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ASYMPTOTIC BEHAVIOR OF SOME NONLOCAL PARABOLIC PROBLEMS

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Abstract. We consider the asymptotic behavior of solutions for a nonlocal quasilinear parabolic equation. We study the case where the associated nonlocal elliptic problem has a unique equilibrium. It is shown that under certain assumptions such an equilibrium is a global attractor.

1. Introduction

We consider the equation

$$\left\{ \begin{array}{ll} \left\{ \begin{array}{rll} \partial_t u - a(r(u))\mathcal{A}u = f(x) & \text{in} & \Omega \\ \mathcal{B}u = 0 & \text{on} & \partial\Omega \end{array} \right\} & \text{for } t \in (0,\infty) \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega & \end{array} \right\}, \tag{1.1}$$

where $r \in C^1(W_p^{2\delta}, \mathbb{R})$ is positively homogeneous, $\delta \geq 0$, $f \in L_p$ with $f \geq 0$, and $a \in C^1(\mathbb{R}, [m,\infty))$, $m > 0$. Throughout this text Ω is a domain in $\mathbb{R}^n$ with compact C^k boundary $\Gamma := \partial\Omega$, $k > 2$, $\mathcal{A}$ is an elliptic second-order differential operator and $\mathcal{B}$ is a boundary operator of order at most one. Many physical and biological processes can be described mathematically by a mass balance law such as

$$u_t + \operatorname{div} \vec{j}(u) = f\,, \tag{1.2}$$

where $u := u(t) := u(t,x)$ is a population density or a temperature at time t and position $x \in \Omega$. Moreover, $\vec{j}$ is the flux, and f is the production rate of the species. The flux is specified by phenomenological laws. In many applications it is of the form

$$\vec{j}(u)(t,x) = a(u(t,x))\nabla u(t,x)\,. \tag{1.3}$$

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These models lead to quasilinear parabolic problems. In this article we consider the situation where

$$\vec{j}(u)(t,x) = a(\textstyle\int_\Omega u(t,x)\,dx)\nabla u(t,x)\,. \tag{1.4}$$

This case was also considered in a series of publications by M. Chipot and coworkers; see for instance [8] and [11]. A main advantage of this equation is that the study of the associated elliptic problem is easier. In [9], [10], it is shown, in the case $p = 2$, $\mathcal{A} = -\triangle$, $\mathcal{B} = \gamma_\partial$ (i.e., the trace operator), that there exists a one-to-one mapping from the set of stationary solutions to (1.1) onto the set of solutions $\mu \in \mathbb{R}$ to

$$a(\mu)\mu = L\,, \quad \text{where} \quad L := r\big((-\triangle|_{H^2\cap \mathring{H}^1_2})^{-1} f\big)\,. \tag{1.5}$$

In this article we study mainly the case where (1.5) has a single solution $\mu_\infty \in \mathbb{R}$. We consider the stability of $u_\infty := \frac{1}{a(\mu_\infty)}(-\triangle|_{H^2\cap \mathring{H}^1_2})^{-1} f$ in (1.1). The main result of this article is:

Theorem 1.1. *Suppose the function a satisfies the condition*

$$\hat{a}\Big(\frac{L}{\hat{a}(\mu)}\Big) \geq \frac{L}{\mu} \quad \textit{for} \quad \mu \in \big(\mu_\infty, \tfrac{L}{m}\big]\,, \tag{1.6}$$

where

$$\hat{a}(\mu) := \begin{cases} \max_{x\in[\mu_\infty,\mu]} a(x)\,, & \mu \geq \mu_\infty\,,\\ \min_{x\in[\mu,\mu_\infty]} a(x)\,, & \mu < \mu_\infty\,. \end{cases}$$

Moreover, suppose Assumptions 4.12 are valid, Ω is bounded and $f \geq 0$, $p > n$. Furthermore, assume the function r is increasing in the sense of (4.11). *Then*

$$u_\infty := \frac{1}{a(\mu_\infty)} A^{-1} f$$

is a global attractor in $W^2_{p,\mathcal{B}}$; that is, for each $u_0 \in W^2_{p,\mathcal{B}}$ we have

$$\big\|u(t;u_0) - u_\infty\big\|_{2,p} \longrightarrow 0\,.$$

Condition (1.6) has an easy graphical interpretation; see Figure 4.3. Assumptions 4.12 are mainly used to guarantee that equation (1.1) has a unique solution. In the class of increasing positively homogeneous functions, assumption (4.11) is not very restrictive. The methods can also be used for the case where f is arbitrary. In a forthcoming paper we will address this issue, which is harder to tackle and requires more assumptions. We have to ensure that the solution $u(t;u_0)$ is in $C^1(\overline{\Omega})$ for $t > 0$ in order to use the Hopf's maximum principle. Therefore we require $p > n$ and Ω bounded. If we have in (1.6) a strict inequality we do not have to require $p > n$ and Ω bounded (see Theorem 4.4). Moreover, many regularity assumptions can be weakened.

This question about stability was first considered in [9, Theorem 13.0]. That work tackles the case $r := l \in \mathcal{L}(L_2(\Omega), \mathbb{R})$. It is proven that if μ_1, μ_2, $\mu_3 \in \mathbb{R}$ are solutions of (1.5) with $\mu_1 < \mu_2 < \mu_3$ and

$$a(\mu_1) \geq a(\mu) > \frac{L}{\mu} \quad \text{for} \quad \mu \in (\mu_1, \mu_2)\,, \tag{1.7}$$

then for each $u_0 \in L_2$ with $u_1 \leq u_0 < u_2$, where $u_i := \frac{1}{a(\mu_i)}(-\triangle|_{\mathring{H}_2^1})^{-1} f$, i=1,2, it follows that $u(t; u_0) \longrightarrow u_1 \quad \text{for} \quad (t \to \infty) \quad \text{in} \quad L_2$. Similarly, if

$$\frac{L}{\mu} > a(\mu) \geq a(\mu_3) \quad \text{for} \quad \mu \in (\mu_2, \mu_3)\,. \tag{1.8}$$

Then for each $u_0 \in L_2$ with $u_2 < u_0 \leq u_3$, where $u_i := \frac{1}{a(\mu_i)}(-\triangle|_{H^2 \cap \mathring{H}_2^1})^{-1} f$, $i = 2, 3$, it follows that $u(t; u_0) \longrightarrow u_3 \quad \text{for} \quad (t \to \infty) \quad \text{in} \quad L_2$.

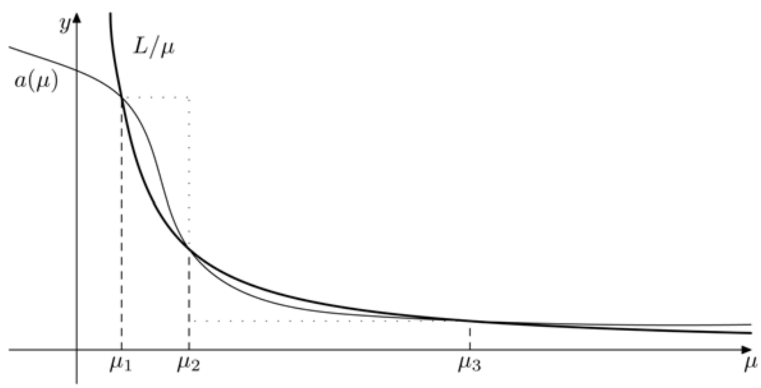

FIGURE 1.1. Conditions (1.7) and (1.8).

These results might be used to address the issue of one stationary solution as well. They allow us to state that for each $u_0 \in L_2$ with $u_0 \leq u_\infty$ or $u_\infty \leq u_0$ we get $u(t; u_0) \longrightarrow u_\infty \quad \text{for} \quad (t \to \infty) \quad \text{in} \quad L_2(\Omega)$, provided we have $r := l \in \mathcal{L}(L_2(\Omega), \mathbb{R})$ and

$$a(\mu) \geq a(\mu_\infty) \quad \text{for} \quad \mu \leq \mu_\infty \quad \text{and} \quad a(\mu_\infty) \geq a(\mu) \quad \text{for} \quad \mu_\infty \leq \mu\,. \tag{1.9}$$

In [11, Theorem 6.1] there is an extension which requires only one condition from (1.9). However, there is for instance no way to tackle an increasing function a. The main problem in this case is that there exists no comparison principle and sets like

$$[u_\infty, \infty) := \{u \in W^2_{p,\mathcal{B}};\ u_\infty \le u\}$$

are not invariant, although the function $\Phi(u) := \langle (-\triangle|_{H^2 \cap \mathring{H}^1_2})^{-1} l, u\rangle$ would be decreasing along a trajectory which stays inside $[u_\infty, \infty)$. We refer to [11, Remark 4.1] for an example where from $v_0 \le v_1$ it follows $u(\cdot; v_0) \not\le u(\cdot; v_1)$.

With the theory presented in this article, we will be able to consider functions a which are bounded below by a linear function on the left of μ_∞ and bounded above by the same linear function on the right of μ_∞. Furthermore, there will be no restrictions on the initial data $u_0 \in L_2$. Moreover, we do not require that r is linear; we just use that r is positively homogeneous and Lipschitz continuous. Thus r may also be a norm. In addition, r does not need to be defined on L_2; e. g. r could be a Dirac measure or could involve the gradient. We also state the convergence in W^2_p, provided that $f \in L_p$, $1 < p < \infty$. We use a method that was introduced in [12]. However, condition (1.6) was not considered in [12]. Moreover, we define in (4.27) a Ljapunov function. This definition is new and published for the first time. The idea of the definition of the Ljapunov function might also be used for a larger class of equations. The methods presented in this article can also deal with the case where equation (1.5) has several solutions.

We also refer to the articles [16], [13] dealing with linearized stability. The disadvantage of this method is that it gives just a local statement. Moreover, it is hard to prove the required estimates on the spectrum of the linearized operator. An advantage is that it gives exponential stability.

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2. The parabolic equation

We give the definition of the Sobolev-Slobodeckii spaces. These spaces can be characterized by a relatively simple intrinsic norm. Setting

$$[u]_{s,p} := \Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, d(x,y) \Big)^{1/p}, \quad \text{for} \quad 1 \le p < \infty,\ 0 < s < 1,$$

and denoting by $\|\cdot\|_{\lfloor s\rfloor,p}$ the usual Sobolev norms, let

$$\|u\|_{s,p} := \Big(\|u\|^p_{\lfloor s\rfloor,p} + \sum_{|\alpha|=\lfloor s\rfloor} [\partial^\alpha u]^p_{s-\lfloor s\rfloor,p}\Big)^{1/p}, \quad \text{for} \quad 1\le p<\infty,\ s\in\mathbb{R}^+\backslash\mathbb{N},$$

where $\lfloor s\rfloor$ is the largest integer smaller than or equal to s. Then, for $1\le p<\infty$, $s\in\mathbb{R}^+\backslash\mathbb{N}$, the **Slobodeckii spaces** are the Banach spaces defined by

$$W^s_p := W^s_p(\Omega,\mathbb{R}) := \big(\{u\in W^{\lfloor s\rfloor}_p;\ \|u\|_{s,p}<\infty\}, \|\cdot\|_{s,p}\big)$$

and

$$W^{-s}_p := W^{-s}_p(\Omega,\mathbb{R}) := \begin{cases} H^{-s}_p & \text{for } s\in\mathbb{N} \\ B^{-s}_{p,p} & \text{for } s\in\mathbb{R}^+\backslash\mathbb{N}\end{cases}.$$

We refer to Triebel's book [19], where a definition of the Bessel potential spaces H^{-s}_p and the Besov spaces $B^{-s}_{p,p}$ can be found. By means of local coordinates, we also define the spaces

$$W^s_p(\Gamma) := W^s_p(\Gamma,\mathbb{R}) \quad \text{for} \quad s\ge 0,\ 1<p<\infty$$

and

$$W^s_p(\Gamma) := W^s_p(\Gamma,\mathbb{R}) \doteq (W^{-s}_{p'}(\Gamma,\mathbb{R}))' \quad \text{for} \quad s<0,\ 1<p<\infty,\ p' := \frac{p}{p-1}.$$

The results stated here hold also for Bessel potential spaces.

In what follows

$$\mathcal{A}u := -\partial_j(a_{jk}\partial_k u + a_j u) + a_0 u, \quad \mathcal{A}^\sharp u := -\partial_k(a_{jk}\partial_j u) + a_j\partial_j u + a_0 u \quad (2.1)$$

denotes a general second-order linear differential operator on Ω and its formal adjoint operator with $a_{jk}\in BUC^2(\overline{\Omega})$, $a_j\in BC^2(\Omega)$, $a_0\in BC^1(\Omega)$. We fix a boundary characterization map, such that $\chi\in C(\Gamma,\{0,1\})$. Then we associate to $\mathcal{A}$ a linear boundary operator of order at most one, defined by

$$\mathcal{B}u := \chi\{\nu^j\gamma_\partial(a_{jk}\partial_k u + a_j u) + c\gamma_\partial u\} + (1-\chi)\gamma_\partial u,$$

where γ_∂ denotes the trace operator for Γ, $\vec{\nu}=(\nu^1,\dots,\nu^n)$ is the outer unit normal vector field to Γ and $c\in C^1(\Gamma,\mathbb{R})$. Furthermore, we set

$$B_0 := \chi\mathcal{B}, \quad \Gamma_0 := \chi^{-1}(0) \quad \text{and} \quad \mathcal{B}_1 := (1-\chi)\mathcal{B}, \quad \Gamma_1 := \chi^{-1}(1).$$

The principal symbol is uniformly strongly elliptic; i.e. there exists $\overline{\alpha}>0$ with

$$a_\pi(x,\xi) := a_{jk}(x)\xi^j\xi^k \ge \overline{\alpha} > 0 \quad \text{for} \quad (x,\xi)\in\overline{\Omega}\times\mathbb{R}^n.$$

Finally, we associate to $(\mathcal{A},\mathcal{B})$ the **Dirichlet form $\mathfrak{a}$**, defined by

$$\mathfrak{a}(v,u) := \langle\partial_j v, a_{jk}\partial_k u + a_j u\rangle + \langle v, a_0 u\rangle + \langle\gamma_\partial v, c\gamma_\partial u\rangle_\partial.$$

For given $1 < p < \infty$ and $(\mathcal{A}, \mathcal{B}) \in \mathcal{L}(W_p^2, L_p \times L_p(\Gamma))$ we define

$$E_1 := W^2_{p,\mathcal{B}} := \ker \mathcal{B} := \{u \in W_p^2;\ \mathcal{B}u = 0\}\,,$$

a closed linear subspace of W_p^2, and

$$E_0 := L_p\,, \quad A := A_0 := A_{0,p} := \mathcal{A}|_{\ker \mathcal{B}} \in \mathcal{L}(E_1, E_0)\,.$$

It is known, see [3, section 3], that $A \in \mathcal{H}(E_1, E_0) := \mathcal{L}(E_1, E_0) \cap \mathcal{H}(E_0)$; i. e. $-A$ is the generator of a strongly continuous analytic semigroup $\{e^{-tA};\ t \geq 0\}$ on E_0 with domain E_1. We assume that

$$\operatorname{type}(-A) := \inf\{\sigma \in \mathbb{R};\ \exists M \geq 1 \text{ with } \|e^{-tA}\| \leq M e^{\sigma t},\ \ t \geq 0\} < 0\,. \tag{2.2}$$

We refer to [5] for the definition of the inter- and extrapolation spaces E_α and the operators A_α with $\alpha \in [-1, 1]$. If the real interpolation functor with parameter p is used, we can characterize them as

$$E_s \doteq W^{2s}_{p,\mathcal{B}} := \begin{cases} \{u \in W_p^{2s};\ \mathcal{B}u = 0\}, & 2s \in (1 + 1/p, 2], \\ \{u \in W_p^{2s};\ (1-\chi)\gamma_\partial u = 0\}, & 2s \in (1/p, 1 + 1/p), \\ W_p^{2s}\,, & 2s \in (-1 + 1/p, 1/p), \\ \{v \in W_{p'}^{-2s};\ (1-\chi)\gamma_\partial v = 0\}', & 2s \in (-2 + 1/p, -1 + 1/p), \\ \{v \in W_{p'}^{-2s};\ \mathcal{B}^\sharp v = 0\}', & 2s \in [-2, -2 + 1/p), \end{cases} \tag{2.3}$$

where $\mathcal{B}^\sharp v := \chi\{\nu^j \gamma_\partial(a_{kj}\partial_k v) + c\gamma_\partial v\} + (1-\chi)\gamma_\partial v$, $2s \in [-2, 2]\backslash\Sigma_p$,

$$\Sigma_p := \begin{cases} \{-1 + 1/p,\ 1/p\} & \text{if } \chi = 0\,, \\ \{-2 + 1/p,\ 1 + 1/p\} & \text{if } \chi = 1\,, \\ (\mathbb{Z} + 1/p) \cap [-2, 2] & \text{otherwise} \end{cases} \quad \text{and} \quad \Sigma_p^+ := \Sigma_p \cap \mathbb{N}\,.$$

We refer to [4] for more details and a description of the missing spaces E_s if $\chi = 0$ or $\chi = 1$ and $2s \notin \Sigma_p$.

In the following we use the convention:
"α" stands for the index of the spaces E_α where the initial data "u_0" is in,
"γ" stands for the index with $f \in E_\gamma$,
"δ" stands for an index such that $r \in C(E_\delta, \mathbb{R})$.

Assumptions 2.1. *Suppose that* $1 < p < \infty$, $f \in W^{2\gamma}_{p,\mathcal{B}}$,

$$2\gamma \in [-2, 0]\backslash\Sigma_p\,, \quad 2\alpha,\, 2\delta \in [2\gamma, 2\gamma + 2)\backslash\Sigma_p\,.$$

(i) $r \in C^{1-}(W^{2\delta}_{p,\mathcal{B}}, \mathbb{R})$, *i.e., is locally Lipschitz continuous, bounded on bounded sets and positively homogeneous, i.e.,*

$$r(su) = s\, r(u) \quad \textit{for} \quad s \in (0, \infty),\ u \in E_1\,. \tag{2.4}$$

Of course, if $r(u) := \langle l, u\rangle$*, then* $l \in W^{-2\delta}_{p',\mathcal{B}^\sharp}$ *is required.*
(ii) $a \in C^{1-}(\mathbb{R}, [m,\infty))$ *with* $m > 0$*.*
In the case $\alpha < \delta$*, there exists a constant* $c > 0$ *with*

$$|a(u)-a(v)| \leq c\,|u-v|(1+|u|^{l-1}+|v|^{l-1}) \quad \textit{for} \quad u \in \mathbb{R} \quad \textit{with} \quad l \in [1, \tfrac{1+\gamma-\alpha}{\delta-\alpha})$$

and in addition, r is subadditive; that is

$$r(u+v) \leq r(u) + r(v) \quad \textit{for} \quad u,\, v \in W^{2\delta}_{p,\mathcal{B}}\,.$$

(iii) $\mathrm{type}(-A) < 0$.

Definition 2.2. *Let*

$$u(\cdot; u_0) \in C((0,\infty), W^{2+2\gamma}_{p,\mathcal{B}}) \cap C^1((0,\infty), W^{2\gamma}_{p,\mathcal{B}}) \cap C([0,\infty), W^{2\alpha}_{p,\mathcal{B}})\,.$$

(i) If $\delta_0 + 1/p < 2 + 2\gamma \leq 2$*, then u is said to be a strong* $(2+2\gamma)$*-solution to* (1.1) *if u satisfies* (1.1) *in the sense of distributions.*
(ii) If $1/p < 2 + 2\gamma \leq 1 + 1/p$*, then u is said to be a weak* $(2+2\gamma)$*-solution to* (1.1) *if u satisfies*

$$\langle v, u_t\rangle + a(r(u))\mathfrak{a}(v,u) = \langle v, f\rangle \quad \textit{and} \quad u(0) = u_0$$

for all $v \in W^{2-2\gamma}_{p',\mathcal{B}^\sharp}$*.*
(iii) If $0 \leq 2 + 2\gamma \leq 1 - \delta_0 + 1/p$*, then is u said to be a very weak* $(2+2\gamma)$*-solution to* (1.1) *if u satisfies*

$$\langle v, u_t\rangle + a(r(u))\langle \mathcal{A}^\sharp v, u\rangle = \langle v, f\rangle \quad \textit{and} \quad u(0) = u_0$$

for all $v \in W^{2-2\gamma}_{p',\mathcal{B}^\sharp}$*.*

Theorem 2.3. *Suppose that Assumptions* 2.1 *are satisfied. Then, for every* $u_0 \in W^{2\alpha}_{p,\mathcal{B}}$*, there exists one and only one global solution to* (1.1) *with*

$$u(\cdot; u_0) \in C^\theta((0,\infty), W^{2+2\gamma}_{p,\mathcal{B}}) \cap C^{1+\theta}((0,\infty), W^{2\gamma}_{p,\mathcal{B}}) \cap C([0,\infty), W^{2\alpha}_{p,\mathcal{B}})\,,$$

where $\theta \in [0,1)$*. Moreover, there exists* $c = c(u_0) > 0$ *such that for* $t \in [1,\infty)$ *we have* $\|u(t;u_0)\|_{W^{2+2\gamma}_{p,\mathcal{B}}} \leq c$*. Furthermore, the map* $(t,u_0) \mapsto u(t;u_0)$ *generates a continuous semiflow, and in the case* $\delta \leq \alpha$ *it is a* C^{1-}*-semiflow, in* $W^{2\alpha}_{p,\mathcal{B}}$*. Moreover, the variation-of-constants formula holds. That is,*

$$u(t;u_0) = U_{a(r(u))}(t,0)u_0 + \int_0^t U_{a(r(u))}(t,s) f\, ds\,, \tag{2.5}$$

where

$$U(t,s) := U_{a(r(u))}(t,s) := e^{-\int_s^t a(r(u(\tau)))\,d\tau\, A_{-1}}. \tag{2.6}$$

For $\omega > \operatorname{type}(-A)$, $2\mu, 2\nu \in [-2,2]\backslash\Sigma_p$, $\mu \le \nu$, *there exists* $c = c(\mu,\nu)$ *with*

$$(t-s)^{\nu-\mu}\big\|e^{-\int_s^t a(r(u(\tau)))\,d\tau A_{-1}}\big\|_{\mathcal{L}(W^{2\mu}_{p,\mathcal{B}},W^{2\nu}_{p,\mathcal{B}})} \le c\, e^{\omega\int_s^t a(r(u(\tau)))\,d\tau} \tag{2.7}$$

for $0 \le s < t \le T$.

Proof. For the definition of the operator A_{-1} we refer to [5, Section V.1.3]. A detailed proof for the statement of the existence of the solution and its continuous dependence on the initial value can be found in [18, Lemma 5.2.2]. The case $\delta < \alpha$ follows also from [2, Theorem 7.1]. The case $\delta = \alpha$, with the additional assumption that A has maximal regularity, follows also from [6]. In [18, Chapter 2] it is explained why the assumption of maximal regularity for A can be omitted. The case $\delta > \alpha$ can be proven with the idea of the next Proposition 2.4. The statement for the evolution operator $U(t,s)$ follows from [18, Proposition 2.2.5] and [5, Theorem V.2.1.3] respectively. From the regularizing effect, it is easy to see in the case $\delta > \alpha$ that $a(r(u(\cdot;u_0))) \in L_{1,loc}(0,\infty)$. □

Let $u(t) := u(t;u_0)$ be the global solution of (1.1) which is bounded for large t. We set

$$\sigma_u(t) := \int_0^t a\big(r(u(\tau;u_0))\big)\,d\tau \quad \text{for} \quad t > 0\,. \tag{2.8}$$

Thus σ is an increasing function on $[0,\infty)$ and we define for $s \ge 0$

$$v_u(s) := u\big(\sigma_u^{-1}(s)\big) \tag{2.9}$$

and get

$$v_u \in C((0,\infty), W^{2+2\gamma}_{p,\mathcal{B}}) \cap C^1((0,\infty), W^{2\gamma}_{p,\mathcal{B}}) \cap C([0,\infty), W^{2\alpha}_{p,\mathcal{B}})\,. \tag{2.10}$$

For $t \in (0,\infty)$ we have

$$v_u\big(\sigma_u(t)\big) = u(t) \quad \text{and} \quad v_u'\big(\sigma_u(t)\big)a\big(r(u(t))\big) = u_t(t)\,.$$

Hence

$$f = u_t(t) + a(r(u(t)))Au(t) = a\big(r(u(t))\big)\big[v_u'\big(\sigma_u(t)\big) + Av_u\big(\sigma_u(t)\big)\big]\,.$$

This means that

$$v_u'(s) + Av_u(s) = \frac{f}{a\big(r(v_u(s))\big)} \quad \text{for} \quad s \in (0,\infty)\,, \quad v_u(0) = u_0\,. \tag{2.11}$$

Conversely, let $v(\cdot) = v(\cdot; u_0)$ be a global solution of (2.11) which is bounded in $W^{2\delta}_{p,\beta}$ for large s. Then we define

$$\varsigma_v(s) = \int_0^s \frac{1}{a\Big(r(v(\tau))\Big)}\, d\tau \tag{2.12}$$

and due to the fact that v is bounded for large s, we can set

$$w_v(t) := v(\varsigma_v^{-1}(t)) \quad \text{for} \quad t \in [0, \infty) \,. \tag{2.13}$$

Consequently,

$$w_v \in C((0,\infty), W^{2+2\gamma}_{p,\mathcal{B}}) \cap C^1((0,\infty), W^{2\gamma}_{p,\mathcal{B}}) \cap C([0,\infty), W^{2\alpha}_{p,\mathcal{B}})$$

is bounded for large t and for $s \in (0, \infty)$

$$w_v'(\varsigma_v(s))\tfrac{1}{a(r(v(s)))} + Aw_v(\varsigma_v(s)) = v'(s) + Av(s) = \tfrac{f}{a(r(v(s)))} \,. \tag{2.14}$$

Hence w_v is a solution of (1.1). Because of

$$(\sigma_u^{-1})'(s) = \tfrac{1}{\sigma_u'(t)} = \frac{1}{a\Big(r(u(t))\Big)} = \frac{1}{a\Big(r(v_u(s))\Big)} = \varsigma_{v_u}'(s) \,,$$

we get $\sigma_u^{-1}(s) = \varsigma_{v_u}(s)$ and $w_{v_u} = u$. Thus we proved

Proposition 2.4. *Problem* (1.1) *and* (2.11) *are equivalent problems under the time transformation* (2.8)–(2.9) *and* (2.12)–(2.13)*, respectively.*

3. The elliptic equation

Nonlinear elliptic problems may have no, one, a finite or infinite number of solutions. Following ideas of [10] and [9] for some nonlinear nonlocal equations, the question about the multiplicity of a solution can be reduced to finding the zeros of a function from $\mathbb{R}$ to $\mathbb{R}$. We consider the equation

$$a(r(u))Au = f \,. \tag{3.1}$$

We repeat the arguments of [9] in an abstract setting, enabling us to state different results and variations.

Assumptions 3.1. (i) *$A \in \mathcal{L}is(F_1, F_0)$, $F_1 \overset{d}{\hookrightarrow} F_0$ are Banach-spaces,*
(ii) *r is a positively homogeneous function from F_1 to $\mathbb{R}$,*
(iii) $a \in C(\mathbb{R}, (0,\infty))$,
(iv) $f \in F_0$.

Of course, a function $u \in F_1$ which satisfies (3.1) is called an F_1-solution of (3.1). Set $L := r(A^{-1}f)$. We also consider the equation

$$a(\mu)\mu = L \,, \quad \mu \in \mathbb{R} \,. \tag{3.2}$$

Proposition 3.2. *The function* $\mu \longmapsto \frac{1}{a(\mu)} A^{-1} f$ *is a one-to-one mapping from the set of solutions to* (3.2) *onto the set of solution to* (3.1).

Proof. Let u_0 be a solution of (3.1). We derive easily that

$$a(r(u_0))\, A^{-1} A u_0 = A^{-1} f\,.$$

Hence,

$$a(r(u_0))\, r(u_0) = r(A^{-1} f) = L\,.$$

Thus, $\mu_0 := r(u_0)$ satisfies (3.2).

Let μ_0 be a solution of (3.2). It follows for $u_0 := \frac{1}{a(\mu_0)} A^{-1} f$ that

$$r(u_0) = \frac{L}{a(\mu_0)} = \mu_0\,.$$

Thus,

$$a(r(u_0))\, A u_0 = a(\mu_0)\, \frac{1}{a(\mu_0)} f = f$$

and Proposition 3.2 follows. □

Proposition 3.2 allows us to consider an equation of the type

$$\begin{cases} a(r(u))\mathcal{A}u = f & \text{in} \quad \Omega\,, \\ \mathcal{B}u = 0 & \text{on} \quad \Gamma\,, \end{cases} \tag{3.3}$$

as the next corollary shows.

Corollary 3.3. *Let* $2\gamma \in [-2,0] \backslash \Sigma_p$, $1 < p < \infty$ *and* $f \in W^{2\gamma}_{p,\mathcal{B}}$. *Suppose that the function* $r : W^{2\gamma-2}_{p,\mathcal{B}} \longrightarrow \mathbb{R}$ *is positively homogeneous. Set*

$$L := r\big(A_{-1}^{-1} f\big)$$

Then there exists a one-to-one mapping from the set of solutions to (3.2) *onto the set of* [*strong*] [*very*] [*weak*] $W^{2\gamma+2}_{p,\mathcal{B}}$*-solutions to* (3.3) (*see Definition* 2.2). *Moreover, if* $\mu_0 \in \mathbb{R}$ *is a solution to* (3.2), *then*

$$u_0 := \frac{1}{a(\mu_0)} \big(A_{-1}^{-1} f\big)$$

is a solution to (3.3).

Proof. For the definition of the operator A_{-1} we refer to [5, Section V.1.3]. The proof follows immediately from Theorem 3.2 and [5, Proposition V.1.5.5]. □

4. Asymptotic Behaviour

In this section we consider the equation

$$u_t + a(r(u))Au = f(x) \quad \text{for} \quad t > 0\,, \quad u(0) = u_0\,. \tag{4.1}$$

In Section 2, we considered the existence and uniqueness of a solution and its regularity. Furthermore, we showed that the solutions generate a semiflow. In Section 3 we considered the associated elliptic problem. We showed that in many situations the question of multiplicity of solutions can be reduced to finding the zeros of a scalar equation. In this section we investigate the stability of stationary solutions. We refer to [16], [14], [12, Theorem 3.1] and [18, Proposition 5.1.1 and 5.4.2], respectively, for results concerning linearized stability.

The next lemma shows that if $u(t;u_0) \rightharpoonup u_\infty$ in L_2 and $r \in L_2(\Omega)'$, u_∞ is a stationary solution of (1.1), then it follows $u(t;u_0) \longrightarrow u_\infty$ in $W^{2+2\gamma}_{p,\mathcal{B}}$. Indeed, the weak convergence leads in that case to condition (4.2).

Lemma 4.1. *Suppose that Assumptions* 2.1 *are satisfied. Let* $u_\infty \in W^{2\gamma+2}_{p,\mathcal{B}}$ *be a stationary solution of* (1.1) *and assume that for the solution* $u(\cdot;u_0)$ *we have:*

$$a(r(u(t;u_0))) \longrightarrow a(r(u_\infty)) \quad \textit{for} \quad (t \to \infty)\,. \tag{4.2}$$

Then it follows that $u(t;u_0) \longrightarrow u_\infty$ *for* $(t \to \infty)$ *in* $W^{2+2\gamma}_{p,\mathcal{B}}$.

Proof. For our global solution $u(\cdot;u_0)$ with $a(r(u(t;u_0))) \longrightarrow a(r(u_\infty))$, we get that the function v_u defined by (2.8) and (2.9) is a global solution of

$$v_t + Av = \frac{f}{a(r(v))} \quad \text{for} \quad t \in (0,\infty) \quad \text{and} \quad v(0) = u_0\,,$$

and

$$a(r(v_u)) \longrightarrow a(r(u_\infty)) \quad \text{for} \quad (t \to \infty)\,.$$

(1) We show that $v_u \longrightarrow u_\infty$ in $W^{2+2\gamma}_{p,\mathcal{B}}$.
With similar techniques as in [5, Theorem II.5.3.1], we can prove that

$$v_u \in BUC^\sigma([1,\infty), W^{2\delta}_{p,\mathcal{B}}) \quad \text{with} \quad \sigma \in (0, 1+\gamma-\delta)\,.$$

We set $g(\tau) := \frac{f}{a(r(v_u(\tau)))} - \frac{f}{a(\mu_\infty)}$ and get, for $t \geq 1$ and $\omega \in (\text{type}(-A), 0)$,

$$\|v_u(2t) - u_\infty\|_{2\gamma+2,p} \leq \|e^{-t\,A}\|_{\mathcal{L}(W^{2\alpha}_{p,\mathcal{B}}, W^{2+2\gamma}_{p,\mathcal{B}})} \|v_u(t) - u_\infty\|_{2\alpha,p}$$
$$+ \Big\| \int_t^{2t} e^{-(2t-\tau)\,A}[g(\tau) - g(2t)]\,d\tau \Big\|_{2\gamma+2,p} + \Big\| \int_t^{2t} e^{-(2t-\tau)\,A}\,g(2t)\,d\tau \Big\|_{2\gamma+2,p}$$

$$\leq c\,e^{\omega\,t} + \max_{\tau\in[t,2t]} \|[g(2t)-g(\tau)]\|_{2\gamma,p}^{1/2}\, c\int_t^{2t}(2t-\tau)^{-1}e^{\omega\,(2t-\tau)}(2t-\tau)^{\sigma/2}\,d\tau$$
$$+\|A^{-1}g(2t)-e^{-t\,A}A^{-1}g(2t)\|_{2\gamma+2,p}$$
$$\leq c\,e^{\omega\,t} + c\max_{\tau\in[t,2t]}|a(r(v(2t)))-a(r(v(\tau)))|^{1/2}$$
$$+c\,|a(r(v(2t)))-a(\mu_\infty)| + c\,e^{\omega\,t}\,|a(r(v(2t)))-a(\mu_\infty)|\,.$$

Hence,

$$\|v_u(2t)-u_\infty\|_{2\gamma+2,p}\longrightarrow 0 \quad \text{for} \quad (t\to\infty)\,.$$

(2) With the (time) back transformation (2.12) and (2.13) the lemma follows. □

4.1. The Case $f \geq 0$.

Assumptions 4.2. *In addition to Assumptions* 2.1, *we assume in this subsection that*

(i) (3.2) *admits just one solution* μ_∞,

(ii) $f\geq 0$, $f\in W^{2\gamma}_{p,\mathcal{B}}$; *i.e., for* $\gamma<0$ *we have* $\langle f,\varphi\rangle\geq 0$ *for* $\varphi\in(W^{-2\gamma}_{p',\mathcal{B}^\sharp})^+ := W^{-2\gamma}_{p',\mathcal{B}^\sharp}\cap L^+_{p'}$,

(iii) r *is increasing; that is* $0\leq u\leq v$ *implies* $0\leq r(u)\leq r(v)$.

We define

$$u_\infty := \frac{1}{a(\mu_\infty)}A_\gamma^{-1}f\,.$$

Definition 4.3. *The non-decreasing version of the function a is defined by*

$$\hat a(\mu) := \begin{cases}\max_{x\in[\mu_\infty,\mu]}a(x)\,, & \mu\geq\mu_\infty\,,\\ \min_{x\in[\mu,\mu_\infty]}a(x)\,, & \mu<\mu_\infty\,.\end{cases}$$

Recall that $0<m\leq a(\cdot)$. *Hence* $0<m\leq\hat a(\cdot)$. *We say that a satisfies condition (4.3) if*

$$\hat a\Big(\frac{L}{\hat a(\mu)}\Big) > \frac{L}{\mu} \quad \textit{for} \quad \mu\in\big(\mu_\infty,\tfrac{L}{m}\big]\,. \tag{4.3}$$

The main theorem of this subsection is

Theorem 4.4. *Suppose that Assumptions* 4.2 *are satisfied. Moreover, suppose that the function a satisfies condition* (4.3). *Then, for every initial value* $u_0\in W^{2\alpha}_{p,\mathcal{B}}$, *we have* $u(t;u_0)\longrightarrow u_\infty$ *for* $(t\to\infty)$ *in* $W^{2\gamma+2}_{p,\mathcal{B}}$. *Furthermore, the difference* $\|u(t;u_0)-u_\infty\|_{2\gamma+2,p}$ *for large t can be estimated uniformly with respect to* $\|u_0\|_{-2,p}$.

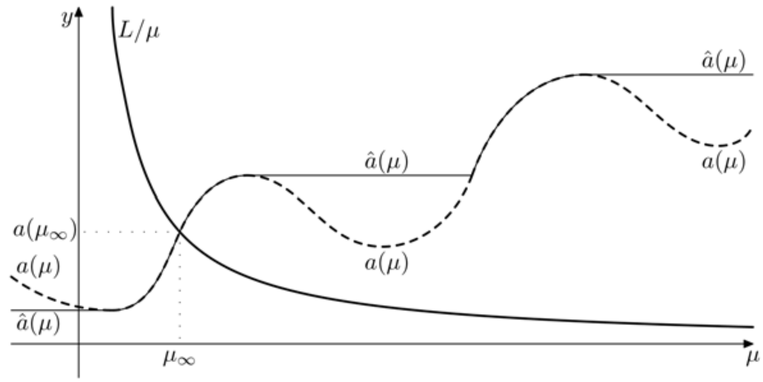

FIGURE 4.1. Definition of the function $\hat{a}$.

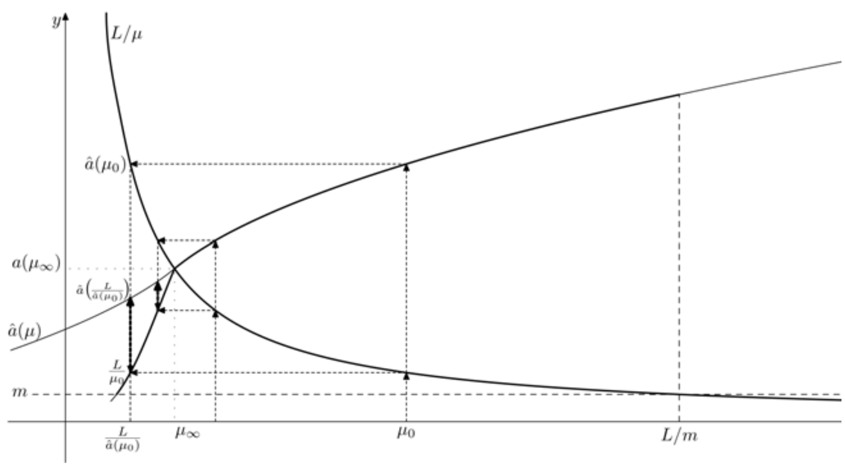

FIGURE 4.2. Graphical interpretation of condition (4.3).

Before proving this theorem we do some preparations which do not require condition (4.3) to be satisfied for a. It follows from Proposition 2.3 that

$$u(t;u_0) = U(t,0)u_0 + \int_0^t U(t,\tau)f\,d\tau\,.$$

It follows (see Theorem 2.3) that there exists a constant $c_0 := c_0(\|u\|_{-2,p}) > 0$ with $|r(u(t;u_0))| \le c_0$ for $t \ge 1$. We define

$$m_0 := \min_{x\in[-c_0,c_0]} a(x)\,, \qquad M_0 := \max_{x\in[-c_0,c_0\vee\frac{L}{m}]} a(x)\,,$$

$$\Phi_0 := \frac{1}{m_0}A^{-1}f\,, \qquad \varphi_0 := \frac{1}{M_0}A^{-1}f\,.$$

Proposition 4.5. *The set* $I_0 := [\varphi_0, \Phi_0] := \{u \in W^{-2\gamma+2}_{p,\mathcal{B}};\ \varphi_0 \le u \le \Phi_0\}$ *is an attractor in the* L_∞*-topology for* $u(\cdot;u_0)$*, and for each* $\varepsilon > 0$ *there exists* $T_0 = T_0(\varepsilon, \|u_0\|_{-2,p}) > 0$ *such that*

$$r(\varphi_0) - \varepsilon \le r(u(t;u_0)) \le r(\Phi_0) + \varepsilon \quad \textit{for} \quad t \ge T_0\,.$$

Proof. Since $1 \le \frac{a(r(u(t;u_0)))}{m_0}$ for $t \ge 1$, we get for $t \ge 1$

$$\begin{aligned} u(t;u_0) \;&\le\; U(t,1)u(1;u_0) + \int_1^t \tfrac{a(r(u(\tau;u_0)))}{m_0} A^{-1}AU(t,\tau)f\,d\tau \\ &=\; U(t,1)u(1;u_0) + \tfrac{1}{m_0}A^{-1}\int_1^t \partial_2 U(t,\tau)f\,d\tau \\ &=\; \underbrace{U(t,1)u(1;u_0)}_{\to 0 \text{ in } L_\infty\cap W^{2\delta}_{p,\mathcal{B}}} + \underbrace{\tfrac{1}{m_0}A^{-1}f}_{=\Phi_0} - \underbrace{\tfrac{1}{m_0}A^{-1}U(t,1)f}_{\to 0 \text{ in } L_\infty\cap W^{2\delta}_{p,\mathcal{B}}}\,. \end{aligned}$$

In an analogous way we estimate

$$u(t;u_0) \ge \underbrace{U(t,1)u(1;u_0)}_{\to 0 \text{ in } L_\infty\cap W^{2\delta}_{p,\mathcal{B}}} + \underbrace{\tfrac{1}{M_0}A^{-1}f}_{=\varphi_0} - \underbrace{\tfrac{1}{M_0}A^{-1}U(t,1)f}_{\to 0 \text{ in } L_\infty\cap W^{2\delta}_{p,\mathcal{B}}}\,.$$

Hence the proposition follows. □

Remark 4.6. From the assumptions that r is monotone and $f \ge 0$ it follows by the maximum principle (e. g. [7, Theorem 41]) that $L \ge 0$. If we do not impose more regularity for f and r than in Assumption 2.1, it is not trivial to introduce additional assumptions to $f \ne 0$, $r \ne 0$ which are not restrictive and which imply $L > 0$. In many situations we can verify $L > 0$. We give an example for the case $2\delta \le 1$. If

$$f \in L_p^+,\ \Omega \text{ bounded},\ p > \frac{n}{2} \quad \text{or} \quad f \in W^{-1+}_{p,\mathcal{B}}\,,\ \mathcal{A} := -\triangle,\ \mathcal{B} := \gamma_\partial,\ p > n,$$

then we can deduce

$$[A^{-1}f](x) > 0 \quad \text{for} \quad x \in \Omega \cup \Gamma_1,\ \text{where } \Gamma_1 := \chi^{-1}(1)\,. \tag{4.4}$$

Indeed, it is well known that if $f \in L_q^+$, $q > n$ we have $[A^{-1}f](x) > 0$ for all $x \in \Omega \cup \Gamma_1$. For $f \in L_p^+$, we take an increasing sequence $(g_n) \in L_q^+$ with $g_n \longrightarrow f$ in L_p^+ and get (4.4) since $W^2_{p,\mathcal{B}} \hookrightarrow C(\overline{\Omega})$. For the second case we refer to [15, Theorem 9.4]. Then

$$L = r(A^{-1}f) > 0$$

follows in the case $r|_{W^1_{p,\mathcal{B}}} \neq 0$. Indeed, if $L = r(A^{-1}f) = 0$, then it follows for each

$$\begin{aligned}\psi \in \mathcal{D}(\Omega \cup \Gamma_1)^+ := \{\, &\psi \in C^\infty(\Omega \cup \Gamma_1); \exists \tilde{\psi} \in \mathcal{D}(\mathbb{R}^n) \text{ with } \tilde{\psi}(x) = \psi(x) \text{ for } \\ &x \in \Omega \cup \Gamma_1,\ \exists \varepsilon > 0 :\ \tilde{\psi}(x) = 0 \text{ for } x \in \mathbb{B}(\Gamma_0, \varepsilon), \\ &\psi(x) \geq 0 \text{ for } x \in \Omega \}\end{aligned}$$

that

$$0 \leq r(\psi) = r(\varphi \tfrac{\psi}{\varphi}) \leq r\big(\varphi \,\big\|\tfrac{\psi}{\varphi}\big\|_\infty\big) = \big\|\tfrac{\psi}{\varphi}\big\|_\infty r(\varphi) = 0\,,$$

where $\varphi := A^{-1}f$. The density of $\mathcal{D}(\Omega \cup \Gamma_1)^+$ in $(W^1_{p,\mathcal{B}})^+$ gives a contradiction to $r \neq 0$. For a function r, which is just defined on spaces with more regularity (i. e. the case where $2\delta > 1$), we also refer to the idea of Lemma 4.14. Anyway, the case $L = 0$ is trivial because it implies $\mu_\infty = 0$ and Proposition 4.5 implies $r(u(t; u_0)) \longrightarrow 0$ since then $r(\varphi_0) = 0$. Thus, by Lemma 4.1, it follows that $u(t; u_0) \longrightarrow u_\infty$ in $W^{2+2\gamma}_{p,\mathcal{B}}$. Hence we can restrict ourselves to the case $L > 0$.

We now define

$$p_0 := r(\varphi_0) = \frac{L}{M_0}\,, \quad P_0 := r(\Phi_0) = \frac{L}{m_0}\,,$$

and iteratively,

$$\begin{aligned} m_{i+1} &:= \min_{x \in [p_i, P_i]} a(x)\,, \quad M_{i+1} := \max_{x \in [p_i, P_i]} a(x)\,, \\ \varphi_{i+1} &:= \frac{1}{M_{i+1}} A^{-1} f\,, \quad p_{i+1} := r(\varphi_{i+1}) = \frac{L}{M_{i+1}}\,, \\ \Phi_{i+1} &:= \frac{1}{m_{i+1}} A^{-1} f\,, \quad P_{i+1} := r(\Phi_{i+1}) = \frac{L}{m_{i+1}}\,. \end{aligned}$$

Inductively, we get

$$\begin{aligned} p_0 \leq p_1 \leq \ldots \leq{} &\mu_\infty \leq \ldots \leq P_1 \leq P_0\,, \\ m_0 \leq m_1 \leq \ldots \leq{} &a(\mu_\infty) \leq \ldots \leq M_1 \leq M_0\,, \\ \varphi_0 \leq \varphi_1 \leq \ldots \leq{} &\varphi_\infty \leq \ldots \leq \Phi_1 \leq \Phi_0\,. \end{aligned} \tag{4.5}$$

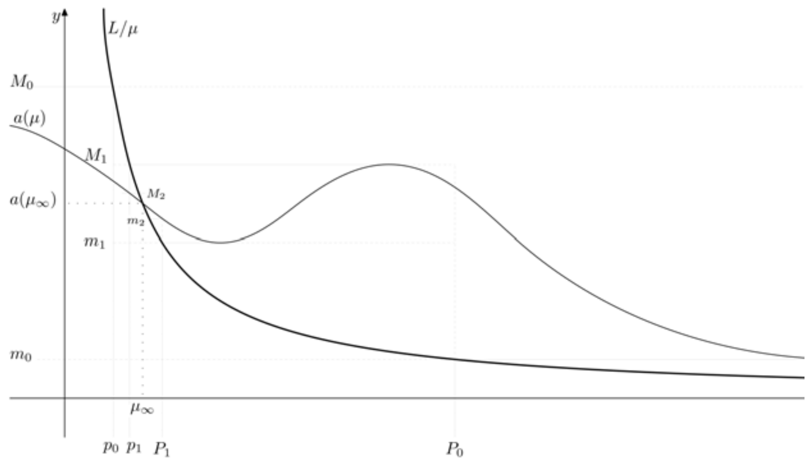

FIGURE 4.3. Graphical interpretation of the iterative definition.

Lemma 4.7. *For each $i \in \mathbb{N}$ the set $I_i := [\varphi_i, \Phi_i] := \{u \in W^{2\gamma+2}_{p,\mathcal{B}};\ \varphi_i \leq u \leq \Phi_i\}$ is an attractor for $u(\cdot; u_0)$ in the $L_\infty + W^{2\gamma+2}_{p,\mathcal{B}}$-topology, and for each $\varepsilon > 0$ there exists $T_i = T_i(\varepsilon, \|u_0\|_{-2,p}) > 0$ such that*

$$r(\varphi_i) - \varepsilon \leq r(u(t; u_0)) \leq r(\Phi_i) + \varepsilon \quad \textit{for} \quad t \geq T_i\,.$$

Proof. We prove this by induction. For $i = 0$, the statement follows from Proposition 4.5. So we just do the induction step $i \mapsto i + 1$. From the continuity of a and the induction assumption it follows that for each $\varepsilon > 0$ there exists $T_i = T_i(\varepsilon, \|u_0\|_{-2,p}) > 0$ such that

$$\begin{aligned} m_{i+1} - \varepsilon = \hat{a}(p_i) - \varepsilon = \hat{a}(r(\varphi_i)) - \varepsilon \ &\leq\ a(r(u(t; u_0))) \leq \hat{a}(r(\Phi_i)) + \varepsilon \\ &=\ \hat{a}(P_i) + \varepsilon = M_{i+1} + \varepsilon \end{aligned}$$

for $t \geq T_i$. As in the proof of Proposition 4.5, we estimate for $t \geq T_i$

$$\begin{aligned} u(t; u_0) \ &\leq\ U(t, T_i)u(T_i; u_0) + \int_{T_i}^{t} \tfrac{a(r(u(\tau;u_0)))}{m_{i+1}-\varepsilon} A^{-1}AU(t,\tau)f\, d\tau \\ &=\ \underbrace{U(t, T_i)u(T_i; u_0)}_{\to 0 \text{ in } L_\infty \cap W^{2\gamma+2}_{p,\mathcal{B}}} + \underbrace{\tfrac{1}{m_{i+1}-\varepsilon} A^{-1} f}_{= \Phi_{i+1} + \frac{\varepsilon}{(m_{i+1}-\varepsilon)m_{i+1}} \Phi_{i+1}} - \underbrace{\tfrac{1}{m_{i+1}-\varepsilon} A^{-1} U(t, T_i) f}_{\to 0 \text{ in } L_\infty \cap W^{2\gamma+2}_{p,\mathcal{B}}}\,. \end{aligned}$$

With the similar lower estimate, the lemma follows. □

It follows also that if, for example,

$$p_i \longrightarrow \mu_\infty \tag{4.6}$$

then $u(t;u_0) \longrightarrow u_\infty$. We will see that (4.6) is satisfied if the function a fulfills condition (4.3).

We formulate some basic consequences of condition (4.3).

Lemma 4.8. *(a) Assume that there exists $i \in \mathbb{N}^*$ such that*

$$\hat{a}(\frac{L}{\hat{a}(\mu)}) > \frac{L}{\mu} \quad \textit{for} \quad \mu \in (\mu_\infty, P_i]\,.$$

Then it follows that $\hat{a}(\frac{L}{\hat{a}(\mu)}) < \frac{L}{\mu}$ for $\mu \in [p_{i-1}, \mu_\infty)$.

(b) Assume that there exists $i \in \mathbb{N}^*$ such that

$$\hat{a}(\frac{L}{\hat{a}(\mu)}) < \frac{L}{\mu} \quad \textit{for} \quad \mu \in [p_i, \mu_\infty)\,.$$

Then it follows that $\hat{a}(\frac{L}{\hat{a}(\mu)}) > \frac{L}{\mu}$ for $\mu \in (\mu_\infty, P_{i-1}]$.

(c) Condition (4.3) is equivalent to

$$\hat{a}(\tfrac{L}{\hat{a}(\mu)}) < \tfrac{L}{\mu} \quad \textit{for} \quad \mu \in (0, \mu_\infty)\,. \tag{4.7}$$

Proof. (a) Suppose there exists $\mu_0 \in [p_{i-1}, \mu_\infty)$ with

$$\hat{a}(\tfrac{L}{\hat{a}(\mu_0)}) \geq \frac{L}{\mu_0}\,. \tag{4.8}$$

Since $m_i \leq \hat{a}(\mu) \leq \hat{a}(\mu_\infty)$ for $\mu \in [p_{i-1}, \mu_\infty]$, we get

$$\mu_1 := \frac{L}{\hat{a}(\mu_0)} \in [\frac{L}{\hat{a}(\mu_\infty)}, \frac{L}{m_i}] = [\mu_\infty, P_i]\,.$$

The case $\mu_1 = \mu_\infty$ is impossible, because in that case

$$\frac{L}{\mu_0} > \frac{L}{\mu_\infty} = a(\mu_\infty) = \hat{a}(\mu_1) = \hat{a}(\tfrac{L}{\hat{a}(\mu_0)}) \overset{(4.8)}{\geq} \frac{L}{\mu_0}$$

would be a contradiction. Thus $\mu_1 \in (\mu_\infty, P_i]$. From the assumption, it follows that

$$\hat{a}(\tfrac{L}{\hat{a}(\mu_1)}) > \frac{L}{\mu_1} = \hat{a}(\mu_0)\,. \tag{4.9}$$

From (4.8) we get $\frac{L}{\hat{a}(\mu_1)} \leq \mu_0$, and since the function $\hat{a}$ is non-decreasing it follows that

$$\hat{a}(\tfrac{L}{\hat{a}(\mu_1)}) \leq \hat{a}(\mu_0)\,,$$

which is a contradiction to (4.9).
(b) The proof can be done as in (a).
(c) The proof can be done as in (a) and (b). □

Now we can prove Theorem 4.4.

Proof. We showed in (4.5) that there exists a non-increasing sequence of intervals $[p_{i+1}, P_{i+1}] \subset [p_i, P_i]$ with $\mu_\infty \in [p_i, P_i]$ for each $i \in \mathbb{N}$. We set

$$p := \lim_{i\to\infty} p_i \quad \text{and} \quad P := \lim_{i\to\infty} P_i\,.$$

We want to show that $p = \mu_\infty = P$, which is done by contradiction.
Case (1): $p < \mu_\infty < P$. We split this case into four cases. Because of

$$\max\{p,\, \tfrac{L}{\hat{a}(P)}\} \le \mu_\infty \le \min\{P,\, \tfrac{L}{\hat{a}(p)}\}\,,$$

it is enough to consider the following cases

$$\begin{array}{llcl} (1.1) & \frac{L}{\hat{a}(P)} \le p & < & P \le \frac{L}{\hat{a}(p)}\,, \\ (1.2) & p \le \frac{L}{\hat{a}(P)} & < & P \le \frac{L}{\hat{a}(p)}\,, \\ (1.3) & \frac{L}{\hat{a}(P)} \le p & < & \frac{L}{\hat{a}(p)} \le P\,, \\ (1.4) & p < \frac{L}{\hat{a}(P)} & \le & \frac{L}{\hat{a}(p)} < P\,. \end{array}$$

First, we verify that cases (1.1)-(1.4) cover case (1). Of course, this is true if we replace '$<$' by '$\le$' in every case. From the assumption of case (1), we see that we can replace the middle '$\le$' symbol by '$<$' in (1.1)-(1.3). The case $p = \frac{L}{\hat{a}(P)}$ and $\frac{L}{\hat{a}(p)} = P$ is covered by (1.1), the case $p = \frac{L}{\hat{a}(P)}$ and $\frac{L}{\hat{a}(p)} \ne P$ by (1.1) or (1.3) and, finally, the case $p \ne \frac{L}{\hat{a}(P)}$ and $\frac{L}{\hat{a}(p)} = P$ by (1.1) or (1.2).
Case (1.1): $\frac{L}{\hat{a}(P)} \le p < \mu_\infty < P \le \frac{L}{\hat{a}(p)}$. By

$$\hat{a}(P) \le \hat{a}\left(\tfrac{L}{\hat{a}(p)}\right) < \tfrac{L}{p} \le \hat{a}(P)$$

we get a contradiction. The first inequality follows because $\hat{a}$ is non-decreasing, the second follows from condition (4.3) and the last from the assumption of case (1.1).
Case (1.2): $p \le \frac{L}{\hat{a}(P)} < P \le \frac{L}{\hat{a}(p)}$. We define the continuous function

$$\delta(p,P) := \frac{L}{p} - \max_{\mu\in[p,P]} a(\mu) = \frac{L}{p} - \max\{\max_{\mu\in[p,\mu_\infty]} a(\mu),\, \hat{a}(P)\}\,.$$

In the case (1.2) we have

$$\delta(p,P) \ge \frac{L}{p} - \max\{\max_{\mu\in[p,\mu_\infty]} a(\mu),\, \hat{a}\left(\tfrac{L}{\hat{a}(p)}\right)\} > 0\,.$$

From the continuity of $\delta(\cdot,\cdot)$ and $p\delta(p,P) < L$, it follows that there exists $\varepsilon > 0$ such that

$$\varrho\,\frac{L}{L-\delta(\varrho,\mathcal{P})\,\varrho} - p > 0 \quad \text{for} \quad \varrho \in (p-\varepsilon, p+\varepsilon),\, \mathcal{P} \in (P-\varepsilon, P+\varepsilon)\,.$$

Thus there exists $i \in \mathbb{N}$ with

$$p_{i+1} = \frac{L}{M_{i+1}} = \frac{L}{L/p_i - \delta(p_i,P_i)} = \frac{p_i\,L}{L-\delta(p_i,P_i)\,p_i} > p\,,$$

which is a contradiction to $p_i \nearrow p$.
Case (1.3): $\frac{L}{\hat{a}(P)} \le p < \frac{L}{\hat{a}(p)} \le P$. The contradiction follows in a similar way. We set

$$\delta(p,P) := \min_{\mu\in[p,P]} a(\mu) - \frac{L}{P} = \min\{\min_{\mu\in[\mu_\infty,P]} a(\mu),\, \hat{a}(p)\} - \frac{L}{P}\,.$$

Again, it follows that we have $\delta(p,P) > 0$ and in analogy to case (1.2), we get that there exists $i \in \mathbb{N}$ such that

$$P_{i+1} = \frac{L}{m_{i+1}} = \frac{L}{\delta(p_i,P_i) + L/P_i} < P\,,$$

which is a contradiction to $P_i \searrow P$.
Case (1.4): $p < \frac{L}{\hat{a}(P)} \le \frac{L}{\hat{a}(p)} < P$. By $\hat{a}(P) < \frac{L}{p}$ it follows that

$$\delta(p,P) := \frac{L}{p} - \max_{\mu\in[p,P]} a(\mu) = \frac{L}{p} - \max\{\max_{\mu\in[p,\mu_\infty]} a(\mu),\, \hat{a}(P)\} > 0\,.$$

As in case (1.2), a contradiction can be obtained.
Case (2): $p < \mu_\infty = P$. Again, the function δ satisfies

$$\delta(p,P) := \frac{L}{p} - \max_{\mu\in[p,P]} a(\mu) > 0\,,$$

and as in case (1.2) a contradiction can be shown.
Case (3): $p = \mu_\infty < P$. As in case (1.3), a contradiction is obtained by using

$$\delta(p,P) := \min_{\mu\in[p,P]} a(\mu) - \frac{L}{P} > 0\,.$$

All the above cases show that $r(u(t;u_0)) \longrightarrow \mu_\infty$ uniformly for all $u_0 \in W^{2\alpha}_{p,\mathcal{B}}$ with $\|u_0\|_{-2,p} \le c$. The statement of Theorem 4.4 now follows from Lemma 4.1. $\square$

Remark 4.9. For the definition of the sequences (p_i), (P_i), (m_i), (M_i) with properties as above, the functions φ_i or Φ_i are not used. This iterative definition can be considered as a numerical method for computing the unique zero μ_∞ of $\mu \mapsto a(\mu) - \frac{L}{\mu}$. From Lemma 4.8 and the proof of Theorem 4.4, it follows easily that the sequence (p_i) converges if and only if the function a satisfies condition (4.3). However, we can not show — nor do we know — whether or not there exists u_0 such that $u(t; u_0) \nrightarrow u_\infty$ if condition (4.3) is not satisfied.

If a is very sharply increasing on one side, it must be plane on the other side in order to satisfy condition (4.3). The next proposition points this out. We define

$$Q(\mu) := Q_a(\mu) := \frac{a(\mu_\infty) - a(\mu)}{\mu_\infty - \mu} \quad \text{for} \quad \mu \neq \mu_\infty \,.$$

Because a is Lipschitz continuous, the following constants are well defined

$$q := \sup_{\mu \in (0,\mu_\infty)} Q_a(\mu) \quad \text{and} \quad d := a(\mu_\infty) - q\,\mu_\infty \,.$$

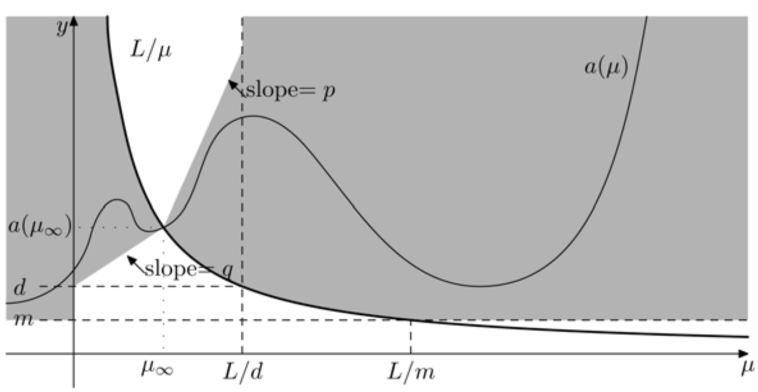

FIGURE 4.4. A sufficient condition for condition (4.3) to hold.

Proposition 4.10. *If one of the conditions*

(i) $q \le 0\,,$

(ii) $d > 0,\ Q_a(\mu) \le p := \frac{a(\mu_\infty)^2}{q\,\mu_\infty^2}$ *for all* $\mu \in (\mu_\infty, \frac{L}{m} \wedge \frac{L}{d}]\,,$

(iii) $d = 0,\ Q_a(\mu) < p := q$ *for all* $\mu \in (\mu_\infty, \frac{L}{m}]\,,$

(iv) $d < 0,\ Q_a(\mu) \le p := \frac{a(\mu_\infty)\,m}{\mu_\infty(m-d)}$ *for all* $\mu \in (\mu_\infty, \frac{L}{m}]$, $a(\frac{m-d}{q}) > m$ *or* $a(\frac{L}{m}) < \frac{Lq}{m-d}\,,$

(v) $d < 0,\ Q_a(\mu) < p := \frac{a(\mu_\infty)\,m}{\mu_\infty(m-d)}$ *for all* $\mu \in (\mu_\infty, \frac{L}{m}]\,,$

is satisfied, then the function a fulfills condition (4.3).

Proof. A detailed proof can be found in [18, Proposition 5.2.14]). □

4.2. **Using Hopf's maximum principle.** In this subsection we prove a generalization of Theorem 4.4. We use Hopf's maximum principle, which requires that our solution is at least in $C^1(\overline{\Omega})$.

Definition 4.11. *We define* $\Gamma_0 := \chi^{-1}(0)$ *and* $\Gamma_1 := \chi^{-1}(1)$. *Furthermore,*

$$C^1_{\mathcal{B}}(\overline{\Omega}) := \{u \in C^1(\overline{\Omega});\ \mathcal{B}u = 0\}\,.$$

For a function $\psi \in C^1_{\mathcal{B}}(\overline{\Omega})$ *we write* $\psi >> 0$ *if and only if*

(i) $\psi(x) > 0$ *for each* $x \in \Omega \cup \Gamma_1$.

(ii) $\partial_\mu \psi(y) < 0$ *for each* $y \in \Gamma_0$, $\mu \in \mathbb{R}^n$ *with* $(\mu|\nu(y)) > 0$,

where $\nu(y)$ *is the outer normal for* Γ *at* $y \in \Gamma$. *Moreover, we write* $v << w$ *for* $v, w \in C^1_{\mathcal{B}}(\overline{\Omega})$ *if and only if* $0 << w - v$. *If* $U, V \subset \mathbb{R}^n$ *are open then*

$$\mathcal{D}iff^m(U,V) := \{\varphi \in C^m(U,V);\ \varphi \text{ is bijective and } \varphi^{-1} \in C^m(V,U)\}$$

is the set of m*-diffeomorphisms from* U *into* V.

Assumptions 4.12. *In addition to Assumptions* 4.2, *we suppose that*

$$f \in L_p \text{ with } p > n, \quad 2\delta \in [0,2) \quad \text{and} \quad \Omega \text{ is bounded.} \tag{4.10}$$

Moreover, the function r *is increasing in the sense that for* $u_1,\, u_2 \in W^2_{p,\mathcal{B}}$ *with*

$$0 \le u_1 << u_2 \quad \text{we have} \quad r(u_1) < r(u_2)\,. \tag{4.11}$$

The next lemma is similar to Lemma 4.8 (c).

Lemma 4.13. *The following two statements are equivalent:*

(i) *The function a satisfies condition* (1.6).

(ii) *The function a satisfies*

$$\hat{a}\Big(\frac{L}{\hat{a}(\mu)}\Big) \le \frac{L}{\mu} \quad \text{for} \quad \mu \in (0,\mu_\infty)\,.$$

Proof. "(i) $\Rightarrow$ (ii)" We suppose that there exists $\mu_0 \in (0, \mu_\infty)$ with

$$\hat{a}\Big(\frac{L}{\hat{a}(\mu_0)}\Big) > \frac{L}{\mu_0}\,.$$

We set $\mu_1 := \frac{L}{\hat{a}(\mu_0)}$ and $\mu_2 := \inf\{\mu \in [\mu_\infty, L/m];\ \hat{a}(\mu) = \frac{L}{\mu_0}\}$ such that we have $\mu_2 < \mu_1$. From the assumption, it follows that

$$\tfrac{L}{\mu_2} \leq \hat{a}\left(\tfrac{L}{\hat{a}(\mu_2)}\right) = \hat{a}(\mu_0) = \tfrac{L}{\mu_1}\,,$$

which is a contradiction to $\frac{L}{\mu_2} > \frac{L}{\mu_1}$.
"(ii) $\Leftarrow$ (i)" This direction can be done in the same way. $\square$

We establish some technical lemmata which are similar to the theorem of de l' Hospital, but in space dimension n.

Lemma 4.14. *Let $\psi \in C^1_{\mathcal{B}}(\overline{\Omega})$ with $\psi >> 0$ and $(v_j) \in C^1_{\mathcal{B}}(\overline{\Omega})^{\mathbb{N}}$ with $v_j \longrightarrow 0$ in $C^1(\overline{\Omega})$. Then $g_j(x) := \frac{v_j(x)}{\psi(x)}$ is in $L_\infty(\Omega)$ and $g_j \longrightarrow 0$ in $L_\infty(\Omega)$.*

Proof. Since $\Gamma_0 \in C^m$ with $m > 2$ for each $y_0 \in \Gamma_0$, there exist open sets U and V in $\mathbb{R}^n$ with $y_0 \in U$ and $0 \in V$ and $\varphi \in \mathcal{D}iff^2(U, V)$ such that

$$\varphi(\Omega \cap U) = \varphi(U) \cap \mathbb{H}^n\,, \quad \varphi(y_0) = 0\,, \quad (\partial_n \varphi^{-1}(0) | \nu(y_0)) < 0\,,$$

where $\mathbb{H}^n := \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n;\ y^n \geq 0\}$. We set $\nu_0 := \partial_n \varphi^{-1}(0)$ and get $\partial_{\nu_0}\psi(y_0) > 0$. By shrinking the neighborhoods U, V and by a continuity argument, we can write for $y \in U \cap \overline{\Omega}$, $\eta \in V$ that

$$(\nabla\psi(y) | \partial_n \varphi^{-1}(\eta)) \geq \tfrac{1}{2}(\nabla\psi(y_0) | \partial_n \varphi^{-1}(0)) = \tfrac{1}{2}\partial_{\nu_0}\psi(y_0)\,.$$

Since Γ_0 is compact, there exists $\alpha > 0$, $n \in \mathbb{N}$, $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ with

$$Z := \bigcup_{i=1}^{n} U_i \supset \Gamma_0\,, \quad \varphi_i \in \mathcal{D}iff^2(U_i, V_i)$$

such that for each $x \in Z \cap \overline{\Omega}$ there exists $i \in \{1, \cdots, n\}$ with

$$x \in U_i \cap \overline{\Omega}\,, \quad (\nabla\psi(x) | \partial_n \varphi_i^{-1}(\varphi_i(x))) \geq \alpha\,.$$

For $x_0 \in U_i \cap \Omega$ we set $\eta_0 := (z_0, t_0) := \varphi_i(x_0)$, where $x_0 \in \mathbb{R}^{n-1}$, $t_0 \in (0, \infty)$. Furthermore, we set for $(z_0, t_0) \in V_i$ with $\{z_0\} \times [0, t_0] \subset V_i$

$$f_j(t_0) := v_j(\varphi_i^{-1}(z_0, t_0)) \quad \text{and} \quad g(t) := \psi(\varphi_i^{-1}(z_0, t_0))\,.$$

From the mean value theorem, it follows that there exists $\tau_j \in (0, t_0)$ with

$$\left|\frac{v_j(x_0)}{\psi(x_0)}\right| = \left|\frac{f_j(t_0)}{g(t_0)}\right| = \left|\frac{f_j'(\tau_j)}{g'(\tau_j)}\right| = \left|\frac{(\nabla v_j(\varphi_i^{-1}(z_0, \tau_j)) | \partial_n \varphi^{-1}(z_0, \tau_j))}{(\nabla\psi(\varphi_i^{-1}(z_0, \tau_j)) | \partial_n \varphi^{-1}(z_0, \tau_j))}\right|$$

$$\leq \frac{1}{\alpha} \sup_{x\in U_i\cap\Omega} \left|\left(\nabla v_j(x)\middle|\partial_n\varphi_i^{-1}(\varphi_i(x))\right)\right|.$$

Hence

$$\tfrac{v_j}{\psi} \in L_\infty(Z\cap\Omega) \quad \text{and} \quad \tfrac{v_j}{\psi} \longrightarrow 0 \text{ in } L_\infty(Z\cap\Omega)\,.$$

The statement in $\Omega\backslash Z$ is trivial because of $\frac{1}{\psi} \in L_\infty(\Omega\backslash Z)$. □

Lemma 4.15. *Suppose that* $u,\, w \in C^1_{\mathcal{B}}(\overline{\Omega})$ *with* $u >> 0,\ w >> 0$. *Then there exists* $a,\, b > 0$ *with* $a\,u \leq w \leq b\,u$.

Proof. In the same way as in the proof of Lemma 4.14, the existence of a neighborhood Z of Γ_0 and $\alpha,\, \beta,\, \gamma,\, \delta > 0$ with

$$\alpha \leq \left(\nabla u(x)\middle|\partial_n\varphi_i^{-1}(\varphi_i(x))\right) \leq \beta\,, \quad \gamma \leq \left(\nabla u(x)\middle|\partial_n\varphi_i^{-1}(\varphi_i(x))\right) \leq \delta$$

for $x \in U_i\cap\overline{\Omega}$, $i \in \{1,\dots,n\}$ can be established. Similarly, as in the proof of Lemma 4.14, we define for $x_0 \in U_i\cap\Omega$, $(z_0,t_0) = \varphi(x_0)$ with $\{z_0\}\times[0,t_0] \subset V_i$

$$f(t_0) := w(\varphi_i^{-1}(z_0,t_0)) \quad \text{and} \quad g(t_0) := u(\varphi_i^{-1}(z_0,t_0))\,.$$

Again, with the mean value theorem, we get the existence of $\tau \in (0,t_0)$ with

$$\frac{w(x_0)}{u(x_0)} = \frac{f(t_0)}{g(t_0)} = \frac{f'(\tau)}{g'(\tau)} = \frac{(\nabla w(\varphi_i^{-1}(z_0,\tau))|\partial_n\varphi^{-1}(z_0,\tau))}{(\nabla u(\varphi_i^{-1}(z_0,\tau))|\partial_n\varphi^{-1}(z_0,\tau))} \leq \frac{\delta}{\alpha}$$

and

$$\frac{w(x_0)}{u(x_0)} \geq \frac{\gamma}{\beta}\,.$$

Setting $a := \min\{\frac{\gamma}{\beta}, \inf_{x\in\Omega\backslash Z}\frac{w(x)}{u(x)}\}$ and $b := \min\{\frac{\delta}{\alpha}, \sup_{x\in\Omega\backslash Z}\frac{w(x)}{u(x)}\}$, the lemma follows. □

Proposition 4.16. *Suppose that* $f \geq 0$ *and* $u \in W^2_{p,\mathcal{B}}$ *with* $u >> 0$ *and set* $\varphi := A^{-1}f$. *Furthermore, let* $(v_j) \in C^1_{\mathcal{B}}(\overline{\Omega})^{\mathbb{N}}$ *with* $v_j \longrightarrow 0$ *for* $j \to \infty$. *Then*

(*i*) *there exists* $N \in \mathbb{N}$: $0 < \inf_{x\in\Omega}\frac{\varphi(x)}{u(x)+v_j(x)} \leq \sup_{x\in\Omega}\frac{\varphi(x)}{u(x)+v_j(x)} < \infty$ *for* $j \geq N$,

(*ii*) $\inf_{x\in\Omega}\frac{\varphi(x)}{u(x)+v_j(x)} \longrightarrow \inf_{x\in\Omega}\frac{\varphi(x)}{u(x)}$,

(*iii*) $\sup_{x\in\Omega}\frac{\varphi(x)}{u(x)+v_j(x)} \longrightarrow \sup_{x\in\Omega}\frac{\varphi(x)}{u(x)}$.

Proof. From the strong maximum principle [1, Theorem 6.1] it follows that $\varphi >> 0$. Also, from Lemma 4.15 it follows that there exist $a,\, b > 0$ with $a\,\varphi \leq u \leq b\,\varphi$. Furthermore it follows from Lemma 4.14 that there exists $N \in \mathbb{N}$ with

$$\left\|\frac{v_j}{\varphi}\right\|_\infty \leq \frac{a}{2} \quad \text{for} \quad j \geq N\,.$$

Hence, for $x \in \Omega$, we have

$$\frac{u(x)+v_j(x)}{\varphi(x)} \geq a + \frac{v_j(x)}{\varphi(x)} \geq a - \left|\frac{v_j(x)}{\varphi(x)}\right| \geq \frac{a}{2} \tag{4.12}$$

for $j \geq N$ and the last inequality of (i) follows. Next observe that for $x \in \Omega$ we have

$$\frac{\varphi(x)}{u(x)+v_j(x)} \geq \frac{1}{\sup_{x\in\Omega}\frac{u(x)}{\varphi(x)} + \sup_{x\in\Omega}\frac{v_j(x)}{\varphi(x)}} \geq \frac{1}{b+\frac{a}{2}} > 0$$

for $j \geq N$, and (i) follows. By

$$\left\|\frac{u(x)+v_j(x)}{\varphi(x)} - \frac{u(x)}{\varphi(x)}\right\|_\infty = \left\|\frac{v_j(x)}{\varphi(x)}\right\|_\infty \longrightarrow 0$$

(see Lemma 4.14), and since

$$\inf_{x\in\Omega}\frac{\varphi(x)}{u(x)+v_j(x)} = \frac{1}{\left\|\frac{u+v_j}{\varphi}\right\|_\infty} \longrightarrow \frac{1}{\left\|\frac{u}{\varphi}\right\|_\infty} = \inf_{x\in\Omega}\frac{\varphi(x)}{u(x)},$$

(ii) follows. Statement (iii) follows since

$$\left|\left\|\tfrac{\varphi}{u+v_j}\right\|_\infty - \left\|\tfrac{\varphi}{u}\right\|_\infty\right| \leq \left\|\tfrac{\varphi}{u+v_j} - \tfrac{\varphi}{u}\right\|_\infty = \left\|\tfrac{-v_j\,\varphi}{u(u+v_j)}\right\|_\infty \overset{j\geq N}{\leq} \tfrac{2}{a}\left\|\tfrac{v_j}{u}\right\|_\infty \longrightarrow 0\,.$$

□

Corollary 4.17. *Let $X := [C^1_{\mathcal{B}}(\overline{\Omega})^+]^o$ be the (open) inner set of the positive cone of $C^1_{\mathcal{B}}(\overline{\Omega})$. Then the functions*

$$I(u) := \inf_{x\in\Omega}\frac{\varphi(x)}{u(x)} \quad \text{and} \quad S(u) := \sup_{x\in\Omega}\frac{\varphi(x)}{u(x)} \tag{4.13}$$

are in $C(X,(0,\infty))$.

For an arbitrary function a which satisfies condition (1.6), the function $\hat{a}$ as introduced by Definition 4.3 is non-decreasing. Unfortunately, it is not injective, meaning that $\hat{a}^{-1}$ is neither defined nor continuous. The next lemmata show that there exists an increasing function $\hat{\hat{a}}$ which is close to $\hat{a}$ and still satisfies condition (1.6). For further consideration, it will be sufficient to work with this 'not well defined' or rather unnatural function $\hat{\hat{a}}$, which has a continuous inverse function. Lemmata 4.18 and 4.19 are presented because we do not want to restrict ourselves to increasing functions a. In fact, if the function a is increasing, the Lemmata 4.18 and 4.19 can be skipped.

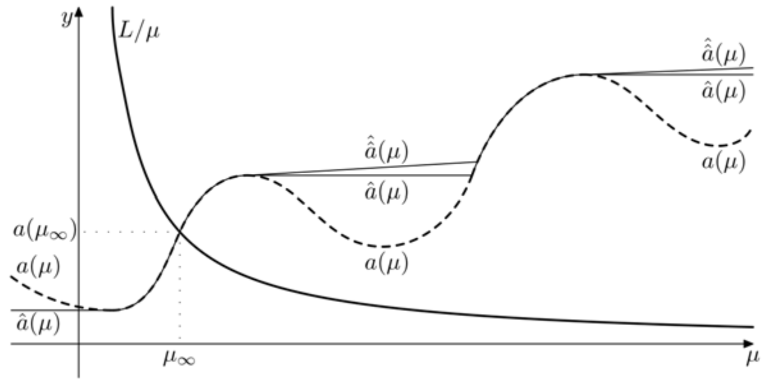

FIGURE 4.5. The definition of the function $\hat{\hat{a}}$.

Lemma 4.18. *Let* $-\infty < \mu_0 < \mu_1 < \infty$ *and* $a \in C^{1-}([\mu_0, \infty), \mathbb{R})$, *i.e., is locally Lipschitz-continuous, be non-decreasing. Let* $b : [\mu_0, \mu_1) \to \mathbb{R}$ *be an increasing function such that there exists* $\beta > 0$ *with*

$$\frac{b(x_1) - b(x_2)}{x_1 - x_2} \geq \beta \quad \text{and} \quad a(x_1) \leq b(x_1) \quad \text{for} \quad x_1,\, x_2 \in [\mu_0, \mu_1)\,.$$

Then there exists an increasing function $c \in C^{1-}([\mu_0, \infty), \mathbb{R})$ *with*

$$a(x) \leq c(x) \leq b(x) \quad \text{for} \quad x \in [\mu_0, \mu_1)\,.$$

Proof. The proof can be found in [18, Lemma 5.2.24]. □

Lemma 4.19. *Let* $a \in C^{1-}(\mathbb{R}, [m, \infty))$ *be a function satisfying condition* (1.6). *Then there exists* $\hat{\hat{a}} \in C^{1-}(\mathbb{R}, [m, \infty))$ *increasing on* $[\mu_\infty, \infty)$ *which also satisfies condition* (1.6). *Moreover,*

$$\hat{\hat{a}}(\mu) = \hat{a}(\mu) \quad \text{for} \quad \mu < \mu_\infty \quad \text{and} \quad a(\mu) \leq \hat{a}(\mu) \leq \hat{\hat{a}}(\mu) \quad \text{for} \quad \mu \geq \mu_\infty\,.$$

Proof. It is easy to see that $\hat{a} \in C^{1-}(\mathbb{R}, [m, \infty))$, introduced in Definition 4.3, is non-decreasing and hence there exists $\alpha > 0$ with

$$0 \leq \frac{\hat{a}(x) - \hat{a}(y)}{x - y} \leq \alpha \quad \text{for} \quad x, y \in [0, \tfrac{L}{m}]\,.$$

We define the function $\hat{a}_{-1} : [\hat{a}(0), \hat{a}(\mu_\infty)] \longrightarrow [0, \mu_\infty]$ by

$$\hat{a}_{-1}(s) := \sup\{t \in [0, \mu_\infty];\ \hat{a}(t) \leq s\}\,.$$

Thus $\hat{a}_{-1}$ is non-decreasing. We show that there exists $\gamma > 0$ such that

$$\frac{\hat{a}_{-1}(s) - \hat{a}_{-1}(t)}{s - t} \geq \gamma \quad \text{for} \quad s, t \in [\hat{a}(0), \hat{a}(\mu_\infty)]\,,\ \ s \neq t\,. \tag{4.14}$$

Suppose that for each $n \in \mathbb{N}$ there exist $s_n,\, t_n \in [\hat{a}(0), \hat{a}(\mu_\infty)]$ with

$$\frac{\hat{a}_{-1}(s_n) - \hat{a}_{-1}(t_n)}{s_n - t_n} < \frac{1}{n}\,.$$

We set $x_n := \hat{a}_{-1}(s_n)$ and $y_n := \hat{a}_{-1}(t_n)$. Thus, by

$$\frac{\hat{a}(x_n) - \hat{a}(y_n)}{x_n - y_n} = \frac{s_n - t_n}{\hat{a}_{-1}(s_n) - \hat{a}_{-1}(t_n)} > n\,,$$

we get a contradiction and (4.14) holds. We define $g : [\mu_\infty, \frac{L}{\hat{a}(0)}] \longrightarrow [0, \mu_\infty]$,

$$x \longmapsto g(x) := \hat{a}_{-1}\big(\tfrac{L}{x}\big)\,.$$

Hence, g is decreasing. Let $x, y \in [\mu_\infty, \frac{L}{\hat{a}(0)}]$ with $x < y$. Thus $\frac{L}{y} < \frac{L}{x}$ and $\hat{a}_{-1}\big(\frac{L}{y}\big) < \hat{a}_{-1}\big(\frac{L}{x}\big)$. Hence,

$$\frac{g(x) - g(y)}{x - y} = \frac{\hat{a}_{-1}(L/x) - \hat{a}_{-1}(L/y)}{x - y} \le \gamma \frac{L/x - L/y}{x - y} = -\frac{\gamma\, L}{xy} \le -\frac{\gamma \hat{a}(0)^2}{L}\,.$$

Finally we define the increasing function $b : [\mu_\infty, \frac{L}{\hat{a}(0)}) \longrightarrow [\hat{a}(\mu_\infty), \infty)$,

$$x \longmapsto b(x) := \frac{L}{g(x)}\,.$$

Again, let $x, y \in [\mu_\infty, \frac{L}{\hat{a}(0)})$ with $x < y$. Since

$$\frac{b(x) - b(y)}{x - y} = \frac{L/g(x) - L/g(y)}{x - y} = \frac{L}{g(x)g(y)} \frac{g(y) - g(x)}{x - y} \ge \frac{\gamma\, \hat{a}(0)^2}{g(\mu_\infty)^2}\,,$$

there exists $\beta > 0$ with

$$\frac{b(x) - b(y)}{x - y} \ge \beta \quad \text{for} \quad x, y \in [\mu_\infty, \tfrac{L}{\hat{a}(0)}),\;\; x \ne y\,. \tag{4.15}$$

We now show that

$$\hat{a}(\mu) \le b(\mu) \quad \text{for} \quad \mu \in [\mu_\infty, \tfrac{L}{\hat{a}(0)})\,. \tag{4.16}$$

Suppose there exists $\mu_0 \in [\mu_\infty, \frac{L}{\hat{a}(0)})$ with $\hat{a}(\mu_0) > b(\mu_0)$. Then $\frac{L}{\mu_0} \in (\hat{a}(0), \hat{a}(\mu_\infty)]$. From the intermediate value theorem follows the existence of $\mu_1 \in (0, \mu_\infty]$ with $\hat{a}(\mu_1) = \frac{L}{\mu_0}$. We set $\mu_2 := \hat{a}_{-1}(\hat{a}(\mu_1)) \in (0, \mu_\infty]$. By Lemma 4.13, it follows that

$$\tfrac{L}{\mu_2} \ge \hat{a}\left(\tfrac{L}{\hat{a}(\mu_2)}\right) = \hat{a}\left(\tfrac{L}{\hat{a}(\mu_1)}\right) = \hat{a}(\mu_0) > b(\mu_0) = \tfrac{L}{\hat{a}_{-1}(L/\mu_0)} = \tfrac{L}{\hat{a}_{-1}(\hat{a}(\mu_1))} = \tfrac{L}{\mu_2}\,,$$

which is a contradiction. Thus (4.16) follows. We extend b with

$$b(\mu) := \begin{cases} b(\mu) & \text{for } \mu \in [\mu_\infty, \frac{L}{\hat{a}(0)})\,, \\ \hat{a}(\mu) & \text{for } \mu \in [0, \mu_\infty)\,. \end{cases}$$

We are going to show that

$$b\left(\tfrac{L}{b(\mu)}\right) = \tfrac{L}{\mu} \quad \text{for} \quad \mu \in [\mu_\infty, \tfrac{L}{\hat{a}(0)})\,. \tag{4.17}$$

Let $\mu \in [\mu_\infty, \frac{L}{\hat{a}(0)})$. Then $b(\mu) \in [\hat{a}(\mu_\infty), \infty)$ and $\frac{L}{b(\mu)} \in (0, \mu_\infty]$. Hence

$$b\left(\tfrac{L}{b(\mu)}\right) = \hat{a}(g(\mu)) = \hat{a}\left(\hat{a}_{-1}\left(\tfrac{L}{\mu}\right)\right) = \tfrac{L}{\mu}$$

follows by (4.17). From Lemma 4.18 it follows that there exists an increasing function $c \in C^{1-}([\mu_\infty, \infty), \mathbb{R})$ with $\hat{a}(x) \le c(x) \le b(x)$ for $x \in [\mu_\infty, \frac{L}{\hat{a}(0)})$. We now define

$$\hat{\hat{a}}(\mu) := \begin{cases} c(\mu) & \text{for } \mu > \mu_\infty\,, \\ \hat{a}(\mu) & \text{for } \mu \le \mu_\infty\,. \end{cases}$$

Because of (4.17) it follows (see [18, Lemma 5.2.18]) that c satisfies condition (1.6), and the lemma follows. □

Remark 4.20. Of course, if the function a is increasing, we have $\hat{\hat{a}} = \hat{a} = a$.

Proposition 4.21. *Suppose that the function a satisfies condition* (1.6). *Let $u_0 \in W^2_{p,\mathcal{B}}$ and suppose that there exists $m_0 \in [m, a(\mu_\infty))$ with*

$$v := \frac{1}{\hat{\hat{a}}(\frac{L}{m_0})} A^{-1} f \le u_0 \le \frac{1}{m_0} A^{-1} f =: w\,.$$

Then

$$v << u(t, u_0) << w \quad \textit{for} \quad t > 0\,. \tag{4.18}$$

Proof. (i) Let $v(x) < u_0(x) < w(x)$ for each $x \in \Omega$. We define

$$\tau := \sup\left\{t > 0;\ r(u(s, u_0)) \in \left[\tfrac{L}{\hat{\hat{a}}(L/m_0)}, \tfrac{L}{m_0}\right],\ 0 \le s \le t\right\} > 0\,.$$

We suppose that $\tau < \infty$. Hence it follows that

$$\frac{L}{\hat{\hat{a}}(L/m_0)} \le r(u(t; u_0)) \le \frac{L}{m_0} \quad \text{for} \quad t \in [0, \tau]\,. \tag{4.19}$$

Let $t \in [0, \tau]$. In the case $r(u(t; u_0)) \le \mu_\infty$, it follows by (4.19) that

$$\hat{\hat{a}}\left(\tfrac{L}{m_0}\right) = \frac{L}{\frac{L}{\hat{\hat{a}}(L/m_0)}} \ge \frac{L}{r(u(t; u_0))} \ge a\big(r(u(t; u_0))\big) \ge \hat{a}\big(r(u(t; u_0))\big)$$

$$= \hat{\hat{a}}\big(r(u(t;u_0))\big) \geq \hat{\hat{a}}\big(\tfrac{L}{\hat{\hat{a}}(L/m_0)}\big) \geq \tfrac{L}{L/m_0} = m_0\,.$$

In the case $r(u(t;u_0)) > \mu_\infty$, it follows by (4.19) that

$$\begin{aligned}\hat{\hat{a}}\big(\tfrac{L}{m_0}\big) \;&\geq\; \hat{\hat{a}}\big(r(u(t;u_0))\big) \geq \hat{a}\big(r(u(t;u_0))\big) \geq a\big(r(u(t;u_0))\big)\\ &\geq\; \frac{L}{r(u(t;u_0))} \overset{(1.6)}{\geq} \frac{L}{L/m_0} = m_0\,.\end{aligned}$$

Thus,

$$m_0 \leq a(r(u(t;u_0))) \leq \hat{\hat{a}}\big(\tfrac{L}{m_0}\big) \quad \text{for} \quad t \in [0,\tau]\,. \tag{4.20}$$

We denote by $U(t,s)$ the evolution operator for $a(r(\cdot;u_0))A$. Note that we have

$$w = U(t,0)w + \int_0^t \underbrace{U(t,s)\tfrac{a(r(u(s,u_0)))}{m_0} f}_{=\partial_2 U(t,s)\frac{1}{m_0}A^{-1}f}\, ds \tag{4.21}$$

and

$$v = U(t,0)v + \int_0^t U(t,s)\tfrac{a(r(u(s,u_0)))}{\hat{\hat{a}}(L/m_0)} f\, ds\,. \tag{4.22}$$

Hence, by the strong maximum principle for parabolic equations (see [17, chapter 3]), it follows that

$$u(t;u_0) - v = \underbrace{U(t,0)(u_0 - v)}_{>>0} + \int_0^t U(t,s)\,\underbrace{\big[1 - \tfrac{a(r(u(s,u_0)))}{\hat{\hat{a}}(L/m_0)}\big]}_{\geq 0} f\, ds \tag{4.23}$$

and, similarly,

$$w - u(t;u_0) = \underbrace{U(t,0)(w - u_0)}_{>>0} + \int_0^t U(t,s)\,\underbrace{\big[\tfrac{a(r(u(s,u_0)))}{m_0} - 1\big]}_{\geq 0} f\, ds\,. \tag{4.24}$$

Thus,

$$v << u(t;u_0) << w \quad \text{for} \quad t \in [0,\tau]\,, \tag{4.25}$$

in contradiction to the definition of τ. Indeed the regularity of the solution implies that (4.25) holds also for t a little bit larger than τ. Hence $\tau = \infty$.

(ii) Let $v \leq u_0 \leq w$. We define

$$z := \tfrac{1}{2}(v + w)\,, \quad u_n := z + \big(1 - \tfrac{1}{n}\big)\,(u_0 - z)\,, \quad n \in \mathbb{N}^*\,.$$

Since $m_0 < \mu_\infty$ it follows that

$$v << u_n << w \quad \text{for} \quad n \in \mathbb{N}^*\,.$$

From (i) we get

$$v \le u(t;u_n) \le w \quad \text{for} \quad t \ge 0,\ n \ge N\,.$$

Since $u_n \longrightarrow u_0$ in $W^{2\alpha}_{p,\mathcal{B}}$, it follows from the continuous dependence of the solution on the initial data that $v \le u(t;u_0) \le w \quad \text{for} \quad t \ge 0$.

(iii) Let $t_0 > 0$. We show that $v << u(t_0;u_0) << w$.
In the case where $u_0 \ne v$ it follows from the strong maximum principle and (4.23) that

$$v << u(t_0;u_0)\,.$$

The case $u_0 \ne w$ is dealt analogously. In the case $v = u_0$ it follows from (ii) $v \le u(t;u_0) \le w$ for each $t \ge 0$. Since v is not a stationary solution, there exists $t_1 \in (0,t_0)$ with $v < u(t_1;u_0)$. As in (4.23), this leads to

$$v << u(t_0 - t_1; u(t_1;u_0)) = u(t_0;u_0)\,.$$

In the same way $u(t_0;u_0) << w$ can be proven. □

Proof of Theorem 1.1. (i) Let $u_0 \in W^{2\alpha}_{p,\mathcal{B}}$. From Theorem 2.3 we see that we have a unique solution in

$$u(\cdot;u_0) \in C^\theta((0,\infty),W^2_{p,\mathcal{B}}) \cap C^{1+\theta}((0,\infty),L_p) \cap C([0,\infty),W^{2\alpha}_{p,\mathcal{B}})\,,$$

where $\theta \in [0,1)$. Moreover, $u_0 \longmapsto u(1;u_0) \in C(W^{2\alpha}_{p,\mathcal{B}}, W^{2-\varepsilon}_{p,\mathcal{B}})\,,\ \varepsilon > 0\,$; see [18, Theorem 2.5.4]. (The case $\varepsilon = 0$ would require maximal regularity and will not be used.) This means we can assume (without loss of generality) that we have a semiflow in $W^{2-\varepsilon}_{p,\mathcal{B}} \hookrightarrow C^1_{\mathcal{B}}(\overline{\Omega})$ and our initial data is in $W^{2-\varepsilon}_{p,\mathcal{B}}$. It is an easy consequence of Proposition 4.5 that there exists $T_0 > 0$ with

$$u(t;u_0) \in X := [C^1_{\mathcal{B}}(\overline{\Omega})^+]^o \quad \text{for} \quad t \ge T_0\,.$$

Thus we can suppose (without loss of generality) that $u(t;u_0) \in X$ for each $t \ge 0$. We define

$$a_{-1}(y) := \begin{cases} \hat{\hat{a}}^{-1}(y) & \text{for } y \ge a(\mu_\infty)\,, \\ \mu_\infty & \text{for } y < a(\mu_\infty)\,. \end{cases} \tag{4.26}$$

Thus, with $I(\cdot),\ S(\cdot)$ as in (4.13), $a_{-1}(S(\cdot)) \in C(X,[\mu_\infty,\infty))$ and

$$\frac{L}{a_{-1}(S(\cdot))} \in C\big(X,(0,a(\mu_\infty)]\big)\,.$$

We set

$$V(\cdot) := -\min\{I(\cdot),\ \tfrac{L}{a_{-1}(S(\cdot))}\} \in C\big(X,[-a(\mu_\infty),0)\big)\,. \tag{4.27}$$

(ii) We show: if $V(w) = -a(\mu_\infty)$ with $w \in X$, then $w = \frac{1}{a(\mu_\infty)} A^{-1} f$. Let $w \in X$ with $V(w) = -a(\mu_\infty)$. This means that

$$I(w) \geq a(\mu_\infty) \quad \text{and} \quad \frac{L}{a_{-1}(S(w))} \geq a(\mu_\infty)\,.$$

From the first inequality it follows that

$$w \leq \frac{1}{a(\mu_\infty)} A^{-1} f\,. \tag{4.28}$$

From the second inequality follows

$$\mu_\infty = \frac{L}{a(\mu_\infty)} \geq a_{-1}(S(w))\,.$$

Thus, from the definition of a_{-1} (see (4.26)), we have $S(w) \leq a(\mu_\infty)$, which means that

$$\frac{1}{a(\mu_\infty)} A^{-1} f \leq w\,. \tag{4.29}$$

Hence (ii) follows by (4.28) and (4.29).

(iii) We show that V is a Ljapunov function on the invariant set $X \cap W^{2-\varepsilon}_{p,\mathcal{B}}$. Let $m_0 := -V(u_0)$ with $u_0 \in X$. From (ii) it follows that we can assume (without loss of generality) $m_0 < a(\mu_\infty)$. So we obtain

$$I(u_0) \geq m_0 \quad \text{and} \quad \frac{L}{a_{-1}(S(u_0))} \geq m_0\,.$$

Thus,

$$u_0 \leq \frac{1}{m_0} A^{-1} f \quad \text{and} \quad \frac{L}{m_0} \geq a_{-1}(S(u_0))\,.$$

Since $\hat{\hat{a}}\big(\frac{L}{m_0}\big) \geq \hat{\hat{a}}(a_{-1}(S(u_0))) \geq S(u_0)$, we get

$$\frac{1}{\hat{\hat{a}}(\frac{L}{m_0})} A^{-1} f \leq u_0\,.$$

Hence we can use Proposition 4.21 to obtain

$$\frac{1}{\hat{\hat{a}}(\frac{L}{m_0})} A^{-1} f << u(t; u_0) << \frac{1}{m_0} A^{-1} f \quad \text{for} \quad t > 0\,.$$

Let $t_0 > 0$. From Lemma 4.15 it follows that there exist $\alpha > \frac{1}{\hat{\hat{a}}(\frac{L}{m_0})}$ and $\beta < \frac{1}{m_0}$ with

$$\alpha\, A^{-1} f \leq u(t_0; u_0) \leq \beta\, A^{-1} f\,.$$

This means that

$$m_0 < \frac{1}{\beta} \le I(u(t_0;u_0)) \quad \text{and} \quad \hat{\hat{a}}\big(\tfrac{L}{m_0}\big) > \tfrac{1}{\alpha} \ge S(u(t_0;u_0))\,. \tag{4.30}$$

Hence,

$$\tfrac{L}{m_0} = a_{-1}\big(\hat{\hat{a}}\big(\tfrac{L}{m_0}\big)\big) > a_{-1}(S(u(t_0;u_0)))$$

or

$$\frac{L}{a_{-1}(S(u(t_0;u_0)))} > m_0\,. \tag{4.31}$$

From (4.30) and (4.31), we obtain

$$V(u(t_0;u_0)) < -m_0 = V(u_0)\,.$$

Hence, it follows that V is decreasing on $X\backslash\{\frac{1}{a(\mu_\infty)}A^{-1}f\}$.

(iv) From LaSalle's invariance principle it follows for $u_0 \in X \cap W^{2-\varepsilon}_{p,\mathcal{B}}$ and its ω-limit set

$$\omega(u_0) := \{w \in X \cap W^{2-\varepsilon}_{p,\mathcal{B}};\ \exists t_n \to \infty \text{ such that } w = \lim_{n\to\infty} u(t_n;u_0)\}\,,$$

that, since $u(\mathbb{R}^+;u_0)$ is relatively compact in $W^{2-\varepsilon}_{p,\mathcal{B}}$ ($u([1,\infty);u_0)$ is bounded in $W^2_{p,\mathcal{B}}$):

(α) $\exists c \in \mathbb{R} : \lim_{t\to\infty} V(u(t;u_0)) = c$,
(β) $V(u(t;y)) = V(y) = c$ for each $y \in \omega(u_0) \neq \emptyset$, $t \ge 0$,
(γ) $\mathrm{d}_{W^{2-\varepsilon}_{p,\mathcal{B}}}(u(t;u_0),\omega(u_0)) \longrightarrow 0$ for $(t \to \infty)$.

From (β) and (ii) we deduce that $\omega(u_0) = \{\frac{1}{a(\mu_\infty)}A^{-1}f\}$ and from (α)

$$V(u(t,u_0)) \longrightarrow -a(\mu_\infty)\,. \tag{4.32}$$

Finally, (γ) shows that $u(t;u_0) \to \frac{1}{a(\mu_\infty)}A^{-1}f$ in $W^{2-\varepsilon}_{p,\mathcal{B}}$. With $r(u(t;u_0)) \to \mu_\infty$ and Lemma 4.1 the theorem follows. □

Remark 4.22. We have not used the fact that r is continuous with respect to the L_∞-topology. Hence functionals like

$$r(u) := \delta_{x_0}(u) := u(x_0) \quad \text{with} \quad x_0 \in \Omega$$

or

$$r(u) := -(\nu(y_0)|\nabla u(y_0)) \quad \text{with} \quad y_0 \in \Gamma_0$$

can be considered as well. Continuity is only used for the existence theory and is satisfied if δ and p are large enough in the above situations.

Examples 4.23. (a) The function

$$a(\mu) := \alpha\, \mu \wedge m \quad \text{for} \quad \mu \in \mathbb{R}$$

with $m > 0$, $\alpha > 0$ satisfies condition (1.6).
(b) The function

$$a(\mu) := \frac{L}{\alpha\mu_\infty - \mu} \wedge m \quad \text{for} \quad \mu < \alpha\mu_\infty$$

with $m > 0$, $\alpha > 0$ satisfies condition (1.6).

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