Advances in Differential Equations Volume 11, Number 4 (2006), 457–480

DERIVATION AND PHYSICAL INTERPRETATION OF GENERAL BOUNDARY CONDITIONS

Gisèle Ruiz Goldstein
Department of Mathematical Sciences, University of Memphis
Memphis, Tennessee 38152

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper we give new derivations of the heat and wave equation which incorporate the boundary conditions into the formulation of the problems. The principle of least action and the inclusion of a kinetic energy contribution on the boundary are used to derive the wave equation together with kinetic boundary conditions. The methods described for both equations admit all of the standard boundary conditions as well as general Wentzell and dynamic boundary conditions; in addition the boundary conditions arise naturally as part of the formulation of the problems. The physical interpretation for general Wentzell boundary conditions is given for both the heat and wave equations.

1. Introduction: The heat equation

The standard derivations of the heat equation are always based on the idea that "heat in equals heat out" over a region. Suppose we consider the heat flow in a metal. Suppose our metal is described by the region $\Omega \subseteq \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathbf{h}$ represent the heat flow vector. Then the amount of heat flowing out of the region is

$$\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} \, dS \tag{1.1}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and dS is the usual surface measure on the boundary. If a heat source is present inside the region, represent it by a function s. Let q be the heat per unit volume. Conservation of heat, when phrased in integral form, says

$$\frac{d}{dt}\int_{\Omega} q \, dx = -\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} \, dS + \int_{\Omega} s \, dx.$$

The term on the left represents the change in heat content in Ω per unit time which must be equal to the flux of heat through the boundary plus the

Accepted for publication: December 2005.
AMS Subject Classifications: 35K20, 35K60, 35L20, 35L70.

contribution by the heat source s in Ω. Using the divergence theorem the above equation can be rewritten in the form

$$\frac{d}{dt}\int_\Omega q\ dx = -\int_\Omega \nabla\cdot\mathbf{h}\ dx + \int_\Omega s\ dx.$$

If q is sufficiently smooth the preceding becomes

$$\int_\Omega q_t\ dx = -\int_\Omega \nabla\cdot\mathbf{h}\ dx + \int_\Omega s\ dx. \tag{1.2}$$

Since (1.2) holds for an arbitrary region Ω, conservation of heat can be written in its differential form

$$q_t = -\nabla\cdot\mathbf{h}\ + s. \tag{1.3}$$

Fourier's law of cooling states that the heat flow is proportional to the temperature gradient. Let $u(x,t)$ represent the temperature at position $x\in\overline{\Omega}$ and time $t\geq 0$. Fourier's law gives the constitutive relationship

$$\mathbf{h} = -\kappa\nabla u, \tag{1.4}$$

where the constant of proportionality κ is called the thermal conductivity, and the minus sign indicates that heat flows from areas of higher temperature towards those of lower temperature. While Fourier's law is not precise, it has been "verified" experimentally for many types of materials such as liquids, gases, metals, and many other materials which conduct heat. It is not a good approximation in substances such as porous media.

Combining (1.3) and (1.4) we obtain

$$q_t = -\nabla\cdot(-\kappa\nabla u) + s = \kappa\ \Delta u + s.$$

If one also makes the assumption that temperature is proportional to the heat content (which is again valid in many materials that conduct heat)

$$q = \rho c u, \tag{1.5}$$

where ρ is the density and c is the heat capacity of the material, we obtain the *heat equation*

$$(\rho c u)_t = \kappa\ \Delta u + s. \tag{1.6}$$

When the density of the material is strictly positive, (1.5) can be written as

$$u_t = \alpha\ \Delta u + \widetilde{s}, \tag{1.7}$$

where $\alpha\,(x) = \frac{\kappa}{\rho(x)c}$ is the so-called diffusion coefficient and $\widetilde{s} = \frac{1}{\rho c}s$. If the source function s depends on the temperature u or its gradient ∇u, (1.6) is a semilinear or quasilinear equation.

Equations of the form (1.6) can also be used to model the diffusion of a chemical in a mixture; in this case Fick's law replaces Fourier's law but nonetheless gives a constitutive relation of the form (1.4). A similar derivation may be given for the mechanics of fluids if we think of temperature being replaced by pressure, heat by the fluid, heat flux by the outflow of fluid, conducting medium by porous medium. The role of the conducting material on the boundary (which will be introduced in the next section) would be replaced by a permeable or semi-permeable membrane.

2. Incorporating the boundary conditions

In the traditional approach equation (1.7) is assumed to hold on the region Ω and the boundary conditions are appended later. There are three standard boundary conditions. The Dirichlet boundary condition specifies the temperature on the boundary

$$u(x,t) = f(x) \text{ for } x \in \partial\Omega \text{ and } t > 0.$$

The Neumann boundary condition specifies the heat flow on the boundary

$$\frac{\partial u}{\partial n}(x,t) = f(x) \quad \text{for } x \in \partial\Omega \text{ and } t > 0.$$

If our material is immersed in a surrounding medium held at temperature T_0, then Newton's law of cooling states that the heat flow through the boundary of Ω is proportional to the temperature difference between the two mediums, that is

$$-\kappa\frac{\partial u}{\partial n} = C(u - T_0) \quad \text{for } x \in \partial\Omega.$$

This is an example of a Robin boundary condition

$$\beta\frac{\partial u}{\partial n}(x,t) + \gamma u(x,t) = f(x) \quad \text{for } x \in \partial\Omega \text{ and } t > 0.$$

The specification of a unique solution to the heat equation also requires an initial condition

$$u(x,0) = u_0(x) \text{ for } x \in \partial\Omega.$$

This method of introducing boundary conditions is ad hoc. It would be more natural if boundary conditions could be derived in the context of the energy balance and constitutive laws given in Section 1. Moreover, neither the standard derivation given in Section 1 nor the standard boundary conditions show how to model a heat source which is located on the boundary. For example suppose we have a metal object described by the region Ω. We wrap the outer surface of the region with conducting material which contains sensors. This material will sense if too much heat is flowing through $\partial\Omega$ or if the temperature becomes too low on $\partial\Omega$, and, if so, the material will heat the surface of our object. In this case the rate of change of heat flow in the region must contain a term to represent the heat generated on the boundary. In particular we should introduce a term of the form

$$\frac{d}{dt}\int_{\partial\Omega} q\ dS.$$

From a mathematical perspective, the boundary is a set of measure zero in $\mathbb{R}^n$, and so whether we consider terms like $\int q_t$ as being integrated over Ω or $\overline{\Omega}$ does not seem significant. But this perspective misses essential connections between the differential equation and the boundary conditions and neglects the contribution of a heat source on the boundary to the total heat content of the region.

Let us rethink the usual derivation of the heat equation. Suppose there is a source on the boundary represented by a function $\phi = \phi(t, x, u, Du, D^2u, \ldots)$. Here $D^ju = \{D^\alpha u : |\alpha| = j\}$ for $j = 1, 2, ...$ Here D^α is the usual multiindex notation for derivatives; if $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_i \in \mathbb{N}_0$, then $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ is a derivative of order $|\alpha| = \alpha_1 + ... + \alpha_n$. (Although in principle ϕ can depend on derivatives of u of third or higher order, we shall restrict our attention to the case of ϕ depending on derivatives of u of at most second order.) The amount of heat leaving the region includes terms like (1.1), but the amount of heat leaving the region must also take into account the action of the heat source on $\partial\Omega$. We use the measure space $(\overline{\Omega}, d\mu)$ which we define as $(\overline{\Omega}, d\mu) = (\Omega,\ dx) \oplus (\partial\Omega,\ dS)$. In this case the rate of change of heat in the region $\overline{\Omega}$ is given by

$$\frac{d}{dt}\int_{\overline{\Omega}} q\ d\mu,$$

while the amount of heat flowing out of the region $\overline{\Omega}$ is still given by (1.1). Again assuming that q is sufficiently smooth and applying the divergence theorem, conservation of heat takes the form

$$\int_{\overline{\Omega}} q_t\ d\mu = -\int_\Omega \nabla\cdot\mathbf{h}\ dx + \int_\Omega s\ dx\ +\ \int_{\partial\Omega}\phi(t, x, u, Du, D^2u)\ dS.$$

Again using Fourier's law and the assumption (1.5) we see

$$\int_\Omega (\rho c u)_t\, dx + \int_{\partial\Omega}(\rho c u)_t\, dS = \int_\Omega \kappa\Delta u\ dx + \int_\Omega s\, dx + \int_{\partial\Omega}\phi\ dS. \tag{2.1}$$

Equating the pieces which hold on Ω and applying the usual argument to show that the equation must hold on any subdomain $\Omega_0 \subset \Omega$, we see that

$$u_t\left(x,t\right) = \alpha\left(x\right)\Delta u\left(x,t\right) + \widetilde{s}\left(x,t,u\right) \qquad \text{in } \Omega \text{ and for } t > 0. \tag{2.2}$$

Then (2.1) reduces to

$$\int_{\partial\Omega} \left[\left(\rho c u\right)_t \ - \phi\right] \, dS = 0. \tag{2.3}$$

Clearly, one sufficient condition for (2.3) to hold is

$$u_t\left(x,t\right) = \frac{1}{\rho\left(x\right)c}\phi(t,x,u,Du,D^2u) \ \text{ on } \partial\Omega \text{ and for } t > 0. \tag{2.4}$$

Next we will show that **all** of the standard boundary conditions can be written in this way for various choices of ϕ. We emphasize that in this derivation the *boundary conditions arise naturally in the formulation of the problem*.

First we consider the case of Dirichlet boundary conditions. Choose

$$\phi \equiv 0.$$

Then

$$u_t \equiv 0 \ \text{ on } \partial\Omega,$$

so $u(x,t) = u_0(x)$ for $t \geq 0,\ x \in \partial\Omega$, where $u_0(x)$ is the initial condition for our problem.

Next we look at Neumann boundary conditions. In this case choose

$$\frac{1}{\rho\left(x\right)c}\phi\left(t,x,u,Du,D^2u\right) = g\left(t\right),$$

where $g\left(t\right)$ is not identically zero. Then

$$u_t\left(x,t\right) = g\left(t\right) \quad \text{on } \partial\Omega,$$

and, if $u\left(x,t\right)$ is sufficiently regular,

$$\left(\nabla u\right)_t = \nabla\left(u_t\right) = 0 \ \text{ on } \partial\Omega.$$

Hence,

$$\nabla u\left(x,t\right) = G\left(x\right) \quad \text{on } \partial\Omega,$$

and consequently,

$$\frac{\partial u}{\partial n}\left(x,t\right) = F\left(x\right) \quad \text{on } \partial\Omega,$$

which is the usual inhomogeneous Neumann boundary condition.

Now let us consider the case of Robin boundary conditions. To that end we choose

$$\frac{1}{\rho c}\phi\left(t,x,u,Du,D^2u\right)=e^{Cr}g\left(t\right)$$

for $C\in\mathbb{R}$ and where r is defined as follows. Let n be the outer unit normal to $\partial\Omega$ at $x\in\partial\Omega$. Let r be a parameter describing the line L which passes through x and contains the vector n such that $r>0$ at all points on $L\cap\Omega$ which are close to x. Then

$$u_t=e^{Cr}g\left(t\right)\ \text{ on }\partial\Omega.$$

Then

$$\frac{\partial u_t}{\partial n}=\left(\frac{\partial u}{\partial n}\right)_t=Ce^{Cr}g(t)\ \text{ on points of } L\cap\partial\Omega \text{ close to } x$$

and

$$\left(\frac{\partial u}{\partial n}\right)_t-Cu_t=0\ \text{ on }\partial\Omega.$$

Hence,

$$\frac{\partial u}{\partial n}-Cu=F\left(x\right)\quad\text{on }\partial\Omega,$$

which is the usual inhomogeneous Robin boundary condition.

In summary the preceding arguments show that when

$$\frac{1}{\rho\left(x\right)c}\phi\left(t,x,u,Du,D^2u\right)=e^{Cr}g\left(t\right),$$

we may obtain all of the standard inhomogeneous boundary conditions. In particular

$$C=\begin{cases}-\infty & \text{Dirichlet boundary conditions}\\ 0 & \text{Neumann boundary conditions}\\ \text{negative} & \text{Dissipative Robin boundary conditions}\\ \text{positive} & \text{Nondissipative Robin boundary condtions.}\end{cases}$$

3. Heat Sources on the Boundary and General Wentzell Boundary Conditions

If there is a heat source acting on the boundary of our region, the source should depend on both the temperature u on the boundary and the heat flux $\frac{\partial u}{\partial n}$ across the boundary. If we assume that dependence is linear in both u and $\frac{\partial u}{\partial n}$, we have

$$\frac{1}{\rho\left(x\right)c}\phi\left(t,x,u,Du,D^2u\right)=-b\left(x\right)\frac{\partial u}{\partial n}\left(x,t\right)+c\left(x\right)u\left(x,t\right)$$

where, as we shall see in the sequel, $b(x) > 0$ represents a heat source and $b(x) < 0$ represents a sink on the boundary. Thus, our boundary condition (2.4) becomes

$$u_t(x,t) = -b(x)\frac{\partial u}{\partial n}(x,t) + c(x)\,u(x,t) \quad \text{on } \partial\Omega. \tag{3.1}$$

For simplicity in the discussion we assume that $s \equiv 0$ in this section.

If u is sufficiently regular and since the differential equation (2.2) holds on $\overline{\Omega}$, we can rewrite the boundary condition as

$$\alpha(x)\,\Delta u + b(x)\frac{\partial u}{\partial n} + c(x)\,u = 0 \quad \text{on } \partial\Omega; \tag{3.2}$$

these are the general Wentzell boundary conditions which were introduced in the context of the heat equation by A. Favini, G. Goldstein, J. Goldstein, and S. Romanelli [10] and subsequently have been studied by many authors (cf. [8], [9], [24], [28], [29], [21]). A recent paper by J.L. Vazquez and E. Vitillaro [26] studied the heat equation with general Wentzell boundary conditions in the form (3.1) where $\alpha \equiv 1$, $c \equiv 0$ and $b(x)$ is constant and $b < 0$. In this case it was shown that the problem is well posed in one dimension and ill posed in two or more dimensions. Hence the model derived in the preceding discussion leads to an ill-posed problem if one has a mechanism on the boundary which removes heat from the region. In the context of control theory, the heat equation with boundary conditions related to (3.1) was introduced by Duvaut and Lions [6].

Another situation we wish to model is when the boundary element contains a thermostat which operates to keep the temperature on the boundary within certain specified bounds. More specifically, suppose the thermostat works to decrease the temperature $u(x,t)$ if $u(x,t) > T_1$, while if the temperature gets too low, say $u(x,t) < T_0$, the thermostat switches the boundary heat source on so that more heat is fed into the system. If the temperature on the boundary stays within certain bounds, $T_0 < u(x,t) \le T_1$, then the thermostat does nothing to change the temperature. Let $-\widetilde{\beta}(u)$ represent the action of the thermostat where

$$\widetilde{\beta}(r) = \begin{cases} C_1 & \text{if} \quad r < T_0 \\ 0 & \text{if} \quad T_0 < r < T_1 \\ C_2 & \text{if} \quad r > T_1. \end{cases}$$

If we fill in the gaps in the graph of $\widetilde{\beta}$, we obtain the graph β where

$$\beta(r) = \begin{cases} C_1 & \text{if } \; r < T_0 \\ [C_1, 0] & \text{if } \; r = T_0 \\ 0 & \text{if } \; T_0 < r < T_1 \\ [0, C_2] & \text{if } \; r = T_1 \\ C_2 & \text{if } \; r > T_1, \end{cases}$$

where $C_1 < 0 < C_2$. Clearly, β is a maximal monotone graph and β is nondecreasing, i.e., $x_1 \leq x_2$ if $x_i \in \beta(r_i)$ and $r_1 < r_2$. In this case our boundary condition becomes

$$u_t + b(x) \frac{\partial u}{\partial n}(x,t) \in -\beta(x, u(x,t)).$$

Notice that if $b(x) \equiv 0$ and if $u(x,t)$ gets too high at some $x \in \partial\Omega$, then β "switches on", and we see that $u_t < 0$ at those points; that is the thermostat is working to decrease the temperature. Again using the fact that the differential equation must hold on $\overline{\Omega}$ (if u is sufficiently regular), we obtain

$$\alpha(x) \Delta u(x,t) + b(x) \frac{\partial u}{\partial n}(x,t) \in -\beta(x, u(x,t)) \quad \text{on } \partial\Omega. \tag{3.3}$$

These are the nonlinear general Wentzell boundary conditions which were introduced by A. Favini, G. Goldstein, J. Goldstein and S. Romanelli [11]. A related nonlinear boundary condition was studied in [27]. More generally, we can allow β to be any nonlinear function or graph in this derivation, although some restrictions will be needed to assure that the resulting problem is well posed.

If we choose

$$\frac{1}{\rho(x) c} \phi\left(t, x, u, Du, D^2 u\right) = a(x) \Delta_{LB} u,$$

where Δ_{LB} is the Laplace-Beltrami operator on $\partial\Omega$ and a is a nonnegative function on $\partial\Omega$, then the boundary conditions (2.4) take the form

$$u_t = a(x) \Delta_{LB} u. \tag{3.4}$$

Since the Laplace-Beltrami operator incorporates tangential derivatives on the boundary, the boundary condition (3.4) may allow for heat flow along the boundary. More generally, we may combine the effects of the above boundary terms if we choose

$$\frac{1}{\rho(x) c} \phi\left(t, x, u, Du, D^2 u\right) = a(x) \Delta_{LB} u - b(x) \frac{\partial u}{\partial n}(x,t) + c(x) u(x,t).$$

This leads to the boundary condition

$$u_t = a(x)\,\Delta_{LB} u - b(x)\,\frac{\partial u}{\partial n}(x,t) + c(x)\,u(x,t) \tag{3.5}$$

which accounts for a heat source on the boundary that can depend on the heat flow along the boundary, the heat flux across the boundary and the temperature at the boundary. The initial-value problem for the heat equation (2.2) with the boundary condition (3.5) is studied by A. Favini, G. Goldstein, J. Goldstein and S. Romanelli [14].

4. Effect of a boundary heat source

Here we give a physical interpretation of the effect of a heat source on the boundary. We begin with the boundary condition (3.1); recall that in this case the heat source depends linearly on both the heat flux and the temperature. Let us begin with the linear general Wentzell boundary conditions in the form

$$u_t + b\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

We work in an infinitesimal region on the boundary. Choose a point $x \in \partial\Omega$ and let $B_\epsilon(x)$ be the ball of radius ϵ about x. Since $\partial\Omega$ is regular, we can choose a coordinate system for $B_\epsilon(x) \cap \overline{\Omega}$ so that the boundary of $B_\epsilon(x) \cap \overline{\Omega}$ in the transformed coordinate system is flat, and x is mapped to $\widetilde{x} = (x_1, ..., x_{n-1}, 0)$, that is, the boundary, at least locally near x, lies in the hyperplane $x_n = 0$. Then the outward unit normal to $\partial\Omega$ at x is the unit vector in the direction of e_n. We will use r as a coordinate on the line containing e_n and denote $\nabla u \cdot e_n$ as $\frac{\partial u}{\partial r} = u_r$. Then locally near x, the boundary condition becomes

$$u_t + bu_r = 0 \tag{4.2}$$

in this transformed local coordinate system. Notice that (4.2) is a one-dimensional unidirectional wave equation whose general solution is given by

$$u(\widetilde{x}, t) = \widetilde{F}(r - bt)$$

for any function $\widetilde{F}$. This is a unidirectional *"heat wave"* which travels **into** the region Ω if $b > 0$. Of course, this wave lives only for an infinitesimally short time, since, once the heat is inside the region, diffusion is the primary process. Mapping back to our original coordinate system, we see that

$$u(x,t) = F(x - btn)$$

is the solution of (4.1). This solution shows that an exterior heat source on the boundary that depends linearly on the heat flux can be thought of as sending a "wave" of heat into an infinitesimal layer near the boundary. Once inside, the diffusion takes over and the wave will cease to exist after some small time.

If we consider the more general linear Wentzell boundary condition

$$u_t + b\frac{\partial u}{\partial n} + cu = 0 \text{ on } \partial\Omega,$$

a similar argument shows that a heat wave of the form

$$u(x,t) = e^{-ct}F(x - btn)$$

is sent into an infinitesimal layer around each point of the boundary. The term e^{-ct} corresponds to some damping if $c > 0$, but since this holds for only a very small time its effect seems not to be of great physical significance. Thus again we see that the effect of the boundary condition (3.1), when the heat source depends linearly on both the heat flux across and the temperature on the boundary, is to send a "heat wave" into the region into an infinitesimal layer near the boundary.

5. Vibrating membranes, the principle of least action and Wentzell boundary conditions

The principle of least action can be used to derive the equation of motion for a vibrating string (in one space dimension) or a vibrating membrane (in two space dimensions). Here we consider the case of a vibrating membrane in two space dimensions. Let $\Omega \subseteq \mathbb{R}^2$ represent our membrane. We assume that Ω is a bounded, connected domain with C^1 boundary $\partial\Omega$. Let $u(x,y,t)$ represent the displacement of the material at the point (x,y) from the equilibrium position at time t. The **action** $\mathcal{S}$ of the system is defined by

$$\mathcal{S} = \int_{t_0}^{t_1} (KE - PE)\, dt \tag{5.1}$$

where KE represents the kinetic energy of the system, PE represents the potential energy of the system and $0 \le t_0 < t_1 \le +\infty$. The *Principle of Least Action* (also known as Hamilton's Principle) states that the physically correct solution is a solution which minimizes the action (or at least is a critical point for $\mathcal{S}$). In the traditional formulation of the aforementioned problems, it is assumed that there is no contribution to the kinetic energy from boundary terms. This assumption makes sense if there is no displacement and, hence, no motion on the boundary. Usually no accounting for boundary contributions to the potential energy is given in any of the standard derivations. If the role of the boundary contributions in the total energy is discussed at all, it is to consider at most the case of a string in which the ends act as a linear spring (cf. [19]). The latter case gives rise to a physical interpretation for the Robin boundary condition for the wave equation.

Let us consider the case of a string. Unless the ends of the string are fixed (corresponding to Dirichlet boundary conditions), there is movement on the boundary. The physical interpretation of the Neumann boundary conditions is that the ends of the string are attached to a frictionless sleeve which can move. For Robin boundary conditions the traditional description is that the ends of the string are acting as a linear spring, so again we have motion on the boundary. In the cases where there is motion on the boundary, we should account for the contribution to the kinetic energy of the system from that motion.

In this section we wish to derive the equation of motion for a vibrating object (in one or two dimensions) in such a way that the boundary conditions arise naturally as part of the formulation of the problem. In doing so we shall recover all the usual boundary conditions. In addition we will be led to important classes of more general boundary conditions and the physical interpretations of these new boundary conditions.

On our region Ω we assume that the boundary $\partial\Omega$ may be affected by vibrations in the region, and thus, the *boundary itself contributes to the kinetic energy of our system*. A well-designed concert hall is a typical example of such a region; the walls of the hall are specifically designed to interact with the sound waves created by the music. It was proposed by Morse and Ingard [23], and further studied by Beale and Rosencrans [4] and Beale [3], that the acoustics of such a hall may be modelled by considering a region (the interior of the concert hall) whose boundary (the walls) behaves as if there is an infinitesimal harmonic oscillator at each point of the boundary. For such a region the boundary clearly makes a contribution to both the kinetic and potential energy of the system; however the contribution to the kinetic energy from the boundary terms is neglected in the mathematical formulation of the problem. In [3], [4] Beale and Rosencrans did use boundary integrals in their energy norm. However, they did not obtain a derivation which furnishes *both* the equation of motion and the boundary conditions simultaneously from the formulation.

In this section we derive the boundary-value problem for our region when the kinetic energy contribution of the boundary condition is incorporated. In mathematical terms, the relevant space is no longer (Ω, dx), but rather $\left(\overline{\Omega}, d\mu\right)$ where $\overline{\Omega} = \Omega \cup \partial\Omega$ and $d\mu = w_1 dA \otimes w_2 \; ds$. Here $dA = dx \; dy$ is Lebesgue measure on Ω , ds is arc length on $\partial\Omega$, and w_1 (respectively w_2) is a weight function which is continuous and strictly positive on Ω (respectively $\partial\Omega$). Here we consider $n = 2$ dimensions. Obvious changes must be made when $n = 1$.

Consider the transverse motion of a flexible membrane Ω whose boundary $\partial\Omega$ may be affected by the vibrations in the region. Let ρ be the density of the membrane; we need only assume that the density is a bounded positive function on $\overline{\Omega}$. In particular in this derivation we can have one material density in the region Ω, a different density on the boundary of the region and each density may be spatially dependent. Let $u(x, y, t)$ be the displacement of the point (x, y) from equilibrium at time t. The kinetic energy can be described by

$$KE = \frac{1}{2}\iint_{\overline{\Omega}} (\rho u)_t^2 \; d\mu = \frac{1}{2}\iint_{\Omega} (\rho u)_t^2 \; w_1 \; dA + \frac{\gamma}{2}\int_{\partial\Omega} (\rho u)_t^2 \; w_2 \; ds. \quad (5.2)$$

Here

$$\gamma = \begin{cases} 0 & \text{if the contribution to } KE \text{ on the boundary is ignored} \\ 1 & \text{if the contribution of } \partial\Omega \text{ to } KE \text{ is considered.} \end{cases} \quad (5.3)$$

We emphasize that the inclusion of the kinetic energy contribution on the boundary is one of the key observations of this work; we introduce the parameter γ so that we may recover the standard results when $\gamma = 0$ and obtain new boundary conditions when $\gamma = 1$.

The potential energy of the solid has several contributions; let

$$PE = U_1 + U_2 + U_3 + U_4. \quad (5.4)$$

Here U_1 represents the amount of work needed to move the solid from its equilibrium position to the position $u(x, y, t)$, U_2 represents the work needed to move the boundary of the solid, U_3 represents any external body forces with density $f(x, y, t)$ which may be acting in the region, and U_4 represents any surface forces with density $g(x, y, t)$ which may be acting on the boundary of the region.

Let $T(x, y)$ denote the tension in the membrane at the point (x, y), and let ΔA represent the area element associated with the perturbed region $[x_0, x_0 + \Delta x] \times [y_0, y_0 + \Delta y]$. Then the amount of work needed to deform ΔA is the product of the tension and the increase in area under the deformation; more specifically,

$$\begin{aligned} \Delta W &= T(x_0 + \Delta x, y_0 + \Delta y)\sqrt{(\Delta x)^2 + (\Delta u)^2}\sqrt{(\Delta y)^2 + (\Delta u)^2} \\ &\quad - T(x_0 + \Delta x, y_0 + \Delta y)(\Delta x)(\Delta y). \end{aligned}$$

Expanding in a Taylor's series we see that

$$\begin{aligned}\Delta W &= \frac{1}{2}T(x_0+\Delta x, y_0+\Delta y)\Big[\Big(\frac{\Delta u}{\Delta x}\Big)^2+\Big(\frac{\Delta u}{\Delta y}\Big)^2\Big]\Delta x\Delta y+\cdots\\ &= \frac{1}{2}T(x_0,y_0)\left[u_x^2(x_0,y_0,t)+u_y^2(x_0,y_0,t)\right]\Delta x\Delta y+\cdots\end{aligned}$$

where the dots represent terms of higher order. If the deformations are relatively small, the work required to deform the solid may be represented by

$$U_1=\frac{1}{2}\iint_\Omega T(x,y)\left[u_x^2(x,y,t)+u_y^2(x,y,t)\right]\ dx\ dy\ . \tag{5.5}$$

The potential energy U_2 reflects the amount of work required to deform the boundary of the region. For now we leave this in the general form

$$U_2=\int_{\partial\Omega}\phi\left(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},t\right)\ ds. \tag{5.6}$$

Later we will show how specific choices of ϕ lead us to all of the standard boundary conditions, as well as some important nonstandard ones.

The effect of the external forces f and g on the region and the boundary respectively give rise to the potential energy terms

$$U_3=\iint_\Omega f(x,y,t)\,u(x,y,t)\ dx\ dy \tag{5.7}$$

and

$$U_4=\int_{\partial\Omega}g(x,y,t)\ \ u(x,y,t)\ ds. \tag{5.8}$$

Combining (5.2), $(5.5)-(5.8)$, we see that the action $\mathcal{S}=\mathcal{S}[u]$ is given by

$$\begin{aligned}\mathcal{S}[u] &= \int_{t_0}^{t_1}(KE-PE)\ dt\\ &= \frac{1}{2}\int_{t_0}^{t_1}\Big\{\iint_\Omega(\rho u)_t^2\,w_1\ dA\Big\}\ dt+\frac{1}{2}\int_{t_0}^{t_1}\Big\{\gamma\int_{\partial\Omega}(\rho u)_t^2\,w_2\ ds\Big\}\ dt\\ &\quad -\frac{1}{2}\int_{t_0}^{t_1}\Big\{\iint_\Omega T\left[u_x^2+u_y^2\right]\ dA\Big\}\ dt\\ &\quad -\int_{t_0}^{t_1}\Big\{\int_{\partial\Omega}\phi\left(x,y,u,Du,D^2u,t\right)\ ds\Big\}\ dt\\ &\quad -\int_{t_0}^{t_1}\Big\{\iint_\Omega fu\ dA\ \Big\}dt-\int_{t_0}^{t_1}\Big\{\int_{\partial\Omega}gu\ ds\ \Big\}dt.\end{aligned}$$

Rearranging we obtain

$$\begin{aligned}\mathcal{S}[u] &= \frac{1}{2}\int_{t_0}^{t_1}\iint_{\Omega}\Big\{(\rho u)_t^2\ \ w_1 - T\left[u_x^2+u_y^2\right]-2fu\Big\}\ dA\ dt \\ &+\frac{1}{2}\int_{t_0}^{t_1}\int_{\partial\Omega}\Big\{\gamma\,(\rho u)_t^2\ \ w_2-2\phi\left(x,y,u,Du,D^2u,t\right)\Big\}\ ds\ dt \\ &-\int_{t_0}^{t_1}\int_{\partial\Omega} gu\ ds\ dt.\end{aligned}$$

We know from the theory of the calculus of variations (cf. [19]) that a function u can be a minimizer of the action only if the the first variation of $\mathcal{S}$, $\delta\mathcal{S}[u]$, vanishes. As in the standard case for the wave equation we actually compute a necessary condition for a stationary point for the action. Notice that we do not assume that the ends of the string are fixed; this fact complicates the computation of the variation. We now give the details.

The variation $\delta\mathcal{S}$ is the linear part in ϵ of $\mathcal{S}[u+\epsilon\xi]-\mathcal{S}[u]$; here $\xi=\xi(x,y,t)$. Then

$$\begin{aligned}&\mathcal{S}[u+\epsilon\xi]-\mathcal{S}[u]=\frac{1}{2}\int_{t_0}^{t_1}\iint_{\Omega}\left\{\rho(u+\epsilon\xi)_t^2\ w_1-2f(u+\epsilon\xi)\ \right\}\ dA\ dt \\ &-\frac{1}{2}\int_{t_0}^{t_1}\iint_{\Omega} T\left[(u+\epsilon\xi)_x^2+(u+\epsilon\xi)_y^2\right]\ dA\ dt \\ &+\frac{1}{2}\int_{t_0}^{t_1}\int_{\partial\Omega}\left\{\gamma\rho(u+\epsilon\xi)_t^2\ w_2-2g(u+\epsilon\xi)\right\}\ ds\ dt \\ &-\int_{t_0}^{t_1}\int_{\partial\Omega}\phi\left(x,y,u+\epsilon\xi,D\left(u+\epsilon\xi\right),D^2\left(u+\epsilon\xi\right),t\right)\ ds\ dt \\ &-\frac{1}{2}\int_{t_0}^{t_1}\iint_{\Omega}\left\{\rho u_t^2\ w_1-T\left[u_x^2+u_y^2\right]-2fu\right\}\ dA\ dt \\ &+\frac{1}{2}\int_{t_0}^{t_1}\int_{\partial\Omega}\{\gamma\rho u_t^2 w_2-2gu\}\,ds\,dt-\int_{t_0}^{t_1}\int_{\partial\Omega}\phi(x,y,u,Du,D^2u,t)\,ds\,dt;\end{aligned}$$

thus,

$$\begin{aligned}\mathcal{S}[u+\epsilon\xi]-\mathcal{S}[u]&=\epsilon\int_{t_0}^{t_1}\iint_{\Omega}\left\{\rho u_t\xi_t w_1-T[u_x\xi_x+u_y\xi_y]-f\xi\right\}\ dA\ dt \\ &+\epsilon\int_{t_0}^{t_1}\int_{\partial\Omega}\left\{\gamma\rho u_t\xi_t w_2-g\xi\right\}\ ds\ dt\end{aligned}$$

$$- \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \{\phi_u \xi + \phi_{u_x} \xi_x + \phi_{u_y} \xi_y\} \; ds \; dt$$

$$+ \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{\phi_{u_{xx}} \xi_{xx} + \phi_{u_{xy}} \xi_{xy} + \phi_{u_{yy}} \xi_{yy}\right\} \; ds \; dt + o\left(\epsilon\right).$$

For simplicity in the remainder of the exposition we assume no dependence of ϕ on the second partial derivatives of u, that is

$$\phi_{u_{xx}} = \phi_{u_{xy}} = \phi_{u_{yy}} = 0. \tag{5.9}$$

Now if we *choose*

$$w_1\left(x, y\right) = \frac{1}{T\left(x, y\right)},$$

then we have

$$\mathcal{S}\left[u + \epsilon\xi\right] - \mathcal{S}\left[u\right] = \epsilon \int_{t_0}^{t_1} \iint_{\Omega} \frac{\rho}{T} u_t \xi_t \; dA \; dt - \epsilon \int_{t_0}^{t_1} \iint_{\Omega} \left(T \nabla u \cdot \nabla \xi\right) \; dA \; dt$$

$$- \epsilon \int_{t_0}^{t_1} \iint_{\Omega} f\xi \; dA \; dt + \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \gamma \rho u_t \xi_t w_2 \; ds \; dt$$

$$- \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \left[\phi_u - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \phi_{u_{x_i}} + g\right] \xi ds \; dt$$

$$- \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\phi_{u_{x_i}} \xi\right) \; ds \; dt + o\left(\epsilon\right);$$

in the last lines we have replaced x by x_1, y by x_2 and used the identity

$$\phi_{u_{x_i}} \xi_{x_i} = \frac{\partial}{\partial x_i} \left[\phi_{u_{x_i}} \xi\right] - \frac{\partial \phi_{u_{x_i}}}{\partial x_i} \xi. \tag{5.10}$$

Applying the divergence theorem to the second term and using the identity $(pq)_r = pq_r + p_r q$ yields

$$\mathcal{S}[u + \epsilon\xi] - \mathcal{S}[u] = \epsilon \int_{t_0}^{t_1} \iint_{\Omega} \left\{-\frac{\rho}{T} u_{tt} + \nabla \cdot \left(T \nabla u\right) - f\right\} \xi \; dA \; dt$$

$$+ \epsilon \int_{t_0}^{t_1} \iint_{\Omega} \frac{\partial}{\partial t} \left[\frac{\rho}{T} u_t \xi\right] \; dA dt + \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{ - \gamma \rho u_{tt} w_2 - T \frac{\partial u}{\partial n}\right\} \xi \; ds \; dt$$

$$+ \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{\frac{\partial}{\partial t} \left[\gamma \rho u_t \xi\right]\right\} w_2 \, ds \, dt - \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \left[\phi_u - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \phi_{u_{x_i}} + g\right] \xi \, ds \, dt$$

$$- \epsilon \int_{t_0}^{t_1} \int_{\partial\Omega} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\phi_{u_{x_i}} \xi\right) \; ds \; dt + o\left(\epsilon\right).$$

Thus, we have shown that

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \iint_\Omega \left\{ -\frac{\rho}{T}u_{tt} + \nabla \cdot (T\nabla u) - f \right\} \xi \, dA \, dt + \int_{t_0}^{t_1} \iint_\Omega \frac{\partial}{\partial t}[\frac{\rho}{T}u_t\xi] dA \, dt$$
$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{ -\gamma\rho u_{tt}w_2 - T\frac{\partial u}{\partial n} \right\} \xi \, ds \, dt + \int_{t_0}^{t_1} \int_{\partial\Omega} \frac{\partial}{\partial t}(\gamma\rho u_t \xi) \, w_2 \, ds \, dt \tag{5.11}$$
$$- \int_{t_0}^{t_1} \int_{\partial\Omega} \left[\phi_u - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\phi_{u_{x_i}} + g \right] \xi \, ds \, dt - \int_{t_0}^{t_1} \int_{\partial\Omega} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\phi_{u_{x_i}} \xi \right) \, ds \, dt.$$

If we assume that $u(x, y, t)$ is not varied at the initial and final *times*, then we have

$$\xi(x, y, t_0) = \xi(x, y, t_1) = 0, \tag{5.12}$$

and hence using Fubini's theorem we have

$$\int_{t_0}^{t_1} \iint_\Omega \frac{\partial}{\partial t} \left[\frac{\gamma\rho}{T} u_t \xi \right] dx \; dy \; dt = \iint_\Omega \int_{t_0}^{t_1} \frac{\partial}{\partial t} \left[\frac{\gamma\rho}{T} u_t \xi \right] dt \; dx \; dy$$
$$= \iint_\Omega \left[\frac{\gamma\rho}{T} u_t \xi \right] \Big|_{t_0}^{t_1} dx \; dy = 0 \tag{5.13}$$

using (5.12). Similarly,

$$\int_{t_0}^{t_1} \int_{\partial\Omega} \left\{ \frac{\partial}{\partial t} [\gamma\rho u_t \xi] \right\} w_2 \; ds \; dt = 0. \tag{5.14}$$

Using (5.13) and (5.14) in (5.11) yields

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \iint_\Omega \left\{ -\frac{\rho}{T}u_{tt} + \nabla \cdot (T\nabla u) - f \right\} \xi \; dA \; dt \tag{5.15}$$
$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{ -\gamma\rho u_{tt}w_2 - T\frac{\partial u}{\partial n} \right\} \xi \; ds \; dt - \int_{t_0}^{t_1} \int_{\partial\Omega} [\phi_u + g] \, \xi \; ds \; dt$$
$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} \Big(\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \phi_{u_{x_i}} \Big) \xi \; ds \; dt - \int_{t_0}^{t_1} \int_{\partial\Omega} \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\phi_{u_{x_i}} \xi \right) \; ds \; dt.$$

The Principle of Least Action says (5.15) must vanish for the function $u(x, y, t)$ which represents the actual motion of the string. In particular it must vanish when we consider functions ξ which vanish on the boundary of the region (but it must also vanish for other choices of ξ as well). First we consider those ξ which satisfy

$$\xi(x, y, t) = 0 \;\text{ for }\; (x, y) \in \partial\Omega, \; t_0 \le t \le t_1. \tag{5.16}$$

For such ξ, (5.15) reduces to

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \iint_\Omega \left\{ -\frac{\rho}{T} u_{tt} + \nabla \cdot (T \nabla u) - f \right\} \xi \; dx \; dy \; dt. \tag{5.17}$$

Recall that if u minimizes the action, then $\delta\mathcal{S} = 0$; moreover, using the fact that the interval $[t_0, t_1]$ and the function ξ for $(x, y) \in \Omega$, $t_0 < t < t_1$, are arbitrary, we see that

$$u_{tt} = c^2 \nabla \cdot (T \nabla u) + \widetilde{f} \quad \text{in } \Omega, \; t > 0. \tag{5.18}$$

Here $c^2(x, y) = \frac{T(x,y)}{\rho(x,y)} > 0$. This is the usual wave equation in two dimensions (when ρ, T are constants) with an external force represented by $f = \frac{-\widetilde{f}\rho}{T}$.

Next we allow for more general ξ by removing the restriction (5.16). Now for simplicity we *assume*

$$\phi_{u_{x_i}} = 0 \;\; \text{for } i = 1, 2. \tag{5.19}$$

By the argument in the preceding paragraph, we know that a minimizer of the action must satisfy (5.18). It follows that (5.15) must reduce to

$$\delta\mathcal{S} = \int_{t_0}^{t_1} \int_{\partial\Omega} \left\{ -\gamma \rho u_{tt} w_2 - T \frac{\partial u}{\partial n} - g - \phi_u \right\} \xi \; ds \; dt.$$

Choose

$$w_2(x, y) = \frac{1}{T(x, y)}. \tag{5.20}$$

Noting that the interval $[t_0, t_1]$ is arbitrary and that ξ is arbitrary, we see that $\delta\mathcal{S} = 0$ if

$$\gamma \frac{\rho}{T} u_{tt} + T \frac{\partial u}{\partial n} + \phi_u + g = 0 \;\; \text{on } \partial\Omega. \tag{5.21}$$

Hence we have derived the wave equation (5.18) and the boundary condition (5.21) which minimizes the action in the case where the contribution of the boundary to the total kinetic energy of the system is included. We shall henceforth call the boundary conditions (5.21) *kinetic boundary conditions.*

If we look at the traditional choice of $\gamma = 0$ together with $\phi \equiv 0$, then (5.21) becomes

$$\frac{\partial u}{\partial n} = \widetilde{g} \quad \text{on } \partial\Omega,$$

where $\widetilde{g} = -\frac{g}{T}$. This is the usual inhomogeneous Neumann boundary condition for the wave equation (5.18). If again we choose $\gamma = 0$, but we choose $\phi = \phi(u) = \frac{\sigma^2}{2} u^2$, then (5.21) becomes

$$\frac{\partial u}{\partial n} + \omega^2 u = \widetilde{g} \;\; \text{on } \partial\Omega, \tag{5.22}$$

where $\omega^2 = \frac{\sigma^2}{T}$ and $\widetilde{g} = -\frac{g}{T}$. This is the usual inhomogeneous Robin boundary conditions. Hence, we can recover these two standard boundary conditions, but it seems as though we cannot recover the Dirichlet boundary conditions. However, that is not the case. To recover the Dirichlet boundary conditions we take $\gamma = 0$. Recall that for Dirichlet boundary conditions, the function u satisfies $u(x,y,t) = G(x,y,t)$ for $(x,y) \in \partial\Omega$, $t > 0$ with G a given function. But then in the computation of $\delta\mathcal{S}$ we must have $\xi(x,y,t) = 0$ for $(x,y) \in \partial\Omega$ since $u + \epsilon\xi$ must also satisfy the boundary condition $u + \epsilon\xi = G$ on $\partial\Omega$. In this case $\delta\mathcal{S}$ reduces precisely to (5.17), and we obtain only the equation $u = G$ for the case of Dirichlet boundary conditions; there is no boundary contribution to $\delta\mathcal{S}$ in the Dirichlet case. Alternatively the homogeneous Dirichlet boundary condition can be viewed as the limiting case of (5.21) as σ (or ω) becomes infinite.

6. Interpretation of the kinetic boundary conditions

In this section we give a physical interpretation of the boundary condition (5.21) in some special cases. For simplicity in the discussion we will restrict our description to one space dimension and constant density and tension; all of the results have similar interpretations in two dimensions and ρ, T being positive functions. In the one-dimensional case $u(x,t)$ is the displacement of the string at position x and time t; let the string have length L so that $\Omega = [0,L]$ and $\partial\Omega = \{0,L\}$.

Recall that the usual homogeneous Neumann boundary conditions for the wave equation

$$u_x(0,t) = u_x(L,t) = 0 \tag{6.1}$$

have the interpretation that the ends of the string are attached to a frictionless sleeve which moves freely along the boundary. That interpretation is valid because $-Tu_x(0,t)$ (respectively $Tu_x(L,t)$) may be regarded as the force the string exerts at position 0 and time t (respectively position L and time t); thus (6.1) says there is no other force acting on the edges of the string ($g = 0$). The homogeneous Robin boundary conditions

$$\begin{aligned} u_x(0,t) - \omega^2 u(0,t) &= 0 \\ u_x(L,t) + \omega^2 u(L,t) &= 0 \end{aligned} \tag{6.2}$$

are often interpreted as the ends of the string being attached to a spring. (The sign difference occurs because we need the directional derivative in the direction of the outward unit normal which points in opposite directions at $x = 0, L$.) That interpretation comes from the view that (6.2) says the force at the ends $-Tu_x(0,t)$ (respectively $Tu_x(L,t)$) is equal to $-\omega^2 u(0,t)$

(respectively $-\omega^2 u(L,t)$); according to Hooke's law the force exerted by a linear spring at $x = j$ is $F_s = -\omega^2 u(j,t)$ where ω^2 is the spring constant. Note that this corresponds to the choice $\phi = \phi(u) = \frac{1}{2}\omega^2 u^2$. Thus, (6.2) can be interpreted as a simple force balance law for the boundary forces; it is perhaps more accurate to say that the ends of the spring are themselves acting like a spring which obeys Hooke's law.

So how do we interpret the boundary conditions (5.21) which incorporate the kinetic energy effects on the boundary? Consider the one-dimensional version of (5.21) with $\phi = \frac{1}{2}w^2 u^2$ on $\partial\Omega$,

$$\begin{aligned} \frac{\rho}{T} u_{tt}(0,t) + T u_x(0,t) + \omega^2 u(0,t) &= 0 \\ \frac{\rho}{T} u_{tt}(L,t) - T u_x(L,t) + \omega^2 u(L,t) &= 0, \end{aligned}$$

which we write as

$$\begin{aligned} \frac{\rho}{T} u_{tt}(0,t) + \omega^2 u(0,t) &= -T u_x(0,t) \\ \frac{\rho}{T} u_{tt}(L,t) + \omega^2 u(L,t) &= T u_x(L,t). \end{aligned} \tag{6.3}$$

The left side of each equation in (6.3) is the equation for a harmonic oscillator. The interpretation of each equation in (6.3) is that at each point of the boundary there is a spring which is acting as a forced harmonic oscillator, and the force driving the spring is the force exerted on the end of the spring by the tension in the spring itself.

We can also discuss the interpretation of the boundary conditions (5.21) with $\phi(u) = \frac{1}{2}\omega^2 u^2$ in two or more dimensions. We pick a point $x_0 \in \partial\Omega$ and "flatten out" the boundary at that point (which we can do since we assume that the boundary of Ω is at least of class C^1). At x_0 the boundary condition (5.21) is acting like a forced harmonic oscillator which oscillates into the region along the line $\mathcal{L}$ where $\mathcal{L}$ is the line that contains the point x_0 and the normal vector to the boundary at x_0. The harmonic oscillator at this point is driven by the force due to the surface tension of the material at the point x_0.

We emphasize that the motion on the boundary given by the kinetic boundary conditions of the form (5.21) is motion into and out of the region at each point of the boundary along the line containing the normal vector at the point. The kinetic boundary conditions of the form (5.21) do not provide for motion *around* the boundary. In many places in the literature such boundary conditions of the form $u_t = Bu$ have been dubbed dynamic boundary conditions, but this is perhaps a misnomer unless B actually depends on tangential derivatives.

Now consider the total energy $\mathcal{E}$ of the system when $\gamma = 1$ and $w_1 = w_2 = T^{-1}$, and suppose that u is a solution of the wave equation (5.18) with the kinetic boundary conditions (5.21). Then

$$\begin{aligned}\mathcal{E}(t) &= KE + PE = \frac{1}{2}\iint_{\Omega}\left\{\frac{\rho}{T}u_t^2 + T\left[u_x^2 + u_y^2\right] + 2fu\right\} dA \\ &+\frac{1}{2}\int_{\partial\Omega}\left\{\frac{\rho}{T}u_t^2 + 2\phi(x,y,u,u_x,u_y,t) + 2gu\right\} ds.\end{aligned}$$

Upon differentiating we see, if u is assumed to be sufficiently regular,

$$\begin{aligned}\mathcal{E}'(t) &= \iint_{\Omega}\left\{\frac{\rho}{T}u_t u_{tt} + T(u_x u_{xt} + u_y u_{yt}) + f\right\} dA \\ &+ \int_{\partial\Omega}\left\{\frac{\rho}{T}u_t u_{tt} + \phi_u u_t + \phi_{u_x} u_{xt} + \phi_{u_y} u_{yt} + \phi_t + g\right\} ds \\ &= \iint_{\Omega}\left\{\frac{\rho}{T}u_t u_{tt} + T(\nabla u\cdot\nabla u_t) + f\right\} dA + \int_{\partial\Omega}\left\{\frac{\rho}{T}u_t u_{tt} + \phi_u u_t\right\} ds \\ &+ \int_{\partial\Omega}\left\{\sum_{i=1}^{2}\left(\frac{\partial}{\partial x_i}(\phi_{u_{x_i}} u_t) - \frac{\partial\phi_{u_{x_i}}}{\partial x_i}u_t\right) + g\right\} ds \qquad (6.4)\end{aligned}$$

where in the last line we have used (5.10). Now *assume*

$$\phi_t = 0; \qquad (6.5)$$

that is, the boundary potential ϕ does not depend on t explicitly. Using the divergence theorem we obtain

$$\begin{aligned}\mathcal{E}'(t) &= \iint_{\Omega}\left\{\frac{\rho}{T}u_{tt} - \nabla\cdot(T\nabla u) + f\right\} u_t\, dx\, dy \\ &+ \int_{\partial\Omega}\left\{\frac{\rho}{T}u_{tt} + T\frac{\partial u}{\partial n} + \phi_u - \sum_{i=1}^{2}\frac{\partial\phi_{u_{x_i}}}{\partial x_i} + g\right\}u_t\, ds + \int_{\partial\Omega}\nabla\cdot\left(\phi_{u_{x_i}}u_t\right) ds;\end{aligned}$$

hence if we again assume that

$$\phi_{u_{x_i}} = 0 \qquad (6.6)$$

we see that

$$\begin{aligned}\mathcal{E}'(t) &= \iint_{\Omega}\left\{\frac{\rho}{T}u_{tt} - \nabla\cdot(T\nabla u) + f\right\} u_t\, dx\, dy \\ &+ \int_{\partial\Omega}\left\{\frac{\rho}{T}u_{tt} + T\frac{\partial u}{\partial n} + \phi_u + g\right\} u_t\, ds. \qquad (6.7)\end{aligned}$$

The first term in (6.7) vanishes since u solves the wave equation. If u satisfies the kinetic boundary conditions, then the term in brackets on $\partial\Omega$ is zero. So for the wave equation with the boundary condition (5.21), we have that energy is conserved whenever (6.5) and (6.6) hold.

7. Nonconservative forces

A conservative force F can be written as $F = -\nabla\psi$ for some scalar function ψ; ψ is the potential for F. Newton believed that some forces, such as friction, are truly nonconservative forces. The modern view is that all forces are conservative when the complete system is viewed. Energy that is observed on a macroscopic level can appear to be lost if one does not consider what is happening at the molecular level. Friction falls in this category of nonconservative forces. The kinetic energy that appears to be lost due to friction is converted to kinetic energy at the molecular level, and that increase can be measured by looking at the change in temperature.

Nevertheless, it is useful to incorporate the effects of friction into many systems. Since the law which governs friction is extremely complicated at best, we must use empirical laws to approximate the effect of friction. It is known that if the velocity is small, then the friction depends linearly on the velocity

$$F_l = d_l v. \tag{7.1}$$

On the other hand if the velocity of our object is large, then the friction is proportional to the square of the velocity

$$F_q = d_q v^2. \tag{7.2}$$

We will incorporate the effect of friction in the boundary condition for the model we derived in the preceding section. Since the empirical laws (7.1) or (7.2) cannot be obtained from a potential, we cannot build them into our system at the level of the action. Instead we look at the equation (5.18) and the boundary conditions (5.21). If we allow the linear friction (7.1) to act at each point of the boundary, we obtain the boundary condition

$$\rho u_{tt} + d_l u_t + \ T\frac{\partial u}{\partial n} + \phi_u + g = 0 \ \text{ on } \partial\Omega. \tag{7.3}$$

If we consider the large velocity approximation for friction (7.2) , we obtain the boundary condition

$$\rho u_{tt} + d_q u_t^2 + \ T\frac{\partial u}{\partial n} + \phi_u + g = 0 \ \text{ on } \partial\Omega. \tag{7.4}$$

In the special case of each point on the boundary acting as a spring, so that $\phi = \phi(u) = \frac{1}{2}\omega^2 u^2$, (5.21) reduces to

$$\rho u_{tt} + d_l u_t + T\frac{\partial u}{\partial n} + \omega^2 u + g = 0 \quad \text{on } \partial\Omega \tag{7.5}$$

in the case of small velocity, while for large velocity (5.21) becomes

$$\rho u_{tt} + d_q u_t^2 + T\frac{\partial u}{\partial n} + \omega^2 u + g = 0 \text{ on } \partial\Omega. \tag{7.6}$$

Notice that the force which drives the harmonic oscillator is the sum of the force generated by the tension of the material on the boundary and the negative of any external force which may be acting on the boundary.

8. General Wentzell boundary conditions and the wave equation

Using the fact that the solution u must satisfy the equation (5.17) as well as the kinetic boundary condition (5.21) and if u is sufficiently regular, we can rewrite the kinetic boundary condition in the form

$$c^2\nabla\cdot(T\nabla u) + T\frac{\partial u}{\partial n} + \phi_u + G = 0 \quad \text{on } \partial\Omega$$

for $G = (g - f|_{\partial\Omega})$. For the case of each boundary point acting as a harmonic oscillator, $\phi = \phi(u) = \frac{\omega^2}{2}u^2$ and $G \equiv 0$, this becomes

$$c^2\nabla\cdot(T\nabla u) + T\frac{\partial u}{\partial n} + \omega^2 u = 0 \quad \text{on } \partial\Omega$$

or

$$\nabla\cdot(a\nabla u) + b\frac{\partial u}{\partial n} + cu = 0 \quad \text{on } \partial\Omega \tag{8.1}$$

where $a(x,y) = T(x,y)$, $b(x,y) = T(x,y)$ and $c(x,y) = \omega^2$. Different choices of the weight functions w_1 and w_2 would lead to different coefficient functions a, b, and c. In general we assume that $a(x,y,z) > 0$, $b(x,y,z) > 0$, and $c(x,y,z) \geq 0$ and that $a \in C(\overline{\Omega})$, $b, c \in C(\partial\Omega)$. Boundary conditions of the form (8.1) are called general Wentzell boundary conditions and were introduced in the context of the wave equation by A. Favini, G. Goldstein, J. Goldstein, and S. Romanelli [13] in one space dimension. The n-dimensional general Wentzell boundary conditions for the wave equation were introduced and studied by C. Gal, G. Goldstein, and J. Goldstein [17]; in this paper connections between the general Wentzell boundary conditions and the acoustic boundary conditions of Morse and Ingard [23] and Beale and Rosencrans [3], [4] were explored. Further mathematical results for general Wentzell boundary conditions for the wave equation were obtained by [2], [21], [24].

In another paper C. Gal, G. Goldstein and J. Goldstein considered the wave equation [18] with general Wentzell boundary conditions of the form

$$\Delta u + \widetilde{b}\frac{\partial u}{\partial n} + \widetilde{c}u \in -\widetilde{\beta}\,(u_t) \quad \text{on } \partial\Omega$$

for $\widetilde{b} = \frac{b}{a}$ and $\widetilde{c} = \frac{c}{a}$. Here we assume that $\widetilde{\beta}$ is a maximal monotone graph, $\widetilde{\beta}\,(0) \ni 0$, and $\widetilde{\beta}$ is nondecreasing; that is, if $\alpha_i \in \widetilde{\beta}\,(r_i)$ and $r_1 \leq r_2$, then $\alpha_1 \leq \alpha_2$. This boundary condition can be thought of as the kinetic boundary condition

$$u_t + b\frac{\partial u}{\partial n} + cu \in -\beta\,(u_t) \quad \text{on } \partial\Omega; \tag{8.2}$$

where $b = T$, $c = \omega^2$ and the term $\beta\,(u_t)$ represents the friction term. In particular,

$$\beta(r) = \begin{cases} 0 & \text{no friction} \\ d_l r & \text{linear friction} \\ d_q r^2 & \text{high velocity friction} \\ \widetilde{\beta}(r) & \text{nonlinear friction.} \end{cases}$$

In [20] the author considers the problem of kinetic boundary conditions for vibrating beams and plates.

References

[1] D.E. Apushkinskaya and A.I. Nazarov, *The Venttsel problem for nonlinear elliptic equations*, J. Math. Sci., 101 (2000), 2861–2880.

[2] A. Batkai and K.-J. Engel, *Abstract wave equations with generalized Wentzell boundary conditions,* preprint.

[3] J.T. Beale, *Spectral properties of an acoustic boundary conditions,* Indiana Univ. Math. J., 25 (1976), 895–917.

[4] J.T. Beale and S.I. Rosencrans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc., 80 (1974), 1276–1278.

[5] J. Cooper, "Introduction to Partial Differential Equations with MATLAB," Birkhauser, Boston, 1998.

[6] G. Duvaut and J.L. Lions, "Inequalities in Mechanics and Physics," (translation) Springer-Verlag, Heildelberg, Berlin, Germany, 1976

[7] K.-J. Engel, *Second order differential operators on* $C[0,1]$ *with Wentzell-Robin boundary conditions*, in *Evolution Equations: Proceedings in honor of J.A. Goldstein's 60th birthday* (ed. by G. Goldstein, R. Nagel and S. Romanelli), Marcel Dekker (2003), 159–165.

[8] K.-J. Engel, *The Laplacian on* $C\left(\overline{\Omega}\right)$ *with generalized Wentzell boundary conditions,* Arch Math. 81 (2003), 548–558.

[9] K.-J. Engel, *Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions*, preprint (2003).

[10] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The heat equation with general Wentzell boundary conditions*, J. Evol. Eq., 2 (2002), 1–19.
[11] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *Nonlinear boundary conditions for nonlinear second order differential operators on* $C[0,1]$, Arch. Math., 76 (2001), 391–400.
[12] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The heat equation with nonlinear general Wentzell boundary conditions*, submitted.
[13] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The one-dimensional wave equation with Wentzell boundary conditions*, in *Differential Equations and Control Theory* (ed. by S. Aizicovici and N. Pavel), M. Dekker (2001), 193–145.
[14] A. Favini, G.R. Goldstein, J.A. Goldstein, and S. Romanelli, *The heat equation with dynamic boundary conditions*, in preparation.
[15] A. Favini, G.R. Goldstein, J.A. Goldstein, E. Obrecht, and S. Romanelli, *General Wentzell boundary conditions and analytic semigroups on* $W^{1.p}(0,1)$, Appl. Anal., 82 (2003), 927–935.
[16] R.P. Feynman, R. B. Leighton, and M. Sands, "The Feynman Lectures on Physics," Addison-Wesley, Reading Massachusetts, 1977.
[17] C. Gal, G.R. Goldstein, and J.A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Eqns., 3 (2003), 623–635.
[18] C. Gal, G.R. Goldstein, and J.A. Goldstein, *Nonlinear kinetic boundary conditions for acoustic wave equations with friction,* preprint.
[19] I.M. Gelfand and S.V. Fomin, "Calculus of Variations," (translated by R. Silverman), Prentice-Hall, New Jersey, 1963.
[20] G.R. Goldstein, *Derivation and interpretation of kinetic boundary conditions for a vibrating beam*, preprint.
[21] J. Liang and T.-J. Xiao, *A solution to an open problem for wave equations with generalized Wentzell boundary conditions*, Math. Ann., 327 (2003), 351–363.
[22] Y. Luo and N.S. Trudinger, *Linear second order elliptic equations with Venttsel boundary conditions*, Proc. Royal Soc. Edinburgh, 118A (1991), 193–207.
[23] P.M. Morse and K.U. Ingard, "Theoretical Acoustics," Mc-Graw Hill, New York, 1968.
[24] D. Mugnolo, *Abstract wave equations with acoustic boundary conditions*, preprint.
[25] M. Pinsky, "Partial Differential Equations and Boundary Value Problems with Applications," Second Edition, McGraw-Hill, New York, 1991.
[26] J.L Vazquez and E. Vitillaro, *Heat equation with dynamical boundary conditions of reactive type,* preprint.
[27] E. Vitillaro, *Global existence for the heat equation with nonlinear dynamical boundary conditions,* Proc. Royal Soc. Edin., 135A (2005), 1–33.
[28] H. Vogt and J. Voigt, *Wentzell boundary conditions in the context of Dirichlet forms*, Adv. Diff. Eqns., 8 (2003), 821–842.
[29] M. Warma, *Wentzell-Robin boundary conditions on* $C[0,1]$*,* Semigroup Forum, 66 (2003), 162–170.