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MEAN FIELD EQUATION FOR THE EQUILIBRIUM TURBULENCE AND A RELATED FUNCTIONAL INEQUALITY

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Abstract. This paper is concerned with the mean field equation for equilibrium turbulence with arbitrarily signed vortices. We develop blow-up analysis and establish a functional inequality of the Trudinger-Moser type.

1. Introduction

This paper is concerned with the equation on a two-dimensional compact orientable Riemannian manifold (M, g) without boundary;

$$-\Delta_g v = \lambda_1\Big(\frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|}\Big) - \lambda_2\Big(\frac{e^{-v}}{\int_M e^{-v} dv_g} - \frac{1}{|M|}\Big)$$
$$\int_M v dv_g = 0, \tag{1.1}$$

where Δ_g, dv_g, and $|M|$ are the Laplace-Beltrami operator, the volume form, and the volume of M, respectively, and λ_1, λ_2 are non-negative constants.

This equation describes the mean field of the equilibrium turbulence with arbitrarily signed vortices [20, 10, 18, 22], and is obtained by Joyce and Montgomery [13] and Pointin and Lundgren [26] from different statistical arguments. Here, these vortices are composed of positive and negative intensities with the same absolute value, and v and $\lambda_1 : \lambda_2$ are associated with

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the stream function of the fluid and the ratio of the numbers of the signed vortices, respectively.

In the previous work [27], we studied the linearized stability and a priori bound of the solution on the bounded domain in $\mathbf{R}^2$. Here, we develop the blow-up analysis and show the existence of the variational solution.

In fact, equation (1.1) is the Euler-Lagrange equation of the functional

$$J_{\lambda_1,\lambda_2}(v) = \frac{1}{2}\int_M |\nabla_g v|^2 \, dv_g - \lambda_1 \log \int_M e^v dv_g - \lambda_2 \log \int_M e^{-v} dv_g$$

defined on

$$E = \Big\{ w \in H^1(M) : \int_M w dv_g = 0 \Big\},$$

which forms a Hilbert space with the inner product $\langle u, v\rangle = \int_M \nabla_g u \cdot \nabla_g v dv_g$. When $(\lambda_1, \lambda_2) = (\lambda, 0)$ or $(\lambda_1, \lambda_2) = (0, \lambda)$, this J_{λ_1,λ_2} is reduced to

$$I_\lambda(v) = \frac{1}{2}\int_M |\nabla_g v|^2 \, dv_g - \lambda \log \int_M e^v dv_g,$$

and it is associated with the Trudinger-Moser inequality [12] indicated by

$$\begin{array}{ll} \inf_{v\in E} I_\lambda(v) > -\infty & \text{if } \lambda \in [0, 8\pi] \\ \inf_{v\in E} I_\lambda(v) = -\infty & \text{if } \lambda > 8\pi. \end{array} \tag{1.2}$$

Here, we have

$$J_{\lambda_1,\lambda_2}(v) = \frac{1}{2}\Big(1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi}\Big)\|v\|_E^2 + \frac{\lambda_1}{8\pi} I_{8\pi}(v) + \frac{\lambda_2}{8\pi} I_{8\pi}(-v),$$

and therefore,

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \text{if } 1 - \frac{\lambda_1}{8\pi} - \frac{\lambda_2}{8\pi} \geq 0. \tag{1.3}$$

However, this trivial inequality is improved as follows; see Figure 1.

Theorem 1.1. *It holds that*

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) > -\infty \quad \textit{if } (\lambda_1, \lambda_2) \in [0, 8\pi] \times [0, 8\pi], \tag{1.4}$$

and in particular, J_{λ_1,λ_2} *has a global minimizer on* E *if* $0 \leq \lambda_1, \lambda_2 < 8\pi$.

Inequality (1.4) is optimal, and it holds that

$$\inf_{v\in E} J_{\lambda_1,\lambda_2}(v) = -\infty \quad \text{if } \lambda_1 > 8\pi \text{ or } \lambda_2 > 8\pi. \tag{1.5}$$

In fact, we have

$$J_{\lambda_1,\lambda_2}(v) = I_{\lambda_1}(v) - \lambda_2 \log \int_M e^{-v} dv_g \tag{1.6}$$

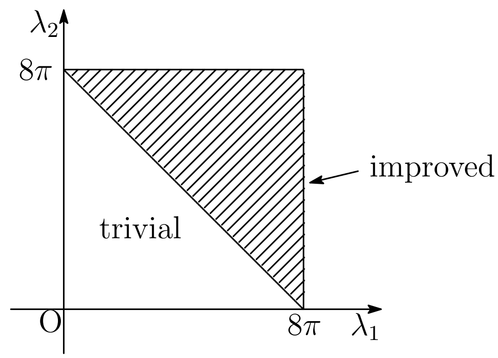

FIGURE 1. The region of the parameters for the Trudinger-Moser inequality.

and from Jensen's inequality

$$\log \int_M e^{-v} dv_g \geq \log |M| . \tag{1.7}$$

Combining (1.2), (1.6), and (1.7) (and similar arguments if $\lambda_2 > 8\pi$), we obtain (1.5).

The conclusion of Theorem 1.1 is equivalent to

$$\inf_{v \in E} J_{8\pi,8\pi}(v) > -\infty. \tag{1.8}$$

In fact, this inequality guarantees

$$\inf_{v \in E} J_{\lambda_1,\lambda_2}(v) > -\infty$$

for $0 \leq \lambda_2 \leq \lambda_1 \leq 8\pi$, $\lambda_2 < 8\pi$, because the second term of the right-hand side of

$$\begin{aligned}
&\frac{1}{2} \|\nabla v\|_2^2 - \lambda_1 \log \int_M e^v - \lambda_2 \log \int_M e^{-v} \\
&= \frac{\lambda_2}{8\pi}\Big(\frac{1}{2} \|\nabla v\|_2^2 - 8\pi \log \int_M e^v - 8\pi \log \int_M e^{-v}\Big) \\
&+\frac{1}{2}\Big(1 - \frac{\lambda_2}{8\pi}\Big) \|\nabla v\|_2^2 - (\lambda_1 - \lambda_2) \log \int_M e^v
\end{aligned}$$

is bounded by $0 \leq \frac{\lambda_1 - \lambda_2}{1 - \frac{\lambda_2}{8\pi}} \leq 8\pi$. Noting $J_{\lambda_1,\lambda_2}(v) = J_{\lambda_2,\lambda_1}(-v)$, we can infer (1.4) from (1.8) for $0 \leq \lambda_1, \lambda_2 \leq 8\pi$ with $\min(\lambda_1, \lambda_2) < 8\pi$. Inequality (1.8), thus, guarantees all the cases of the conclusion of Theorem 1.1.

Since $J_{8\pi,8\pi}(v+c)=J_{8\pi,8\pi}(v)$ for $v\in H^1(M)$ and $c\in\mathbf{R}$, inequality (1.8) implies

$$\inf_{v\in H^1(M)} J_{8\pi,8\pi}(v) > -\infty. \tag{1.9}$$

If $\Omega\subset\mathbf{R}^2$ is a bounded domain, we can take a flat torus M whose cell domain $\hat{\Omega}$ contains Ω. Then, each $v\in H_0^1(\Omega)$ is regarded as an element in $H^1(M)$, denoted by $\hat{v}$, by taking zero extensions to $\hat{\Omega}$ and then periodic extensions to $\mathbf{R}^2$. In this case, inequality (1.9) implies

$$\inf_{v\in H_0^1(\Omega)} \tilde{J}_{8\pi}(v) > -\infty \tag{1.10}$$

for

$$\tilde{J}_\lambda(v)=\frac{1}{2}\left\|\nabla v\right\|_{L^2(\Omega)}^2-\lambda\Big(\log\int_\Omega e^v dx+\log\int_\Omega e^{-v}dx\Big),$$

because of $J_{8\pi,8\pi}(\hat{v})\le\tilde{J}_{8\pi}(v)$. Similarly to (1.8), this inequality (1.10) guarantees

$$\inf_{v\in H_0^1(\Omega)} \tilde{J}_{\lambda_1,\lambda_2}(v) > -\infty \qquad \text{for } (\lambda_1,\lambda_2)\in[0,8\pi]\times[0,8\pi],$$

where

$$\tilde{J}_{\lambda_1,\lambda_2}(v)=\frac{1}{2}\left\|\nabla v\right\|_{L^2(\Omega)}^2-\lambda_1\log\int_\Omega e^v-\lambda_2\log\int_\Omega e^{-v}.$$

Inequality (1.10) is optimal, because the optimality of the standard Trudinger-Moser inequality is improved by

$$\inf_{v\in H_0^1(\Omega),v\ge 0} \tilde{I}_\lambda(v)=-\infty \qquad \text{for } \lambda>8\pi,$$

where

$$\tilde{I}_\lambda(v)=\frac{1}{2}\left\|\nabla v\right\|_{L^2(\Omega)}^2-\lambda\log\int_\Omega e^v.$$

If $\Omega\subset\mathbf{R}^2$ is a simply connected domain with C^1 boundary, we have a conformal mapping $\varphi:\Omega\to S_+^2$, where $S_+^2\subset\mathbf{R}^3$ is a hemi-sphere. In this case, each $v\in H^1(\Omega)$ is regarded as an element in $H^1(S^2)$ by even extension, and then it holds that

$$\inf_{v\in H^1(\Omega)} \tilde{J}_{4\pi}(v) > -\infty \tag{1.11}$$

by (1.8). This inequality, however, is proven for general domains $\Omega\subset\mathbf{R}^2$ with smooth boundary $\partial\Omega$, because we can control the behavior of the solution sequence to

$$-\Delta v=\lambda_1\Big(\frac{e^v}{\int_\Omega e^v dx}-\frac{1}{|\Omega|}\Big)-\lambda_2\Big(\frac{e^{-v}}{\int_\Omega e^{-v}dx}-\frac{1}{|\Omega|}\Big)$$

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \qquad \int_\Omega v dx = 0$$

similarly to that of (1.1) by the method of reflection [23, 34]. Inequality (1.11) is generalized as

$$\inf_{v\in H^1(\Omega), \int_\Omega v dx=0} \tilde{J}_{\lambda_1,\lambda_2}(v) > -\infty \qquad \text{for } (\lambda_1,\lambda_2) \in [0,4\pi]\times[0,4\pi]$$

similarly, using Chang-Yang's inequality [7]:

$$\inf_{v\in H^1(\Omega), \int_\Omega v dx=0} \tilde{I}_{4\pi}(v) > -\infty.$$

Inequality (1.11) is also optimal by itself because of the optimality of (1.8).

Actually, although Shafrir and Wolansky [31] obtained a related result that leads to (1.4), our method for the proof is completely different. We develop blow-up analysis for the solution sequence to (1.1), and apply the argument of Jost and Wang [14] concerning $SU(3)$ Toda systems.

For this purpose, we put

$$\mu_1 \equiv \lambda_1 \frac{e^v}{\int_M e^v dv_g}, \qquad \mu_2 \equiv \lambda_2 \frac{e^{-v}}{\int_M e^{-v} dv_g} \tag{1.12}$$

and identify them with $\mu_i dv_g$ $(i = 1, 2)$ in the space of measures $\mathcal{M}(M) = C(M)^*$ in (1.1). Moreover, we define $(u_1, u_2) \in E \times E$ by

$$u_i(x) \equiv \int_M G(x,y)\mu_i(y) dv_g \quad \text{for } i = 1, 2, \tag{1.13}$$

where $G = G(x, y)$ indicates the Green's functions of $-\Delta_g$; that is,

$$-\Delta_g G(\cdot, y) = \delta_x - \frac{1}{|M|} \quad \text{in } M, \qquad \int_M G(\cdot, y) dv_g = 0.$$

Given $x_0 \in M$, further, we take an iso-thermal chart (Ψ, U) satisfying

$$\Psi(x_0) = 0, \quad \Psi(x) = X \in \mathbf{R}^2, \quad g = e^{\xi(X)}(dX_1^2 + dX_2^2). \tag{1.14}$$

Then, each function $f(x)$ defined on M induces $f \circ \Psi^{-1}(X)$ defined on $\Omega \equiv \Psi(U) \subset \mathbf{R}^2$, which we denote simply $f(X)$. Here, we always assume that $\partial\Omega$ is smooth. Finally, the regular part $H_\Psi(x, y)$ of $G(x, y)$ relative to the iso-thermal chart (Ψ, U) satisfying (1.14) is defined as follows:

$$H_\Psi(x,y) = G(x,y) - \frac{1}{2\pi} \log |\Psi(x) - \Psi(y)|^{-1}.$$

Under these preparations, our result of the blow-up analysis is stated as follows.

Theorem 1.2. *Let $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ be sequences of non-negative constants satisfying*

$$\lambda_{i,n} \longrightarrow \lambda_i (\geq 0) \quad \text{as } n \longrightarrow \infty \text{ for } i = 1, 2 \tag{1.15}$$

and $\{v_n\} \subset E$ be a sequence of solutions to (1.1) *corresponding to $(\lambda_{1,n}, \lambda_{2,n})$. Moreover, let $\mu_{i,n}$ and $u_{i,n}$ be the functions determined from v_n and $\lambda_{i,n}$ by* (1.12) *and* (1.13) *for $i = 1, 2$, respectively. Without loss of generality, we may assume that*

$$\mu_{i,n} \longrightarrow \mu_i \quad \text{weakly} * \text{ in } \mathcal{M}(M) \text{ for } i = 1, 2. \tag{1.16}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the blow-up sets of (*this subsequence of*) *$\{v_n\}$ and $\{-v_n\}$, respectively, that is,*

$$\mathcal{S}_1 = \{x \in M : \exists x_n \longrightarrow x \text{ s.t. } v_n(x_n) \longrightarrow +\infty\},$$
$$\mathcal{S}_2 = \{x \in M : \exists x_n \longrightarrow x \text{ s.t. } v_n(x_n) \longrightarrow -\infty\}.$$

Then the following alternatives hold:

(1) (*compactness*) *We have $\mathcal{S}_1 \cup \mathcal{S}_2 = \emptyset$ and there exist $v \in E$ and a subsequence of $\{v_n\}$* (*denoted by the same symbol, and also hereafter*) *such that*

$$v_n \longrightarrow v \quad \text{in } E,$$

where this v is a solution to (1.1) *for those λ_1 and λ_2.*

(2) (*one-sided concentration*) *There exist $i \in \{1, 2\}$ such that $\mathcal{S}_i \neq \emptyset$ and $\mathcal{S}_j = \emptyset$ for $j \in \{1, 2\} \backslash \{i\}$. Moreover, it holds that*

$$\mu_i = \sum_{x_0 \in \mathcal{S}_i} 8\pi \delta_{x_0} \tag{1.17}$$

and

$$\mu_{i,n} \longrightarrow 0 \quad \text{in } L^\infty(\omega) \tag{1.18}$$

for every $\omega \Subset M \backslash \mathcal{S}_i$. On the other hand, there exists $u_j \in E$ and a subsequence of $\{u_{j,n}\}$ such that

$$u_{j,n} \longrightarrow u_j \qquad \text{in } E,$$

where this u_j is a solution to

$$-\Delta_g v = \lambda \Big(\frac{K(x) e^v}{\int_M K(x) e^v dv_g} - \frac{1}{|M|} \Big), \quad \int_M v dv_g = 0, \tag{1.19}$$

with $K(x) = e^{-\sum_{x_0 \in \mathcal{S}_i} 8\pi G(x, x_0)}$. Furthermore, for each $x_0 \in \mathcal{S}_i$, the following relation holds in the iso-thermal chart satisfying (1.14) *around* x_0*:*

$$\nabla_X \left(8\pi H_\Psi(X, x_0) + \sum_{x_0' \in \mathcal{S}_j \setminus \{x_0\}} 8\pi G(X, x_0') - u_j(X) + \xi(X) \right) \Big|_{X=0} = 0. \quad (1.20)$$

(3) (*concentration*) *For each* $i = 1, 2$*, we have* $\mathcal{S}_i \neq \emptyset$ *and there exists a positive constant* $m_i(x_0) \geq 4\pi$ *for each* $x_0 \in \mathcal{S}_i$*. We have, furthermore, a non-negative function* $r_i(x) \in L^1(M) \cap L^\infty_{\mathrm{loc}}(M \setminus \mathcal{S}_i)$ *such that*

$$\mu_i = r_i + \sum_{x_0 \in \mathcal{S}_i} m_i(x_0) \delta_{x_0}$$

and

$$\mu_{i,n} \longrightarrow r_i \quad \text{in } L^p(\omega) \quad (1.21)$$

for every $p \in [1, \infty)$ *and every* $\omega \Subset M \setminus \mathcal{S}_i$*. Finally, the following facts hold*:

3-i) *If there exists* $x_0 \in \mathcal{S}_i \setminus \mathcal{S}_j$ *for* $i \neq j$*, then we have* $m_i(x_0) = 8\pi$*,* $r_i \equiv 0$*, and* (1.20) *at* x_0*.*

3-ii) *For every* $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$*, we have*

$$(m_1(x_0) - m_2(x_0))^2 = 8\pi \left(m_1(x_0) + m_2(x_0) \right). \quad (1.22)$$

Moreover, if $\mathcal{S}_i \subset \mathcal{S}_j$ *and there exists* $x_0 \in \mathcal{S}_i$ *satisfying*

$$m_i(x_0) - m_j(x_0) > 4\pi, \quad (1.23)$$

then we have $r_i \equiv 0$*; see Figure* 2.

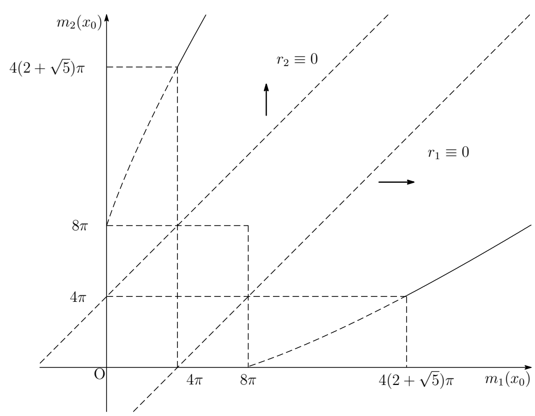

FIGURE 2. The mass of concentration at $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$.

3-iii) *If $r_1 \equiv 0$ and $r_2 \equiv 0$, the following relation holds at each $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$:*

$$\begin{aligned} & m_1(x_0)\nabla_X \Big(8\pi H_\Psi(X,x_0) + \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0')G(X,x_0') \\ & - \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0')G(X,x_0') + \xi(X)\Big)\Big|_{X=0} \\ & + m_2(x_0)\nabla_X \Big(8\pi H_\Psi(X,x_0) - \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0')G(X,x_0') \\ & + \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0')G(X,x_0') + \xi(X)\Big)\Big|_{X=0} = 0. \end{aligned} \tag{1.24}$$

Possible values of $(m_1(x_0), m_2(x_0))$ for $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ are more restrictive, and we expect

$$(m_1(x_0), m_2(x_0)) = 8\pi\Big(\frac{(\ell-1)\ell}{2}, \frac{\ell(\ell+1)}{2}\Big), \quad 8\pi\Big(\frac{\ell(\ell+1)}{2}, \frac{(\ell-1)\ell}{2}\Big)$$

for $\ell = 1, 2, 3, \ldots$. This problem will be studied in our forthcoming papers.

Now, we refer to related work. First, if the vortices are provided with the same sign, then equation (1.1) is reduced to

$$-\Delta_g v = \lambda\Big(\frac{e^v}{\int_M e^v dv_g} - \frac{1}{|M|}\Big), \qquad \int_M v dv_g = 0. \tag{1.25}$$

This equation, and its generalization (1.19) with the inhomogeneous coefficient $K(x)$, appear in the self-dual gauge field theory [35], stationary system of chemotaxis or self-interacting particles [34], and the prescribing Gaussian curvature problem [1]. It has been studied widely in recent years [21, 33, 3, 17, 16, 4, 5, 15, 32, 11, 29, 23, 25, 2, 8, 9], and especially, we have the quantization of λ in $8\pi\mathbf{N}$ for the non-compact sequence of solutions [16] (based on [3, 17], see also [21, 33, 29] for another method) with the classification of the singular limit using the Green's function [21, 19, 25]. This result was a fundamental tool or motivation for the variational method [32, 11], the singular perturbation of the solution (see [2] for bounded domains), and the calculation of the topological degree [16, 9].

Second, specifying the singular limit [25] for (1.19) is used in the proof of the existence of a solution of $SU(3)$ Toda systems [6], that is, the system of 2×2 Liouville equations on (M, g):

$$-\Delta_g u_1 = \lambda_1\Big(\frac{e^{a_{11}u_1 + a_{12}u_2}}{\int_M e^{a_{11}u_1 + a_{12}u_2} dv_g} - \frac{1}{|M|}\Big)$$

$$-\Delta_g u_2 = \lambda_2 \Big(\frac{e^{a_{21}u_1 + a_{22}u_2}}{\int_M e^{a_{21}u_1+a_{22}u_2} dv_g} - \frac{1}{|M|} \Big)$$
$$\int_M u_1 dv_g = 0, \qquad \int_M u_2 dv_g = 0 \tag{1.26}$$

with

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

For u_i $(i = 1, 2)$ defined by (1.13), we have $v = u_1 - u_2$ in (1.1). Roughly speaking, u_1 and u_2 correspond to the positive and the negative parts of v, respectively, and we can regard (1.1) as the Liouville system (1.26) with

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{1.27}$$

Furthermore, we can regard each u_i as a solution to the mean field equation (1.19), putting

$$v = a_{ii} u_i, \quad \lambda = a_{ii}\lambda_i, \quad K = e^{a_{ij}u_j} \tag{1.28}$$

for $j \neq i$.

Following these observations, in section 2 we prepare some results on (1.19) from [25]. Then, in section 3, we complete the proof of Theorem 1.2 by the method of symmetrization [30, 23, 34]. It is employed in the study of the system of chemotaxis, while similar techniques were adopted in the weak formulation of the Euler equation of incompressible ideal fluid by [28] and in the study of the $SU(3)$ Toda system [6, 24]. Finally, in section 4, we prove Theorem 1.1.

2. Preliminaries

Let v be a solution to (1.1), and (u_1, u_2) be determined by (1.12-1.13). Then, it holds that

$$-\Delta_g u_1 = \lambda_1 \Big(\frac{e^{u_1 - u_2}}{\int_M e^{u_1-u_2} dv_g} - \frac{1}{|M|} \Big)$$
$$-\Delta_g u_2 = \lambda_2 \Big(\frac{e^{-u_1 + u_2}}{\int_M e^{-u_1+u_2} dv_g} - \frac{1}{|M|} \Big)$$
$$\int_M u_1 dv_g = 0, \quad \int_M u_2 dv_g = 0. \tag{2.1}$$

Given $i = 1, 2$, we see that u_i satisfies (1.19) for $\lambda = \lambda_i$ and $K = K_i = e^{-u_j}$, where $j \in \{1, 2\} \setminus \{i\}$. Therefore, for the sequence of solutions $\{v_n\}$ to (1.1) with $(\lambda_1, \lambda_2) = (\lambda_{1,n}, \lambda_{2,n})$, we can apply [25] to $\{(u_{1,n}, u_{2,n})\}$ for $(u_{1,n}, u_{2,n})$ defined by (1.12-1.13), putting

$$K_{i,n} = e^{-u_{j,n}}.$$

First, since $G(x, y) \geq -A$ holds with some $A \in \mathbf{R}$, we have

$$u_{j,n} = \lambda_{j,n} \int_M G(\cdot, y)\mu_{j,n}(y)dv_g \geq -\lambda_{j,n}A,$$

namely, there is $C > 0$ independent of n such that

$$u_{j,n} \geq -C \quad \text{for } j = 1, 2. \tag{2.2}$$

See [1]. This implies

$$\limsup_{n\longrightarrow\infty} \|K_{i,n}\|_{L^\infty(M)} = \limsup_{n\longrightarrow\infty} \|e^{-u_{j,n}}\|_{L^\infty(M)} < +\infty$$

for each $i = 1, 2$. Next, from Jensen's inequality we have

$$\liminf_{n\longrightarrow\infty} \int_M K_{i,n}(x)dv_g = \liminf_{n\longrightarrow\infty} \int_M e^{-u_{j,n}(x)}dv_g \geq |M| > 0.$$

Finally, from the elliptic L^1 estimate we have $\limsup_{n\longrightarrow\infty} \|u_{j,n}\|_{W^{1,q}(M)} < +\infty$ for $q \in [1, 2)$ and therefore, passing to a subsequence, $u_{j,n} \to u_j$ follows in $L^p(M)$ for $p \in [1, \infty)$ and for almost every $x \in M$ for some $u_j \in W^{1,q}(M)$. This implies

$$K_{i,n} = e^{-u_{j,n}} \to e^{-u_j} (\equiv K_i) \qquad \text{in } L^p(M)$$

and for almost every $x \in M$. We hereby can apply Theorem 2.1 of [25] and obtain the following.

Lemma 2.1. *Under the assumptions and notations of Theorem* 1.2, *let* $\mathcal{S}_{u_i}$ *be the blow-up set of the sequence* $\{u_{i,n}\}$, *that is,*

$$\mathcal{S}_{u_i} \equiv \{x \in M : \exists x_n \longrightarrow x \ s.t. \ u_{i,n}(x_n) \longrightarrow +\infty\}.$$

Then, the following alternatives hold for each $i = 1, 2$*:*

(1) (*compactness*) *We have* $\mathcal{S}_{u_i} = \emptyset$, $u_i \in E$, *and there exists a subsequence of* $\{u_{i,n}\}$ *such that*

$$u_{i,n} \longrightarrow u_i \quad \text{in } E,$$

where u_i *is a solution to* (1.19) *for* $\lambda = \lambda_i$ *and* $K = K_i$.

(2) (*concentration*) *We have* $\mathcal{S}_{u_i} \neq \emptyset$ *and there exist*

$$m_i(x_0) \geq 4\pi \quad \text{for each } x_0 \in \mathcal{S}_{u_i}$$

and non-negative $r_i(x) \in L^1(M) \cap L^\infty_{\text{loc}}(M \backslash \mathcal{S}_{u_i})$ *such that*

$$\mu_i = r_i + \sum_{x_0 \in \mathcal{S}_{u_i}} m_i(x_0)\delta_{x_0},$$

where

$$\mu_{i,n} \longrightarrow r_i \quad in\ L^p(\omega) \tag{2.3}$$

for every $p \in [1,\infty)$ *and every* $\omega \Subset M\backslash \mathcal{S}_{u_i}$.

We note that $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the blow-up sets of $\{v_n\}$ and $\{-v_n\}$, respectively, which, however, coincide with $\mathcal{S}_{u_1}$ and $\mathcal{S}_{u_2}$, the blow-up sets of $\{u_{1,n}\}$ and $\{u_{2,n}\}$, respectively.

Lemma 2.2. *It holds that* $\mathcal{S}_{u_i} = \mathcal{S}_i$ *for* $i = 1, 2$.

Proof. We shall show the relation for $i = 1$. In fact, we have

$$v_n = u_{1,n} - u_{2,n} \leq u_{1,n} - C$$

by (2.2), and it holds that $\mathcal{S}_1 \subset \mathcal{S}_{u_1}$. Therefore, we have only to show $\mathcal{S}_{u_1} \subset \mathcal{S}_1$.

In fact, the blow-up set $\mathcal{S}_{u_1}$ coincides with the singular support of μ_1, and

$$\mu_{1,n} = \lambda_{1,n} \frac{e^{v_n}}{\int_M e^{v_n}} \left(= \lambda_{1,n} \frac{e^{u_{1,n}-u_{2,n}}}{\int_M e^{u_{1,n}-u_{2,n}}} \right)$$

is L^∞ unbounded around $x_0 \in \mathcal{S}_{u_1}$. Therefore, we may suppose

$$\lim_{n\to\infty} \sup_{B(x_0,r_0)} \left(v_n - \log \int_M e^{v_n} \right) = +\infty$$

for any $r_0 > 0$. Then, we obtain $r_0 > 0$ and $x_n \in \overline{B(x_0, r_0)}$ satisfying $\overline{B(x_0, r_0)} \cap \mathcal{S}_{u_1} = \{x_0\}$ and

$$v_n(x_n) - \log \int_M e^{v_n} = \max_{x \in \overline{B(x_0,r_0)}} \left(v_n(x) - \log \int_M e^{v_n} \right) \to +\infty.$$

On the other hand, we have

$$\log \left(\frac{1}{|M|} \int_M e^{v_n} \right) \geq \frac{1}{|M|} \int_M v_n = 0$$

by Jensen's inequality, and hence $v_n(x_n) \to +\infty$ follows from

$$v_n(x_n) - \log \int_M e^{v_n} \leq v_n(x_n) - \log |M| .$$

Therefore, if $x_n \to x_0$ is proven, then we have $x_0 \in \mathcal{S}_1$.

Suppose the contrary. Then, choosing a subsequence if necessary, we may assume $x_n \to \overline{x} \neq x_0$. This means $\overline{x} \notin \mathcal{S}_{u_1}$, and hence $\limsup_{n\longrightarrow\infty} u_{1,n}(x_n) < +\infty$. Then, it holds that

$$\limsup_{n\longrightarrow\infty} \left(v_n(x_n) - \log \int_M e^{v_n} \right) \leq \limsup_{n\longrightarrow\infty} v_n(x_n) - \log |M|$$

$$\leq \limsup_{n \longrightarrow \infty} u_{1,n}(x_n) - \log|M| - C < +\infty,$$

a contradiction. □

We can also apply Theorem 2.2 of [25] when one of the sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$ is relatively compact and the other concentrates. In combination with the residual vanishing criterion of Theorem 2.1 of [25], we obtain the following:

Lemma 2.3. *Suppose that there exists $i \in \{1,2\}$ such that the sequences $\{u_{i,n}\}$ satisfy the concentration case and $\{u_{j,n}\}$ for $j \neq i$ satisfies the compact case of Lemma 2.1. Then we have $m_i(x_0) = 8\pi$, $r_i \equiv 0$, and (1.20) for each $x_0 \in \mathcal{S}_{u_i}$. Moreover, the convergence (2.3) is improved as in $L^\infty(\omega)$. These are also true around $x_0 \in \mathcal{S}_{u_i} \setminus \mathcal{S}_{u_j}$ if both sequences $\{u_{i,n}\}$ for $i = 1,2$ concentrate in Lemma 2.1.*

From the above argument we get Theorem 1.2 (1)-(3-i). Concerning the residual vanishing when $\mathcal{S}_i \backslash \mathcal{S}_j = \emptyset$ we have the following:

Lemma 2.4. *When $\mathcal{S}_{u_i} \subset \mathcal{S}_{u_j}$ holds, $r_i = 0$ if there exists $x_0 \in \mathcal{S}_{u_i} \cap \mathcal{S}_{u_j}$ such that $m_i(x_0) - m_j(x_0) > 4\pi$. The last condition is relaxed as $m_i(x_0) - m_j(x_0) \geq 4\pi$ if $r_j = 0$ is known.*

The last statement of the above lemma is a direct consequence of Theorem 2.1 of [25]. On the other hand, the lack of summability of $r_j \neq 0$ around x_0 is compensated by the strict inequality, $m_i(x_0) - m_j(x_0) > 4\pi$, from the same argument for the $SU(3)$ Toda system [6, Lemma 12]. Thus, we obtain the above lemma.

3. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2 by the method of symmetrization [30, 23, 34].

In fact, we have

$$\nabla \mu_{i,n} = \mu_{i,n} \nabla \left(u_{i,n} - u_{j,n}\right)$$
$$\Delta \mu_{i,n} = \nabla \cdot \left(\mu_{i,n} \nabla \left(u_{i,n} - u_{j,n}\right)\right),$$

and hence it holds that

$$-\int_M \mu_{i,n} \Delta \psi = \int_M \int_M \nabla_x G(x,y) \cdot \nabla \psi(x) \mu_{i,n}(x) \mu_{i,n}(y)$$
$$-\int_M \int_M \nabla_x G(x,y) \cdot \nabla \psi(x) \mu_{j,n}(x) \mu_{i,n}(y)$$

for any $\psi \in C^2(M)$. Adding these equalities for $(i,j) = (1,2), (2,1)$, we have

$$\begin{aligned}
&-\int_M (\mu_{1,n} + \mu_{2,n})\,\Delta\psi \\
&= \int_M \int_M \nabla_x G(x,y)\cdot\nabla\psi(x)\left\{\mu_{1,n}(x)\mu_{1,n}(y) + \mu_{2,n}(x)\mu_{2,n}(y)\right\} \\
&- \int_M \int_M \nabla_x G(x,y)\cdot\nabla\psi(x)\mu_{1,n}(x)\mu_{2,n}(y) \\
&- \int_M \int_M \nabla_x G(x,y)\cdot\nabla\psi(x)\mu_{2,n}(x)\mu_{1,n}(y),
\end{aligned}$$

where the last term is equal to

$$\int_M \int_M \nabla_y G(x,y)\cdot\nabla\psi(y)\mu_{1,n}(x)\mu_{2,n}(y)$$

by $G(x,y) = G(y,x)$. The first term is also symmetrized, and we have

$$\begin{aligned}
&-\int_M (\mu_{1,n} + \mu_{2,n})\Delta\psi \\
&= \int_M \int_M \rho_\psi(x,y)\left\{\mu_{1,n}(x)\mu_{1,n}(y) - 2\mu_{1,n}(x)\mu_{2,n}(y) + \mu_{2,n}(x)\mu_{2,n}(y)\right\},
\end{aligned}$$

where

$$\rho_\psi(x,y) = \frac{1}{2}\left(\nabla_x G(x,y)\cdot\nabla\psi(x) + \nabla_y G(x,y)\cdot\nabla\psi(y)\right).$$

All the results in this section are obtained by this relation. First, we note the following.

Lemma 3.1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain containing the origin with smooth boundary $\partial\Omega$, and $\{g_{1,n}\}, \{g_{2,n}\}$ be sequences in $W^{1,\infty}(\Omega)$ satisfying*

$$\nabla g_{i,n} \to G_i \qquad \text{in } L^\infty(\Omega)^2$$

with $G_1, G_2 \in C(\overline{\Omega})^2$. Let $\{w_{1,n}\}$ and $\{w_{2,n}\}$ be sequences in $H_0^1(\Omega)$ satisfying

$$\begin{aligned}
-\Delta w_{i,n} &= e^{w_{i,n} - w_{j,n} + g_{i,n}} \qquad \text{in } \Omega \\
w_{i,n} &= 0 \qquad \text{on } \partial\Omega
\end{aligned}$$

for $i, j = 1, 2$ with $i \neq j$, and suppose that

$$\begin{aligned}
e^{w_{i,n} - w_{j,n} + g_{i,n}} &\to m_i\delta_0 + r_i(x) \qquad * \text{ weakly in } \mathcal{M}(\overline{\Omega}) \\
e^{w_{i,n} - w_{j,n} + g_{i,n}} &\to r_i \qquad \text{in } L^1_{loc}(\overline{\Omega} \setminus \{0\})
\end{aligned}$$

for $i = 1, 2$, where $r_i \in L^1(\Omega)$ and $m_i > 0$. Then, we have

$$(m_1 - m_2)^2 = 8\pi(m_1 + m_2). \tag{3.1}$$

If $r_1 = r_2 = 0$, furthermore, then it holds that

$$\frac{m_1 G_1(0) + m_2 G_2(0)}{m_1 + m_2} = -8\pi \nabla_x H_\Omega(x,0)|_{x=0}\,, \tag{3.2}$$

where

$$H_\Omega(x,y) = G_\Omega(x,y) + \frac{1}{2\pi}\log|x-y|$$

with $G_\Omega = G_\Omega(x,y)$ standing for the Green's function of $-\Delta$ in Ω under the Dirichlet boundary condition.

Proof. Letting $\mu_{i,n} = e^{w_{i,n} - w_{j,n} + g_{i,n}}$, we have

$$\Delta\mu_{i,n} = \nabla\cdot\left(\mu_{i,n}\nabla\left(w_{i,n} - w_{j,n} + g_{i,n}\right)\right),$$

similarly. Therefore, it holds that

$$\begin{aligned}
&-\int_\Omega(\mu_{1,n}+\mu_{2,n})\Delta\psi - \int_\Omega\int_\Omega\left((\nabla g_{1,n}\cdot\nabla\psi)\mu_{1,n} + (\nabla g_{2,n}\cdot\nabla\psi)\mu_{2,n}\right)\\
&= \int_\Omega\int_\Omega \rho_\psi(x,y)\left\{\mu_{1,n}(x)\mu_{1,n}(y) - 2\mu_{1,n}(x)\mu_{2,n}(y) + \mu_{2,n}(x)\mu_{2,n}(y)\right\},
\end{aligned}$$

where $\psi \in C_0^2(\Omega)$. We take $\psi(x) = |x-a|^2\varphi(x)$ for $\varphi \in C_0^2(\Omega)$ with $\varphi(x) \equiv 1$ near 0 and $a \in \mathbf{R}^2$. In this case we have

$$\nabla\psi(x) = 2(x-a), \qquad \Delta\psi = 4 \qquad \text{near } 0,$$

and hence

$$\begin{aligned}
\int_\Omega(\mu_{1,n}+\mu_{2,n})\,\Delta\psi &\to 4(m_1+m_2) + \int_\Omega(r_1+r_2)\Delta\psi\\
\int_\Omega(\nabla g_{i,n}\cdot\nabla\psi)\mu_{i,n} &\to -2m_i a\cdot G_i(0) + \int_\Omega(G_i\cdot\nabla\psi)\,r_i
\end{aligned}$$

from the assumption. Furthermore,

$$\begin{aligned}
\rho_\psi(x,y) &= \frac{1}{2}\left\{\nabla_x G_\Omega(x,y)\cdot\nabla\psi(x) + \nabla_y G_\Omega(x,y)\cdot\nabla\psi(y)\right\}\\
&= -\frac{1}{4\pi}\frac{(x-y)\cdot\{\nabla\psi(x)-\nabla\psi(y)\}}{|x-y|^2}\\
&\quad+\frac{1}{2}\left\{\nabla_x H_\Omega(x,y)\cdot\nabla\psi(x) + \nabla_y H_\Omega(x,y)\cdot\nabla\psi(y)\right\}\\
&= -\frac{1}{2\pi} + \left\{(x-a)\cdot\nabla_x H_\Omega(x,y) + (y-a)\cdot\nabla_y H_\Omega(x,y)\right\}
\end{aligned}$$

holds near $(x,y) = (0,0)$, and therefore, we have

$$\int_\Omega\int_\Omega \rho_\psi(x,y)\mu_{i,n}(x)\mu_{i,n}(y) \to -\frac{m_i^2}{2\pi} + m_i^2(-a)\cdot\nabla_x H_\Omega(0,0)$$

$$
\begin{aligned}
&+ m_i^2(-a)\cdot\nabla_y H_\Omega(0,0) + m_i\int_\Omega \rho_\psi(0,y)r_i(y) + m_i\int_\Omega \rho_\psi(x,0)r_i(x)\\
&+ \int_\Omega\int_\Omega \rho_\psi(x,y)r_i(x)r_i(y)\\
=& -\frac{m_i^2}{2\pi} - 2m_i^2 a\cdot\nabla_x H_\Omega(0,0)\\
&+ 2m_i\int_\Omega \rho_\psi(x,0)r_i(x) + \int_\Omega\int_\Omega \rho_\psi(x,y)r_i(x)r_i(y).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_\Omega\int_\Omega \rho_\psi(x,y)\mu_{1,n}(x)\mu_{2,n}(y) \to -\frac{m_1m_2}{2\pi} - 2m_1m_2 a\cdot\nabla_x H_\Omega(0,0)\\
&\quad + m_1\int_\Omega \rho_\psi(x,0)r_2(x) + m_2\int_\Omega \rho_\psi(x,0)r_1(x) + \int_\Omega\int_\Omega \rho_\psi(x,y)r_1(x)r_2(y).
\end{aligned}
$$

In this way, we obtain

$$
\begin{aligned}
&-4(m_1+m_2) - \int_\Omega (r_1+r_2)\Delta\psi + 2a\cdot[m_1G_1(0)+m_2G_2(0)]\\
&-\int_\Omega (G_1\cdot\nabla\psi)r_1 - \int_\Omega (G_2\cdot\nabla\psi)r_2\\
&= -\frac{1}{2\pi}(m_1^2+m_2^2-2m_1m_2) - 2(m_1^2+m_2^2-2m_1m_2)a\cdot\nabla_x H_\Omega(0,0)\\
&\quad +2\Big((m_1-m_2)\int_\Omega \rho_\psi(x,0)r_1(x) + (-m_1+m_2)\int_\Omega \rho_\psi(x,0)r_2(x)\Big)\\
&\quad + \int_\Omega\int_\Omega \rho_\psi(x,y)\left\{r_1(x)r_1(y) - 2r_1(x)r_2(y) + r_2(x)r_2(y)\right\},
\end{aligned}
$$

and therefore, we can apply the argument in the proof of Lemma 4.1 of [25]. Namely, first, we put $a=0$ and shrink the diameter of the support of ψ. This implies

$$
-4(m_1+m_2) = -\frac{1}{2\pi}(m_1^2+m_2^2-2m_1m_2),
$$

or equivalently, (3.1). Next, from the arbitrariness of a we get

$$
m_1G_1(0)+m_2G_2(0) = -(m_1^2+m_2^2-2m_1m_2)\nabla_x H_\Omega(0,0)
$$

in the case of $r_1=r_2=0$, which is equivalent to (3.2). □

Now, we show the following.

Lemma 3.2. *In Lemma* 2.1, *we have* (1.22) *for each* $x_0\in\mathcal{S}_{u_1}\cap\mathcal{S}_{u_2}$. *Furthermore,* (1.24) *holds true if* $r_1=r_2=0$.

Proof. Given $x_0 \in \mathcal{S}_{u_1} \cap \mathcal{S}_{u_2}$, we take an iso-thermal chart (Ψ, U) satisfying (1.14) and $\overline{U} \cap (\mathcal{S}_{u_1} \cup \mathcal{S}_{u_2}) = \{x_0\}$. Then, $u_{i,n}(X) = u_{i,n} \circ \Psi^{-1}(X)$ is a solution of

$$-\Delta u_{i,n} = \lambda_{i,n}\Big(\frac{e^{u_{i,n}-u_{j,n}}}{\int_M e^{u_{i,n}-u_{j,n}}} - \frac{1}{|M|}\Big)e^{\xi}.$$

Taking $h_{i,n}, h_\xi$ by

$$\begin{aligned} \Delta h_{i,n} &= 0 \quad \text{in } \Omega \qquad h_{i,n} = u_{i,n} \quad \text{on } \partial\Omega \\ \Delta h_\xi &= e^{\xi} \quad \text{in } \Omega \qquad h_\xi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

we put $\tilde{u}_{i,n} = u_{i,n} - h_{i,n} - \frac{\lambda_{i,n}}{|M|}h_\xi$. Then, it holds that

$$\begin{aligned} -\Delta \tilde{u}_{i,n} &= e^{\tilde{u}_{i,n}-\tilde{u}_{j,n}+g_{i,n}} \quad \text{in } \Omega \\ \tilde{u}_{i,n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

$$g_{i,n} = h_{i,n} - h_{j,n} + \frac{\lambda_{i,n} - \lambda_{j,n}}{|M|}h_\xi + \xi + \log\lambda_{i,n} - \log\int_M e^{u_{i,n}-u_{j,n}}$$

belongs to $W^{1,\infty}(\Omega)$. Furthermore, the elliptic regularity guarantees

$$\begin{aligned} \nabla g_{i,n} &= \nabla\Big(h_{i,n} - h_{j,n} + \frac{\lambda_{i,n} - \lambda_{j,n}}{|M|}h_\xi + \xi\Big) \\ &\to \nabla\Big(h_i - h_j - \frac{\lambda_i - \lambda_j}{|M|}h_\xi + \xi\Big) \quad (\equiv G_i(\cdot)) \quad \text{in } L^\infty(\Omega) \end{aligned}$$

by $\overline{U} \cap (\mathcal{S}_1 \cup \mathcal{S}_2) = \{x_0\}$, where h_i is a solution to

$$\Delta h_i = 0 \quad \text{in } \Omega, \qquad h_i = u_i \quad \text{on } \partial\Omega.$$

It is obvious that

$$\nabla\Big(h_i - h_j - \frac{\lambda_i - \lambda_j}{|M|}h_\xi + \xi\Big) \in C(\overline{\Omega})^2,$$

and Lemma 3.1 is applicable. Therefore, (1.22) holds true.

If $r_1 = r_2 = 0$, then we get (3.2). In this case we have

$$u_i = \sum_{x_0' \in \mathcal{S}_i} m_i(x_0')G(\cdot, x_0')$$

from the assumption, and therefore, the relation

$$\begin{aligned} -\Delta\Big(h_i - h_j + \frac{\lambda_i - \lambda_j}{|M|}h_\xi\Big) &= -\frac{\lambda_i - \lambda_j}{|M|}e^{\xi} \quad \text{in } \Omega \\ h_i - h_j + \frac{\lambda_i - \lambda_j}{|M|}h_\xi &= u_i - u_j \qquad \text{on } \partial\Omega \end{aligned}$$

implies

$$h_i - h_j + \frac{\lambda_i - \lambda_j}{|M|} h_\xi = \sum_{x_0' \in \mathcal{S}_i} m_i(x_0') G(\cdot, x_0')$$
$$- \sum_{x_0' \in \mathcal{S}_j} m(x_0') G(\cdot, x_0') - \{m_i(x_0) - m_j(x_0)\} G_\Omega(X, 0).$$

The right-hand side is equal to

$$\{m_i(x_0) - m_j(x_0)\} H_\Psi(X, x_0) + \sum_{x_0' \in \mathcal{S}_i \setminus \{x_0\}} m_i(x_0') G(\cdot, x_0')$$
$$- \sum_{x_0' \in \mathcal{S}_j \setminus \{x_0\}} m_j(x_0') G(\cdot, x_0') - (m_i(x_0) - m_j(x_0)) H_\Omega(X, 0),$$

and hence it holds that

$$\begin{aligned} & m_1(x_0) G_1(0) + m_2(x_0) G_2(0) \\ &= \nabla_X \{ [m_1(x_0)(m_1(x_0) - m_2(x_0)) + m_2(x_0)(m_2(x_0) - m_1(x_0))] \\ &\qquad [H_\Psi(X, x_0) - H_\Omega(X, 0)] \\ &\quad + (m_1(x_0) - m_2(x_0)) \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0') G(\cdot, x_0') \\ &\quad + (m_2(x_0) - m_1(x_0)) \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0') G(\cdot, x_0') \\ &\quad + \ (m_1(x_0) + m_2(x_0)) \xi(X) \} \big|_{X=0} \,. \end{aligned}$$

Since we have

$$\begin{aligned} & m_1(x_0)\left(m_1(x_0) - m_2(x_0)\right) + m_2(x_0)\left(m_2(x_0) - m_1(x_0)\right) \\ &= 8\pi \left(m_1(x_0) + m_2(x_0)\right), \end{aligned}$$

relation (3.2) is equivalent to

$$\nabla_X \Big[8\pi H_\Psi(X, x_0) + \frac{m_1(x_0) - m_2(x_0)}{m_1(x_0) + m_2(x_0)} \sum_{x_0' \in \mathcal{S}_1 \setminus \{x_0\}} m_1(x_0') G(X, x_0')$$
$$+ \frac{-m_1(x_0) + m_2(x_0)}{m_1(x_0) + m_2(x_0)} \sum_{x_0' \in \mathcal{S}_2 \setminus \{x_0\}} m_2(x_0') G(X, x_0') + \xi(X) \Big] \Big|_{X=0} = 0.$$

This implies (1.24) and the proof is complete. □

4. Proof of Theorem 1.1

We prove Theorem 1.1, namely,

$$\Lambda = [0, 8\pi] \times [0, 8\pi], \tag{4.1}$$

where

$$\Lambda \equiv \{(\lambda_1, \lambda_2) \in [0, \infty) \times [0, \infty) : \inf_{v \in E} J_{\lambda_1, \lambda_2}(v) > -\infty\}.$$

We have already $\Lambda \neq \emptyset$ from (1.3) and also

$$t(\lambda_1, \lambda_2) \in \Lambda \quad \text{for all } t \in [0, 1] \text{ if } (\lambda_1, \lambda_2) \in \Lambda \tag{4.2}$$

by

$$J_{t\lambda_1, t\lambda_2}(v) = \frac{1}{2}(1 - t)\|v\|_E^2 + tJ_{\lambda_1, \lambda_2}(v) \geq tJ_{\lambda_1, \lambda_2}(v).$$

First, we prove a weak version of Theorem 1.1:

Lemma 4.1. *It hold that*

$$[0, 8\pi) \times [0, 8\pi) \subset \Lambda.$$

In fact, if this is not the case, there exists $(\lambda_1^0, \lambda_2^0) \in [0, 8\pi) \times [0, 8\pi)$ satisfying

$$t(\lambda_1^0, \lambda_2^0) \in \Lambda \qquad \forall t \in [0, 1) \tag{4.3}$$

$$t(\lambda_1^0, \lambda_2^0) \notin \Lambda \qquad \forall t > 1. \tag{4.4}$$

Then, we obtain the following.

Proposition 4.2. *There exists a sequence $\{v_n\} \subset E$ such that*

$$\lim_{n \longrightarrow \infty} \|v_n\|_E = \infty \quad \textit{and} \quad \lim_{n \longrightarrow \infty} \frac{J_{\lambda_1^0, \lambda_2^0}(v_n)}{\|v_n\|_E^2} \leq 0.$$

Proof. First, given $\delta > 0$ and $C > 0$, we have $v \in E$ satisfying

$$J_{\lambda_1^0, \lambda_2^0}(v) < \frac{\delta}{2}\|v\|_E^2 - C. \tag{4.5}$$

In fact, if this is not the case, we have $\delta > 0$ and $C > 0$ such that

$$J_{\lambda_1^0, \lambda_2^0}(v) \geq \frac{\delta}{2}\|v\|_E^2 - C$$

for every $v \in E$. Since we may choose $0 < \delta \ll 1$, the above relation implies

$$\inf_{v \in E} J_{\frac{\lambda_1^0}{1-\delta}, \frac{\lambda_2^0}{1-\delta}}(v) > -\infty,$$

which contradicts (4.4).

Now we choose v_n satisfying (4.5) with $\delta = \delta_n = \frac{1}{n}$ and $C = C_n = n$ and prove that $\|v_n\|_E \longrightarrow \infty$ (taking a subsequence if necessary).

For this purpose, we suppose $\limsup_{n\longrightarrow\infty}\|v_n\|_E = M < \infty$ and fix $t \in [0,1)$. From (4.3), we have

$$\inf_{v\in E} J_{t\lambda_1^0, t\lambda_2^0}(v) \equiv C(t) > -\infty,$$

and therefore,

$$\begin{aligned} J_{\lambda_1^0,\lambda_2^0}(v_n) &= \frac{1}{t} J_{t\lambda_1^0,t\lambda_2^0}(v_n) + \frac{1}{2}\Big(1 - \frac{1}{t}\Big)\|v_n\|_E^2 \\ &\geq \frac{1}{t} C(t) + \frac{1}{2}\Big(1 - \frac{1}{t}\Big)\|v_n\|_E^2. \end{aligned}$$

Combining this inequality and (4.5), we have

$$\frac{1}{2}\Big(1 - \frac{1}{t}\Big)\|v_n\|_E^2 + \frac{1}{t}C(t) < \frac{1}{2n}\|v_n\|_E^2 - n,$$

that is,

$$n + \frac{1}{t}C(t) < \frac{1}{2}\Big(\frac{1}{n} + \frac{1}{t} - 1\Big)\|v_n\|_E^2 \leq \Big(\frac{1}{t} - 1\Big)M^2$$

for sufficiently large n, which leads to a contradiction. □

Here we use the following lemma.

Lemma 4.3 ([14, Lemma 4.4]). *For any two sequences $\{a_n\}$ and $\{b_n\}$ satisfying*

$$\lim_{n\longrightarrow\infty} a_n = \infty \quad \textit{and} \quad \lim_{n\longrightarrow\infty} \frac{b_n}{a_n} \leq 0,$$

there exists a smooth function $F : [0,\infty) \longrightarrow \mathbf{R}$ satisfying

$$0 < F'(t) < 1 \quad \textit{and} \quad F'(t) \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow \infty$$

and

$$b_{n_k} - F(a_{n_k}) \longrightarrow -\infty \quad \textit{as} \quad k \longrightarrow \infty$$

for some subsequence $\{n_k\}$.

We apply the above lemma for

$$a_n = \frac{1}{2}\|v_n\|_E^2, \qquad b_n = J_{\lambda_1^0,\lambda_2^0}(v_n)$$

and obtain $F(t)$, where $\{v_n\}$ is a sequence in E of Proposition 4.2. Now, we define

$$I_\varepsilon(v) = J_{(1-\varepsilon)\lambda_1^0,(1-\varepsilon)\lambda_2^0}(v) - F\big(\frac{1}{2}\|v\|_E^2\big).$$

Then, it holds that

$$\inf_{v\in E} I_0(u) = -\infty. \tag{4.6}$$

On the other hand, we have the following.

Lemma 4.4. *For every* $0 < \varepsilon \ll 1$, *there exists* $v_\varepsilon \in E$ *satisfying*

$$\inf_{v \in E} I_\varepsilon(v) = I_\varepsilon(v_\varepsilon) > -\infty.$$

Proof. Fix $\sigma > 0$, and we have $M > 0$ such that

$$F'(t) \leq \sigma \quad \text{for every } t \geq M,$$

which implies

$$F(t) = F(M) + \int_M^t F'(s) ds \leq F(M) + \sigma t \quad \text{for every } t \geq M.$$

Consequently, there exists $C > 0$ such that

$$F(t) \leq \sigma t + C \quad \text{for every } t \geq 0,$$

and therefore,

$$I_\varepsilon(v) \geq (1-\sigma) J_{\frac{1-\varepsilon}{1-\sigma}\lambda_1^0, \frac{1-\varepsilon}{1-\sigma}\lambda_2^0}(v) - C$$

for any $v \in E$.

Taking σ in $0 < \sigma < \varepsilon$, we see that $I_\varepsilon(v)$ is bounded from below on E. It is moreover coercive together with $J_{\frac{1-\varepsilon}{1-\sigma}\lambda_1^0, \frac{1-\varepsilon}{1-\sigma}\lambda_2^0}(v)$, and the conclusion is obtained from a standard argument. □

Proof of Lemma 4.1. We take $\varepsilon_n \downarrow 0$ and a minimizer $v_n \in E$ of $I_{\varepsilon_n}(\cdot)$, which is a solution to (1.1) for

$$\lambda_{i,n} = \frac{1-\varepsilon_n}{1-\mu_n}\lambda_i^0,$$

where $\mu_n = F'\left(\frac{1}{2}\|v_n\|_E^2\right)$. If

$$\limsup_{n \longrightarrow \infty} \|v_n\|_E = \infty, \tag{4.7}$$

then we have

$$\mu_n \longrightarrow 0 \quad \text{and} \quad \lambda_{i,n} \longrightarrow \lambda_i^0$$

for $i = 1, 2$, taking a subsequence if necessary. Thus, $(\lambda_1^0, \lambda_2^0) \notin [0, 8\pi) \times [0, 8\pi)$ from Theorem 1.2, which contradicts the choice of $(\lambda_1^0, \lambda_2^0)$.

Suppose that (4.7) does not hold. Then, there exist $v_\infty \in E$ and a subsequence of $\{v_n\}$ such that $v_n \longrightarrow v_\infty$ weakly in E. From the standard argument, we have

$$\liminf_{n \longrightarrow \infty} I_{\varepsilon_n}(v_n) \geq I_0(v_\infty) > -\infty$$

and this v_∞ must be a minimizer of $I_0(\cdot)$ on E. In fact, if there exists $v \in E$ such that $I_0(v) < I_0(v_\infty)$, then we have

$$I_{\varepsilon_n}(v) < I_0(v_\infty) \leq I_{\varepsilon_n}(v_n)$$

for sufficiently large n by

$$I_\varepsilon(v) = I_0(v) + \varepsilon\Big\{ - I_0(v) + \frac{1}{2}\|v\|_E^2 - F\Big(\frac{1}{2}\|v\|_E^2\Big)\Big\}.$$

This contradicts the minimality of $I_{\varepsilon_n}(v_n)$.

On the other hand, $I_0(\cdot)$ does not attain the minimum by (4.6), and this is a contradiction. $\square$

Proof of Theorem 1.1. If (4.1) is false, then we have $(\lambda_1^0, \lambda_2^0) \notin \Lambda$ such that

$$(\lambda_1^0, \lambda_2^0) \in [0, 8\pi] \times [0, 8\pi] \backslash [0, 8\pi) \times [0, 8\pi)$$

by Lemma 4.1. Then, we obtain

$$\inf_{v \in E} J_t(v) > -\infty \quad \text{for every } t \in [0, 1)$$

with the minimum attained, where

$$J_t(v) \equiv J_{t\lambda_1^0, t\lambda_2^0}(v).$$

Now, we take $t_n \uparrow 1$ and a minimizer $v_n \in E$ of $J_{t_n}(\cdot)$. If

$$\liminf_{n \longrightarrow \infty} J_{t_n}(v_n) > -\infty \tag{4.8}$$

is proven, then we obtain $(\lambda_1^0, \lambda_2^0) \in \Lambda$ by

$$J_0(v) = \lim_{n \longrightarrow \infty} J_{t_n}(v) \geq \liminf_{n \longrightarrow \infty} J_{t_n}(v_n) > -\infty,$$

a contradiction.

When $\{v_n\}$ is relatively compact in E, the limit of the converging subsequence of $\{v_n\}$ becomes a minimizer of J_0 (see the argument in the proof of Lemma 4.1). This implies $(\lambda_1^0, \lambda_2^0) \in \Lambda$, a contradiction. If this is not the case, we can apply Theorem 1.2, and obtain the following two alternatives, passing to a subsequence, where $\mu_{i,n}$ and $u_{i,n}$ denote the functions defined by (1.12-1.13):

(1) There exist $x_1 \in M$ and $u_2 \in E$ such that

$$\mu_{1,n} \longrightarrow 8\pi\delta_{x_1} \quad \text{weakly} * \text{ in } \mathcal{M}(M) \quad \text{and} \quad u_{2,n} \longrightarrow u_2 \quad \text{in } E.$$

(2) There exist x_1, $x_2 \in M$ satisfying $x_1 \neq x_2$ and

$$\mu_{1,n} \longrightarrow 8\pi\delta_{x_1} \quad \text{and} \quad \mu_{2,n} \longrightarrow 8\pi\delta_{x_2} \quad \text{weakly} * \text{ in } \mathcal{M}(M).$$

Case (1). In this case, we have $\lambda_1^0 = 8\pi$ and

$$\begin{aligned} J_{t_n}(v_n) &= J_{t_n}(u_{1,n} - u_{2,n}) \\ &\geq \frac{1}{2}\|u_{1,n}\|_E^2 - \int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g + \frac{1}{2}\|u_{2,n}\|_E^2 \\ &\qquad - 8\pi t_n \log \int_M e^{u_{1,n}+C} dv_g - t_n \lambda_2^0 \log \int_M e^{C+u_{2,n}} dv_g \\ &= J_{8\pi t_n}(u_{1,n}) + J_{t_n\lambda_2^0}(u_{2,n}) - \int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g - t_n(8\pi + \lambda_2^0)C \end{aligned}$$

by (2.2). On the other hand, we have

$$J_{8\pi t_n}(u_{1,n}) = t_n J_{8\pi}(u_{1,n}) + \frac{1}{2}(1-t_n)\|u_{1,n}\|_E \geq t_n J_{8\pi}(u_{1,n})$$

and therefore,

$$\liminf_{n\longrightarrow\infty} J_{8\pi t_n}(u_{1,n}) > -\infty$$

from (1.2). We have also

$$\lim_{n\longrightarrow\infty} J_{t_n\lambda_2^0}(u_{2,n}) = J_{\lambda_2^0}(u_2) > -\infty$$

by $u_{2,n} \longrightarrow u_2$ in E, and thus, all we have to do is to control the cross term $\int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g$.

In fact, using the elliptic regularity and (2.2), we are able to conclude

$$u_{2,n} \longrightarrow u_2 \quad \text{in } C(M)$$

from $u_{2,n} \longrightarrow u_2$ in E, and then it follows that

$$\int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g = \int_M \mu_{1,n} u_{2,n} dv_g \longrightarrow 8\pi u_2(x_1) < \infty.$$

Thus, we obtain (4.8).

Case (2). Similar to the case (1), we have only to control the cross term $\int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g$.

For this purpose, we take a cut-off function $\rho \in C^\infty(M)$ around x_1, satisfying

$$\begin{aligned} &0 \leq \rho(x) \leq 1 \quad \text{for every } x \in M \\ &\rho \equiv 1 \quad \text{near } x_1 \\ &x_2 \notin \operatorname{supp} \rho \end{aligned} \tag{4.9}$$

and use the relation

$$\int_M \nabla_g u_{1,n} \cdot \nabla_g u_{2,n} dv_g = \int_M \mu_{1,n} u_{2,n} dv_g$$

$$= \int_M \rho\mu_{1,n}u_{2,n}dv_g + \int_M (1-\rho)\mu_{1,n}u_{2,n}dv_g. \quad (4.10)$$

Here (1.21) (with $r_i = 0$), (4.9), and the elliptic regularity assure

$$\rho(x)u_{2,n}(x) \longrightarrow 8\pi\rho(x)G(x,x_2) \quad \text{in } C(M)$$
$$(1-\rho)\,\mu_{1,n} \longrightarrow 0 \quad \text{in } L^p(M) \qquad \text{for any } p \in [1,\infty)$$

and consequently,

$$u^0_{1,n} = \int_M G(\cdot,y)(1-\rho(y))\mu_{1,n}(y)dv_g \longrightarrow 0 \quad \text{in } C(M).$$

Thus, we obtain

$$(4.10) = \int_M \mu_{1,n}(\rho u_{2,n})dv_g + \int_M u^0_{1,n}\mu_{2,n}dv_g \longrightarrow (8\pi)^2 G(x_1,x_2) < \infty,$$

and hence (4.8). $\square$

REFERENCES

[1] T. Aubin, "Some Nonlinear Problems in Riemannian Geometry," Springer, Berlin, 1998.

[2] S. Baraket and F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension* 2, Calc. Var. Partial Differential Equations, 6 (1998), 1–38.

[3] H. Brezis and F. Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations, 16 (1991), 1223–1253.

[4] E. Caglioti, P. L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description*, Comm. Math. Phys., 143 (1992), 501–525.

[5] ———, *A special class of stationary flows for two-dimensional Euler equations: A statistical mechanics description. part II*, Comm. Math. Phys., 174 (1995), 229–260.

[6] D. Chae, H. Ohtsuka, and T. Suzuki, *Some existence results for* $SU(3)$ *Toda system*, to appear in; Calc. Var. Partial Differential Equations.

[7] S.-Y.-A. Chang and P.-C. Yang, *Conformal deformation of metrics on* S^2, J. Differential Geometry, 27 (1988), 259-296.

[8] C.-C. Chen and C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemannian surfaces*, Comm. Pure Appl. Math., 55 (2002), 728–771.

[9] ———, *Topological degree for a mean field equations on Riemann surfaces*, Comm. Pure Appl. Math., 56 (2003), 1667–1803

[10] A.J. Chorin, "Vorticity and Turbulence," Springer, New York, 1994.

[11] W. Ding, J. Jost, J. Li, and G. Wang, *Existence results for mean field equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 16 (1999), 653–666.

[12] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv., 68 (1993), 415–454.

[13] G. Joyce and D. Montgomery, *Negative temperature states for the two-dimensional guiding-centre plasma*, J. Plasma Phys., 10 (1973), 107–121.

[14] J. Jost and G. Wang, *Analytic aspects of the Toda system: I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math., 54 (2001), 1289–1319.

[15] M.K.H. Kiessling, *Statistical mechanics of classical particles with logarithmic interactions*, Comm. Pure Appl. Math., 46 (1993), 27–56.

[16] Y.Y. Li, *Harnack type inequality: the method of moving planes*, Comm. Math. Phys., 200 (1999), 421–444.

[17] Y. Y. Li and I. Shafrir, *Blow-up analysis for solutions of* $-\Delta u = Ve^u$ *in dimension two*, Indiana Univ. Math. J., 43 (1994), 1255–1270.

[18] P. -L. Lions, "On Euler Equations and Statistical Physics," Scuola Normale Superiore, Pisa, 1997.

[19] L. Ma and J. C. Wei, *Convergence for a Liouville equation*, Comment. Math. Helv., 76 (2001), 506–514.

[20] C. Marchioro and M. Pulvirenti, "Mathematical Theory of Incompressible Nonviscous Fluids," Springer, New York, 1994.

[21] K. Nagasaki and T. Suzuki, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially-dominated nonlinearities*, Asymptotic Analysis, 3 (1990), 173–188.

[22] P. K. Newton, "The N-Vortex Problem: Analytical Techniques," Springer, New York, 2001.

[23] H. Ohtsuka and T. Suzuki, *Palais-Smale sequence relative to the Trudinger-Moser inequality*, Calc. Var. Partial Differential Equations, 17 (2003), 235–255.

[24] ———, *Blowup analysis for* $SU(3)$ *Toda systems and applications*, RIMS Kokyuroku, 1405 (2004), 73–99.

[25] ———, *Blow-up analysis for Liouville type equation in self-dual gauge field theories*, Comm. Contemp. Math., 7 (2005), 177–205.

[26] Y. B. Pointin and T. S. Lundgren, *Statistical mechanics of two-dimensional vortices in a bounded container*, Phys. Fluids, 19 (1976), 1459–1470.

[27] K. Sawada, T. Suzuki, and F. Takahashi, *Mean field equation for equilibrium vortices with neutral orientation*, preprint.

[28] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Comm. Partial Differential Equations, 20 (1995), 1077–1104.

[29] T. Senba and T. Suzuki, *Some structures of the solution set for a stationary system of chemotaxis*, Adv. Math. Sci. Appl., 10 (2000), 191–224.

[30] ———, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations, 6 (2001), 21–50.

[31] I. Shafrir and G. Wolansky, *Moser-Trudinger and logarithmic HLS inequalities for systems*, preprint.

[32] M. Struwe and G. Tarantello, *On multivortex solutions in Chern-Simons gauge theory*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 1 (1998), 109–121.

[33] T. Suzuki, "Semilinear Elliptic Equations," Gakkotosho, Tokyo, 1994.

[34] ———, "Free Energy and Self-Interacting Particles," Birkhäuser, Boston, 2005.

[35] Y. Yang, "Solitons in Field Theory and Nonlinear Analysis," Springer, New York, 2001.