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FIRST ORDER ASYMPTOTICS FOR THE TRAVELLING WAVES IN THE GROSS-PITAEVSKII EQUATION

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Abstract. In a previous paper [7], we investigated the asymptotic behaviour of subsonic travelling waves of finite energy for the Gross-Pitaevskii equation in every dimension $N \geq 2$. In particular, we gave their first-order asymptotics in case they were axisymmetric. In the present paper, we compute their first-order asymptotics at infinity in the general case.

Introduction

1. **Motivation and main result.** The Gross-Pitaevskii equation is a relevant model in several domains of physics (Bose-Einstein condensation, superconductivity, superfluidity, nonlinear optics...). This nonlinear Schrödinger equation is written

$$i\partial_t u = \Delta u + u(1 - |u|^2), \tag{0.1}$$

for a function u defined from $\mathbb{R} \times \mathbb{R}^N$ (with $N \geq 2$) to $\mathbb{C}$. It conserves (at least formally) two integral quantities which play a role in the asymptotic description of subsonic travelling waves of finite energy: the so-called Ginzburg-Landau energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1 - |u|^2)^2, \tag{0.2}$$

and the momentum

$$\vec{P}(u) = \frac{1}{2}\int_{\mathbb{R}^N} i\nabla u.(u - 1). \tag{0.3}$$

A travelling wave v for the Gross-Pitaevskii equation is a particular solution of equation (0.1) of the form $u(t, x) = v(x_1 - ct, x_2, \dots, x_N)$. The parameter $c \geq 0$ is the speed of the travelling wave v, which moves in the direction

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x_1. The equation for the profile v, which we will consider from now on, is written

$$ic\partial_1 v + \Delta v + v(1 - |v|^2) = 0. \tag{0.4}$$

The travelling waves of finite energy are supposed to play a major role in the long-time dynamics of the Gross-Pitaevskii equation. C.A. Jones, S.J. Putterman and P.H. Roberts [9, 10] consequently considered their existence and qualitative properties by means of numerical simulations and formal computations. They conjectured that there exist non-constant travelling waves of finite energy if and only if their speed c satisfies

$$0 < c < c_s = \sqrt{2}, \tag{0.5}$$

which means that all non-constant travelling waves of finite energy are subsonic. Indeed, the characteristic speed $c_s = \sqrt{2}$ is the speed of sound waves for equation (0.1) near the constant solution $u = 1$. To our knowledge, their conjecture remains an open problem [1]. However, many recent papers [1, 2, 3, 4, 5, 6] partly confirm its validity. That is the reason why we will only consider subsonic travelling waves of finite energy for which inequality (0.5) is valid.

Under this assumption, C.A. Jones, S.J. Putterman and P.H. Roberts [9, 10] described the asymptotic behaviour of travelling waves of finite energy which are axisymmetric around axis x_1. They computed their first-order asymptotics (up to a multiplicative constant of modulus one) in dimension two,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2}, \tag{0.6}$$

and in dimension three,

$$v(x) - 1 \underset{|x| \to +\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{\frac{3}{2}}}. \tag{0.7}$$

Here, the constant α is the stretched dipole coefficient linked to the energy $E(v)$ and to the scalar momentum $p(v) = P_1(v)$ in the direction x_1, by the formula

$$2\pi\alpha\sqrt{1 - \frac{c^2}{2}} = cE(v) + 2\Big(1 - \frac{c^2}{4}\Big)p(v) \tag{0.8}$$

in dimension two, and

$$4\pi\alpha = \frac{c}{2}E(v) + 2p(v) \tag{0.9}$$

[1]The non-existence of non-constant travelling waves of finite energy for $c = \sqrt{2}$ in every dimension $N \geq 3$, and their existence for every speed $0 < c < \sqrt{2}$ in every dimension $N \geq 2$ are not yet established.

in dimension three. In a previous paper [7], we derived rigorously formulae (0.6), (0.7), (0.8) and (0.9). More precisely, we established the existence of first-order asymptotics for any subsonic travelling wave of finite energy, before computing explicitly formulae (0.6), (0.7), (0.8) and (0.9) in the axisymmetric case.

Theorem 1 ([7])**.** *Let $N \geq 2$ and $0 < c < \sqrt{2}$. Consider a travelling wave v of finite energy and of speed c for the Gross-Pitaevskii equation. There exist a complex number λ_∞ of modulus one and a smooth function v_∞ defined from the sphere $\mathbb{S}^{N-1}$ to $\mathbb{R}$ such that*

$$|x|^{N-1}\big(v(x) - \lambda_\infty\big) - i\lambda_\infty v_\infty\Big(\frac{x}{|x|}\Big) \underset{|x|\to+\infty}{\to} 0 \text{ uniformly.} \tag{0.10}$$

Assume moreover that the function v is axisymmetric around the axis x_1; i.e., it only depends on the variables x_1 and $|x_\perp| = \sqrt{\sum_{i=2}^N x_i^2}$. The function v_∞ then is written

$$\forall \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{\big(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}\big)^{\frac{N}{2}}}, \tag{0.11}$$

where the constant α is equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1 - \frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Big(\frac{4-N}{2}cE(v) + \Big(2 + \frac{N-3}{2}c^2\Big)p(v)\Big). \tag{0.12}$$

Remark. We also computed explicitly the function v_∞ for every subsonic travelling wave v in dimension two. In this case, there exist some constants α and β such that the function v_∞ is written

$$\forall \sigma = (\sigma_1, \sigma_2) \in \mathbb{S}^1, v_\infty(\sigma) = \alpha \frac{\sigma_1}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}} + \beta \frac{\sigma_2}{1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}}. \tag{0.13}$$

Moreover, the constants α and β are linked to the energy $E(v)$ and the momentum $\vec{P}(v)$ by the formulae

$$\alpha = \frac{1}{2\pi\sqrt{1 - \frac{c^2}{2}}}\Big(cE(v) + \Big(2 - \frac{c^2}{2}\Big)p(v)\Big), \tag{0.14}$$

$$\beta = \frac{1}{\pi}\sqrt{1 - \frac{c^2}{2}}P_2(v). \tag{0.15}$$

However, we were not able to compute explicitly the value of the function v_∞ in the general case. We only conjectured its value in Conjecture 1 of [7]. The goal of the present paper is to fill this gap by confirming the validity of this conjecture.

Theorem 2. *Let $N \geq 2$ and $0 < c < \sqrt{2}$. Consider a travelling wave v of finite energy and of speed c for the Gross-Pitaevskii equation. There exist some constants α, β_2, ..., β_N such that the function v_∞ defined by statement* (0.10) *of Theorem 1 is equal to*

$$\forall \sigma \in \mathbb{S}^{N-1}, v_\infty(\sigma) = \alpha \frac{\sigma_1}{\left(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}\right)^{\frac{N}{2}}} + \sum_{j=2}^{N} \beta_j \frac{\sigma_j}{\left(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2}\right)^{\frac{N}{2}}}. \quad (0.16)$$

Moreover, the constants α and β_j are equal to

$$\alpha = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1 - \frac{c^2}{2}\Big)^{\frac{N-3}{2}}\Big(\frac{4-N}{2} cE(v) + \Big(2 + \frac{N-3}{2}c^2\Big)p(v)\Big), \quad (0.17)$$

$$\beta_j = \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}}\Big(1 - \frac{c^2}{2}\Big)^{\frac{N-1}{2}} P_j(v). \quad (0.18)$$

Remarks. 1. Formulae (0.16), (0.17) and (0.18) are identically equal to formulae (0.13), (0.14) and (0.15) in dimension two.

2. Theorem 2 is also consistent with formulae (0.11) and (0.12) in the axisymmetric case. Indeed, a travelling wave v, which is axisymmetric around axis x_1, is an even function of each variable x_j for $j \in \{2, \dots, N\}$. Therefore, the functions $v - 1$ and $\partial_j v$ are respectively even and odd functions of each variable x_j. By definition (0.3), the scalar momentum $P_j(v)$ in direction x_j, and consequently the constant β_j, vanish for every $j \in \{2, \dots, N\}$. As a consequence, formulae (0.16), (0.17) and (0.18) are identically equal to formulae (0.11) and (0.12) in the axisymmetric case.

3. The first-order term v_∞ of the asymptotics of v is completely determined by some integral quantities α, β_2, ..., β_N, linked to the energy $E(v)$ and the momentum $\vec{P}(v)$ by formulae (0.17) and (0.18). As mentioned in [7], this raises an interesting question. Consider N real numbers a_1, ..., a_N: is it possible to construct a travelling wave v such that the values of the integral quantities α, β_2, ..., β_N are exactly equal to a_1, ..., a_N ? In other words, is it possible to construct travelling waves v whose asymptotics correspond to any possible one given by Theorem 2, or are there other restrictions for admissible asymptotics? To our knowledge, these questions remain open problems. Indeed, the existence results of F. Béthuel and J.C. Saut [2, 3], F. Béthuel, G. Orlandi and D. Smets [1] and D. Chiron [4] assert the existence of presumably axisymmetric travelling waves, for which the constants β_2, ..., β_N are equal to 0. Therefore, we do not know of any travelling wave for which the values of β_2, ..., β_N are not 0. Thus, a first step to answer the

questions above seems to be the proof of the existence of travelling waves which are not axisymmetric.

The present paper focuses on the proof of Theorem 2. However, this requires many arguments from the proof of Theorem 1, which forms the core of an earlier paper [7]. Therefore, we first recall some notation and results of [7] before establishing Theorem 2.

2. **Preliminaries.** In [7], Theorem 1 results from a new formulation of equation (0.4), which relies on a polar form of the function v. Indeed, there is some positive real number R_0 and some functions $\rho := |v|$ and θ in $C^\infty(B(0,R_0)^c,\mathbb{R})$ such that

$$v = \rho e^{i\theta}$$

on $B(0,R_0)^c$. By introducing a cut-off function $\psi \in C^\infty(\mathbb{R}^N,[0,1])$ such that

$$\begin{cases} \psi = 0 & \text{on } \mathrm{B}(0, 2\mathrm{R}_0), \\ \psi = 1 & \text{on } \mathrm{B}(0, 3\mathrm{R}_0)^{\mathrm{c}}, \end{cases}$$

we compute new equations for the new variables $\eta := 1-\rho^2$ and $\psi\theta$,

$$\Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}), \tag{0.19}$$

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(\mathrm{G}), \tag{0.20}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta),$$
$$G = i\nabla v.v + \nabla(\psi\theta).$$

We then derive our new formulation by transforming equations (0.19) and (0.20) in the convolution equations

$$\eta = K_0 * F + 2c\sum_{k=1}^N K_k * G_k, \tag{0.21}$$

$$\forall j \in \{1,\ldots,N\}, \partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^N L_{j,k} * G_k + \sum_{k=1}^N R_{j,k} * G_k, \tag{0.22}$$

where K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are the kernels of the Fourier transform

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{0.23}$$

$$\forall j \in \{1,\ldots,N\}, \widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4+2|\xi|^2-c^2\xi_1^2}, \tag{0.24}$$

$$\forall (j,k) \in \{1,\ldots,N\}^2, \widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4+2|\xi|^2-c^2\xi_1^2)}, \tag{0.25}$$

$$\forall (j,k) \in \{1,\ldots,N\}^2, \widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}. \tag{0.26}$$

Theorem 1 then results from equations (0.21) and (0.22). Indeed, the asymptotics of K_0, K_j, $L_{j,k}$ and $R_{j,k}$ give the asymptotics of η and $\psi\theta$ (which yield the asymptotics of v). More precisely, Proposition 5 of [7] asserts that there exist some functions $\eta_\infty \in C^1(\mathbb{S}^{N-1})$ and $(\theta_\infty, v_\infty) \in C^2(\mathbb{S}^{N-1})^2$ such that

$$\begin{aligned} R^N\eta(R\sigma) &\underset{R\to+\infty}{\to} \eta_\infty(\sigma) \text{ in } C^1(\mathbb{S}^{N-1}),\\ R^{N-1}(\psi\theta)(R\sigma) &\underset{R\to+\infty}{\to} \theta_\infty(\sigma) \text{ in } C^2(\mathbb{S}^{N-1}),\\ R^{N-1}\big(v(R\sigma)-1\big) &\underset{R\to+\infty}{\to} iv_\infty(\sigma) \text{ in } C^1(\mathbb{S}^{N-1}). \end{aligned}$$

Moreover, equations (70) and (72) of [7] give expressions of η_∞, θ_∞ and v_∞ for every $\sigma \in \mathbb{S}^{N-1}$,

$$\eta_\infty(\sigma) = K_{0,\infty}(\sigma)\int_{\mathbb{R}^N} F(x)dx + 2c\sum_{j=1}^N K_{j,\infty}(\sigma)\int_{\mathbb{R}^N} G_j(x)dx, \tag{0.27}$$

$$\begin{aligned} \theta_\infty(\sigma) = v_\infty(\sigma) = &\frac{c}{2(N-1)}\Big(\sum_{j=1}^N \sigma_j K_{j,\infty}(\sigma)\Big)\int_{\mathbb{R}^N} F(x)dx + \sum_{k=1}^N \Big(\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\sigma_k \\ &- \frac{c^2}{N-1}\sum_{j=1}^N \sigma_j L_{j,k,\infty}(\sigma)\Big)\int_{\mathbb{R}^N} G_k(x)dx. \end{aligned} \tag{0.28}$$

Here, $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ are bounded functions on $\mathbb{S}^{N-1}$, which give the asymptotics of K_0, K_j and $L_{j,k}$ as claimed above. More precisely, they are defined by the following theorem.

Theorem 3 ([7])**.** *Consider the space of functions*

$$\widehat{\mathcal{K}}(\mathbb{R}^N) := \{u \in C^\infty(\mathbb{R}^N\setminus\{0\},\mathbb{C}) : \forall i \in \mathbb{N}, d^iu \in M_i^\infty(\mathbb{R}^N)\cup M_{i+2}^\infty(\mathbb{R}^N)\},$$

where $M_\alpha^\infty(\mathbb{R}^N) := \{u : \mathbb{R}^N \mapsto \mathbb{C} : \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\{|x|^\alpha|u(x)|, x \in \mathbb{R}^N\} < +\infty\}$ *for every* $\alpha > 0$. *Assume that* K *is a tempered distribution whose*

Fourier transform

$$\widehat{K} = \frac{P}{Q}$$

is a rational fraction which belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ *and such that*

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, Q(\xi) \neq 0.$$

Then, there exists a measurable function $K_\infty \in L^\infty(\mathbb{S}^{N-1}, \mathbb{C})$ *such that*

$$\forall \sigma \in \mathbb{S}^{N-1}, R^N K(R\sigma) \underset{R \to +\infty}{\to} K_\infty(\sigma). \tag{0.29}$$

As mentioned in the proof of Corollary 3 of [7], the kernels K_0, K_j and $L_{j,k}$ satisfy all the assumptions of Theorem 3. The functions $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ are equal to the function K_∞ defined by assertion (0.29) for each kernel K_0, K_j or $L_{j,k}$.

Actually, the proof of Theorem 3 yields integral expressions of K_∞, and consequently, of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$. Indeed, consider some distribution K which verifies all the assumptions of Theorem 3. The function $\widehat{K}$ as well as all its derivatives are rational fractions only singular at the origin. Therefore, they may be written for every $j \in \{1, \ldots, N\}$ and $p \in \mathbb{N}$,

$$\partial_j^p \widehat{K} = \frac{P_p}{Q_p} = \frac{\sum_{k=0}^{d_p} P_{k,p}}{\sum_{k=0}^{d'_p} Q_{k,p}}, \tag{0.30}$$

where $P_{k,p}$ and $Q_{k,p}$ are homogeneous polynomial functions either equal to 0 or of degree k, and the polynomial functions P_p and Q_p are inductively defined by

$$P_0 = P \text{ and } P_{p+1} = \partial_j P_p Q_p - P_p \partial_j Q_p, \tag{0.31}$$

$$Q_0 = Q \text{ and } Q_{p+1} = Q_p^2. \tag{0.32}$$

Now, denote for every $i \in \{0, 1, 2\}$, and $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$l_i(\xi) = \begin{cases} \min\{k \in \{0, \ldots, d_p\}, P_{k,N+i-1}(\xi) \neq 0\}, \\ \qquad \text{if } \exists k \in \{0, \ldots, d_p\}, P_{k,N+i-1}(\xi) \neq 0, \\ +\infty, \qquad \text{otherwise}, \end{cases}$$

$$l'_i(\xi) = \min\{k \in \{0, \ldots, d'_p\}, Q_{k,N+i-1}(\xi) \neq 0\}.$$

Since Q does not vanish on $\mathbb{R}^N \setminus \{0\}$, the polynomial function Q_p does not vanish on $\mathbb{R}^N \setminus \{0\}$, so, the functions l_i and l'_i are well defined on $\mathbb{R}^N \setminus \{0\}$. Moreover, Claim 1 of [7] states that for every $i \in \{0,1,2\}$,

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \frac{\partial_j^{N+i-1} \widehat{K}\left(\frac{\xi}{R}\right)}{R^{N+i-1}} \underset{R \to +\infty}{\to} R_i(\xi), \tag{0.33}$$

where the function R_i is written, for all $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$R_i(\xi) = \begin{cases} \delta_{l'_i(\xi), l_i(\xi)+N-1+i} \frac{P_{l_i(\xi),N+i-1}(\xi)}{Q_{l'_i(\xi),N+i-1}(\xi)}, \text{ if } l_i(\xi) \neq +\infty, \\ 0, \text{ otherwise.} \end{cases} \tag{0.34}$$

The function K_∞ now may be written as a function of R_0, R_1 and R_2. Indeed, formula (61) of [7] asserts that for every $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_j \neq 0$,

$$\begin{aligned} K_\infty(\sigma) = & \frac{i^N}{(2\pi\sigma_j)^N} \Bigg(\int_{B(0,1)} R_1(\xi)(e^{i\xi.\sigma} - 1) d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_0(\xi) d\xi \\ & - \frac{1}{i\sigma_j} \left(\int_{B(0,1)^c} R_2(\xi) e^{i\xi.\sigma} d\xi + \int_{\mathbb{S}^{N-1}} \xi_j R_1(\xi) e^{i\xi.\sigma} d\xi \right) \Bigg). \end{aligned} \tag{0.35}$$

Thus, formula (0.35) yields some integral expressions of K_∞, which only depend on the value of $\widehat{K}$ (through R_0, R_1 and R_2). However, formulae (0.23), (0.24) and (0.25) give the explicit values of $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$, so, it seems possible to compute explicitly $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ by formula (0.35). This computation is the key ingredient of the proof of Theorem 2 as mentioned below.

3. **Sketch of the proof of Theorem 2.** Theorem 2 specifies the asymptotics of v by giving the value of v_∞. In [7], we already computed this value in dimension two and in the axisymmetric case (Cf Theorem 1). In both cases, we derived a linear partial differential equation for v_∞ on $\mathbb{S}^{N-1}$, and solved it to get the value of v_∞. However, we were only able to solve such an equation when it reduces to an ordinary differential equation, i.e., in dimension two and in the axisymmetric case. Moreover, such a resolution presents a major drawback: the considered equation may have some "parasite" solutions which do not correspond to the asymptotics of any travelling wave. Consequently, Theorem 2 relies on a completely different argument: the direct computation of v_∞ by formula (0.28). Indeed, equations (0.27) and (0.28) reduce the computation of η_∞, θ_∞ and v_∞ to the computation of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ on one hand, and of $\int_{\mathbb{R}^N} F(x)dx$ and $\int_{\mathbb{R}^N} G_k(x)dx$

on the other hand. However, we already computed these integrals in [7] (see the remark of Subsection 3.3). They are equal to

$$\int_{\mathbb{R}^N} F(x)dx = 2\Big((4-N)E(v) + c(N-3)p(v)\Big), \tag{0.36}$$

$$\int_{\mathbb{R}^N} G_k(x)dx = 2P_k(v). \tag{0.37}$$

Therefore, it only remains to compute $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ by using formula (0.35) as mentioned above. However, this computation may be quite involved because of the anisotropy of K_0, K_j and $L_{j,k}$, so, we do not proceed by a direct computation. Instead, we compute formula (0.35) for some simple distribution which presents the same asymptotics as K_0, K_j or $L_{j,k}$. Indeed, consider for instance the kernel K_0. Its behaviour at infinity heuristically depends on the behaviour near the origin of its Fourier transform $\widehat{K_0}$. By formula (0.23), the function $\widehat{K_0}$ behaves near the origin like the function $\widehat{R_0^c}$ defined by

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \widehat{R_0^c}(\xi) = \frac{|\xi|^2}{2|\xi|^2 - c^2\xi_1^2}. \tag{0.38}$$

Thus, the kernel K_0 presumably has the same asymptotics as the tempered distribution R_0^c whose Fourier transform is equal to $\widehat{R_0^c}$. However, the computation of the asymptotics of R_0^c is much easier. Indeed, the function $\widehat{R_0^c}$ may be written

$$\forall \xi = (\xi_1, \xi_\perp) \in \mathbb{R}^N \setminus \{0\}, \widehat{R_0^c}(\xi) = \sum_{j=1}^N \frac{1}{2 - c^2\delta_{j,1}} \widehat{R_{j,j}}\bigg(\sqrt{1 - \frac{c^2}{2}}\xi_1, \xi_\perp\bigg), \tag{0.39}$$

where $R_{j,j}$ is the so-called composed Riesz operator defined by formula (0.26). By standard Riesz operator theory, the distribution $R_{j,k}$ is actually given by

$$R_{j,k} = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(PV(\tilde{R}_{j,k}1_{B(0,1)}) + \tilde{R}_{j,k}1_{B(0,1)^c}\Big) + \frac{\delta_{j,k}}{N}\delta_0, \tag{0.40}$$

where δ_0 denotes the Dirac mass at the origin, and $PV(\tilde{R}_{j,k}1_{B(0,1)})$ denotes the principal value at the origin of the function $\tilde{R}_{j,k} : x \mapsto \frac{\delta_{j,k}|x|^2 - Nx_jx_k}{|x|^{N+2}}$, defined by, for all $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\Big\langle PV(\tilde{R}_{j,k}1_{B(0,1)}), \phi\Big\rangle = \int_{B(0,1)} \frac{\delta_{j,k}|x|^2 - Nx_jx_k}{|x|^{N+2}}\Big(\phi(x) - \phi(0)\Big)dx.$$

In particular, formula (0.40) give the asymptotics of $R_{j,k}$, which are equal to

$$\forall \sigma \in \mathbb{S}^{N-1}, R^N R_{j,k}(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k).$$

By formula (0.39), the asymptotics of R_0^c are then given by, for all $\sigma \in \mathbb{S}^{N-1}$,

$$R^N R_0^c(R\sigma) \underset{R\to+\infty}{\to} \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-3}{2}}c^2}{8\pi^{\frac{N}{2}}\left(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}\right)^{\frac{N}{2}}}\left(1-\frac{N\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\right). \tag{0.41}$$

The formal simplification above then yields the value of $K_{0,\infty}$, which is presumably equal to the second member of equation (0.41). The same argument also yields the values of $K_{j,\infty}$ and $L_{j,k,\infty}$, and consequently, explicit expressions of η_∞, θ_∞ and v_∞ by equations (0.27), (0.28), (0.36) and (0.37).

Now, in order to complete the proof of Theorem 2, we must justify rigorously the strategy above. The first step is to establish that the kernels K_0, K_j and $L_{j,k}$ really have the same asymptotics as the tempered distributions R_0^c, $R_{1,j}^c$ and $S_{j,k}^c$, whose Fourier transforms are given by formula (0.38) and

$$\widehat{R_{1,j}^c}(\xi) = \frac{\xi_1\xi_j}{2|\xi|^2 - c^2\xi_1^2}, \tag{0.42}$$

$$\widehat{S_{j,k}^c}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{2|\xi|^4 - c^2\xi_1^2|\xi|^2}. \tag{0.43}$$

This claim results from integral expression (0.35). More precisely, we prove the next proposition for a more general class of kernels.

Proposition 1. *Let $j \in \{1,\dots,N\}$, and $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_j \neq 0$. Consider a tempered distribution K whose Fourier transform*

$$\widehat{K} = \frac{P}{Q}$$

is a rational fraction which belongs to $\widehat{\mathcal{K}}(\mathbb{R}^N)$ and such that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, Q(\xi) \neq 0.$$

Assume moreover that the degrees of the homogeneous polynomial components of P and Q of lower degree (respectively denoted S_0 and T_0) are equal, and denote

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \widehat{R}(\xi) = \frac{S_0(\xi)}{T_0(\xi)}. \tag{0.44}$$

Then, the function K_∞ *defined by formula* (0.29) *may be written*

$$\begin{aligned}K_\infty(\sigma) =&\frac{i^N}{(2\pi\sigma_j)^N}\Big(\int_{B(0,1)}\partial_j^N\widehat{R}(\xi)(e^{i\xi.\sigma}-1)d\xi+\int_{\mathbb{S}^{N-1}}\xi_j\partial_j^{N-1}\widehat{R}(\xi)d\xi\\&-\frac{1}{i\sigma_j}\Big(\int_{B(0,1)^c}\partial_j^{N+1}\widehat{R}(\xi)e^{i\xi.\sigma}d\xi+\int_{\mathbb{S}^{N-1}}\xi_j\partial_j^N\widehat{R}(\xi)e^{i\xi.\sigma}d\xi\Big)\Big).\end{aligned} \tag{0.45}$$

Proposition 1 results from the strategy above. The functions R_i of formula (0.34) only depend on the behaviour near the origin of $\widehat{K}$. In turn, this behaviour only depends on the homogeneous polynomial components of lowest degree of the numerator and denominator of $\widehat{K}$, i.e. on $\widehat{R}$. More precisely, we will establish that the functions R_i are identically equal to $\partial_j^{N+i-1}\widehat{R}$. Proposition 1 will then result from equation (0.35).

We now apply Proposition 1 to link the asymptotics of K_0, K_j and $L_{j,k}$ to the asymptotics of R_0^c, $R_{1,j}^c$ and $S_{j,k}^c$.

Corollary 1. *Let* $(j,k,l)\in\{1,\dots,N\}^3$, *and* $\sigma\in\mathbb{S}^{N-1}$ *such that* $\sigma_l\neq 0$. *Then, the functions* $K_{0,\infty}$, $K_{j,\infty}$ *and* $L_{j,k,\infty}$ *respectively may be written*

$$\begin{aligned}K_{0,\infty}(\sigma) =&\frac{i^N}{(2\pi\sigma_l)^N}\Big(\int_{B(0,1)}\partial_l^N\widehat{R_0^c}(\xi)(e^{i\xi.\sigma}-1)d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^{N-1}\widehat{R_0^c}(\xi)d\xi\\&-\frac{1}{i\sigma_l}\Big(\int_{B(0,1)^c}\partial_l^{N+1}\widehat{R_0^c}(\xi)e^{i\xi.\sigma}d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^N\widehat{R_0^c}(\xi)e^{i\xi.\sigma}d\xi\Big)\Big),\end{aligned} \tag{0.46}$$

$$\begin{aligned}K_{j,\infty}(\sigma) =&\frac{i^N}{(2\pi\sigma_l)^N}\Big(\int_{B(0,1)}\partial_l^N\widehat{R_{1,j}^c}(\xi)(e^{i\xi.\sigma}-1)d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^{N-1}\widehat{R_{1,j}^c}(\xi)d\xi\\&-\frac{1}{i\sigma_l}\Big(\int_{B(0,1)^c}\partial_l^{N+1}\widehat{R_{1,j}^c}(\xi)e^{i\xi.\sigma}d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^N\widehat{R_{1,j}^c}(\xi)e^{i\xi.\sigma}d\xi\Big)\Big),\end{aligned} \tag{0.47}$$

and

$$\begin{aligned}L_{j,k,\infty}(\sigma) =&\frac{i^N}{(2\pi\sigma_l)^N}\Big(\int_{B(0,1)}\partial_l^N\widehat{S_{j,k}^c}(\xi)(e^{i\xi.\sigma}-1)d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^{N-1}\widehat{S_{j,k}^c}(\xi)d\xi\\&-\frac{1}{i\sigma_l}\Big(\int_{B(0,1)^c}\partial_l^{N+1}\widehat{S_{j,k}^c}(\xi)e^{i\xi.\sigma}d\xi+\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^N\widehat{S_{j,k}^c}(\xi)e^{i\xi.\sigma}d\xi\Big)\Big).\end{aligned} \tag{0.48}$$

The second step is to compute explicitly the second members of equations (0.46), (0.47) and (0.48). This gives the explicit values of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$.

Proposition 2. *Let $(j,k) \in \{1,\dots,N\}^2$ and $\sigma \in \mathbb{S}^{N-1}$. The functions $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$ are respectively equal to*

$$K_{0,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-3}{2}}c^2}{8\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Big(1-\frac{N\sigma_1^2}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\Big), \tag{0.49}$$

$$K_{j,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Big(\delta_{j,1}\big(1-\frac{c^2}{2}\big)^{-\frac{\delta_{j,1}+1}{2}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}}\sigma_1\sigma_j}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\Big), \tag{0.50}$$

$$L_{j,k,\infty}(\sigma) = \frac{\Gamma(\frac{N}{2})}{2c^2\pi^{\frac{N}{2}}}\Big(\big(1-\frac{c^2}{2}\big)^{\frac{N}{2}}\Big(\frac{\delta_{j,k}(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N(1-\frac{c^2}{2})^{-\delta_{j,1}-\delta_{k,1}+\frac{1}{2}}\sigma_j\sigma_k}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big) - \delta_{j,k} + N\sigma_j\sigma_k\Big). \tag{0.51}$$

Proposition 2 results from formula (0.40). Indeed, by equations (0.38), (0.42) and (0.43), the distributions R_0^c, $R_{1,j}^c$ and $S_{j,k}^c$ are expressed as functions of $R_{j,k}$. Therefore, the computation of the second member of equations (0.46), (0.47) and (0.48) reduces to the computation of the integrals

$$I_{j,k}(\sigma) = \frac{i^N}{(2\pi\sigma_l)^N}\Big(\int_{B(0,1)}\partial_l^N\widehat{R_{j,k}}(\xi)(e^{i\xi.\sigma}-1)d\xi + \int_{\mathbb{S}^{N-1}}\xi_l\partial_l^{N-1}\widehat{R_{j,k}}(\xi)d\xi - \frac{1}{i\sigma_l}\Big(\int_{B(0,1)^c}\partial_l^{N+1}\widehat{R_{j,k}}(\xi)e^{i\xi.\sigma}d\xi + \int_{\mathbb{S}^{N-1}}\xi_l\partial_l^N\widehat{R_{j,k}}(\xi)e^{i\xi.\sigma}d\xi\Big)\Big), \tag{0.52}$$

for every $(j,k) \in \{1,\dots,N\}^2$ and $\sigma \in \mathbb{S}^{N-1}$ (with $l \in \{1,\dots,N\}$ such that $\sigma_l \neq 0$). Actually, we already computed such integrals in [8] in the case $j = k = 1$ (cf Theorem 6 of [8]). The same argument yields the following lemma in the general case.

Lemma 1. *Let $1 \leq j,k \leq N$ and $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_j \neq 0$. Then, the following equality holds*

$$I_{j,k}(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}(\delta_{j,k} - N\sigma_j\sigma_k). \tag{0.53}$$

Lemma 1 finally gives the values of $K_{0,\infty}$, $K_{j,\infty}$ and $L_{j,k,\infty}$. Formulae (0.27), (0.28), (0.36) and (0.37) then give the values of η_∞, θ_∞ and v_∞, which completes the proof of Theorem 2.

4. **Plan of the paper.** The paper splits into three parts. The first part is devoted to the proofs of Proposition 1 and Corollary 1, in which the asymptotic study of K_0, K_j and $L_{j,k}$ is reduced to the study of simplified kernels. The second part deals with the proof of Proposition 2, which brings explicit asymptotics for K_0, K_j and $L_{j,k}$. In the last part, Theorem 2 is deduced from Proposition 2 and formulae (0.27), (0.28), (0.36) and (0.37) by some algebraic computations.

1. Reduction to simplified kernels

In the first part, we reduce the computation of the asymptotics of K_0, K_j and $L_{j,k}$ to the computation of the asymptotics of R_0^c, $R_{1,j}^c$ and $S_{j,k}^c$. This simplification yields integral formulae (0.46), (0.47) and (0.48) of Corollary 1. However, we first compute a similar formula for a more general class of kernels in Proposition 1.

Proof of Proposition 1. Let $\sigma \in \mathbb{S}^{N-1}$ and consider some integer $j \in \{1,\dots,N\}$ such that $\sigma_j \neq 0$. Using notation (0.34), we claim that

Claim 1. *Let $i \in \{0,1,2\}$. The following equality holds for almost every $\xi \in \mathbb{R}^N \setminus \{0\}$,*

$$R_i(\xi) = \partial_j^{N+i-1}\widehat{R}(\xi). \tag{1.1}$$

Proof of Claim 1. Indeed, consider some integer $p \in \mathbb{N}$. By definition (0.44), the function $\widehat{R}$ is a homogeneous rational fraction, so, its partial derivative $\partial_j^p\widehat{R}$ may be written

$$\partial_j^p\widehat{R} = \frac{S_p}{T_p},$$

where S_p and T_p are homogeneous polynomial functions inductively defined by

$$\begin{aligned} S_{p+1} &= \partial_j S_p T_p - S_p \partial_j T_p, \\ T_{p+1} &= T_p^2. \end{aligned} \tag{1.2}$$

Moreover, since Q does not vanish on $\mathbb{R}^N\setminus\{0\}$, T_0 is not identically equal to 0. Therefore, by a straightforward inductive argument, T_p does not identically vanish too, and its degree is equal to $2^p d^o(T_0)$. On the other hand, either S_p vanishes, or its degree is equal to $d^o(S_0) + (2^p - 1)d^o(T_0) - p$.

Now, consider the partial derivative $\partial_j^p \widehat{K}$, and denote by v_p and v'_p, the valuations of the polynomial functions P_p and Q_p defined by formulae (0.31) and (0.32). On one hand, S_0 and T_0 are by definition the homogeneous polynomial components of P and Q of lower degree. Hence, v_0 and v'_0 are respectively equal to $d^o(S_0)$ and $d^o(T_0)$, and with notation (0.30),

$$P_{v_0,0} = S_0 \text{ and } Q_{v'_0,0} = T_0.$$

On the other hand, by inductive equations (0.31) and (0.32), the homogeneous polynomial components of lower degree of P_{p+1} and Q_{p+1} which may not vanish, are respectively equal to $\partial_j P_{v_p,p} Q_{v'_p,p} - P_{v_p,p} \partial_j Q_{v'_p,p}$ and $Q^2_{v'_p,p}$. For the denominator Q_p, it follows from this inductive property, equations (1.2), and the non-vanishing of the polynomial functions T_p that for every $p \in \mathbb{N}$,

$$v'_p = d^o(T_p) = 2^p d^o(T_0), \tag{1.3}$$

$$Q_{v'_p,p} = T_p. \tag{1.4}$$

Likewise, for the numerator P_p, either the polynomial function S_p does not vanish, and consequently, v_p and $P_{v_p,p}$ are respectively equal to $d^o(S_p) = d^o(S_0) + (2^p - 1)d^o(T_0) - p$ and S_p, or S_p is identically equal to 0, and subsequently, v_p is either equal to $-\infty$ or strictly more than $d^o(S_0) + (2^p - 1)d^o(T_0) - p$. In short, we obtain

$$v_p = -\infty \text{ or } v_p \geq d^o(S_0) + (2^p - 1)d^o(T_0) - p. \tag{1.5}$$

Moreover, if $v_p = d^o(S_0) + (2^p - 1)d^o(T_0) - p$, then,

$$P_{v_p,p} = S_p \neq 0. \tag{1.6}$$

Consider finally the set $\Omega_p := \{\xi \in \mathbb{R}^N \setminus \{0\} : S_p(\xi) \neq 0\}$. On one hand, if $S_p = 0$, by assertions (1.5) and (1.6), either P_p is identically equal to 0, or its valuation v_p is strictly more than $d^o(S_0) + (2^p - 1)d^o(T_0) - p$. When $P_p = 0$, we obtain that

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \lim_{R \to +\infty} \frac{\partial_j^p \widehat{K}\left(\frac{\xi}{R}\right)}{R^p} = \partial_j^p \widehat{R}(\xi) = 0. \tag{1.7}$$

Likewise, when $v_p > d^o(S_0) + (2^p - 1)d^o(T_0) - p$, we compute by definition (0.30) and statements (1.3) and (1.4) that for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$R^{-p}\partial_j^p \widehat{K}\Big(\frac{\xi}{R}\Big) = \frac{\sum_{k=v_p}^{d_p} R^{-k} P_{k,p}(\xi)}{\sum_{k=2^p d^o(T_0)}^{d'_p} R^{p-k} Q_{k,p}(\xi)}$$

$$= R^{2^p d^o(T_0)-p-v_p}\Big(\frac{P_{v_p,p}(\xi)}{T_p(\xi)} + \underset{R\to+\infty}{o}(1)\Big).$$

However, since $d^o(S_0) = d^o(T_0)$, we have

$$2^p d^o(T_0) - p - v_p < d^o(T_0) - d^o(S_0) = 0,$$

so,

$$R^{2^p d^o(T_0)-p-v_p} \underset{R\to+\infty}{\to} 0,$$

and assertion (1.7) also holds when $v_p > d^o(S_0) + (2^p - 1)d^o(T_0) - p$. Thus, it holds as soon as $S_p = 0$.

On the other hand, if $S_p \neq 0$, the set Ω_p is the non-vanishing set of a non-vanishing polynomial function. Therefore, Ω_p is a set of full measure. Moreover, we compute by definition (0.30) and assertions (1.3), (1.4) and (1.6) that for every $\xi \in \Omega_p$,

$$R^{-p}\partial_j^p \widehat{K}\Big(\frac{\xi}{R}\Big) = R^{v'_p-v_p-p}\Big(\frac{S_p(\xi)}{T_p(\xi)} + \underset{R\to+\infty}{o}(1)\Big) = \partial_j^p \widehat{R}(\xi) + \underset{R\to+\infty}{o}(1).$$

Hence, we have for almost every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$R^{-p}\partial_j^p \widehat{K}\Big(\frac{\xi}{R}\Big) \underset{R\to+\infty}{\to} \partial_j^p \widehat{R}(\xi). \tag{1.8}$$

Thus, by equation (1.7), assertion (1.8) holds almost everywhere in any case. By choosing p equal to $N+i-1$, and by invoking property (0.33), we conclude that formula (1.1) holds almost everywhere. □

End of the proof of Proposition 1. Proposition 1 follows from equation (0.35) and Claim 1. Indeed, formula (0.45) is a direct consequence of equation (0.35) and assertion (1.1). □

Corollary 1 then specifies the results of Proposition 1 to K_0, K_j and $L_{j,k}$.

Proof of Corollary 1. Indeed, the kernels K_0, K_j and $L_{j,k}$ satisfy all the assumptions of Proposition 1. By formulae (0.23), (0.24) and (0.25), their

Fourier transforms are rational fractions in $\widehat{\mathcal{K}}(\mathbb{R}^N)$ (see the proof of Corollary 3 of [7]), whose denominators only vanish at the origin. Moreover, the degrees of the homogeneous components of lower order of the numerator and denominator of their Fourier transforms are equal. Thus, formula (0.45) holds for K_0, K_j and $L_{j,k}$. This formula gives equations (0.46), (0.47) and (0.48) by noticing that the functions $\widehat{R}$ associated to K_0, K_j and $L_{j,k}$ are respectively $\widehat{R_0^c}$, $\widehat{R_{1,j}^c}$ and $\widehat{S_{j,k}^c}$. □

2. Explicit asymptotics of the kernels

In order to obtain explicit asymptotics of K_0, K_j and $L_{j,k}$, we now compute explicitly formulae (0.46), (0.47) and (0.48). As mentioned in the introduction, this computation results from explicit expression (0.40) for $R_{j,k}$. This expression gives formula (0.53) of Lemma 1, which yields formulae (0.49), (0.50) and (0.51) of Proposition 2 by standard algebraic computations. Actually, Lemma 1 is reminiscent of [8] where it is proved for $j = k = 1$. However, we mention its proof for the sake of completeness.

Proof of Lemma 1. The proof of Lemma 1 relies on the following lemma which is reminiscent of [8].

Lemma 2 ([8])**.** *Let $1 \leq j \leq N$ and $\lambda > 0$. Consider a tempered distribution f on $\mathbb{R}^N$ such that its Fourier transform belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$. Assume moreover that there exist some integers $1 \leq p \leq m$ and some positive real number A such that*

(i) $\forall \xi \in \mathbb{R}^N \setminus \{0\}, |\widehat{f}(\xi)| \leq A(|\xi|^{-r} + |\xi|^s)$,
(ii) $\forall (k, \xi) \in \{0, \ldots, p\} \times B(0,1), |\xi|^{N-p+k}|\partial_j^k \widehat{f}(\xi)| \leq A$,
(iii) $\partial_j^m \widehat{f} \in L^1(B(0,1)^c)$,
(iv) $\forall k \in \{0, \ldots, m-1\}, \partial_j^k \widehat{f} \in L^{q_{m-k}}(B(0,1)^c)$,

where $r < N$, $s \geq 0$, $1 < q_k < \frac{N}{N-k}$ if $1 \leq k \leq N-1$, and $1 < q_k \leq +\infty$ if $k > N$. Then, the function $x \mapsto x_j^p f(x)$ is continuous on $\Omega_j = \{x \in \mathbb{R}^N : x_j \neq 0\}$ and satisfies for every $x \in \Omega_j$,

$$x_j^p f(x) = \frac{i^p}{(2\pi)^N}\Big((-ix_j)^{p-m} \int_{B(0,\lambda)^c} \partial_j^m \widehat{f}(\xi) e^{ix.\xi} d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^{p-1} \widehat{f}(\xi) d\xi$$
$$+ \sum_{k=p}^{m-1} \frac{(-ix_j)^{p-k-1}}{\lambda} \int_{S(0,\lambda)} \xi_j \partial_j^k \widehat{f}(\xi) e^{ix.\xi} d\xi + \int_{B(0,\lambda)} \partial_j^p \widehat{f}(\xi)(e^{ix.\xi} - 1) d\xi\Big).$$

Lemma 2 commonly yields integral expressions of some tempered distribution f as a function of some partial derivatives of its Fourier transform $\widehat{f}$, which presents the advantage of being known explicitly. On the contrary, in this paper, it will be used to compute the explicit value of some integral expressions like (0.52). Indeed, consider the composed Riesz kernel $R_{j,k}$. By formula (0.26), its Fourier transform $\widehat{R_{j,k}}$ belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$. Moreover, $\widehat{R_{j,k}}$ is a homogeneous rational fraction of degree 0. Therefore, its partial derivative of order α is a homogeneous rational fraction of order $-|\alpha|$. In particular, $R_{j,k}$ satisfies all the assumptions of Lemma 2 with $p = N$, $m = N + 1$, $r = s = 0$ and $q_k = \frac{N}{N+1-k}$ for every $1 \leq k \leq N$. Hence, the function $x \mapsto x_l^p R_{j,k}(x)$ is continuous on $\Omega_l = \{x \in \mathbb{R}^N : x_l \neq 0\}$ for every $l \in \{1, \ldots, N\}$, and satisfies for every $\lambda > 0$ and every $x \in \Omega_l$,

$$
\begin{aligned}
x_l^N R_{j,k}(x) = \frac{i^N}{(2\pi)^N} \Big(& \frac{i}{x_l} \int_{B(0,\lambda)^c} \partial_l^{N+1} \widehat{R_{j,k}}(\xi) e^{ix.\xi} d\xi \\
& + \frac{i}{\lambda x_l} \int_{S(0,\lambda)} \xi_l \partial_l^N \widehat{R_{j,k}}(\xi) e^{ix.\xi} d\xi \\
& + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_l \partial_l^{N-1} \widehat{R_{j,k}}(\xi) d\xi + \int_{B(0,\lambda)} \partial_l^N \widehat{R_{j,k}}(\xi)(e^{ix.\xi} - 1) d\xi \Big).
\end{aligned}
$$

On the other hand, the restriction of $R_{j,k}$ to $\mathbb{R}^N \setminus \{0\}$ may be written, by formula (0.40),

$$
\forall x \in \mathbb{R}^N \setminus \{0\}, R_{j,k}(x) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{j,k}|x|^2 - N x_j x_k}{|x|^{N+2}},
$$

which gives for every $x \in \Omega_l$,

$$
\begin{aligned}
& \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{\delta_{j,k}|x|^2 - N x_j x_k}{|x|^{N+2}} = \frac{i^N}{(2\pi x_l)^N} \Big(\frac{i}{x_l} \int_{B(0,\lambda)^c} \partial_l^{N+1} \widehat{R_{j,k}}(\xi) e^{ix.\xi} d\xi \\
& + \frac{i}{\lambda x_l} \int_{S(0,\lambda)} \xi_l e^{ix.\xi} \partial_l^N \widehat{R_{j,k}}(\xi) d\xi + \frac{1}{\lambda} \int_{S(0,\lambda)} \xi_l \partial_l^{N-1} \widehat{R_{j,k}}(\xi) d\xi \\
& + \int_{B(0,\lambda)} \partial_l^N \widehat{R_{j,k}}(\xi)(e^{ix.\xi} - 1) d\xi \Big).
\end{aligned}
$$

By writing $x = R\sigma$, where $R > 0$ and $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_l \neq 0$, and choosing $\lambda = \frac{1}{R}$, the change of variables $u = R\xi$ leads to

$$\frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(\delta_{j,k} - N\sigma_j\sigma_k\Big) = \frac{i^N}{(2\pi\sigma_l)^N}\Big(\frac{i}{\sigma_l}\int_{B(0,1)^c} \frac{\partial_l^{N+1}\widehat{R_{j,k}}(\frac{u}{R})}{R^{N+1}} e^{i\sigma.u} du + \int_{\mathbb{S}^{N-1}} \frac{\partial_l^{N-1}\widehat{R_{j,k}}(\frac{u}{R})}{R^{N-1}} u_l du + \frac{i}{\sigma_l}\int_{\mathbb{S}^{N-1}} \frac{\partial_l^{N}\widehat{R_{j,k}}(\frac{u}{R})}{R^{N}} e^{i\sigma.u} u_l du + \int_{B(0,1)} \frac{\partial_l^{N}\widehat{R_{j,k}}(\frac{u}{R})}{R^{N}} (e^{i\sigma.u} - 1) du\Big). \tag{2.1}$$

However, the partial derivative of order α of $\widehat{R_{j,k}}$ is a homogeneous rational fraction of degree $-|\alpha|$. Therefore, for every $n \in \mathbb{N}$ and $u \in \mathbb{R}^N \setminus \{0\}$,

$$\frac{\partial_l^n \widehat{R_{j,k}}(\frac{u}{R})}{R^n} = \partial_l^n \widehat{R_{j,k}}(u).$$

Consequently, by definition (0.52) and equation (2.1), formula (0.53) holds, which completes the proof of Lemma 1. □

Proposition 2 then follows from Lemma 1 by some algebraic computations.

Proof of Proposition 2. Let $(j,k,l) \in \{1,\dots,N\}^3$. Consider the tempered distribution $R_{j,k}^c$ whose Fourier transform is

$$\forall \xi \in \mathbb{R}^N \setminus \{0\}, \widehat{R_{j,k}^c}(\xi) := \frac{\xi_j\xi_k}{2|\xi|^2 - c^2\xi_1^2}. \tag{2.2}$$

The function $\widehat{R_{j,k}^c}$ is a homogeneous rational fraction of degree 0 only singular at the origin. Therefore, its partial derivative $\partial^\alpha \widehat{R_{j,k}^c}$ is a homogeneous rational fraction of degree $-|\alpha|$, which is smooth on $\mathbb{R}^N \setminus \{0\}$. Consequently, the functions $\xi \mapsto \partial_l^{N+1}\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}$ and $\xi \mapsto \partial_l^N \widehat{R_{j,k}^c}(\xi)(e^{i\sigma.\xi} - 1)$ belong to $L^1(B(0,1)^c)$, respectively $L^1(B(0,1))$, for every $\sigma \in \mathbb{S}^{N-1}$. Thus, the function $I_{j,k}^c$ defined by

$$I_{j,k}^c(\sigma) := \frac{i^N}{(2\pi\sigma_l)^N}\Big(\frac{i}{\sigma_l}\int_{B(0,1)^c} \partial_l^{N+1}\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}d\xi + \frac{i}{\sigma_l}\int_{\mathbb{S}^{N-1}} \xi_l\partial_l^N\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}d\xi + \int_{\mathbb{S}^{N-1}} \xi_l\partial_l^{N-1}\widehat{R_{j,k}^c}(\xi)d\xi + \int_{B(0,1)} \partial_l^N\widehat{R_{j,k}^c}(\xi)(e^{i\sigma.\xi} - 1)d\xi\Big), \tag{2.3}$$

is well defined for every $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_l \neq 0$. Moreover, we claim that

Claim 2. *Let $(j,k,l) \in \{1,\dots,N\}^3$ and $\sigma \in \mathbb{S}^{N-1}$ such that $\sigma_l \neq 0$. Then,*

$$I_{j,k}^c(\sigma) = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{\frac{N-1-\delta_{j,1}-\delta_{k,1}}{2}}}{4\pi^{\frac{N}{2}}(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Big(\delta_{j,k} - \big(1-\frac{c^2}{2}\big)^{1-\frac{\delta_{j,1}+\delta_{k,1}}{2}}\frac{\sigma_j\sigma_k}{1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2}}\Big). \tag{2.4}$$

Proof of Claim 2. By definitions (0.26) and (2.2), we compute

$$\forall \xi = (\xi_1,\xi_\perp) \in \mathbb{R}^N \setminus \{0\}, \widehat{R_{j,k}^c}(\xi) = \frac{1}{2(1-\frac{c^2}{2})^{\frac{\delta_{j,1}+\delta_{k,1}}{2}}}\widehat{R_{j,k}}\Big(\sqrt{1-\frac{c^2}{2}}\xi_1,\xi_\perp\Big). \tag{2.5}$$

Therefore, the first integral of $I_{j,k}^c(\sigma)$ may be written

$$\begin{aligned}
&\int_{B(0,1)^c} \partial_l^{N+1}\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}d\xi \\
&= \frac{\big(1-\frac{c^2}{2}\big)^{\frac{N\delta_{j,1}-\delta_{k,1}}{2}}}{2}\int_{B(0,1)^c} \partial_l^{N+1}\widehat{R_{j,k}}\Big(\sqrt{1-\frac{c^2}{2}}\xi_1,\xi_\perp\Big)e^{i\sigma.\xi}d\xi \\
&= \frac{\big(1-\frac{c^2}{2}\big)^{\frac{N\delta_{j,1}-\delta_{k,1}-1}{2}}}{2}\int_{|\xi|^2-\frac{c^2}{2}|\xi_\perp|^2>1-\frac{c^2}{2}} \partial_l^{N+1}\widehat{R_{j,k}}(\xi)e^{ir_\sigma\sigma'.\xi}d\xi,
\end{aligned}$$

where

$$r_\sigma = \sqrt{\frac{2-c^2+c^2\sigma_1^2}{2-c^2}}, \tag{2.6}$$

and

$$\sigma' = \frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{1}{2}}}\Big(\sigma_1,\sqrt{1-\frac{c^2}{2}}\sigma_\perp\Big). \tag{2.7}$$

However, the function $\partial_l^{N+1}\widehat{R_{j,k}}$ is a homogeneous rational fraction of degree $-N-1$, so, the change of variables $u = r_\sigma\xi$ gives

$$\begin{aligned}
&\frac{i^{N+1}}{(2\pi)^N\sigma_l^{N+1}}\int_{B(0,1)^c} \partial_l^{N+1}\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}d\xi \\
&= \frac{i^{N+1}(1-\frac{c^2}{2})^d}{(\pi r_\sigma)^N(2\sigma_l')^{N+1}}\int_{\Omega_{c,\sigma}} \partial_l^{N+1}\widehat{R_{j,k}}(u)e^{i\sigma'.u}du,
\end{aligned} \tag{2.8}$$

where $\Omega_{c,\sigma} = \{u \in \mathbb{R}^N : |u|^2 - \frac{c^2}{2}|u_\perp|^2 > r_\sigma^2(1-\frac{c^2}{2})\}$ and $d = -\frac{\delta_{j,1}+\delta_{k,1}+1}{2}$. Likewise, the second integral, by formula (2.5) and the change of variables above, may be written

$$\begin{aligned}&\frac{i^{N+1}}{(2\pi)^N\sigma_l^{N+1}}\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^N\widehat{R_{j,k}^c}(\xi)e^{i\sigma.\xi}d\xi\\&=\frac{i^{N+1}(1-\frac{c^2}{2})^d}{(\pi r_\sigma)^N(2\sigma_l')^{N+1}}\int_{\Lambda_{c,\sigma}}\nu_l(u)\partial_l^N\widehat{R_{j,k}}(u)e^{i\sigma'.u}du,\end{aligned}\tag{2.9}$$

where $\Lambda_{c,\sigma} = \partial\Omega_{c,\sigma}$ and ν_l is the l^{th}-component of the outward normal of $\Lambda_{c,\sigma}$. The third and fourth integrals respectively become by the same arguments,

$$\frac{i^N}{(2\pi\sigma_l)^N}\int_{\mathbb{S}^{N-1}}\xi_l\partial_l^{N-1}\widehat{R_{j,k}^c}(\xi)d\xi=\frac{i^N(1-\frac{c^2}{2})^d}{2^{N+1}(\pi r_\sigma\sigma_l')^N}\int_{\Lambda_{c,\sigma}}\nu_l(u)\partial_l^{N-1}\widehat{R_{j,k}}(u)du,\tag{2.10}$$

and

$$\begin{aligned}&\frac{i^N}{(2\pi\sigma_l)^N}\int_{B(0,1)}\partial_l^N\widehat{R_{j,k}^c}(\xi)(e^{i\sigma.\xi}-1)d\xi\\&=\frac{i^N(1-\frac{c^2}{2})^d}{2^{N+1}(\pi r_\sigma\sigma_l')^N}\int_{\Omega_{c,\sigma}^c}\partial_l^N\widehat{R_{j,k}}(u)(e^{i\sigma'.u}-1)du.\end{aligned}\tag{2.11}$$

Hence, by equations (2.3), (2.8), (2.9), (2.10) and (2.11),

$$\begin{aligned}I_{j,k}^c(\sigma)=&\frac{i^N(1-\frac{c^2}{2})^d}{2^{N+1}(\pi r_\sigma\sigma_l')^N}\Big(\int_{\Omega_{c,\sigma}^c}\partial_l^N\widehat{R_{j,k}}(u)(e^{i\sigma'.u}-1)du\\&+\int_{\Lambda_{c,\sigma}}\nu_l(u)\partial_l^{N-1}\widehat{R_{j,k}}(u)du\\&+\frac{i}{\sigma_l'}\Big(\int_{\Lambda_{c,\sigma}}\nu_l(u)\partial_l^N\widehat{R_{j,k}}(u)e^{i\sigma'.u}du+\int_{\Omega_{c,\sigma}}\partial_l^{N+1}\widehat{R_{j,k}}(u)e^{i\sigma'.u}du\Big)\Big),\end{aligned}$$

so, by integrating by parts,

$$\begin{aligned}I_{j,k}^c(\sigma)=&\frac{i^N(1-\frac{c^2}{2})^d}{2^{N+1}(r_\sigma\sigma_l')^N}\Big(\int_{B(0,1)}\partial_l^N\widehat{R_{j,k}}(u)(e^{i\sigma'.u}-1)du\\&+\int_{\mathbb{S}^{N-1}}\nu_l(u)\partial_l^{N-1}\widehat{R_{j,k}}(u)du\\&+\frac{i}{\sigma_l'}\Big(\int_{\mathbb{S}^{N-1}}\nu_l(u)\partial_l^N\widehat{R_{j,k}}(u)e^{i\sigma'.u}du+\int_{B(0,1)^c}\partial_l^{N+1}\widehat{R_{j,k}}(u)e^{i\sigma'.u}du\Big)\Big).\end{aligned}$$

Finally, definition (0.52) and formula (0.53) of Lemma 1 yield

$$I^c_{j,k}(\sigma) = \frac{(1-\frac{c^2}{2})^d}{2r_\sigma^N} I_{j,k}(\sigma') = \frac{\Gamma(\frac{N}{2})(1-\frac{c^2}{2})^{-\frac{\delta_{j,1}+\delta_{k,1}+1}{2}}}{4r_\sigma^N \pi^{\frac{N}{2}}}(\delta_{j,k} - \sigma'_k\sigma'_l),$$

which is exactly formula (2.4) by definitions (2.6) and (2.7). □

End of the proof of Proposition 2. The proof of formulae (0.49), (0.50) and (0.51) which give the asymptotics of K_0, K_j and $L_{j,k}$ now follows from Claim 2. Indeed, by definitions (0.38),(0.43) and (2.2), the Fourier transforms $\widehat{R^c_0}$ and $\widehat{S^c_{j,k}}$ may be written

$$\widehat{R^c_0}(\xi) = \sum_{j=1}^N \widehat{R^c_{j,j}}(\xi), \qquad \widehat{S^c_{j,k}}(\xi) = \frac{2}{c^2}\widehat{R_{j,k}}(\xi) - \frac{1}{c^2}\widehat{R^c_{j,k}(\xi)},$$

so, by formulae (0.46), (0.47) and (0.48),

$$\begin{aligned} K_{0,\infty}(\sigma) =& \sum_{l=1}^N I^c_{l,l}(\sigma), \qquad K_{j,\infty}(\sigma) = I^c_{1,j}(\sigma), \\ L_{j,k,\infty}(\sigma) =& \frac{2}{c^2} I_{j,k}(\sigma) - \frac{1}{c^2} I^c_{j,k}(\sigma). \end{aligned}$$

Formulae (0.49), (0.50) and (0.51) then result from formulae (0.53) and (2.4). □

3. Explicit asymptotics of the travelling waves

By formulae (0.49), (0.50) and (0.51), we are now in position to compute v_∞ and to end the proof of Theorem 2.

Proof of Theorem 2. Indeed, equation (0.28) gives the value of the function v_∞ as a function of the integrals of F and G_k on $\mathbb{R}^N$, and of the values of $K_{j,\infty}$ and $L_{j,k,\infty}$. However, we now know all these quantities by formulae (0.36), (0.37), (0.50) and (0.51). This yields formulae (0.16), (0.17) and (0.18) of Theorem 2. Moreover, the same argument (based on formula (0.27) for η_∞) also yields the values of η_∞ and θ_∞ which are equal to

$$\begin{aligned} \eta_\infty(\sigma) = & \frac{c\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}\Big(1-\frac{c^2}{2}\Big)^{\frac{N-3}{2}} \Big(\Big(\frac{4-N}{2}cE(v) + \Big(2+\frac{N-3}{2}c^2\Big)p(v)\Big) \\ & \times \Big(\frac{1}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} - \frac{N\sigma_1^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big) \end{aligned}$$

$$- 2\Big(1 - \frac{c^2}{2}\Big) \sum_{j=2}^{N} P_j(v) \frac{N\sigma_1\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N+2}{2}}}\Big),$$

and

$$\theta_\infty(\sigma) = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \Big(1 - \frac{c^2}{2}\Big)^{\frac{N-3}{2}} \Big(\Big(\frac{4-N}{2} cE(v) + \Big(2 + \frac{N-3}{2}c^2\Big)p(v)\Big)$$
$$\times \frac{\sigma_1}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}} + 2\Big(1 - \frac{c^2}{2}\Big) \sum_{j=2}^{N} P_j(v) \frac{\sigma_j}{(1 - \frac{c^2}{2} + \frac{c^2\sigma_1^2}{2})^{\frac{N}{2}}}\Big).$$

□

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