ on $I = [-\frac{1}{2}, \frac{1}{2}]$ are given, related to α_j, β_j via (4.6). In terms of $\alpha_j, \beta_j, w_j, \mathcal{V}_j$, $j \leq \mathbb{N}$ one defines $\phi_j^k = \phi_{\alpha_j \beta_j}^k$ via (4.5)–(4.7); the space L_0 is then given by

$$L_0 = \operatorname{span}(\phi_1^1, \phi_1^2, \ldots, \phi_N^1, \phi_N^2). \qquad (4.9)$$

We note:

Proposition 4.6.
(a) $\langle \phi_j^k, \phi_m^n \rangle = 0$ *if* $k \neq n$ *or* $j \neq m$.
(b) $\langle \mathcal{N}(\phi_j^k, v), \phi_m^n \rangle = 0$ *if* $j \neq m$.

The proof is by straightforward computation and omitted. In addition to this summary we recall a fact on $-A + \lambda \mathcal{P} M$:

Proposition 4.7. *Let* $\lambda = \lambda_L$*; there is* $k > 0$ *such that*

$$\langle (-A + \lambda \mathcal{P} M) g, g \rangle \leq -k \|g\|^2 \quad \text{for } g \text{ in } \operatorname{rg}(-A + \lambda \mathcal{P} M) \cap \operatorname{dom}(A).$$

Proof. $B(\lambda) = -A + \lambda \mathcal{P} M$ is selfadjoint on $\mathcal{E}$ with compact resolvents and has $\sigma = 0$ as rightmost eigenvalue. It follows that the restriction of $B(\lambda)$ to its invariant subspace $\operatorname{rg}(B(\lambda))$ has its spectrum strictly to the left of $\sigma = 0$. From these facts, the proposition follows. □

The last preparatory remark concerns the quadratic form

$$F(\eta) = A \sum_1^n \eta_j^2 + \sum_1^n B_j \eta_j, \quad A > 0. \qquad (4.10)$$

Proposition 4.8.

$$F(\eta) \geq -\Big(\sum_1^n B_j^2 \Big) (4A)^{-1}.$$

The proof, by linear algebra, is omitted.

Proof of Theorem 1. We first simplify the notation by numbering the list of eigenfunctions ϕ_j^1, ϕ_j^2, $j \leq \mathbb{N}$ consecutively: $\phi_1 = \phi_1^1$, $\phi_2 = \phi_1^2, \ldots, \phi_{2N-1} =$

ϕ_N^1 $\phi_{2N} = \phi_N^2$ and set

$$\eta\phi = \sum_1^{2N} \eta_j \phi_j, \quad \eta = (\eta_1, \ldots, \eta_{2N}), \quad |\eta|^2 = \sum_1^{2N} \eta_j^2. \tag{4.11}$$

Next we note that each product $\psi_j = w_j \mathcal{V}_j$, $j \leq \mathbb{N}$ has property (C) in Definition 1. In fact, by the second equation in (4.6) and the boundary conditions we have that

$$\lambda \int_I w\mathcal{V}\, dz = \kappa \int_I ((\partial_z \mathcal{V})^2 + \gamma^2 \mathcal{V}^2)\, dz > 0. \tag{4.12}$$

On the other hand, since $w, \mathcal{V}$ are necessarily holomorphic on $I = [-\frac{1}{2}, \frac{1}{2}]$ we infer from the boundary conditions at $+\frac{1}{2}$:

$$w(\tfrac{1}{2} - h) = ah^p + O(h^{p+1}), \quad \mathcal{V}(\tfrac{1}{2} - h) = bh^q + \mathcal{V}(h^{q+1}) \tag{4.13}$$

for some $p \geq 2$, $q \geq 1$, $a \neq 0$, $b \neq 0$. From (4.12), (4.13) it follows that $\psi_j = w_j \mathcal{V}_j$, $j \leq \mathbb{N}$ have indeed property (C). By Proposition 4.3 there is a smooth (even holomorphic) $v_4 \in H_0^1(I)$ such that

$$A_j = -2L_1 L_2 \int_I w_j \mathcal{V}_j \partial_z v_4\, dz > 0, \quad j \leq \mathbb{N}. \tag{4.14}$$

We assume that v_4 has been chosen so as to satisfy (4.14). We also note that by (4.9), (4.11) every $\psi \in \mathcal{E}$ has a unique representation

$$\psi = \eta\phi + r, \quad r \in \mathrm{rg}(B), \quad B = -A + \lambda \mathcal{P} M \tag{4.15}$$

with $r \in \mathrm{dom}(A)$ if and only if $\psi \in \mathrm{dom}(A)$. Besides (4.14) we also assume for what follows

$$-\delta = |\delta|, \quad \text{i.e., } \delta \leq 0. \tag{4.16}$$

In order to find values ε, δ for which the weak energy inequality (4.3) holds we pick $\psi = \eta\phi + r$ in $\mathrm{dom}(A)$ arbitrarily and evaluate (4.3) for such ψ; i.e., we consider

$$\mathcal{D} = \langle B(\eta\phi + r), \eta\phi + r\rangle + \varepsilon \langle \mathcal{P}M(\eta\phi + r), (\eta\phi + r)\rangle + \delta \langle \mathcal{N}(\eta\phi + r, v), \eta\phi + r)\rangle.$$

Now $\mathcal{P}M$ and B are symmetric, moreover,

$$\langle B\eta\phi, r\rangle = \langle B\eta\phi, \eta\phi\rangle = \langle Br, \eta\phi\rangle = 0.$$

On the basis of Propositions 4.4–4.6, we thus obtain:

$$\begin{aligned} \mathcal{D} &= \langle Br, r\rangle + \varepsilon \langle \mathcal{P}M\eta\phi, \eta\phi\rangle + 2\varepsilon \langle \mathcal{P}M\eta\phi, r\rangle + \varepsilon \langle \mathcal{P}Mr, r\rangle \\ &+ \delta \Sigma \eta_j^2 \langle \mathcal{N}(\phi_j, v), \phi_j\rangle + \delta \langle \mathcal{N}(r, v), r\rangle + \delta \Sigma \eta_j \{\langle \mathcal{N}(\phi_j, v), r\rangle + \langle \mathcal{N}(r, v), \phi_j\rangle\}. \end{aligned}$$

Setting

$$B_j = \langle \mathcal{N}(\phi_j, v), r\rangle + \langle \mathcal{N}(r, v), \phi_j\rangle, \quad j \le 2N, \tag{4.17}$$

we obtain, after a rearrangement of terms based on (4.17):

$$\mathcal{D} = \langle Br, r\rangle + \varepsilon\langle \mathcal{P}M\eta\phi, \eta\phi\rangle + 2\varepsilon\langle \mathcal{P}M\eta\phi, r\rangle + \varepsilon\langle \mathcal{P}Mr, r\rangle$$
$$- \tfrac{1}{2}|\delta|\Sigma\eta_j^2\langle \mathcal{N}(\phi_j, v), \phi_j\rangle - |\delta|\{\tfrac{1}{2}\Sigma\eta_j^2\langle \mathcal{N}(\phi_j, v), \phi_j\rangle + \Sigma\eta_j B_j + \langle \mathcal{N}(r, v), r\rangle\}.$$

We recall (4.14), Proposition 4.5, set $A_j = \langle \mathcal{N}(\phi_j, v), \phi_j\rangle$ and $2A = \min A_j$. By Proposition 4.7 and since $A > 0$ we obtain from the last equation the estimate:

$$\mathcal{D} \le -k\|r\|^2 + \varepsilon\langle \mathcal{P}M\eta\phi, \eta\phi\rangle + 2\varepsilon\langle \mathcal{P}M\eta\phi, r\rangle + \varepsilon\langle \mathcal{P}Mr, r\rangle$$
$$- |\delta|A|\eta|^2 - |\delta|\{|\eta|^2 A + \Sigma\eta_j B_j + \langle \mathcal{N}(r, v), r\rangle\}.$$

Now set $B^2 = \Sigma B_j^2$ and recall Proposition 4.8; we get

$$\mathcal{D} \le -k\|r\|^2 + \varepsilon\langle \mathcal{P}M\eta\phi, \eta\phi\rangle + 2\varepsilon\langle \mathcal{P}M\eta\phi, r\rangle + \varepsilon\langle \mathcal{P}Mr, r\rangle$$
$$- |\delta|A|\eta|^2 + |\delta|B^2(4A)^{-1} + |\delta|\ |\ \langle \mathcal{N}(r, v), r\rangle|.$$

Prior to the last step we note that there are constants $a, b, c, d, e > 0$ such that:

$$|\langle \mathcal{P}M\eta\phi, \eta\phi\rangle| \le a|\eta|^2, \quad 2|\langle \mathcal{P}M\eta\phi, r\rangle| \le b(|\eta|^2 + \|r\|^2),$$
$$|\langle \mathcal{P}Mr, r\rangle| \le c\|r\|^2, \quad B^2 4A^{-1} \le d\|r\|^2, \quad |\langle \mathcal{N}(r, v), r\rangle| \le e\|r\|^2.$$

By combining these inequalities with the last estimate for D we get, after some rearrangement of terms:

$$\mathcal{D} \le \{-k + |\varepsilon|(b + c) + |\delta|(d + e)\}\|r\|^2 + \{|\varepsilon|(a + b) - |\delta|A\}|\eta|^2.$$

The right-hand side is ≤ 0 for every choice of η, r if $|\varepsilon|, |\delta|$ satisfy the inequalities

$$|\varepsilon| \le \tfrac{k}{4}(b + c)^{-1}, \quad |\delta| \le q\tfrac{k}{4}(d + e)^{-1}, \quad |\varepsilon| \le (a + b)^{-1}A|\delta|. \tag{4.18}$$

From this the theorem follows. □

Now we come to the proof of Theorem 2. Here we are content to stress the essential points, since the proof, based on a bootstrap argument used in [10], follows closely the arguments in [15] (proofs of Lemma 2 and Theorem 4).

Proof of Theorem 2. We confine the parameters ε, δ provisionally to the range

$$|\varepsilon| \le a_0, \text{ some } a_0 \in (0, \lambda); \quad |\delta| \le 1. \tag{4.19}$$

Next we let $w \in \operatorname{dom}(A)$ and list some inequalities, all of which are standard:

$$\|\mathcal{P}Mw\| \le a\|w\|, \quad \|\mathcal{N}(w, v)\| \le b\|w\|, \quad \|\mathcal{N}(v, w)\| \le c\|A^{\frac{1}{2}}w\|, \tag{4.20}$$

$$|\langle \mathcal{N}(w,w), Aw\rangle| \le d\|A^{\frac{1}{2}}w\|^{\frac{3}{2}}\ \|Aw\|^{\frac{3}{2}},\ \|A^{\frac{1}{2}}w\| \le e\|Aw\|.$$

Here $a, b, \dots$ are constants depending on the setting and partly on v in (1.2). The inequality next to the last one follows from (3.10); see e.g. [7], or in our context [15], section 5. Now let $w(t)$ with $w(0) \in \mathrm{dom}(A)$ be a strong solution of (4.1), with $[0, T)$ ($T \le \infty$) its maximal interval of existence. Thus $w \in C^1([0,T),\mathcal{E})$, $w(t) \in \mathrm{dom}(A)$ for $t \in [0,T)$ and $Aw(\ ) \in C^0([0,T),\mathcal{E})$, and (4.1) holds pointwise. One then has

$$\partial_t\|A^{\frac{1}{2}}w(t)\|^2 = 2\langle Aw(t), \partial_t w(t)\rangle, \quad t \in [0,T). \tag{4.21}$$

([15], Proposition 5.2). Now we multiply (4.1) scalarly with $Aw(t)$ so as to get

$$\begin{aligned}\tfrac{1}{2}\,\partial_t\|A^{\frac{1}{2}}w\|^2 + \|Aw\|^2 = (\lambda+\varepsilon)\langle \mathcal{P}Mw, Aw\rangle + \delta\langle \mathcal{N}(w,v), Aw\rangle \\ + \delta\langle \mathcal{N}(v,w), Aw\rangle + \langle \mathcal{N}(w,w), Aw\rangle.\end{aligned} \tag{4.22}$$

In order to pass to estimates we set

$$X = \|A^{\frac{1}{2}}w(t)\|, \quad Y = \|w(t)\|, \quad Z = \|Aw(t)\|. \tag{4.23}$$

Based on (4.19), (4.20), (4.23) we infer from (4.22):

$$\partial_t X^2 + 2Z^2 \le 2((\lambda+a_0)a+b)YZ + 2c|\delta|XZ + 2d\,X^{\frac{3}{2}}Z^{\frac{3}{2}}.$$

Setting $C_0 = 2((\lambda+a_0)a+b)$ and recalling the last inequality in (4.20) we get

$$\partial_t X^2 + 2Z^2 \le C_0YZ + 2ce|\delta|Z^2 + 2dX^{\frac{3}{2}}Z^{\frac{3}{2}}. \tag{4.24}$$

We now restrict δ by the condition

$$2ce|\delta| \le 1 \tag{4.25}$$

so as to infer from (4.24)

$$\partial_t X^2 + Z^2 \le C_0YZ + 2dX^{\frac{3}{2}}Z^{\frac{3}{2}}. \tag{4.26}$$

By two applications of Young's inequality and by the last inequality in (4.20) we find $k, C > 0$ so as to obtain

$$\partial_t\|A^{\frac{1}{2}}w(t)\|^2 + k\|A^{\frac{1}{2}}w(t)\|^2 \le C\|A^{\frac{1}{2}}w(t)\|^6 + C\|w(t)\|^2 \tag{4.27}$$

for $t \in [0,T)$ whereby the substitutions in (4.23) have been undone. From this point on we follow the arguments in the proof of Theorem 4 in [15] in a verbatim way so as to conclude: (*) given $\varepsilon > 0$ there is $\mu > 0$ such that if $\|A^{\frac{1}{2}}w(0)\| \le \mu$ then $\|A^{\frac{1}{2}}w(t)\| \le \varepsilon$ on $t \in [0,T)$. From (*) it follows: (**) if $\|A^{\frac{1}{2}}w(0)\| \le \mu$ then $T = \infty$ and $\|A^{\frac{1}{2}}w(t)\| \le \varepsilon$ on $[0,\infty)$. This is the desired stability result. Stronger forms of stability, involving fractional power spaces $\mathcal{E}_\gamma$, $\gamma \in (\frac{3}{4}, 1)$, then follow from (**) and the singular Gronwall inequality; see [15] for details. □

Remark. The weak energy stability asserted by Theorem 1 is used at the beginning of the proof of Theorem 4 in [15] and here it enters when it comes to exploit (4.27) via Gronwall techniques. That is, if $\|A^{\frac{1}{2}}w(0)\|$ is small, then $\|w(0)\|$ is small via the Poincaré inequality, and then Theorem 1 guarantees that $\|w(t)\| \le \|w(0)\|$ for all $t < T$ (T = maximal existence time). Gronwall techniques (see proof of Theorem 4 in [15] for details) applied to (4.27) then yield $T = \infty$ and that $\|A^{\frac{1}{2}}w(t)\|$, $\|w(t)\|$ remain small for all $t < \infty$.

With c, δ_2 given by Corollary 1, and with G as in (4.2) one has:

Corollary 2. *If* $|\varepsilon| \le c|\delta|$, $-\delta = |\delta| \in [0, \delta_2]$, *then*

$$\sigma(-A + (\lambda + \varepsilon)\mathcal{P}M - \delta G) \subseteq \{z : \mathrm{re}(z) \le 0\}. \tag{4.28}$$

Proof. If (4.28) would fail, Lyapounov instability of the solution $w = 0$ of (4.1) would follow along standard arguments, in contradiction to Corollary 1. □

Comment. A proof of Corollary 2 by perturbation techniques seems to be very difficult in view of the eventual high dimension of the eigenspace L_0 of $0 \in \sigma(-A + \lambda\mathcal{P}M)$. While Theorem 1.2 and the corollaries express a property of the critical Rayleigh number λ_L, there are connections between λ_L and the critical Rayleigh number λ_ω in (3.16). In order to state it and for later use we stipulate:

If L_1, L_2 are periods, N_1, N_2 integers > 0, then $L = (L_1, L_2)$, $N = (N_1, N_2)$ and $NL = (N_1L_1, N_2L_2)$; if $N' = (N_1', N_2')$ then $N' \le N$ means $N_1' \le N_1$, $N_2' \le N_2$. We also recall the notational conventions fixed subsequent to (3.6). Specifically

$$G(v, \cdot) = G_L(v, \cdot) = -\{\mathcal{N}_L(v, \cdot) + \mathcal{N}_L(\cdot, v)\}. \tag{4.29}$$

The following is obvious. For any pair $N = (N_1, N_2)$ of integers > 0, $\mathcal{E}_L$ is a closed subspace of $\mathcal{E}_{NL}$, invariant under A_{NL}, $\mathcal{P}_{NL}$, $\mathcal{N}_{NL}(\,,\,)$ and hence under $G_{NL}(v, \cdot)$; the restriction of $A_{NL}, \mathcal{P}_{NL}, \mathcal{N}_{NL}$ to the invariant subspace then coincides with A_L, $\mathcal{P}_L$, $\mathcal{N}_L$.

Theorem 3. *Let* $\lambda = \lambda_L = \lambda_\omega$. *Given* $N^0 = (N_1^0, N_2^0)$ *there are smooth fields* $v(z) = (v_1, v_2, 0, v_4)$ *in* $H_0^1(I)^4$ *and constants* $c = c(N^0, v)$, $\delta_0 = \delta_0(N^0, v) > 0$ *as follows: if* $-\delta = |\delta| \in [0, \delta_0]$, $|\varepsilon| \le c|\delta|$, *then* $w = 0$ *is an equilibrium of*

$$\partial_t w = (-A_{NL} + \lambda\mathcal{P}_{NL}M)w + \varepsilon\mathcal{P}_{NL}Mw - \delta G_{NL}(v, w) + \mathcal{N}_{NL}(w, w) \tag{4.30$_\mathrm{N}$}$$

Lyapounov stable under NL-periodic perturbations for all $N \leq N^0$.

Proof. Recalling (3.7), (3.15) we invoke Lemma 2.1 in [14] according to which

$$\sigma(L,\lambda) \leq \sigma_{\mathcal{L}^2}(\lambda) \text{ for any period pair } L = (L_1, L_2). \tag{4.31}$$

Since $\lambda = \lambda_L = \lambda_\omega$ by assumption and hence $\sigma(L,\lambda) = 0$ we have that $0 \in \sigma(-A_L + \lambda\mathcal{P}_L M)$, whence by the preceding remarks $0 \in \sigma(-A_{NL} + \lambda\mathcal{P}_{NL}M)$ for any pair $N = (N_1, N_2)$. By (3.7) this implies $\sigma(NL,\lambda) \geq 0$. By (4.31) on the other hand we have that $\sigma(NL,\lambda) \leq \sigma_{\mathcal{L}^2}(\lambda)$. By combining these inequalities we get

$$0 \leq \sigma(NL,\lambda) \leq \sigma_{\mathcal{L}^2}(\lambda) \leq \sigma_{\mathcal{L}^2}(\lambda_\omega) = 0 \tag{4.32}$$

which by (3.8) and the arbitrariness of $N = (N_1, N_2)$ implies:

$$\lambda = \lambda_{NL} = \lambda_\omega \quad \text{for all } N = (N_1, N_2), \quad N_j > 0. \tag{4.33}$$

We now fix $N^0 = (N_1^0, N_2^0)$ and apply Theorem 1 to all λ_{N_L}, $N \leq N_0$ so as to find lists of holomorphic functions on $I = [-\frac{1}{2}, \frac{1}{2}]$

$$\psi_1^N, \psi_2^N, \ldots, \psi_{n_N}^N, \quad N \leq N^0 \tag{4.34}$$

all whose members have property (C). By Proposition 4.3 there exist smooth functions $v_4 \in H_0^1(I)$ such that

$$\int_I \psi_j^N \partial_z v_4 \, dz < 0 \quad \text{for } N \leq \mathbb{N}^0, \quad j \leq n_N. \tag{4.35}$$

We now pick any smooth functions $v_1, v_2 \in H_0^1(I)$ and set $v = (v_1, v_2, 0, v_4)$. We then apply Theorem 1 to each λ_{NL}, $N \leq \mathbb{N}^0$, taking (4.35) into account. By taking the minimum of all constants $\delta_0(N)$, $c_0(N)$ which emerge from Theorem 1 one finds constants c_0, δ_0 such that $-\delta = |\delta| \in [0, \delta_0]$ and $|\varepsilon| \leq c_0|\delta|$ implies that $(4.30)_N$ is weakly energy stable in $\mathcal{E}_{NL}$. By Theorem 2, under an eventual shrinking of c_0, δ_0, it then follows that if $-\delta = |\delta| \in [0, \delta_0]$ and $|\varepsilon| \leq c_0|\delta|$, then $w = 0$ is an equilibrium of $(4.30)_N$, $N \leq N_0$, Lyapounov stable under NL-periodic perturbations. □

Remark. The assertion of Theorem 3 holds precisely if $\lambda_L = \lambda_\omega$; if $\lambda_L \neq \lambda_\omega$, the assertion is false. In order to prove this we invoke Theorem 1 in [14], which when specialized to the present case ($T_1 = 0$ in Theorem 1) states:

Proposition 4.9. *There is $\zeta \in \sigma(-A_\omega + \lambda\mathcal{P}_\omega M)$ with $\zeta > 0$ if and only if for all sufficiently large integers $N_1 > 0$ there is $\sigma \in \sigma(-A_{NL} + \lambda\mathcal{P}_{NL}M)$ with $\sigma > 0$, where $N = (N_1, N_1)$.*

The counterpart to Theorem 3 is Theorem 4 below, in which equation $(4.30)_{\mathrm{N}}$ is considered for $\lambda = \lambda_L$.

Theorem 4. *Let $\lambda_L \neq \lambda_\omega$. For every integer $N_1 > 0$ sufficiently large and every smooth vector field $v = (v_1, v_2, 0, v_4) \in H_0^1(I)^4$ there are $\varepsilon_0 = \varepsilon_0(N_1, v)$, $\delta_0 = \delta_0(N_1, v) > 0$ as follows: if $|\varepsilon| \leq \varepsilon_0$, $|\delta| \leq \delta_0$, then $w = 0$ is a solution of* $(4.30)_N$*, Lyapounov unstable under NL-periodic perturbations* $(N = (N_1, N_1))$.

Proof. Since $\sigma(L, \lambda_L) = 0$ and according to (4.31) we have

$$0 = \sigma(L, \lambda_L) \leq \sigma_{\mathcal{L}^2}(\lambda_L). \tag{4.36}$$

In case of an equality sign, $\lambda_L = \lambda_\omega$ would follow via Proposition 3.3 and (3.16), contrary to the assumption. Thus

$$\sigma_{\mathcal{L}^2}(\lambda_L) > 0. \tag{4.37}$$

Since $\sigma_{\mathcal{L}^2}(\lambda_L) \in \sigma(-A_\omega + \lambda_L \mathcal{P}_\omega M)$ and by virtue of (4.37) we can apply Proposition 4.9 so as to find an integer $N_1 > 0$ and $\sigma = \sigma(N_1)$ such that

$$\sigma = \sigma(-A_{NL} + \lambda_L \mathcal{P}_{NL} M) \text{ and } \sigma > 0, \quad N = (N_1, N_1). \tag{4.38}$$

We now pick a smooth vector field $v(z) = (v_1, v_2, 0, v_4)$ in $H_0^1(I)^4$ arbitrarily and set

$$T_{\varepsilon\delta}^{NL} = -A_{NL} + \lambda_L \mathcal{P}_{NL} + \varepsilon \mathcal{P}_{NL} M - \delta G_{NL}(v, \cdot). \tag{4.39}$$

Since $G_{NL}(v, \cdot)$ (given by (4.2)) is bounded relative to $-A_{NL}$ with relative bound zero, it follow from standard perturbation theory ([5], Theorem 1.7, page 368) that there are $\varepsilon_0 = \varepsilon_0(N_1, v)$, $\delta_0 = \delta_0(N_1, v) > 0$ such that $|\varepsilon| \leq \varepsilon_0$, $|\delta| \leq \delta_0$ implies that there is an eigenvalue $\sigma' \in \sigma(T_{\varepsilon\delta}^{NL})$ with $\mathrm{re}(\sigma') > 0$. From [2], 5.1.3, page 102 it then follows that $w = 0$ is a Lyapounov unstable equilibrium of $(4.30)_N$. □

5. An instability result

The positive results in the last section depend to a large extent on the fact that the exterior force $f = f(z)$ and the associated equilibria $v = v(z)$ in (1.2) depend on $z \in [-\frac{1}{2}, \frac{1}{2}]$ only. Below we prove an instability result which exhibits the special role of the Rayleigh number λ_ω and which holds for a much larger class of exterior forces f and corresponding equilibria v. Let $L = (L_1, L_2)$ be a period pair; let $\mathcal{U} \subseteq \mathbb{R}^2$ be a neighbourhood of $\sigma \in \mathbb{R}^2$. We are given a holomorphic mapping

$$f(\varepsilon, \delta) \in \mathcal{E}_L, \quad \| \; \| \text{ for } (\varepsilon, \delta) \in \mathcal{U};\; f(0,0) = 0 \tag{5.1}$$

and a holomorphic mapping

$$\text{(a)} \quad v(\varepsilon, \delta) \in \mathrm{dom}(A_L), \; \| \; \|_{H^2} \quad \text{for} \quad (\varepsilon, \delta) \in \mathcal{U};\; v(0,0) = 0 \tag{5.2}$$

(b) $v(\varepsilon,\delta) \in H_L^3(Q)^4,\ (\varepsilon,\delta) \in \mathcal{U}$.

We assume that $v(\varepsilon,\delta)$ is an equilibrium solution of (1.1):

$$(-A_L + (\lambda_L + \varepsilon)\mathcal{P}_L M)v + \mathcal{N}_L(v,v) = f(\varepsilon,\delta), \quad v = v(\varepsilon,\delta), \tag{5.3}$$

with λ_L as in (3.8). We study the stability of $v(\varepsilon,\delta)$ under $\mathcal{L}^2(\Omega)$ perturbations, $\Omega = R^2 \times (-\frac{1}{2},\frac{1}{2})$ and are thus led to investigate the evolution equation

$$\partial_t w = (-A_\omega + (\lambda_L + \varepsilon)\mathcal{P}_\omega M)w - G_\omega(\varepsilon,\delta;w) + \mathcal{N}_\omega(w,w), \tag{5.4}$$
$$G_\omega(\varepsilon,\delta;w) = -\mathcal{N}_\omega(v(\varepsilon,\delta),w) - \mathcal{N}_\omega(w,v(\varepsilon,\delta)), \quad \mathcal{N}_\omega(u,w) = -\mathcal{P}_\omega(u\nabla)w.$$

This leads to the study of the spectrum of the operators

$$T(\varepsilon,\delta) = (-A_\omega - (\lambda_L + \varepsilon)\mathcal{P}_\omega M) - G_\omega(\varepsilon,\delta;\ ), \quad (\varepsilon,\delta) \in \mathcal{U}. \tag{5.5}$$

A remark as to (5.2)(b) is appropriate. In [16], page 66, one requires that $v(x,y,z)$ be L_1, L_2 periodic in x, y and in $C^m(\overline{\Omega})$ for some $m > 0$. In fact $m = 1$ suffices; this requires $v \in H_L^3(Q)^4$. Now an analysis shows that $v = H_L^2(Q)^4$ suffices for the considerations in [16], but in order to avoid a discussion of this point we retain (5.2)(b). Our result is:

Theorem 5. *Assume $\lambda_\omega \neq \lambda_L$. Then there are $\varepsilon_2, \delta_2 > 0$ as follows: if $|\varepsilon| \le \varepsilon_2$, $|\delta| \le \delta_2$, then there is $\sigma \in \sigma(T(\varepsilon,\delta))$ with $\mathrm{re}(\sigma) > 0$.*

Corollary. *If $|\varepsilon| \le \varepsilon_2$, $|\delta| \le \delta_2$, then $w = 0$ is a Lyapounov unstable equilibrium of (5.4); i.e., $v(\varepsilon,\delta)$ is Lyapounov unstable under $\mathcal{L}^2(\Omega)$ perturbations.*

Proof. This is a consequence of Theorem 5 in combination with Theorem 12.10 and 12.11 in [16].

The proof of Theorem 5 splits into two steps. In the first we exploit a spectral formula, given by Theorem 12.1 in [16], which holds in the present setting on the basis of assumptions (5.2)(a), (b) and which assumes the form:

$$\sigma(-A_\omega + \lambda_L \mathcal{P}_\omega M - G_\omega(\varepsilon,\delta;\ )) = \text{closure of} \bigcup_{\theta \in \dot{M}_0} \sigma(-A_\theta + \lambda_L \mathcal{P}_\theta M - G_\theta(\varepsilon,\delta;\ ) \tag{5.6}$$

with $\dot{M}_0$ as in (3.2), while $G_\omega(\varepsilon,\delta;\ )$ is given by (5.4); we have set

$$G_\theta(\varepsilon,\delta;\ ) = -\mathcal{N}_\theta(v(\varepsilon,\delta),\cdot) - \mathcal{N}_\theta(\cdot,v(\varepsilon,\delta)), \quad \mathcal{N}_\theta(u,w) = -\mathcal{P}_\theta(u\nabla)w.$$

A special case of (5.6) is

$$\sigma(-A_\omega + \lambda_L \mathcal{P}_\omega M) = \text{ closure of } \bigcup_{\theta \in \dot{M}_0} \sigma(-A_\theta + \lambda_L \mathcal{P}_\theta M). \tag{5.7}$$

Proposition 5.1. *There is* $\theta_0 \in \dot{M}_0$ *and* $\sigma \in \sigma(-A_{\theta_0} + \lambda_L \mathcal{P}_{\theta_0} M)$ *with* $\sigma > 0$.

Proof. Since $\lambda_L \neq \lambda_\omega$ by assumption we conclude as in the proof of Theorem 4 that $\sigma_{\mathcal{L}^2}(\lambda_L) > 0$. Since $\sigma_{\mathcal{L}^2}(\lambda_L) \in \sigma(-A_\omega + \lambda_L \mathcal{P}_\omega M)$ it follows from (5.7) and since $-A_\theta + \lambda_L \mathcal{P}_\theta M$ is selfadjoint that there is $\theta_0 \in \dot{M}_0$ and $\sigma \in R$ with:

$$\sigma \in \sigma(-A_{\theta_0} + \lambda_L \mathcal{P}_{\theta_0} M). \tag{5.8}$$

Remark. We note that (5.6), (5.7) are valid for any period pair $(L_1, L_2) = L$. Next we need the notion of holomorphic family of type (A) ([5], 375–381). In [5], this notion is explained for one variable κ; but an inspection shows that the considerations extend as they stand to two and more variables.

Proposition 5.2. *Let* $\theta \in \dot{M}_0$*; there are* $\varepsilon_1, \delta_1 > 0$ *such that the operator family*

$$T_\theta(\varepsilon, \delta) = -A_\theta + \lambda_L \mathcal{P}_\theta M - G_\theta(\varepsilon, \delta;\ ), \quad |\varepsilon| \le \varepsilon_1,\ |\delta| \le \delta_1 \text{ is of type (A).} \tag{5.9}$$

Proof. Below c_0, c_1, c_2 are constants with different meanings at different places. We introduce two operator families via

$$H(\varepsilon, \delta)w = \mathcal{N}_\theta(v(\varepsilon, \delta), w), \quad L(\varepsilon, \delta)w = \mathcal{N}_\theta(w, v(\varepsilon, \delta)), \tag{5.10}$$

where $w \in \operatorname{dom}(A_\theta)$. A glance at [5], page 375, shows that the proposition is proved if we can show that $H(\varepsilon, \delta)$, $L(\varepsilon, \delta)$, $|\varepsilon| < \varepsilon_1$, $|\delta| \le \delta_1$ are holomorphic families of type (A) for ε_1, δ_1 sufficiently small. Now by (5.2)(a) there are $a, b > 0$ such that $v(\varepsilon, \delta)$ has the representation:

$$v(\varepsilon, \delta) = \sum a_{jk} \varepsilon^j \delta^k, \quad \text{with } a_{jk} \in \operatorname{dom}(A_L) \text{ and} \tag{5.11}$$
$$\|a_{jk}\|_{H^2} \le c_0 a^j b^k, \quad |a\varepsilon| < 1,\ |b\delta| < 1.$$

Next we define operators H_{jk} from $\operatorname{dom}(A_\theta)$ to $\mathcal{E}_\theta$ via

$$H_{jk} w = \mathcal{N}_\theta(a_{jk}, w), \quad w \in \operatorname{dom}(A_\theta). \tag{5.12}$$

From (3.10), (5.11), (5.12) we get the estimates:

$$\|H_{jk} w\| \le c_1 \|a_{jk}\|_{H^2}\ \|\nabla w\| \le c_2 a^j b^k \|w\|_{H^1}$$

whence

$$\|\Sigma H_{jk} w \varepsilon^j \delta^k\| \le \sum c_2 \|w\|_{H^1} |a\varepsilon|^j\ |b\delta|^k < \infty \text{ for } |\varepsilon a|, |b\delta| < 1.$$

Thus,

$$H(\varepsilon, \delta)w = \Sigma H_{jk} w \varepsilon^j \delta^k \text{ strongly for } |\varepsilon a|, |\delta b| < 1;$$

i.e., $H(\varepsilon, \delta)$, $|\varepsilon| < a^{-1}$, $|\delta| < b^{-1}$ is indeed holomorphic of type (A).

In order to handle $L(\varepsilon,\delta)$ we define operators L_{jk} via

$$L_{jk}w = \mathcal{N}_\theta(w, a_{jk}), \quad w \in \mathrm{dom}(A_\theta). \tag{5.13}$$

From (3.10) and standard embedding theorems we get:

$$\|L_{jk}w\| \le c_0\|w\|_6 \;\|\nabla a_{jk}\|_3 \le c_1\|w\|_{H^1}\|a_{jk}\|_{H^2} \le c_2\|w\|_{H^1}\; a^j b^k$$

whence

$$\|\sum L_{jk}w\varepsilon^j\delta^k\| \le \Sigma c_2\|w\|_{H^1}|a\varepsilon|^j|b\delta|^k < 1 \text{ for } |a\varepsilon|, |b\delta| < 1.$$

Thus,

$$L(\varepsilon,\delta)w = \Sigma L_{jk}w\varepsilon^j\delta^k \text{ strongly for } |a\varepsilon|, |b\delta| < 1;$$

i.e., $L(\varepsilon,\delta)$, $|\varepsilon| < a^{-1}$, $|\delta| < b^{-1}$ is a holomorphic family of type (A), which proves the proposition for $\varepsilon_1 = \frac{a^{-1}}{2}$, $\delta_1 = \frac{b^{-1}}{2}$. □

Proof of Theorem 5. We recall $\theta_0 \in \dot{M}_0$ and $\sigma > 0$ in Proposition 5.1. Since $T_{\theta_0}(\varepsilon,\delta)$, $|\varepsilon| \le \varepsilon_1$, $|\delta| \le \delta_1$ is a holomorphic family of type (A) by Proposition 5.2, it follows from Proposition 5.1 above, from Theorem 1.7, page 368 and the remarks on page 379 in [5] that given $\mu > 0$ there are $\varepsilon_2 \le \varepsilon_1$, $\delta_2 \le \delta_1$ such that $|\varepsilon| \le \varepsilon_2$, $|\delta| \le \delta_2$ implies that there is $\sigma^* = \sigma^*(\varepsilon,\delta)$ with

$$\sigma^* \in \sigma(T_{\theta_0}(\varepsilon,\delta)) \text{ and } |\sigma^* - \sigma| < \mu. \tag{5.14}$$

Since $\sigma > 0$ we can choose μ so small that $\mathrm{re}(\sigma^*) > 0$ follows. We now fix ε, δ with $|\varepsilon| \le \varepsilon_2$, $|\delta| \le \delta_2$ and apply again the spectral formula (5.6), which in terms of (5.5), (5.9), reads:

$$\sigma(T(\varepsilon,\delta)) = \text{closure}\Big\{\bigcup_{\theta\in\dot{M}_0}\sigma(T_\theta(\varepsilon,\delta))\Big\}. \tag{5.15}$$

From (5.15) we infer in particular that $\sigma(T_{\theta_0}(\varepsilon,\delta)) \subseteq \sigma(T(\varepsilon,\delta))$ holds, which by virtue of (5.14) and our choice of μ implies:

$$\sigma^* \in \sigma(T(\varepsilon,\delta)), \quad \mathrm{re}(\sigma^*) > 0 \quad (\sigma^* = \sigma^*(\varepsilon,\delta)). \tag{5.16}$$

Theorem 5 now follows from (5.16) and the arbitrariness of ε,δ. □

Comment. Theorems 1, 2,and 5 show, generally speaking, that if $\lambda_L \ne \lambda_\omega$, a family of equilibria (or of bifurcating solutions) $v(\varepsilon,\delta)$ of (5.3) can be Lyapounov stable under L-periodic perturbations (as a solution to (5.3)) for some $\varepsilon \ne 0$, $\delta \ne 0$ arbitrary small, but is necessarily Lyapounov unstable under $\mathcal{L}^2(\Omega)$ perturbations for all ε,δ sufficiently small. A case with which one is precisely in this situation is given by the equilibria $\delta v(z)$ related to $\delta f(z)$ via (1.2). For suitably chosen $v(z) \ne 0$, δv is an equilibrium of (5.3), Lyapounov stable under L-periodic perturbations for $|\varepsilon| \le c_0|\delta|$, $-\delta = |\delta| \in$

$[0, \delta_0]$, but Lyapounov unstable under $\mathcal{L}^2(\Omega)$ perturbations for $|\varepsilon| \le \varepsilon_2$, $|\delta| \le \delta_2$. Further examples which fall under the scope of Theorem 5 (after some preparatory steps) are provided by the bifurcating solutions of the Bénard problem ([6],[13]) and by the traveling waves constructed in [17].

6. The critical Rayleigh number λ_ω

By its definition via (3.15) and Proposition 3.3, λ_ω is the uniquely determined λ such that $\sigma(-A_\omega + \lambda \mathcal{P}_\omega M)$ has $\sigma = 0$ as its rightmost point. The existence is secured by Proposition 3.3. In order to obtain a more constructive characterization of λ_ω we recall the Orr-Sommerfeld equations (4.6) and list a number of results proved in [3]; see also [6], [7], [1].

Proposition 6.1. *There are real analytic functions $\lambda_n(\gamma^2)$, cf. $\gamma^2 > 0$, $n \ge 1$ such that:*

(a) $0 < \lambda_1(\gamma^2) < \lambda_2(\gamma^2) < \dots$

(b) $\lim_n \lambda_n(\gamma^2) = \infty$, $\lim_{\gamma^2 \to \infty} \lambda_n(\gamma^2) = \infty$, $\lim_{\gamma^2 \to 0} \lambda_n(\gamma^2) = \infty$.

(c) If $\lambda = \lambda_n(\gamma^2)$, for some $n \ge 1$, $\gamma^2 > 0$ there are (up to scalar multiples) uniquely determined real holomorphic functions $w(z), \mathcal{V}(z) \ne 0$, $z \in [-\frac{1}{2}, \frac{1}{2}]$ which satisfy (4.6),

(d) if (4.6) holds for some $\lambda > 0$, $\gamma^2 > 0$, $w, \mathcal{V} \ne 0$ then $\lambda = \lambda_n(\gamma^2)$ for some $n \ge 1$,

(e) if $\gamma^2 = \left(\frac{2\pi\alpha}{L_1}\right)^2 + \left(\frac{2\pi\beta}{L_2}\right) > 0$ for some $\alpha, \beta \in \mathbb{Z}$ and period pair $L = (L_1, L_2)$, then $0 \in \sigma(-A_L + \lambda_n(\gamma^2)\mathcal{P}_L M)$ for all $n \ge 1$.

Properties (a)–(e) are proved in [3]. A further property can be read off from figure 2 in [3]; since we do not know of any proof we state it as a conjecture

$$\lambda_1(\gamma^2), \quad \gamma^2 > 0 \text{ is convex.} \tag{6.1}$$

The Rayleigh number λ_ω is now characterized by:

Theorem 6. *Let $\mathcal{M} = \{\zeta :$ there is $\gamma^2 > 0$ with $\lambda_1(\gamma^2) = \zeta\}$. Then (a) $\lambda_\omega = \inf \mathcal{M}$, (b) $\lambda_\omega \in \mathcal{M}$.*

The proof relies on Proposition 6.1 above, and on Lemma 2.2 and Proposition 2.2 in [14]. Lemma 2.2 in particular, which implies (4.31) as a special case, is as follows:

Proposition 6.2. *Let $L = (L_1, L_2)$ be any period pair and set*

$$\mathcal{M}_L = \{\sigma(\theta, \lambda) : \theta \in \dot{M}_\theta\} \cup \{\sigma(L, \lambda)\}, \quad \dot{M}_0 \text{ via } (3.2);$$

$\mathcal{M}_L$ has a maximum σ_∞ and $\sigma_\infty = \sigma_{\mathcal{L}^2}(\lambda)$.

Remark. While the set $\mathcal{M}_L$ depends on the periods $L = (L_1, L_2)$ and on λ, the supremum is independent of L.

Proof of Theorem 6. We first prove (b) and set $\lambda = \lambda_\omega$. Fix any period pair $L = (L_1, L_2)$. Since $\sigma_{\mathcal{L}^2}(\lambda_\omega) = 0$ by definition, $\sigma = 0$ is the rightmost point in $\mathcal{M}_L$ by Proposition 6.2, whence $\sigma(\theta, \lambda_\omega) = 0$ for some $\theta \in \dot{M}_0$ ((2.2)) or $\sigma(L, \lambda_\omega) = 0$. In either case we can invoke Proposition 2.2 in [14] which asserts that there are holomorphic functions $w, \mathcal{V} \neq 0$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\gamma^2 > 0$ such that the Orr-Sommerfeld equations hold for $w, \mathcal{V}, \lambda_\omega, \gamma^2$. By Proposition 6.1 (d) there is $n \geq 1$ such that $\lambda_n(\gamma^2) = \lambda_\omega$. Now $n \geq 2$ cannot hold. Otherwise we have $\lambda_1(\gamma^2) < \lambda_n(\gamma^2)$. Now fix a period pair $\ell = (\ell_1, \ell_2)$, $\ell_j > 0$ and integers $\alpha, \beta \in \mathbb{Z}$ such that $\gamma^2 = \left(\frac{2\pi\alpha}{\ell_1}\right)^2 + \left(\frac{2\pi\beta}{\ell_2}\right)^2$. By Proposition 6.1 (e) we then have

$$0 \in \sigma(-A_\ell + \lambda_1(\gamma^2)\mathcal{P}_\ell M). \tag{6.2}$$

From (6.2), Proposition 3.2 and from (4.30) we infer

$$0 \leq \sigma(\ell, \lambda_1(\gamma^2)) < \sigma(\ell, \lambda_n(\gamma^2)) \leq \sigma_{\mathcal{L}^2}(\lambda_n(\gamma^2)) = \sigma_{\mathcal{L}^2}(\lambda_\omega) \tag{6.3}$$

which contradicts $\sigma_{\mathcal{L}^2}(\lambda_\omega) = 0$; thus $n = 1$ and hence (b) follows.

It remains to prove (a). Thus let $\lambda_\omega = \lambda_1(\gamma^2)$ and assume $\lambda_1(\widetilde{\gamma}^2) < \lambda_1(\gamma^2)$, for some $\gamma^2, \widetilde{\gamma}^2 > 0$. Fix periods $L = (L_1, L_2)$ and integers $\alpha, \beta \in \mathbb{Z}$, $\widetilde{\gamma}^2 = \left(\frac{2\pi\alpha}{L_1}\right)^2 + \left(\frac{2\pi\beta}{L_2}\right)^2$. Then

$$0 \in \sigma(-A_L + \lambda_1(\widetilde{\gamma}^2)\mathcal{P}_L M) \tag{6.4}$$

by Proposition 6.1 (e), whence $\sigma(L_1, \lambda_1(\widetilde{\gamma}^2)) \geq 0$.

Since $\lambda_1(\widetilde{\gamma}^2) < \lambda_1(\gamma^2)$ and by Proposition 6.2 we have

$$0 \leq \sigma(L, \lambda_1(\widetilde{\gamma}^2)) < \sigma(L, \lambda_1(\gamma^2)) \leq \sigma_{\mathcal{L}^2}(\lambda_1(\gamma^2)) = \sigma_{\mathcal{L}^2}(\lambda_\omega)$$

where (4.31) has been used, contradicting $\sigma_{\mathcal{L}_2}(\lambda_\omega) = 0$. □

Corollary. *If conjecture* (6.1) *holds, there is a unique* $\gamma_0^2 > 0$ *such that* $\lambda_1(\gamma^2) > \lambda_1(\gamma_0^2)$ *for* $\gamma^2 \neq \gamma_0^2$ *and* $\lambda_\omega = \lambda_1(\gamma_0^2)$.

Remark. If (6.1) is false then by Proposition 6.1 (a), (b) there are finitely many $\gamma_j^2 > 0$, $j \leq n$ such that $\lambda_1(\gamma^2) > \lambda_1(\gamma_j^2)$, if $\gamma^2 \neq \gamma_j^2$, $j \leq n$ and $\lambda_\omega = \lambda_1(\gamma_j^2)$. It follows that the set of critical periods $L = (L_1, L_2)$ with $\lambda_L = \lambda_\omega$ is not empty; it suffices to fix periods L_1, L_2 such that $\left(\frac{2\pi\alpha}{L_1}\right)^2 + \left(\frac{2\pi\beta}{L_2}\right)^2 = \gamma_j^2$, for some $j \leq n$, $\alpha, \beta \in \mathbb{Z}$. On the other hand, the set of these critical periods has measure zero.

7. Outlook

Guided by Theorem 6 one may try to characterize a critical Rayleigh number λ_ω associated with the linear part $T'(\lambda,\delta)$ of the right-hand side of the evolution equation

$$\partial_t w = (-A_\omega + \lambda \mathcal{P}_\omega M)w - \delta G_\omega(v,w) + \mathcal{N}_\omega(w,w). \tag{7.1}$$

Theorems 1, 2 suggest that it might be advantageous to replace $T'(\lambda,\delta)$ by the operator $T(\lambda,\delta)$, related to the weak energy stability of (7.1). To define it we let $H(v)$ be the 4×4 matrix given as follows: (a) the third row is $(\partial_z v_1, \partial_z v_2, 0, \partial_z v_4)$, (b) the third column is $(\partial_z v_1, \partial_z v_2, 0, \partial_z v_4)^t$, (c) all other entries of $H(v)$ are equal to 0. We also set $\frac{\delta}{2} = \tau$ and stipulate

$$T(\lambda,\tau) = -A_\omega + \lambda\mathcal{P}_\omega M - \tau \mathcal{P}_\omega H(v), \quad \mathrm{dom}(T(\lambda,\tau)) = \mathrm{dom}(A). \tag{7.2}$$

It then follows that (7.1) is weakly energy stable if and only if:

$$\langle T(\lambda,\tau)w, w\rangle \le 0 \text{ for } w \in \mathrm{dom}(A). \tag{7.3}$$

$T(\lambda,\tau)$ is selfadjoint on $\mathcal{E}$ and bounded from above; let $\sigma_{\mathcal{L}^2}(\lambda,\tau)$ be the rightmost point in its spectrum. By arguments similar to those in [14], section 2 one proves:

Proposition 7.1. *There is $\tau_0 > 0$ as follows: if $|\tau| \le \tau_0$, then there is a unique λ_τ with $\sigma(\lambda_\tau,\tau) = 0$.*

One may try to characterize λ_τ in a way similar to λ_ω via Theorem 6. This seems difficult since the Orr-Sommerfeld equations are now replaced by two complicated ODEs which do not admit any simple theory. In the special case where $f = (0,0,0,f_4)$ or equivalently $v = (0,0,0,v_4)$, equations (4.6) are replaced by the system:

$$\begin{aligned} &\nu(\partial_z^2 - \gamma^2)^2 w = (\lambda - \tau\partial_z v_4)\gamma^2 \mathcal{V} \qquad (7.4)\\ &\kappa(\partial_z^2 - \gamma^2)\mathcal{V} = (\lambda - \tau\partial_z v_4)w, \quad w(\pm\tfrac12) = \partial_z w(\pm\tfrac12) = \mathcal{V}(\pm\tfrac12) = 0. \end{aligned}$$

In order to characterize λ_τ we stipulate

$$\lambda \in \mathcal{M}_\tau \text{ iff there is } \gamma^2 > 0 \text{ such that (7.4) has a solution } w, \mathcal{V} \ne 0. \tag{7.5}$$

Theorem 7. *Let $v = (0,0,0,v_4)$. There is $\tau_0 > 0$ such that $|\tau| \le \tau_0$ implies: (a) $\lambda_\tau = \inf \mathcal{M}_\tau$, (b) $\lambda_\tau \in \mathcal{M}_\tau$.*

The proof, which is similar to that of Theorem 6, is omitted for reasons of space. Although (7.4) looks similar to (4.6), one cannot apply the methods leading to Proposition 6.1 and to Theorem 6, since the method of oscillation kernels, basic to Proposition 6.1, is lacking; see [11] in this respect.

These remarks indicate the difficulty in obtaining stability results via Bloch space techniques. We mention two papers in this respect: in [9], sideband instability via Bloch space techniques is proved for a setting somewhat different from ours, while in [4] stability results in the sense of Eckhaus for a Bénard problem on a strip are obtained by rather involved Bloch space techniques.

Appendix A. Appendix

For $\beta > \alpha > 0$ we set $\tau(\alpha,\beta) = \frac{1}{2}(\beta-\alpha)(\beta+\alpha)^{-1}$ and $\mu(\alpha,\beta) = \frac{1}{2} - \tau(\alpha,\beta) = \alpha(\alpha+\beta)^{-1}$. A step function $t_{\alpha\beta}(z)$, $z \in [-\frac{1}{2}, \frac{1}{2}] = I$ is defined as follows:

$$\begin{aligned} t(z) &= \alpha \text{ for } -\tfrac{1}{2} \le z \le t(\alpha,\beta) \\ &= -\beta \text{ for } \tau(\alpha,\beta) < z \le \tfrac{1}{2}. \end{aligned} \tag{A.1}$$

Now let $f(z)$, $z \in I$ have property (C). Thus,

$$f(\tfrac{1}{2} - h) = a_0 h^p + \cdots \text{, for some } p \ge 1,\ a_0 \ne 0 \text{ for } h \in [\tfrac{1}{2} - \delta], \tag{A.2}$$

$$\text{for some } \delta > 0; \text{ and } \int_I f\, dz > 0.$$

Lemma A1.

$$\lim_{\beta\to\infty} \int_I f t_{\alpha\beta}\, ds = \alpha \int_I f\, dz > 0.$$

Proof. We may assume that δ in (A.2) is such that

$$|f(\tfrac{1}{2} - h)| \le 2|a_0| h^P,\ h \in [0,\delta]. \tag{A.3}$$

By straightforward computation one shows:

$$\int_I t_{\alpha\beta}\, dz = 0. \tag{A.4}$$

We then have

$$\int_I f t_{\alpha\beta}\, dz = \alpha \int_{-\frac{1}{2}}^{\tau(\alpha,\beta)} f\, dz - \beta \int_{\tau(\alpha,\beta)}^{\frac{1}{2}} f\, dz = \mathrm{I}_{\alpha\beta} - \mathrm{II}_{\alpha\beta}.$$

Since $\lim_{\beta\to\infty} = \tau(\alpha,\beta) = \frac{1}{2}$ we have

$$\lim_{\beta\to\infty} \mathrm{I}_{\alpha\beta} = \alpha \int_I f\, dz. \tag{A.5}$$

Since $\lim_{\beta\to\infty} \mu(\alpha,\beta) = 0$ we may take β so large that

$$\tfrac{1}{2} - \tau(\alpha,\beta) = \mu(\alpha,\beta) \le \delta. \tag{A.6}$$

We then have

$$|\mathrm{II}(\alpha,\beta)| \leq \beta \int_{\tau(\alpha,\beta)}^{\frac{1}{2}} |f(z)|\,dz = \beta \int_0^{\mu(\alpha,\beta)} |f(\tfrac{1}{2}-h)|\,dh$$
$$\leq 2|a_0|\beta \int_0^{\mu(\alpha,\beta)} h^p\,d = \beta\mu(\alpha,\beta)^{p+1}(p+1)^{-1}$$

whence $\lim_{\beta\to\infty} \mathrm{II}(\alpha,\beta) = 0$. To sum up

$$\lim_{\beta\to\infty} (\mathrm{I}(\alpha,\beta) - \mathrm{II}(\alpha,\beta)) = \alpha \int_I f\,ds > 0.$$

Corollary. *Let $f_1,\ldots,f_N$ have property (C). For $\beta > 0$ sufficiently large, $\int_I f_p t_{\alpha\beta}\,dz > 0$, $j \leq N$.*

Next we stipulate: $\psi \in \widehat{L}^2(I)$ if and only if $\psi \in L^2(I)$ and $\int_I \psi\,dz = 0$. We define a bounded linear operator T from $H_0^1(I)$ to $\widehat{L}^2(I)$ by $T_\psi = \partial_z\psi$. Without proof we note:

Lemma A2. *T is one-to-one and onto.*

Now let f_j, $j \leq \mathbb{N}$ have property (C). By the corollary there is $\beta > 0$ so large that $\int_I f_j t_{\alpha\beta}\,dz > 0$, $j \leq \mathbb{N}$.

By Lemma A2 and since $t_{\alpha\beta} \in \widehat{L}^2(I)$,there is $v_0 \in H_0^1(I)$ with $Tv_0 = t_{\alpha\beta}$ whence

$$\int_I f_j \partial_z v_0 > 0, \quad j \leq \mathbb{N}. \tag{A.7}$$

By choosing $v \in H_0^1(I)$ holomorphic with $\|v - v_0\|_{H^1}$ sufficiently small we infer $\int_I f_j \partial_z v\,dz > 0$, $j \leq N$ from (A.7).

REFERENCES

[1] J. Denzler, *A study of spectral theory of the Orr-Sommerfeld equation for plane Couette flow,* Habilitationsschrift, Technische Universität München, September 1999.
[2] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lect. Notes in Math., 840, Springer, Berlin 1981.
[3] V.I. Judovič, *On the origin of convection,* J. Appl. Math. Mech., 30 (1966), 1193–1199.
[4] Y. Kagei and W. von Wahl, *The Eckhaus criterion for convection solutions of the Boussinesq equation,* Int. J. Nonlinear Mechanics, 32 (1997), 563–620.
[5] T. Kato, "Perturbation Theory for Linear Operators," Springer, Berlin 1966.
[6] K. Kirchgässner and H. Kielhöfer, *Stability and bifurcation in fluid dynamics,* Rocky Mt. J. of Math., 3, N (1973), 275–318.
[7] K. Kirchgässner, *Bifurcation in nonlinear hydrodynamic stability,* SIAM Rev. 17 (1975), 452–483.

[8] A. Mielke, *Instability and stability of rolls in the Swift-Hohenberg equation,* Comm. Math. Phys., 189 (1997), 829–853.

[9] A. Mielke, *Mathematical analysis of sideboard instabilities with applications to Rayleigh-Bénard convection,* J. Nonlinear Sci., 7 (1997), 57–99.

[10] H. Ponce, R. Racke, T. Sideris, and E. Titi, *Global stability of large solutions to 3D-Navier-Stokes equations,* Comm. Math. Phys., 159 (1994), 329–341.

[11] P. Rabinowitz, *Existence and nonuniqueness of rectangular solutions of the Bénard problem,* Arch. Rat. Mech. Anal., 29 (1958), 32–57.

[12] P. Roberts, "An Introduction to Magneto-Hydrodynamics," Longmans 1974.

[13] B. Scarpellini, *Equilibrium solutions of the Bénard equations with an exterior force,* Diff. and Integr. Eq., 16 (2003), 129–158.

[14] B. Scarpellini and W. von Wahl, *Stability properties of the Boussinesq equations,* J. of Appl. Math. and Physics (ZAMP) 49 (1998), 294–321.

[15] B. Scarpellini, *Large solutions of the magnetic Bénard problem on the infinite layer,* Nonlinear Analysis, 48 (2002), 95–116.

[16] B. Scarpellini, "Stability, Instability and Direct Integrals," CRCPress, Boca-Raton, Florida 1999.

[17] B. Scarpellini, *Bifurcation of traveling waves related to the Bénard equations with an exterior force,* To appear in "Nonlinear elliptic and parabolic problems", Birkhäuser, 2005.

[18] M. Reed and B. Simon, "Methods of Math. Physics," Vol. IV, Academic Press, New York, 1978.

[19] A. Pazy, "Semigroups of Linear Operators and Applications to PDE's," Appl. Math. Sci., Vol 44, Springer, New York 1983.