Advances in Differential Equations Volume 11, Number 4 (2006), 419–456

ASYMPTOTIC BEHAVIOR FOR ABSTRACT WAVE EQUATIONS WITH DECAYING DISSIPATION

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(Submitted by: Yoshikazu Giga)

Abstract. We consider the initial-value problem of the abstract wave equation with dissipation whose coefficient tends to 0 as $t \to \infty$. In the case that the coefficient of the dissipation is a positive constant, Ikehata-Nishihara and Chill-Haraux obtained the decay estimate of the difference between the solution of this equation and the solution of the corresponding abstract heat equation. The purpose of this paper is to show the decay estimate of the difference between the solution of the above equation and the solution of the corresponding abstract parabolic equation. We also show the decay estimate for the solution of this dissipative wave equation.

Introduction

Let H be a separable Hilbert space with norm $\|\cdot\|$. Let A be a non-negative self-adjoint operator with domain $\mathcal{D}(A)$. Then, for a non-negative number γ, the space $\mathcal{D}(A^\gamma)$ becomes a Hilbert space with the graph norm of A^γ denoted by $\|u\|_\gamma = (\|u\|^2 + \|A^\gamma u\|^2)^{1/2}$. We consider the initial-value problem of the abstract dissipative wave equation

$$u'' + b(t)u' + Au = 0, \quad u(0) = u_0, \quad u'(0) = u_1. \tag{0.1}$$

Here $b(t)$ is a C^1 function on $[0, \infty)$ satisfying the following:

$$b_0(1+t)^{-\alpha} \le b(t) \le b_1(1+t)^{-\alpha}, \tag{0.2}$$

$$\left|b'(t)\right| \le b_2(t+1)^{-\alpha-1}, \tag{0.3}$$

where $0 < \alpha < 1$, $b_0, b_1, b_2 > 0$ are constants.

Accepted for publication: January 2006.
AMS Subject Classifications: 35L90, 35L15.
Partly supported by Grant-in-Aid for Scientific Research (C) 17540173, Ministry of Education, Culture, Sports, Science and Technology, Japan.

Remark 1. Though we put the assumption $0 < \alpha < 1$ in the sequel, all results in this paper hold in the special case $b(t) \equiv b_0$, since $b'(t) = 0$ and we can proceed in the same way.

If

$$H = L^2(\Omega),$$

$$A = -\Delta \quad \text{with domain} \quad \mathcal{D}(A) = \{u \in H^2(\Omega);\ u(t,x) = 0 \ \text{ on } \partial\Omega\},$$

where Ω is an exterior domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, or $\Omega = \mathbb{R}^n$, then (0.1) becomes the following initial-boundary-value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b(t)\frac{\partial u}{\partial t} = 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{0.4}$$

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = u_1(x) \text{ in } \Omega, \tag{0.5}$$

$$u(t,x) = 0 \qquad \text{on } [0,\infty) \times \partial\Omega. \tag{0.6}$$

Matsumura [15] and Mochizuki [18] considered this equation with $b(t)$ replaced by $b(t) = b(t,x)$ satisfying

$$b_0(1 + |x| + t)^{-1} \leq b(x,t) \leq b_1 \tag{0.7}$$

for some $b_0, b_1 > 0$, with some assumption on $\partial_t b(x,t)$, and showed the energy decay of the solution of the dissipative wave equation (0.4)–(0.6):

$$\|\nabla u(t)\|^2 + \|\partial_t u(t)\|^2 < C(t+1)^{-\mu}, \tag{0.8}$$

where $\mu \in (0,1]$ is a constant depending on $b(t,x)$ and C is a constant depending on the initial data and $b(t,x)$. Mochizuki-Nakazawa [19], [20] obtained more precise estimates. See also, Uesaka [27] and Hirosawa-Nakazawa [7]. On the other hand, Mochizuki [17] proved that the energy does not decay in general and that every finite energy solution behaves like a solution of the free wave equation as $t \to \infty$, under the assumption that the dissipative term is sufficiently weak in the sense that $0 \leq b(x,t) \leq C(1+|x|)^{-\alpha}$ for some $\alpha > 1$. Hence the energy of the solution of (0.4)–(0.6) decays only when the dissipation is strong enough.

In the case that $b(t)$ is a positive constant, Nishihara [22] ($n = 1$) and Han–Milani [6] considered the Cauchy problem for quasilinear hyperbolic equations in $\mathbb{R}^n$, and showed that the L^∞ norm of the difference between the solution of the quasilinear hyperbolic equation and the solution of the corresponding parabolic equation decays faster than each of the solutions does (diffusion phenomenon). For diffusion phenomenon, there are many results for (0.4)–(0.6) with $b(t) \equiv 1$ and some results for (0.1) with $b(t) \equiv 1$

as follows. Karch [12] obtained the decay estimate of the L^p norm ($2 \leq p < 2n/(n-1)$) of the difference between the solution of (0.4)–(0.5) in $\mathbb{R}^n$ and the self-similar solution of the heat equation for initial data $(u_0, u_1) \in (W_p^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$. Ikehata [8] obtained the decay estimate of the L^2 norm of the difference between the solution of the dissipative wave equation (0.4)–(0.6) and the solution of the heat equation, in exterior domains. Ikehata-Nishihara [11] considered the abstract hyperbolic equation (0.1) and obtained the decay estimate of the difference between the solution u of (0.1) with $b(t) \equiv 1$, and the solution v of the corresponding abstract heat equation

$$v' + Av = 0, \quad v(0) = u_0 + u_1. \tag{0.9}$$

Chill-Haraux [3] improved their estimate to the following one:

$$\|u(t) - v(t)\|_{1/2} \leq Ct^{-1}(\|u_0\|_{1/2} + \|u_1\|) \quad \text{for every } t \geq 1. \tag{0.10}$$

When 0 belongs to the essential A [3] also showed that the estimate is optimal in the sense that

$$\limsup_{t\to\infty} \sup_{\|u_0\|_{1/2}, \|u_1\| \leq 1} t\|u(t) - v(t)\|_{1/2} =: m > 0.$$

(See also Nishihara [23] and Narazaki [21] for L^p-L^q estimates of the difference for the solution of (0.4)–(0.5) with $b(t) \equiv 1$ and the solution of the corresponding heat equation in $\mathbb{R}^n$ with $n = 3$ and $n \geq 2$, respectively.)

The purpose of this paper is to show the decay estimates of the difference between the solution $u(t)$ of (0.1) and the solution $v(t)$ of the corresponding abstract parabolic equation:

$$b(t)v' + Av = 0, \quad v(0) = v_0, \tag{0.11}$$

where

$$v_0 = u_0 + \frac{u_1}{b(0)} - u_1 \int_0^\infty \frac{b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma)ds, \tag{0.12}$$

for a decaying coefficient $b(t)$ of the dissipative term which satisfies the assumption (0.2) and (0.3).

Chill-Haraux [3] obtained the estimate (0.10) as follows: They showed that the restriction of each of the solutions of (0.1) with $b(t) \equiv 1$ and (0.9) for a high frequency region decays exponentially for $t \geq 1$, and estimated the difference between solutions of (0.1) with $b(t) \equiv 1$ and (0.9) for a low frequency region by using an explicit formula of the solution of the dissipative abstract wave equation (0.1). In our situation $b(t)$ decays as $t \to \infty$, and hence the separation of the spectrum depends on t. Separation of this type is used by the author in [26], [29] in order to prove the $H^\infty(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ well posedness for weak hyperbolic equations. Then we show that the restriction of each solution of (0.1) and (0.11) for a high frequency region decays exponentially for $t \geq 1$, as well as [3]. We cannot use their method for the estimate of the difference between the solutions of (0.1) and (0.11)–(0.12) for a low frequency region, since we do not have the direct representation formula of the solution of (0.1) for a general function $b(t)$. Here, we express the solution of (0.1) as $u = u_+ + u_-$, where u_- decays faster than u_+. Then by transforming the equation (0.1) into an integral equation, we give the decay estimate of the difference between u_+ and the solution of the parabolic equation, and the decay estimate of u_- itself.

As an application, we obtain the estimate of the difference between the solution of the initial-boundary-value problem (0.4)–(0.6) and the solution of the corresponding parabolic equation:

$$b(t)\frac{\partial v}{\partial t} - \Delta v = 0 \qquad \text{in } [0,\infty)\times\Omega, \tag{0.13}$$

$$v(0,x) = v_0(x), \qquad \text{in } \Omega, \tag{0.14}$$

$$v(t,x) = 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{0.15}$$

where

$$v_0(x) = u_0(x) + \frac{u_1(x)}{b(0)} - u_1(x)\int_0^\infty \frac{b'(s)}{b(s)^2}\exp(-\int_0^s b(\sigma)d\sigma)ds, \tag{0.16}$$

and Ω is an exterior domain in $\mathbb{R}^n$ or $\mathbb{R}^n$. We can also obtain similar results with the Dirichlet conditions (0.6) and (0.15) replaced by the Robin boundary-value conditions:

$$\frac{\partial u}{\partial \nu}(t,x) + \sigma(x)u(t,x) = 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{0.17}$$

and

$$\frac{\partial v}{\partial \nu}(t,x) + \sigma(x)v(t,x) = 0 \qquad \text{on } [0,\infty)\times\partial\Omega, \tag{0.18}$$

respectively, where ν denotes the outer unit normal to $\partial\Omega$, and $\sigma(x)$ is a non-negative smooth function on $\partial\Omega$. Here we note that if we take $\sigma \equiv 0$, then (0.17) and (0.18) are the Neumann boundary conditions.

We also show the decay estimate of the solution of (0.1) itself. In the case that $b(t)$ is a positive constant, there are many results about the decay of the $L^2(\Omega)$ norm of the solution of (0.4)–(0.6). For the whole domain, Matsumura [14] showed the decay of the $L^2(\mathbb{R}^n)$ norm of the solution of (0.4)–(0.5), provided initial data u_0, u_1 belong to $L^p(\mathbb{R}^n)$ for $1 \leq p < 2$ (see also Kawashima–Nakao–Ono [13] and references therein). For an exterior domain Ω, Ikehata [9] showed the decay of the $L^2(\Omega)$ norm of the solution of (0.4)–(0.6), provided initial data u_0, u_1 belong to $L^{2n/(n+2)}(\Omega)$ (see also Ono [24]). In the case that $b = b(x)(\geq 0)$ such that $b(x) \geq b_0(> 0)$ for large x, Ikehata [10] showed the decay of the $L^2(\Omega)$ norm of the solution of (0.4)–(0.6) for an exterior domain Ω, provided initial data u_0, u_1 belong to some weighted Sobolev spaces. In the case that the coefficient of the dissipative term is possibly close to 0 as $t \to \infty$ or $|x| \to \infty$, [15], [27], [18], [19], [20], [7] showed the decay of the energy of the solution of the wave equation, as is stated above. However, the decay of the $L^2(\mathbb{R}^n)$ norm of the solution itself was not shown for such a dissipative wave equation. We shall show the decay of the norm $\|\cdot\|$ of the solution of (0.1). Then the decay of the $L^2(\mathbb{R}^n)$ norm of the solution of (0.4)–(0.5) with $\Omega = \mathbb{R}^n$ follows as an application. We also obtain estimates of the decay of the energy which are better than the corresponding estimates in [15], [27], [18], [19], [20], [7] provided initial data belongs to homogeneous Sobolev spaces of negative order.

After submitting the first version of this paper, the author was informed that Wirth [28] independently obtained the decay estimate of the difference between the solution of (0.4)–(0.5) and the solution of the parabolic equation (0.13)–(0.14) with (0.16), and the decay estimate of the solution of (0.4)–(0.5) itself, in the case where $\Omega = \mathbb{R}^n$ and $b(t)$ is a monotone function satisfying suitable assumption. He proved them by WKB-representation of the solutions.

Throughout this paper, we assume that $b(t)$ satisfies (0.2) and (0.3), and consider mild solutions of (0.1), the unique existence of which is well known. As is well known, if the initial data (u_0, u_1) belongs to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, then the solution becomes a strong solution.

This paper is organized as follows: In Section 1, we state our main results: we state abstract theorems in Subsection 1.1, and we apply our theorems to the dissipative wave equation in exterior domains, the whole space and the half space in Subsection 1.2. In Section 2, we reduce (0.11) to ordinary differential equations, and separate high frequency parts and low frequency parts for each t. In Section 3, we estimate the difference between the solution of the dissipative wave equation and the solution of the corresponding parabolic equation for low frequency. In Section 4, we estimate each solution of the dissipative wave equation and the corresponding parabolic equation for high frequency. We prove Theorem 1 in Section 5, and prove Theorem 2 in Section 6.

1. Main results

1.1. **Abstract theorems.** We obtain the following theorem concerning the difference between the solution $u(t)$ of (0.1) and the solution $v(t)$ of the corresponding abstract parabolic equation (0.11)–(0.12):

Theorem 1. *Let β and γ be arbitrary non-negative numbers. Let T_0 be an arbitrary positive number. Then there exists a positive constant C depending only on α, β, γ, b_0, b_1, b_2, and T_0 such that the following estimates hold for every $(u_0, u_1) \in (\mathcal{D}(A^{\beta+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^\beta) \cap \mathcal{R}(A^\gamma))$:*

$$\begin{aligned} &\left\| A^\beta(u(t) - v(t)) \right\|_{1/2} \\ &\leq C(t+1)^{\alpha-1-(\alpha+1)(\beta+\gamma)} \left(\|u_0\|_{\beta+1/2} + \|\tilde{u_0}\| + \|u_1\|_\beta + \|\tilde{u}_1\| \right), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\|A^\beta(u'(t) - v'(t))\| \\ &\leq C(t+1)^{\alpha-2-(\alpha+1)(\beta+\gamma)} \left(\|u_0\|_{\beta+1/2} + \|\tilde{u_0}\| + \|u_1\|_\beta + \|\tilde{u}_1\| \right), \end{aligned} \tag{1.2}$$

for every $t \geq T_0$, where u and v are the solution of the equation (0.1) and (0.11)–(0.12) respectively, and $\tilde{u_0}$ and $\tilde{u}_1$ are elements of H such that $u_0 = A^\gamma \tilde{u_0}$ and $u_1 = A^\gamma \tilde{u_1}$ respectively. Here $\mathcal{R}(A^\gamma)$ denotes the range of A^γ.

Remark 2. If 0 is an eigenvalue of A and $\gamma > 0$, then the elements $\tilde{u_0}$ and $\tilde{v_0}$ are not determined uniquely.

Remark 3. Chill-Haraux [3, Theorem 1.3] asserts that

$$\lim_{t\to\infty} t\|u(t) - v(t)\|_{1/2} = 0$$

for each fixed initial data $(u_0, u_1) \in \mathcal{D}(A^{1/2}) \times H$. In the same way, we have the following estimate by using the estimate (1.1),

$$\lim_{t\to\infty} (t+1)^{1-\alpha+(\alpha+1)(\beta+\gamma)} \left\| A^\beta(u(t) - v(t)) \right\|_{1/2} = 0,$$

for each fixed initial data $(u_0, u_1) \in (\mathcal{D}(A^{\beta+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^\beta) \cap \mathcal{R}(A^\gamma))$. Similar estimates hold in Theorem 2 and the corollaries.

Taking $\beta = \gamma = 0$ in Theorem 1, we obtain the following:

Corollary 1. *Let T_0 be an arbitrary positive number. Then there exists a positive constant C depending only on α, b_0, b_1, b_2 and T_0 such that the following estimates hold for every $(u_0, u_1) \in \mathcal{D}(A^{1/2}) \times H$:*

$$\|u(t) - v(t)\|_{1/2} \leq C(t+1)^{\alpha-1} \left(\|u_0\|_{1/2} + \|u_1\| \right), \tag{1.3}$$

$$\|u'(t) - v'(t)\| \le C(t+1)^{\alpha-2}\left(\|u_0\|_{1/2} + \|u_1\|\right), \tag{1.4}$$

for every $t \ge T_0$, where u is the solution of the equation (0.1) *and v is the solution of the equation* (0.11)–(0.12).

Remark 4. When $0 \in \sigma(A)$, we easily see that

$$\limsup_{t\to\infty} \sup_{\|u_0\|_{1/2}, \|u_1\| \le 1} \|v(t)\| = 1.$$

Hence the solution of (0.11)–(0.12) and therefore the solution of (0.1) itself does not satisfy the estimate (1.3). The fact that the difference of the solution of (0.1) and that of (0.11) decays faster than each solution is called 'diffusion phenomenon'.

If the initial data belong to $\mathcal{R}(A^\gamma)$, the solution decays faster accordingly to γ. Namely, we have the following theorem:

Theorem 2. *Let $\tilde\beta$ and γ be arbitrary non-negative numbers. Then there exists a positive constant C depending only on α, $\tilde\beta$, γ, b_0, b_1, and b_2 such that the following estimates hold for every $(u_0, u_1) \in (\mathcal{D}(A^{\tilde\beta+1/2}) \cap \mathcal{R}(A^\gamma)) \times (\mathcal{D}(A^{\tilde\beta}) \cap \mathcal{R}(A^\gamma))$:*

$$\begin{aligned} &\|A^\beta u(t)\| \\ &\le C(t+1)^{-(\alpha+1)(\beta+\gamma)}\left(\|u_0\|_\beta + \|\tilde u_0\| + \|\tilde u_1\|_{\max\{\beta+\gamma-\frac{1}{2},0\}}\right) \qquad (1.5)\\ &\le C(t+1)^{-(\alpha+1)(\beta+\gamma)}\left(\|u_0\|_\beta + \|\tilde u_0\| + \|u_1\|_{\max\{\beta-\frac{1}{2},0\}} + \|\tilde u_1\|\right) \qquad (1.6)\\ &\qquad \text{for every } \beta \text{ such that } 0 \le \beta \le \tilde\beta + \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} &\|A^\beta u'(t)\| \\ &\le C(t+1)^{-1-(\alpha+1)(\beta+\gamma)}\left(\|u_0\|_{\beta+\frac{1}{2}} + \|\tilde u_0\| + \|u_1\|_\beta + \|\tilde u_1\|\right) \qquad (1.7)\\ &\qquad \text{for every } \beta \text{ such that } 0 \le \beta \le \tilde\beta \end{aligned}$$

for every $t \ge 0$, where u is the solution of equation (0.1) *and $\tilde u_0$ and $\tilde u_1$ are elements of H such that $u_0 = A^\gamma \tilde u_0$ and $u_1 = A^\gamma \tilde u_1$ respectively.*

1.2. **Application to dissipative wave equations in exterior domains.** Throughout this subsection, let Ω be an exterior domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ or the whole space or the half space. We apply our abstract theorems to the problems for dissipative wave equations (0.4)–(0.6) and (0.4)–(0.5) with (0.17) in Ω. Although the results in this subsection hold also for a bounded domain Ω, there is no interest in those results, since both solutions (modulo constants for the Neumann boundary condition) of the dissipative wave equation and of the corresponding parabolic equation decay exponentially. In the case that Ω is an exterior domain in $\mathbb{R}^n$ or the whole space or the half space, our results imply the diffusion phenomenon (see Remark 4).

Before stating our results, we introduce some notation. For $s \geq 0$, let

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^n) \right\},$$

with the norm $\|f\|_{H^s} = \|\langle \xi \rangle^s \hat{f}\|_{L^2}$, where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. For $s < \frac{n}{2}$, let

$$\dot{H}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : |\xi|^s \hat{f} \in L^2(\mathbb{R}^n) \right\},$$

with the norm $\|f\|_{\dot{H}^s} = \||\xi|^s \hat{f}\|_{L^2}$ (see Bergh-Löfström [1, Chapter 6] and Bourdaud [2]). Let

$$H^s(\Omega) = \{ f : \exists g \in H^s(\mathbb{R}^n) \text{ such that } g|_\Omega = f \},$$

with the norm $\|f\|_{H^s(\Omega)} = \inf \left\{ \|g\|_{H^s(\mathbb{R}^n)} : g|_\Omega = f \right\}$.

Remark 5. If $0 \leq \gamma < n/2$, Hardy's inequality implies the following continuous embedding:

$$L^p_\mu(\mathbb{R}^n) \subset \dot{H}^{-\gamma}(\mathbb{R}^n) \qquad (\, \gamma = \mu + n(\frac{1}{p} - \frac{1}{2})\,), \tag{1.8}$$

for $p \in (1, 2]$ and $\mu \geq 0$, where

$$L^p_\mu(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n); \|u\|_{L^p_\mu} = \|(1+|x|)^\mu u(x)\|_{L^p} < \infty \}.$$

Let $H = L^2(\Omega)$ and let $A_D = A_R = -\Delta$ with domain

$$\mathcal{D}(A_D) = \{ u \in H^2(\Omega);\ u(t,x) = 0 \ \text{ on } \partial\Omega \},$$
$$\mathcal{D}(A_R) = \{ u \in H^2(\Omega);\ \frac{\partial}{\partial \nu} u(t,x) + \sigma(x) u(t,x) = 0 \ \text{ on } \partial\Omega \},$$

where $\sigma(x)$ is a non-negative smooth function on $\partial\Omega$. Then A_D and A_R become non-negative self-adjoint operators (see Mizohata [16, Chapter 3, section 16] for A_R).

The characterization of the fractional powers of A_D and A_R given by Fujiwara [4] and Grisvard [5] yields the following:

(i) (The Dirichlet boundary condition)

$$\begin{aligned}\mathcal{D}(A_D^\beta) =&\{u \in H^{2\beta}(\Omega);\ (-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ &\qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \tfrac{1}{4}\} \\ &\qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \tfrac{1}{4} \notin \mathbb{N} \cup \{0\}); \\ \mathcal{D}(A_D^\beta) =&\{u \in H^{2\beta}(\Omega);\ (-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ &\qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \tfrac{1}{4}, \\ &\qquad \text{and } \int_\Omega \frac{1}{\zeta(x)} |((-\Delta)^{\beta-\frac{1}{4}} u(x)|^2 dx < \infty\} \\ &\qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \tfrac{1}{4} \in \mathbb{N} \cup \{0\}),\end{aligned}$$

where $\zeta(x)$ denotes the distance from $x \in \mathbb{R}^n$ to $\partial\Omega$.

(ii) (The Robin boundary condition)

$$\begin{aligned}\mathcal{D}(A_R^\beta) =&\{u \in H^{2\beta}(\Omega);\ (\frac{\partial}{\partial\nu} u + \sigma(x))(-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ &\qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \tfrac{3}{4}\} \\ &\qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \tfrac{3}{4} \notin \mathbb{N} \cup \{0\}); \\ \mathcal{D}(A_R^\beta) =&\{u \in H^{2\beta}(\Omega);\ (\frac{\partial}{\partial\nu} u + \sigma(x))(-\Delta)^k u(x) = 0 \text{ on } \partial\Omega \\ &\qquad \text{for every non-negative integer } k \text{ such that } k < \beta - \tfrac{3}{4}, \\ &\qquad \text{and } \int_\Omega \frac{1}{\zeta(x)} |(\frac{\partial}{\partial\zeta} + \sigma(x))(-\Delta)^{\beta-\frac{3}{4}} u(x)|^2 dx < \infty\} \\ &\qquad\quad (\text{for } \beta \geq 0 \text{ such that } \beta - \tfrac{3}{4} \in \mathbb{N} \cup \{0\}).\end{aligned}$$

Taking $A = A_D$ or A_R, Theorem 1 with $\gamma = 0$ yields the decay estimate of the difference between the solutions of the dissipative wave equation and the solution of the corresponding parabolic equation in exterior domains.

Corollary 2. *Let $A = A_D$ or $A = A_R$. Let β be an arbitrary non-negative number. Let T_0 be an arbitrary positive number. Then there exists a positive constant C depending only on α, β, T_0, b_0, b_1 and b_2 such that the following estimates hold for every $(u_0, u_1) \in \mathcal{D}(A^{\max\{\beta+1/2,1\}}) \times \mathcal{D}(A^{\max\{\beta,1/2\}})$:*

$$\begin{aligned}\left\|(-\Delta)^\beta (u(t) - v(t))\right\|_{H^1} &\leq C(t+1)^{\alpha-1-(\alpha+1)\beta} \left(\|u_0\|_{H^{2\beta+1}} + \|u_1\|_{H^{2\beta}}\right) \\ \left\|(-\Delta)^\beta (u'(t) - v'(t))\right\|_{L^2} &\leq C(t+1)^{\alpha-2-(\alpha+1)\beta} \left(\|u_0\|_{H^{2\beta+1}} + \|u_1\|_{H^{2\beta}}\right)\end{aligned}$$

for every $t \ge T_0$*, where* $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{\max\{2\beta+1,2\}-i}(\Omega))$ *is the solution of the equation* (0.4)–(0.5) *and* $v \in C^1((0,\infty); H^2(\Omega)) \cap C([0,\infty); L^2(\Omega))$ *is the solution of the parabolic equation* (0.13)–(0.14) *and* (0.16) *with the boundary conditions* (0.5) *and* (0.15) *in case* $A = A_D$*, and with the boundary conditions* (0.17) *and* (0.18) *in case* $A = A_R$ *respectively.*

Next we consider the problem in the whole space. Let

$$H = L^2(\mathbb{R}^n), \quad A = -\Delta \quad \text{with domain} \quad \mathcal{D}(A) = H^2(\mathbb{R}^n). \tag{1.9}$$

Then $D(A^\gamma) = H^{2\gamma}(\mathbb{R}^n)$ and $\mathcal{R}(A^\gamma) = \dot{H}^{-2\gamma}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $\gamma \ge 0$. Hence, Theorem 1 implies the decay estimate of the difference between the solutions of the dissipative wave equation and the solution of the corresponding parabolic equation in the whole space.

Corollary 3. *Let* β *and* γ *be arbitrary non-negative numbers. Let* T_0 *be an arbitrary positive number. Then there exists a positive constant* C *depending only on* α*,* β*,* γ*,* T_0*,* b_0*,* b_1 *and* b_2 *such that the following estimates hold for every* $(u_0, u_1) \in (H^{2\beta+1}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n)) \times (H^{2\beta}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n))$*:*

$$\begin{aligned}
&\left\|(-\Delta)^\beta (u(t) - v(t))\right\|_{H^1} \\
&\le C(t+1)^{\alpha-1-(\alpha+1)(\beta+\gamma)} \left(\|u_0\|_{H^{2\beta+1}} + \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{H^{2\beta}} + \|u_1\|_{\dot{H}^{-2\gamma}}\right), \\
&\left\|(-\Delta)^\beta (u'(t) - v'(t))\right\|_{L^2} \\
&\le C(t+1)^{\alpha-2-(\alpha+1)(\beta+\gamma)} \left(\|u_0\|_{H^{2\beta+1}} + \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{H^{2\beta}} + \|u_1\|_{\dot{H}^{-2\gamma}}\right),
\end{aligned}$$

for every $t \ge T_0$*, where* $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\beta+1-i}(\mathbb{R}^n))$ *is the solution of the equation* (0.4)–(0.5) *with* $\Omega = \mathbb{R}^n$*, and* $v \in C^1((0,\infty); H^2(\mathbb{R}^n)) \cap C([0,\infty); L^2(\mathbb{R}^n))$ *is the solution of the parabolic equation* (0.13)–(0.14) *and* (0.16) *with* $\Omega = \mathbb{R}^n$*.*

By taking $A = A_D$ or A_R, Theorem 2 with $\gamma = 0$ implies the following estimate of the solution of the dissipative wave equation.

Corollary 4. *Let* $A = A_D$ *or* $A = A_R$*. Let* $\tilde{\beta}$ *be an arbitrary number such that* $\tilde{\beta} \ge \frac{1}{2}$*. Then there exists a positive constant* C *depending only on* α*,* $\tilde{\beta}$*,* b_0*,* b_1 *and* b_2 *such that the following estimates hold for every* $(u_0, u_1) \in \mathcal{D}(A^{\tilde{\beta}+1/2}) \times \mathcal{D}(A^{\tilde{\beta}})$*:*

$$\left\|(-\Delta)^\beta u(t)\right\|_{L^2} \le C(t+1)^{-(\alpha+1)\beta} (\|u_0\|_{H^{2\beta}} + \|u_1\|_{H^{\max\{2\beta-1,0\}}}),$$
$$\text{for every } \beta \text{ such that } 0 \le \beta \le \tilde{\beta} + \tfrac{1}{2},$$

$$\left\|(-\Delta)^{\beta}\partial_t u(t)\right\|_{L^2} \le C(t+1)^{-1-(\alpha+1)\beta}(\|u_0\|_{H^{2\beta+1}} + \|u_1\|_{H^{2\beta}}),$$
$$\textit{for every } \beta \textit{ such that } 0 \le \beta \le \tilde{\beta},$$

for every $t \ge 0$*, where* $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\tilde{\beta}+1-i}(\Omega))$ *is the solution of the equation* (0.4)–(0.5)*, with the boundary condition* (0.5) *in case* $A = A_D$*, and with the boundary condition* (0.17) *in case* $A = A_R$ *respectively.*

In the same way as in Corollary 3, Theorem 2 yields the following decay estimate of the solution of (0.4)–(0.5) with $\Omega = \mathbb{R}^n$.

Corollary 5. *Let* $\tilde{\beta}$ *be an arbitrary number such that* $\tilde{\beta} \ge \frac{1}{2}$*. Let* γ *be arbitrary non-negative numbers. Then there exists a positive constant* C *depending only on* α, β, γ, b_0, b_1 *and* b_2 *such that the following estimates hold for every* $(u_0, u_1) \in (H^{2\tilde{\beta}+1}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n)) \times (H^{2\tilde{\beta}}(\mathbb{R}^n) \cap \dot{H}^{-2\gamma}(\mathbb{R}^n))$*:*

$$\begin{aligned}\left\|(-\Delta)^{\beta}u(t)\right\|_{L^2} \le C(t+1)^{-(\alpha+1)(\beta+\gamma)}(&\|u_0\|_{H^{2\beta}} + \|u_0\|_{\dot{H}^{-2\gamma}} \\ &+ \|u_1\|_{H^{\max\{2\beta-1,0\}}} + \|u_1\|_{\dot{H}^{-2\gamma}}),\end{aligned} \tag{1.10}$$
$$\textit{for every } \beta \textit{ such that } 0 \le \beta \le \tilde{\beta} + \tfrac{1}{2},$$

$$\begin{aligned}\left\|(-\Delta)^{\beta}\partial_t u(t)\right\|_{L^2} \le C(t+1)^{-1-(\alpha+1)(\beta+\gamma)}(&\|u_0\|_{H^{2\beta+1}} \\ &+ \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{H^{2\beta}} + \|u_1\|_{\dot{H}^{-2\gamma}}),\end{aligned} \tag{1.11}$$
$$\textit{for every } \beta \textit{ such that } 0 \le \beta \le \tilde{\beta},$$

for every $t \ge 0$*, where* $u \in \bigcap_{i=0,1,2} C^i([0,\infty); H^{2\tilde{\beta}+1-i}(\mathbb{R}^n))$ *is the solution of the equation* (0.4)–(0.5) *with* $\Omega = \mathbb{R}^n$*.*

Remark 6. If we take $\beta = 0$ in (1.10), we have

$$\|u(t)\|_{L^2} \le C(t+1)^{-(\alpha+1)\gamma}\left(\|u_0\|_{L^2} + \|u_0\|_{\dot{H}^{-2\gamma}} + \|u_1\|_{L^2} + \|u_1\|_{\dot{H}^{-2\gamma}}\right)$$

for every $t \ge 0$, where u is the solution of the equation (0.4)–(0.5) with $\Omega = \mathbb{R}^n$. Hence by (1.8) with γ replaced by 2γ, if u_0 and u_1 belong to $L^p_\mu(\mathbb{R}^n)$ for $p \in (1,2]$ and $\mu \ge 0$ such that $\mu + n(\frac{1}{p} - \frac{1}{2}) < \frac{n}{4}$, we have

$$\|u(t)\|_{L^2} \le C(1+t)^{(\alpha+1)(n/4-n/(2p)-2\mu)}(\|u_0\|_{L^p} + \|u_0\|_{L^2} + \|u_1\|_{L^p} + \|u_1\|_{L^2}).$$

If we take $\mu = 0$, this order is the same in the decay estimate of [14] in the case that $b(t) \equiv$ *constant*.

Remark 7. If we take $\beta = \frac{1}{2}$ in (1.10) and $\beta = 0$ in (1.11), we have the energy decay:

$$\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \tag{1.12}$$

$$\le C(t+1)^{-(\alpha+1)(1+2\gamma)}\left(\|u_0\|_{H^1}+\|u_0\|_{\dot H^{-2\gamma}}+\|u_1\|_{L^2}+\|u_1\|_{\dot H^{-2\gamma}}\right)^2$$

for every $t \ge 0$, where u is the solution of the equation (0.4)–(0.5) with $\Omega = \mathbb{R}^n$. Matsumura [15], Mochizuki [18], Mochizuki-Nakazawa [19], [20], Uesaka [27] showed the polynomial energy decay with order ≤ 1 (see (0.8)). Hirosawa-Nakazawa [7] showed the energy decay (1.12) with $\gamma = 0$, under the assumption that $b = b(t)$ is a non-increasing function with respect to t and satisfies (0.2) and (0.3) with $\alpha b_1 \le b_2$ and $1/2 < \alpha \le 1$. We removed these assumptions for $\gamma = 0$ and $0 < \alpha < 1$. If we take $\gamma > 0$, we obtain decay of the energy faster than previous results.

2. Reduction of the equations to ordinary differential equations

Chill-Haraux [3] derived an ordinary differential equation from (0.1) by using the spectral theorem for self-adjoint operators. We also use the spectral theorem as in [3]. The self-adjoint operator A is unitarily equivalent to a multiplication operator on some L^2 space, by the spectral theorem (see Reed-Simon [25, Theorem VIII.4, page 260]). Namely, we can identify H with $L^2(E, d\mu)$ on a measure space (E, μ) and A with the multiplication operator as follows:

$$(Au)(\xi) = a(\xi)u(\xi) \qquad (\xi \in E, \quad u \in \mathcal{D}(A)),$$

where a is a nonnegative μ-measurable function, and

$$D(A^\gamma) = L^2(E; (1 + a^{2\gamma})d\mu) \quad \text{ for } \gamma \ge 0.$$

Then the equation (0.1) is equivalent to

$$\begin{cases} u''(t,\xi) + b(t)u'(t,\xi) + a(\xi)u(t,\xi) = 0, \\ u(0,\xi) = u_0(\xi), \quad u'(0,\xi) = u_1(\xi), \end{cases} \tag{2.1}$$

for every fixed $\xi \in E$, where $'$ means the derivative with respect to t.

Also, the equation (0.11)–(0.12) is equivalent to

$$\begin{cases} b(t)v'(t,\xi) + a(\xi)v(t,\xi) = 0, \\ v(0,\xi) = v_0(\xi) = u_0(\xi) + \dfrac{u_1(\xi)}{b(0)} \\ \qquad\qquad - u_1(\xi)\displaystyle\int_0^\infty \frac{b'(s)}{b(s)^2}\exp\Big(-\int_0^s b(\sigma)d\sigma\Big)ds, \end{cases} \tag{2.2}$$

for every fixed $\xi \in E$.

Chill-Haraux [3] showed that the restrictions of the solutions of (2.1) and (2.2) with $b(t) \equiv 1$ to the region $\{a(\xi) \ge 1/16\}$ decay exponentially for

$t \geq 1$, and showed the estimate of the difference between solutions of (2.1) and (2.2) with $b(t) \equiv 1$ restricted to the region $\{a(\xi) < 1/16\}$. Here, since $b(t)$ decays as $t \to \infty$, the separating point of the spectrum depends on t.

We define the mapping T from $(0, \infty)$ to $(0, \infty]$ as follows:

$$T(\lambda) = \begin{cases} \left(\frac{3}{16} b_0^2 \lambda^{-1}\right)^{1/2\alpha} - 1 \quad \text{for} \quad 0 < \lambda < \frac{3}{64} b_0^2 (:= \lambda_0), \\ +\infty \qquad \text{for} \quad \lambda = 0. \\ 2^{\frac{1}{\alpha}} - 1 (:= t_0) \qquad \text{for} \quad \lambda \geq \frac{3}{64} b_0^2. \end{cases}$$

Then, T is strictly increasing from $(0, \lambda_0]$ to $[t_0, \infty)$; it has the inverse $\Lambda : [t_0, \infty) \to (0, \lambda_0]$. We extend $\Lambda : [0, \infty) \to (0, \lambda_0]$ by

$$\Lambda(t) = \begin{cases} \lambda_0 \quad \text{for} \quad 0 \leq t \leq t_0 \\ \dfrac{3}{16} b_0^2 (t+1)^{-2\alpha} \quad \text{for} \quad t_0 < t < \infty. \end{cases} \tag{2.3}$$

We define the sets $\mathcal{G}_+$, $\mathcal{G}_-$ and $\mathcal{G}_0$ as follows:

$$\begin{aligned} \mathcal{G}_- :&= \{(t, \xi) \in [0, \infty) \times E; a(\xi) \leq \lambda_0, 0 \leq t < T(a(\xi))\} \\ &= \{(t, \xi) \in [t_0, \infty) \times E; a(\xi) < \Lambda(t)\} \\ &\qquad \cup \{(t, \xi) \in [0, t_0) \times E; a(\xi) \leq \lambda_0\} \\ \mathcal{G}_+ :&= \{(t, \xi) \in [0, \infty) \times E; t \geq T(a(\xi))\} \\ &= \{(t, \xi) \in [t_0, \infty) \times E; \Lambda(t) \leq a(\xi)\} \\ \mathcal{G}_0 :&= \{(t, \xi) \in [0, \infty) \times E; (t, a(\lambda)) \in [0, t_0] \times [\lambda_0, \infty)\}. \end{aligned} \tag{2.4}$$

For each $t \geq 0$, put

$$\mathcal{G}_-(t) := \{\xi \in E; 0 \leq a(\xi) < \Lambda(t)\}, \quad \mathcal{G}_+(t) := \{\xi \in E; a(\xi) \geq \Lambda(t)\}. \tag{2.5}$$

3. Estimate for low frequency

In this section, we estimate the difference between the solutions of (2.1) and (2.2) for $(t, \xi) \in \mathcal{G}_-$.

From the assumption (0.2) and the definition of $T(\lambda)$, it follows that

$$\frac{3}{4} b(t)^2 \geq \frac{3}{4} b_0^2 (t+1)^{-2\alpha} \geq \frac{3}{4} b_0^2 (T(a(\xi)) + 1)^{-2\alpha} = 4a(\xi)$$

for $(t, \xi) \in \mathcal{G}_-$. Thus, we have

$$b(t)^2 - 4a(\xi) \geq \frac{1}{4} b(t)^2 \geq \frac{b_0^2}{4} (t+1)^{-2\alpha}; \tag{3.1}$$

that is,

$$\sqrt{b(t)^2 - 4a(\xi)} \geq \frac{1}{2} b(t) \geq \frac{b_0}{2} (t+1)^{-\alpha}, \tag{3.2}$$

for $(t,\xi) \in \mathcal{G}_-$, and we also have

$$b(t) - \sqrt{b(t)^2 - 4a(\xi)} = \frac{4a(\xi)}{\sqrt{b(t)^2 - 4a(\xi)} + b(t)} \le \frac{4a(\xi)}{b(t)} \le \frac{4a(\xi)}{b_0}(t+1)^\alpha \tag{3.3}$$

for $(t,\xi) \in \mathcal{G}_-$. Put

$$w_+(t,\xi) := u'(t,\xi) + \frac{1}{2}(b(t) + \sqrt{b^2(t) - 4a(\xi)})u(t,\xi). \tag{3.4}$$

$$\begin{aligned} w_-(t,\xi) :&= u'(t,\xi) + \frac{1}{2}(b(t) - \sqrt{b^2(t) - 4a(\xi)})u(t,\xi), \\ &= u'(t,\xi) + \frac{2a(\xi)u(t,\xi)}{b(t) + \sqrt{b^2(t) - 4a(\xi)}}. \end{aligned} \tag{3.5}$$

Here we note that

$$u(t,\xi) = \frac{w_+(t,\xi) - w_-(t,\xi)}{\sqrt{b^2(t) - 4a(\xi)}}, \tag{3.6}$$

$$\begin{aligned} u'(t,\xi) = &\Big(\frac{b(t)}{2\sqrt{b^2(t) - 4a(\xi)}} + \frac{1}{2}\Big)w_-(t,\xi) \\ &- \frac{2a(\xi)}{\sqrt{b^2(t) - 4a(\xi)}(b(t) + \sqrt{b^2(t) - 4a(\xi)})}w_+(t,\xi). \end{aligned} \tag{3.7}$$

We see that (2.1) is equivalent to

$$\begin{cases} w_-'(t,\xi) + b_+(t,\xi)w_-(t,\xi) = \phi_-(t,\xi)(w_+(t,\xi) - w_-(t,\xi)) & t > 0, \\ w_+'(t,\xi) + b_-(t,\xi)w_+(t,\xi) = \phi_+(t,\xi)(w_+(t,\xi) - w_-(t,\xi)) & t > 0, \\ w_-(0,\xi) = u_1(\xi) + \frac{1}{2}(b(0) - \sqrt{b^2(0) - 4a(\xi)})u_0(\xi), \\ w_+(0,\xi) = u_1(\xi) + \frac{1}{2}(b(0) + \sqrt{b^2(0) - 4a(\xi)})u_0(\xi), \end{cases} \tag{3.8}$$

where

$$b_+(t,\xi) := \frac{1}{2}\left(b(t) + \sqrt{b^2(t) - 4a(\xi)}\right), \tag{3.9}$$

$$b_-(t,\xi) := \frac{1}{2}\left(b(t) - \sqrt{b^2(t) - 4a(\xi)}\right) = \frac{2a(\xi)}{b(t) + \sqrt{b^2(t) - 4a(\xi)}} \tag{3.10}$$

$$\phi_\pm(t,\xi) := \frac{b_\pm'(t,\xi)}{\sqrt{b^2(t) - 4a(\xi)}} \tag{3.11}$$

$$= \frac{1}{2}\Big(1 \pm \frac{b(t)}{\sqrt{b^2(t) - 4a(\xi)}}\Big)\frac{b'(t)}{\sqrt{b^2(t) - 4a(\xi)}}. \tag{3.12}$$

Here we note that

$$\phi_-(t,\xi) = -\frac{2a(\xi)b'(t)}{(b^2(t)-4a(\xi))(\sqrt{b^2(t)-4a(\xi)}+b(t))}. \tag{3.13}$$

By assumption (0.2), we have

$$b_+(t,\xi) \geq \frac{1}{2}b(t) \geq \frac{b_0}{2}(1+t)^{-\alpha}, \tag{3.14}$$

$$b_-(t,\xi) \geq \frac{a(\xi)}{b(t)} \geq \frac{a(\xi)}{b_1}(t+1)^{\alpha} \tag{3.15}$$

for every $(t,\xi) \in \mathcal{G}_-$. By (3.2) and (0.3), we have

$$|\phi_+(t,\xi)| \leq \frac{3b_2}{b_0}(t+1)^{-1} \tag{3.16}$$

for every $(t,\xi) \in \mathcal{G}_-$. By (3.13) and (3.2), we obtain

$$|\phi_-(t,\xi)| \leq \frac{8a(\xi)|b'(t)|}{b^3(t)} \leq \frac{8a(\xi)b_2}{b_0^3}(t+1)^{2\alpha-1} \tag{3.17}$$

for every $(t,\xi) \in \mathcal{G}_-$. Put

$$B_\pm(t,\xi) := \int_0^t b_\pm(s,\xi)ds, \tag{3.18}$$

$$\begin{aligned}\Phi_\pm(t,\xi) &:= \int_0^t \phi_\pm(s,\xi)ds \\ &= \frac{1}{2}\left[\log(\tau+\sqrt{\tau^2-4a(\xi)}) \pm \log\sqrt{\tau^2-4a(\xi)}\right]_{b(0)}^{b(t)},\end{aligned}$$

$$\Phi(t,\xi) := (\Phi_+ + \Phi_-)(t,\xi) = \left[\log(\tau+\sqrt{\tau^2-4a(\xi)})\right]_{b(0)}^{b(t)}, \tag{3.19}$$

and put

$$W_+(t,\xi) := \exp(B_-(t,\xi) - \Phi_+(t,\xi))w_+(t,\xi). \tag{3.20}$$

$$W_-(t,\xi) := \exp(B_+(t,\xi) + \Phi_-(t,\xi))w_-(t,\xi). \tag{3.21}$$

Then by (3.15), we have

$$\exp(-B_-(t,\xi)) \leq e^{c_1\lambda_0}\exp(-c_1a(\xi)(t+1)^{\alpha+1}) \tag{3.22}$$

for every $(t,\xi) \in \mathcal{G}_-$, where $c_1 = \frac{1}{b_1(\alpha+1)}$. By using (3.18), $W_+(t,\xi)$ is expressed as

$$W_+(t,\xi) = \frac{h(0,a(\xi))}{h(t,a(\xi))}\exp(B_-(t,\xi))w_+(t,\xi), \tag{3.23}$$

where

$$h(t,\lambda) = \left(b(t)\sqrt{b(t)^2-4\lambda} + b(t)^2 - 4\lambda\right)^{1/2}. \tag{3.24}$$

Here we note that by (3.2), we have

$$h(t,a(\xi)) \geq \frac{1}{2}b(t) \tag{3.25}$$

for every $(t,\xi) \in \mathcal{G}_-$. Equation (3.8) is equivalent to

$$\begin{cases} W_-'(t,\xi) = \phi_-(t,\xi)\exp((B_+ - B_- + \Phi)(t,\xi))W_+(t,\xi) & t \geq 0, \\ W_+'(t,\xi) = -\phi_+(t,\xi)\exp((B_- - B_+ - \Phi)(t,\xi))W_-(t,\xi) & t \geq 0, \\ W_-(0,\xi) = w_-(0,\xi) = u_1(\xi) + b_-(0,\xi)u_0(\xi), \\ W_+(0,\xi) = w_+(0,\xi) = u_1(\xi) + b_+(0,\xi)u_0(\xi). \end{cases} \tag{3.26}$$

From (3.26), it follows that

$$\begin{aligned} W_-(t,\xi) =& w_-(0,\xi) \\ &+ \int_0^t \phi_-(s,\xi)\exp((B_+ - B_- + \Phi)(s,\xi))W_+(s,\xi)ds, \end{aligned} \tag{3.27}$$

$$\begin{aligned} W_+(t,\xi) =& w_+(0,\xi) \\ &- \int_0^t \phi_+(s,\xi)\exp((B_- - B_+ - \Phi)(s,\xi))W_-(s,\xi)ds \\ =& G(t,\xi) + F(t,\xi), \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} G(t,\xi) :=& w_+(0,\xi) \\ &- \int_0^t \phi_+(s,\xi)\exp((B_- - B_+ - \Phi)(s,\xi))ds\, w_-(0,\xi), \end{aligned} \tag{3.29}$$

$$\begin{aligned} F(t,\xi) :=& -\int_0^t\int_0^s \phi_+(s,\xi)\phi_-(\sigma,\xi)\exp((B_- - B_+ - \Phi)(s,\xi) \\ &\times \exp((B_+ - B_- + \Phi)(\sigma,\xi)W_+(\sigma,\xi)d\sigma ds. \end{aligned} \tag{3.30}$$

We show that the following estimates hold for $F(t,\xi)$ and $G(t,\xi)$ for $(t,\xi) \in \mathcal{G}_-$:

Lemma 1. *There exists a positive constant C depending only on α, b_0, b_1 and b_2 such that the following estimates hold for every $(t,\xi) \in \mathcal{G}_-$.*

$$|G(t,\xi)| \leq C(|u_0(\xi)| + |u_1(\xi)|), \tag{3.31}$$

$$|F(t,\xi)| \leq Ca(\xi)(t+1)^{2\alpha}(|u_0(\xi)| + |u_1(\xi)|), \tag{3.32}$$

and especially

$$|F(t,\xi)| \le C(|u_0(\xi)| + |u_1(\xi)|). \tag{3.33}$$

Proof. In the proof, C denotes various positive constants depending only on α, b_0, b_1 and b_2.

Let ξ be an arbitrary fixed element such that $a(\xi) \le \lambda_0$. Then we shall estimate $|F(t,\xi)|$ for every $0 \le t \le T(a(\xi))$. It follows from the definitions of F and G that

$$\begin{aligned} |F(t,\xi)| &\le \int_0^t \int_0^s |\phi_+(s,\xi)||\phi_-(\sigma,\xi)| \exp(-[(B_+ - B_- + \Phi)(\cdot,\xi)]_\sigma^s) \\ &\qquad \times |F(\sigma,\xi) + G(\sigma,\xi)| d\sigma ds \\ &= \int_0^t \int_\sigma^t |\phi_+(s,\xi)||\phi_-(\sigma,\xi)| \exp(-[(B_+ - B_- + \Phi)(\cdot,\xi)]_\sigma^s) \\ &\qquad \times |F(\sigma,\xi) + G(\sigma,\xi)| ds d\sigma. \end{aligned} \tag{3.34}$$

By (3.2), we have

$$[(B_+ - B_-)(\cdot,\xi)]_\sigma^s = \int_\sigma^s \sqrt{b^2(\tau) - 4a(\xi)} d\tau \ge c_2[(\tau+1)^{1-\alpha}]_\sigma^s \tag{3.35}$$

for every $0 \le \sigma \le s \le t < T(a(\xi))$ since $\{(t,\xi); 0 \le t < T(a(\xi))\} \subset \mathcal{G}_-$, where $c_2 = \frac{b_0}{2(1-\alpha)}$. By (0.3) and (3.2), we have

$$\begin{aligned} |\Phi(s,\xi) - \Phi(\sigma,\xi)| &= \Big| \int_\sigma^s (\phi_+ + \phi_-)(\tau,\xi) d\tau \Big| \\ &= \Big| \int_\sigma^s \frac{b'(\tau)}{\sqrt{b^2(\tau) - 4a(\xi)}} d\tau \Big| \le \int_\sigma^s \frac{2b_2}{b_0} (\tau+1)^{-1} d\tau, \end{aligned} \tag{3.36}$$

for every $0 \le \sigma \le s \le t < T(a(\xi))$. Then dividing the interval of the last integral at $c_3 := (\frac{b_0^2}{8b_2})^{\frac{1}{1-\alpha}}$, we have

$$\begin{aligned} |\Phi(s,\xi) - \Phi(\sigma,\xi)| &\le \frac{2b_2}{b_0} \int_{[\sigma,s]\cap[c_3,\infty]} (\tau+1)^{-1} d\tau + \frac{2b_2}{b_0} \int_0^{c_3} (\tau+1)^{-1} d\tau \\ &\le \frac{2b_2 c_3^{\alpha-1}}{b_0} \int_\sigma^s (\tau+1)^{-\alpha} d\tau + \frac{2b_2}{b_0} \log(c_3+1) \\ &\le \frac{b_0}{4(1-\alpha)} [(\tau+1)^{1-\alpha}]_\sigma^s + \frac{2b_2}{b_0} \log(c_3+1) \end{aligned} \tag{3.37}$$

for $0 \le \sigma \le s \le T(a(\xi))$. Inequalities (3.35) and (3.37) imply

$$[(B_+ - B_- + \Phi)(\cdot,\xi)]_\sigma^s \ge \frac{b_0}{4(1-\alpha)} [(\tau+1)^{1-\alpha}]_\sigma^s - \frac{2b_2}{b_0} \log(c_3+1),$$

and thus we obtain

$$\exp(-[(B_+ - B_- + \Phi)(\cdot,\xi)]_\sigma^s) \le C_1 \exp(-c_4(s+1)^{1-\alpha} + c_4(\sigma+1)^{1-\alpha}), \quad (3.38)$$

for $0 \le \sigma \le s \le T(a(\xi))$, where $C_1 := (c_3+1)^{\frac{2b_2}{b_0}}$ and $c_4 := \frac{c_2}{2} = \frac{b_0}{4(1-\alpha)}$. Since

$$\begin{aligned} &\int_\sigma^\infty (s+1)^{-1} \exp(-c_4(s+1)^{1-\alpha}) ds = \frac{1}{1-\alpha} \int_{(\sigma+1)^{1-\alpha}}^\infty \rho^{-1} \exp(-c_4\rho) d\rho \\ &\le \frac{1}{c_4(1-\alpha)} (\sigma+1)^{\alpha-1} \exp\left(-c_4(\sigma+1)^{1-\alpha}\right), \end{aligned} \quad (3.39)$$

for every $\sigma \ge 0$, inequalities (3.38) and (3.16) yield

$$\begin{aligned} &\int_\sigma^t |\phi_+(s,\xi)| \exp(-[(B_+ - B_- + \Phi)(\cdot,\xi)]_\sigma^s) ds \\ &\le \frac{3b_2C_1}{b_0} \exp(c_4(\sigma+1)^{1-\alpha}) \int_\sigma^t (s+1)^{-1} \exp(-c_4(s+1)^{1-\alpha}) ds \\ &\le \frac{3b_2C_1}{(1-\alpha)b_0c_4} (\sigma+1)^{\alpha-1} \end{aligned} \quad (3.40)$$

for $0 \le \sigma \le t < T(a(\xi))$. Substituting (3.17) and (3.40) into (3.34), we obtain

$$|F(t,\xi)| \le Ca(\xi) \int_0^t (\sigma+1)^{3\alpha-2} (|F(\sigma,\xi)| + |G(\sigma,\xi)|) d\sigma, \quad (3.41)$$

for every $0 \le t < T(a(\xi))$.

Since $\alpha \le 1$, we have

$$\int_0^t (\sigma+1)^{3\alpha-2} d\sigma \le C(t+1)^{2\alpha} \quad (3.42)$$

for every $t \ge 0$. From (3.29) and (3.40) with $\sigma = 0$, it follows that

$$|G(t,\xi)| \le |w_+(0,\xi)| + \frac{3b_2C_1}{(1-\alpha)b_0c_4} |w_-(0,\xi)| \quad (3.43)$$

for every $0 \le t < T(a(\xi))$. Since

$$|w_+(0,\xi)| \le |u_1(\xi)| + b_1|u_0(\xi)| \quad \text{and} \quad |w_-(0,\xi)| \le |u_1(\xi)| + \frac{|u_0(\xi)|}{b_0}, \quad (3.44)$$

the inequality (3.43) implies that (3.31) holds for every $0 \le t < T(a(\xi))$. Substituting (3.42) and (3.31) into (3.41), we have

$$|F(t,\xi)| \le Ca(\xi)(t+1)^{2\alpha}(|u_0(\xi)| + |u_1(\xi)|) + Ca(\xi)\int_0^t (\sigma+1)^{3\alpha-2}|F(\sigma,\xi)|d\sigma \tag{3.45}$$

for every $0 \le t < T(a(\xi))$. Applying Gronwall's inequality to (3.45), we obtain

$$\begin{aligned}|F(t,\xi)| &\le Ca(\xi)(t+1)^{2\alpha}(|u_0(\xi)| + |u_1(\xi)|)\\ &\quad + C\int_0^t a(\xi)^2(s+1)^{2\alpha}(s+1)^{3\alpha-2}\exp\Big(a(\xi)\int_s^t(\sigma+1)^{3\alpha-2}d\sigma\Big)ds\\ &\quad \times(|u_0(\xi)| + |u_1(\xi)|)\end{aligned} \tag{3.46}$$

for every $0 \le t < T(a(\xi))$. By the definition of $T(a(\xi))$, we have

$$a(\xi)(s+1)^{2\alpha} \le C \tag{3.47}$$

for every $0 \le s < T(a(\xi))$. By (3.42) and (3.47), we have

$$\exp\Big(a(\xi)\int_s^t(\sigma+1)^{3\alpha-2}d\sigma\Big) \le C. \tag{3.48}$$

Substituting (3.47) and (3.48) into the second term of the right-hand side of (3.46), and using (3.42) again, we obtain

$$\begin{aligned}|F(t,\xi)| &\le C\Big(a(\xi)(t+1)^{2\alpha} + a(\xi)\int_0^t(s+1)^{3\alpha-2}ds\Big)(|u_0(\xi)| + |u_1(\xi)|)\\ &\le Ca(\xi)(t+1)^{2\alpha}(|u_0(\xi)| + |u_1(\xi)|)\end{aligned} \tag{3.49}$$

for every $0 \le t < T(a(\xi))$. That is, (3.32) holds for every $0 \le t < T(a(\xi))$. By (3.32) and (3.47), we especially see that (3.33) holds for every $0 \le t < T(a(\xi))$. Since ξ is an arbitrary fixed element such that $a(\xi) \le \lambda_0$, and since

$$\mathcal{G}_- = \bigcup_{a(\xi)\le\lambda_0}\{(t,\xi); 0 \le t < T(a(\xi))\},$$

(3.31), (3.32) and (3.33) holds for every $(t,\xi) \in \mathcal{G}_-$. □

We shall estimate the difference between $w_+(t,x)$ and the solution of the parabolic equation on $\mathcal{G}_-$:

Lemma 2. *There exists a positive constant C depending only on α, b_0, b_1 and b_2 such that*

$$\begin{aligned}&\Big|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}}-\exp\Big(\int_0^t\frac{-a(\xi)}{b(s)}ds\Big)v_0(\xi)\Big|\\&\le C\Big(a(\xi)(t+1)^{2\alpha}+a(\xi)^2(t+1)^{3\alpha+1}\Big)\exp(-c_1a(\xi)(t+1)^{\alpha+1})\\&\qquad\times(|u_0(\xi)|+|u_1(\xi)|)+C\exp\Big(-\frac{b_0}{1-\alpha}(t+1)^{1-\alpha}\Big)|u_1(\xi)|\end{aligned}\tag{3.50}$$

for every $(t,\xi)\in\mathcal{G}_-$.

Proof. In the proof, C denotes various positive constants depending only on α, b_0, b_1 and b_2.

We prove (3.50) by dividing the left-hand side of it into three terms;

$$\begin{aligned}&\Big|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}}-\exp\Big(-\int_0^t\frac{a(\xi)}{b(s)}ds\Big)v_0(\xi)\Big|\\&\le\Big|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}}-\exp(-B_-(t,\xi))\frac{G(t,\xi)}{b(0)}\Big|\\&\quad+\exp(-B_-(t,\xi))\Big|\frac{1}{b(0)}G(t,\xi)-v_0(\xi)\Big|\\&\quad+\Big|\Big(\exp(-B_-(t,\xi))-\exp(-\int_0^t\frac{a(\xi)}{b(s)}ds\Big)v_0(\xi)\Big|\\&:=I_1+I_2+I_3.\end{aligned}\tag{3.51}$$

First we estimate I_1. By (3.23) and (3.28), we have

$$\begin{aligned}I_1&=\exp(-B_-(t,\xi))\\&\quad\times\Big|\Big(\frac{h(t,a(\xi))}{h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}}-\frac{1}{b(0)}\Big)W_+(t,\xi)+\frac{1}{b(0)}F(t,\xi)\Big|\\&\le\exp(-B_-(t,\xi))\Big(\Big|\frac{h(t,a(\xi))}{h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}}-\frac{1}{b(0)}\Big||W_+(t,\xi)|+\frac{1}{b(0)}|F(t,\xi)|\Big).\end{aligned}\tag{3.52}$$

By (3.28), (3.31) and (3.33), we have

$$|W_+(t,\xi)|\le|F(t,\xi)|+|G(t,\xi)|\le C(|u_0(\xi)|+|u_1(\xi)|)\tag{3.53}$$

for every $(t,\xi)\in\mathcal{G}_-$.

By using (3.2) and (3.25), we have

$$\Big|\frac{h(t,a(\xi))}{h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}}-\frac{1}{b(0)}\Big|\tag{3.54}$$

$$= \frac{\left|h(t,a(\xi))^2 b(0)^2 - h(0,a(\xi))^2(b(t)^2-4a(\xi))\right|}{h(0,a(\xi))b(0)\sqrt{b(t)^2-4a(\xi)}\left(h(t,a(\xi))b(0)+h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}\right)}$$
$$\leq \frac{8}{b(0)^3 b(t)^2}\left(\left|h(t,a(\xi))^2 b(0)^2 - h(0,a(\xi))^2 b(t)^2\right| + 4a(\xi)h(0,a(\xi))^2\right)$$

for every $(t,\xi) \in \mathcal{G}_-$. By the definition of h (see (3.24)), (0.2) and (3.2), we have

$$\begin{aligned}
&\left|h(t,a(\xi))^2 b(0)^2 - h(0,a(\xi))^2 b(t)^2\right| \\
&= \left|b(0)^2 b(t)\sqrt{b(t)^2-4a(\xi)} - b(0)b(t)^2\sqrt{b(0)^2-4a(\xi)} - 4a(\xi)(b(0)^2-b(t)^2)\right| \\
&= \left|b(0)b(t)\frac{4a(\xi)(b(t)^2-b(0)^2)}{b(0)\sqrt{b(t)^2-4a(\xi)}+b(t)\sqrt{b(0)^2-4a(\xi)}} - 4a(\xi)(b(0)^2-b(t)^2)\right| \\
&\leq Ca(\xi)
\end{aligned}$$

for every $(t,\xi) \in \mathcal{G}_-$. Substituting this inequality into (3.54), and using (0.2), we obtain

$$\left|\frac{h(t,a(\xi))}{h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}} - \frac{1}{b(0)}\right| \leq Ca(\xi)(t+1)^{2\alpha} \tag{3.55}$$

for every $(t,\xi) \in \mathcal{G}_-$. The inequalities (3.55) and (3.53) yield

$$\begin{aligned}
&\left|\frac{h(t,a(\xi))}{h(0,a(\xi))\sqrt{b(t)^2-4a(\xi)}} - \frac{1}{b(0)}\right||W_+(t,\xi)| \\
&\leq Ca(\xi)(t+1)^{2\alpha}(|u_0(\xi)|+|u_1(\xi)|)
\end{aligned} \tag{3.56}$$

for every $(t,\xi) \in \mathcal{G}_-$. Substituting (3.22), (3.32) and (3.56) into (3.52), we obtain

$$I_1 \leq Ca(\xi)(t+1)^{2\alpha}\exp\left(-c_1 a(\xi)(t+1)^{\alpha+1}\right)(|u_0(\xi)|+|u_1(\xi)|) \tag{3.57}$$

for every $(t,\xi) \in \mathcal{G}_-$.

Next we estimate I_2. By definition (see (0.12), (3.4), (3.5), and (3.29)), we have

$$\begin{aligned}
\frac{1}{b(0)}G(t,\xi) - v_0(\xi) = \frac{1}{2}\Big(&\frac{\sqrt{b(0)^2-4a(\xi)}}{b(0)} - 1\Big)u_0(\xi) \\
&- \int_0^t \phi_+(s,\xi)\exp((B_- - B_+ - \Phi)(s,\xi))ds \\
&\quad \times \left(\frac{u_1}{b(0)} + \frac{1}{2}\Big(1 - \frac{\sqrt{b(0)^2-4a(\xi)}}{b(0)}\Big)u_0(\xi)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^\infty \frac{b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma) ds u_1(\xi) \\
&= \Big(-1 - \int_0^t \phi_+(s,\xi) \exp((B_- - B_+ - \Phi)(s,\xi)) ds \Big) \\
&\qquad \times \frac{2a(\xi)}{b(0)(b(0) + \sqrt{b(0)^2 - 4a(\xi)})} u_0(\xi) \\
&- \int_0^t \phi_+(s,\xi) \exp((B_- - B_+ - \Phi)(s,\xi)) ds \frac{u_1(\xi)}{b(0)} \\
&+ b(0) \int_0^\infty \frac{b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma) ds \frac{u_1(\xi)}{b(0)}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\Big| \frac{1}{b(0)} G(t,\xi) - v_0(\xi) \Big| \\
&\le \frac{2a(\xi)}{b(0)^2} \Big(1 + \int_0^t |\phi_+(s,\xi)| \exp((B_- - B_+ - \Phi)(s,\xi)) ds \Big) |u_0(\xi)| \\
&+ \int_0^t \Big| \phi_+(s,\xi) \exp((B_- - B_+ - \Phi)(s,\xi)) \\
&\qquad - \frac{b(0)b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)) d\sigma \Big| ds \frac{|u_1(\xi)|}{b(0)} \\
&+ \int_t^\infty \frac{|b'(s)|}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma) ds |u_1(\xi)| := J_1 + J_2 + J_3.
\end{aligned} \tag{3.58}
$$

By (3.40) with $\sigma = 0$, we have

$$
J_1 \le Ca(\xi)|u_0(\xi)|. \tag{3.59}
$$

By (3.12), (3.19) and (3.35), we see that

$$
\begin{aligned}
&\phi_+(s,\xi) \exp((B_- - B_+ - \Phi)(s,\xi)) \\
&\quad = \frac{(b(0) + \sqrt{b(0) - 4a(\xi)})b'(s)}{2(b(s)^2 - 4a(\xi))} \exp\Big(-\int_0^s \sqrt{b(\sigma)^2 - 4a(\xi)} d\sigma \Big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big| \phi_+(s) \exp((B_- - B_+ - \Phi)(s,\xi)) - \frac{b(0)b'(s)}{b(s)^2} \exp(-\int_0^s b(\sigma)d\sigma) \Big| \\
&\le \Big| \frac{b(0) + \sqrt{b(0) - 4a(\xi)}}{2(b(s)^2 - 4a(\xi))} - \frac{b(0)}{b(s)^2} \Big| |b'(s)| \exp\Big(-\int_0^s \sqrt{b(\sigma)^2 - 4a(\xi)} d\sigma \Big)
\end{aligned}
$$

$$+ b(0)\frac{|b'(s)|}{b(s)^2}\Big|\exp\Big(-\int_0^s\sqrt{b(\sigma)^2-4a(\xi)}d\sigma\Big)-\exp\Big(-\int_0^s b(\sigma)d\sigma\Big)\Big|$$
$$:= J_{2,1}(s,\xi)+J_{2,2}(s,\xi). \tag{3.60}$$

By using (3.2) and (3.3), we have

$$\begin{aligned}
&\Big|\frac{b(0)+\sqrt{b(0)^2-4a(\xi)}}{2(b(s)^2-4a(\xi))}-\frac{b(0)}{b(s)^2}\Big| \\
&= \Big|\frac{b(s)^2\sqrt{b(0)^2-4a(\xi)}-b(s)^2b(0)+8a(\xi)b(0)}{2(b(s)^2-4a(\xi))b(s)^2}\Big| \\
&\le \Big|\frac{\sqrt{b(0)^2-4a(\xi)}-b(0)}{2(b(s)^2-4a(\xi))}\Big|+\Big|\frac{4a(\xi)b(0)}{(b(s)^2-4a(\xi))b(s)^2}\Big| \\
&\le \frac{8a(\xi)}{b(0)b(s)^2}+\frac{16a(\xi)b(0)}{b(s)^4}\le Ca(\xi)(s+1)^{4\alpha}.
\end{aligned}$$

By this inequality, (0.3) and (3.35) with $\sigma=0$, we see that

$$J_{2,1}(s,\xi)\le Ca(\xi)(s+1)^{3\alpha-1}\exp\Big(-\frac{b_0}{2(1-\alpha)}(s+1)^{1-\alpha}\Big).$$

Thus, we obtain

$$\int_0^t J_{2,1}(s,\xi)ds\le Ca(\xi)\int_0^\infty(s+1)^{3\alpha-1}\exp\Big(-\frac{b_0}{2(1-\alpha)}(s+1)^{1-\alpha}\Big)ds\le Ca(\xi). \tag{3.61}$$

Next we estimate $J_{2,2}(s,\xi)$. There is $\theta\in(0,1)$ such that

$$\begin{aligned}
&\exp\Big(-\int_0^s\sqrt{b(\sigma)^2-4a(\xi)}d\sigma\Big)-\exp\Big(-\int_0^s b(\sigma)d\sigma\Big) \\
&=\exp\Big(-\theta\int_0^s\sqrt{b(\sigma)^2-4a(\xi)}d\sigma-(1-\theta)\int_0^s b(\sigma)d\sigma\Big) \\
&\quad\times\int_0^s\Big(-\sqrt{b(\sigma)^2-4a(\xi)}+b(\sigma)\Big)d\sigma.
\end{aligned} \tag{3.62}$$

Since $b(\sigma)\ge\sqrt{b(\sigma)^2-4a(\xi)}\ge b_0(\sigma+1)^{-\alpha}/2$ by (3.2), and since

$$\int_0^s\Big(-\sqrt{b(\sigma)^2-4a(\xi)}+b(\sigma)\Big)d\sigma\le\frac{4a(\xi)}{b_0(\alpha+1)}(s+1)^{\alpha+1}$$

by (3.3), it follows from (3.62) that

$$\Big|\exp\Big(-\int_0^s\sqrt{b(\sigma)^2-4a(\xi)}d\sigma\Big)-\exp\Big(-\int_0^s b(\sigma)d\sigma\Big)\Big|$$

$$\leq \frac{4a(\xi)}{b_0(\alpha+1)}(s+1)^{\alpha+1}\exp\Big(-\frac{b_0}{2(1-\alpha)}((s+1)^{1-\alpha}-1)\Big).$$

This inequality together with (0.2) and (0.3) imply

$$\begin{aligned}&\int_0^t J_{2,2}(s,\xi)ds\\ &\leq \frac{4b_1b_2}{b_0^3(\alpha+1)}a(\xi)\int_0^\infty (s+1)^{2\alpha}\exp\Big(-\frac{b_0}{2(1-\alpha)}((s+1)^{1-\alpha}-1)\Big)ds\\ &\leq Ca(\xi).\end{aligned} \tag{3.63}$$

From (3.60), (3.61) and (3.63), it follows that

$$J_2 \leq \Big(\int_0^t J_{2,1}(s,\xi)ds + \int_0^t J_{2,2}(s,\xi)ds\Big)\frac{|u_1(\xi)|}{b(0)} \leq Ca(\xi)|u_1(\xi)|. \tag{3.64}$$

By the assumptions (0.2) and (0.3), we have

$$\begin{aligned}J_3 &\leq \frac{b_1}{b_0^2}\int_t^\infty (s+1)^{\alpha-1}\exp(-\frac{b_0}{1-\alpha}(s+1)^{1-\alpha})ds|u_1(\xi)|\\ &\leq C\exp(-\frac{b_0}{1-\alpha}(t+1)^{1-\alpha})|u_1(\xi)|.\end{aligned} \tag{3.65}$$

Inequality (3.58) together with inequalities (3.59), (3.64) and (3.65) yield

$$\begin{aligned}\Big|\frac{1}{b(0)}G(t,\xi)-v_0(\xi)\Big| &\leq Ca(\xi)(|u_0(\xi)|+|u_1(\xi)|)\\ &\quad + C\exp(-\frac{b_0}{1-\alpha}(t+1)^{1-\alpha})|u_1(\xi)|.\end{aligned}$$

From this inequality and (3.22), it follows that

$$\begin{aligned}I_2 &\leq Ca(\xi)\exp(-c_1a(\xi)(t+1)^{\alpha+1})(|u_0(\xi)|+|u_1(\xi)|)\\ &\quad + C\exp\Big(-\frac{b_0}{1-\alpha}(t+1)^{1-\alpha}\Big)|u_1(\xi)|\end{aligned} \tag{3.66}$$

for every $(t,\xi)\in\mathcal{G}_-$.

Last we estimate I_3. In the same way as in the estimate of $J_{2,2}(s,\xi)$, we have

$$\begin{aligned}&\Big|\exp(-B_-(t,\xi)) - \exp\Big(-\int_0^t \frac{a(\xi)}{b(s)}ds\Big)\Big|\\ &\leq a(\xi)\exp\Big(-\int_0^t\frac{a(\xi)}{b(s)}ds\Big)\Big|\int_0^t \frac{2}{b(s)+\sqrt{b(s)^2-4a(\xi)}}-\frac{1}{b(s)}ds\Big|.\end{aligned} \tag{3.67}$$

Since

$$\Big|\frac{2}{b(s)+\sqrt{b(s)^2-4a(\xi)}}-\frac{1}{b(s)}\Big|=\frac{4a(\xi)}{b(s)(b(s)+\sqrt{b(s)^2-4a(\xi)})^2}\le\frac{4a(\xi)}{b(s)^3}, \tag{3.68}$$

we have the following by using (0.2)

$$\Big|\int_0^t\frac{2}{b(s)+\sqrt{b(s)^2-4a(\xi)}}-\frac{1}{b(s)}ds\Big|\le 4a(\xi)b_0^{-3}(t+1)^{3\alpha+1}. \tag{3.69}$$

From (3.67), (0.2), (3.69) and (0.12), we obtain

$$I_3\le 4b_0^{-3}a(\xi)^2(t+1)^{3\alpha+1}\exp\left(-c_1a(\xi)(t+1)^{\alpha+1}\right)(|u_0(\xi)|+|u_1(\xi)|) \tag{3.70}$$

for every $(t,\xi)\in\mathcal{G}_-$. Substituting (3.57), (3.66) and (3.70) into (3.51), we obtain (3.50). □

We shall estimate the decay of $w_-(t,\xi)$ on $\mathcal{G}_-$:

Lemma 3. *There exists a positive constant C depending only on α, b_0, b_1 and b_2 such that*

$$\begin{aligned}|w_-(t,\xi)|\le C\Big(&\exp(-c_2(t+1)^{1-\alpha}) \\ &+a(\xi)\exp(-c_1a(\xi)(t+1)^{\alpha+1})(t+1)^{2\alpha-1}\Big)(|u_0(\xi)|+|u_1(\xi)|)\end{aligned} \tag{3.71}$$

for every $(t,\xi)\in\mathcal{G}_-$.

Proof. In the proof, C denotes various positive constants depending only on α, b_0, b_1 and b_2.

Put

$$\widetilde{W}_-(t,\xi):=e^{B_-(t,\xi)-B_+(t,\xi)}W_-(t,\xi)=e^{B_-(t,\xi)+\Phi_-(t,\xi)}w_-(t,\xi) \tag{3.72}$$

for $(t,\xi)\in\mathcal{G}_-$. Multiplying by $\exp(B_-(t,\xi)-B_+(t,\xi))$ on both sides of (3.27), we obtain

$$\begin{aligned}\widetilde{W}_-(t,\xi)=&e^{B_-(t,\xi)-B_+(t,\xi)}w_-(0,\xi) \\ &+e^{B_-(t,\xi)-B_+(t,\xi)}\int_0^t\phi_-(s,\xi)e^{(B_+-B_-+\Phi)(s,\xi)}W_+(s,\xi)ds\end{aligned} \tag{3.73}$$

for $(t,\xi)\in\mathcal{G}_-$. By (3.19), we have

$$\exp(\Phi(s,\xi))=\frac{b(s)+\sqrt{b(s)^2-4a(\xi)}}{b(0)+\sqrt{b(0)^2-4a(\xi)}}\le\frac{2b(s)}{b(0)} \tag{3.74}$$

for every $0 \le s \le t$, since $(s,\xi) \in \mathcal{G}_-$. From (3.73) together with (3.17), (3.74), (3.35), (3.53) and the definitions (3.4) and (3.5), it follows that

$$\begin{aligned}\left|\widetilde{W}_-(t,\xi)\right| &\le e^{c_2-c_2(t+1)^{1-\alpha}}|w_-(0,\xi)| \\ &\quad + e^{c_2-c_2(t+1)^{1-\alpha}} \int_0^t e^{c_2(s+1)^{1-\alpha}} \frac{16a(\xi)|b'(s)|}{b(s)^2 b(0)} |W_+(s,\xi)| ds \\ &\le C\Big(e^{-c_2(t+1)^{1-\alpha}} + a(\xi) e^{-c_2(t+1)^{1-\alpha}} \int_0^t e^{c_2(s+1)^{1-\alpha}} (s+1)^{\alpha-1} ds\Big) \\ &\quad \times (|u_0(\xi)| + |u_1(\xi)|)\end{aligned} \tag{3.75}$$

for $(t,\xi) \in \mathcal{G}_-$. We prove

$$\begin{aligned}\int_0^t e^{c_2(s+1)^{1-\alpha}} (s+1)^{\alpha-1} ds &= \frac{1}{1-\alpha} \int_1^{(t+1)^{1-\alpha}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho \\ &\le C(t+1)^{2\alpha-1} \exp\left(c_2(t+1)^{1-\alpha}\right).\end{aligned} \tag{3.76}$$

We first consider the case $0 < \alpha < 1/2$. Then we have

$$\begin{aligned}&\int_{\frac{1}{2}(t+1)^{1-\alpha}}^{(t+1)^{1-\alpha}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho \le \left(\frac{1}{2}(t+1)^{1-\alpha}\right)^{(2\alpha-1)/(1-\alpha)} \int_{\frac{1}{2}(t+1)^{1-\alpha}}^{(t+1)^{1-\alpha}} e^{c_2\rho} d\rho \\ &\le \frac{2}{c_2}(t+1)^{2\alpha-1} \exp\left(c_2(t+1)^{1-\alpha}\right)\end{aligned}$$

for every $t \ge 0$. We have

$$\begin{aligned}&\int_1^{\frac{1}{2}(t+1)^{1-\alpha}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho \\ &\le \frac{1}{2}(t+1)^{1-\alpha} \sup_{1 \le \rho \le \frac{1}{2}(t+1)^{1-\alpha}} \left(e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)}\right) \\ &\le C(t+1)^{1-\alpha} \exp\left(\frac{c_2}{2}(t+1)^{1-\alpha}\right) \le C(t+1)^{2\alpha-1} \exp\left(c_2(t+1)^{1-\alpha}\right)\end{aligned}$$

for every t such that $(t+1)^{1-\alpha} > 2$, since $\sup_{t \ge 0}(t+1)^{2-3\alpha} \exp(-c_2(t+1)^{1-\alpha}/2) < \infty$. Thus, we obtain

$$\begin{aligned}&\int_1^{(t+1)^{1-\alpha}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho \\ &= \int_1^{\max\{\frac{1}{2}(t+1)^{1-\alpha},1\}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho + \int_{\frac{1}{2}(t+1)^{(1-\alpha)}}^{(t+1)^{1-\alpha}} e^{c_2\rho} \rho^{(2\alpha-1)/(1-\alpha)} d\rho \\ &\le C(t+1)^{2\alpha-1} \exp\left(c_2(t+1)^{1-\alpha}\right).\end{aligned}$$

In the case that $\alpha \geq 1/2$, we have

$$\int_1^{(t+1)^{1-\alpha}} e^{c_2\rho}\rho^{(2\alpha-1)/(1-\alpha)}d\rho \leq (t+1)^{2\alpha-1}\int_1^{(t+1)^{1-\alpha}} e^{c_2\rho}d\rho \\ \leq \frac{1}{c_2}(t+1)^{2\alpha-1}\exp(c_2(t+1)^{1-\alpha}),$$

and thus, (3.76) holds in both cases.

Substituting (3.76) into (3.75), we obtain

$$\left|\widetilde{W}_-(t,\xi)\right| \leq C\left(\exp(-c_2(t+1)^{1-\alpha}) + a(\xi)(t+1)^{2\alpha-1}\right)(|u_0(\xi)| + |u_1(\xi)|) \tag{3.77}$$

for $(t,\xi) \in \mathcal{G}_-$. By (3.18) and (3.2), we have

$$\exp(-\Phi_-(t)) = \Big(\frac{b(t)+\sqrt{b(t)^2-4a(\xi)}}{\sqrt{b(t)^2-4a(\xi)}}\frac{\sqrt{b(0)^2-4a(\xi)}}{b(0)+\sqrt{b(0)^2-4a(\xi)}}\Big)^{1/2} \\ \leq 3. \tag{3.78}$$

Since $|w_-(t,\xi)| = \exp(-\Phi_-(t,\xi) - B_-(t,\xi))\left|\widetilde{W}_-(t,\xi)\right|$ by definition (3.72), inequality (3.77) together with (3.22), (3.78) imply (3.71). □

We obtain the following corollary from Lemmas 2 and 3.

Corollary 6. *Let β be an arbitrary non-negative constant. There exists a positive constant C depending only on α, β, b_0, b_1 and b_2 such that*

$$a(\xi)^\beta\Big|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}} - \exp\left(-\int_0^t \frac{a(\xi)}{b(s)}ds\right)v_0(\xi)\Big| \\ \leq C(t+1)^{\alpha-1-(\alpha+1)\beta}(|u_0(\xi)| + |u_1(\xi)|) \tag{3.79}$$

and

$$a(\xi)^\beta\Big|\frac{w_-(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}}\Big| \leq C(t+1)^{2(\alpha-1)-(\alpha+1)\beta}(|u_0(\xi)| + |u_1(\xi)|) \tag{3.80}$$

for every $(t,\xi) \in \mathcal{G}_-$.

Proof. Multiplying by $a(\xi)^\beta$ on both sides of (3.50) and using the assumption that $a(\xi) \leq \lambda_0$, we obtain

$$a(\xi)^\beta\Big|\frac{w_+(t,\xi)}{\sqrt{b(t)^2-4a(\xi)}} - \exp\Big(-\int_0^t \frac{a(\xi)}{b(s)}ds\Big)v_0(\xi)\Big| \\ \leq C(t+1)^{\alpha-1-\beta(\alpha+1)}\left(a(\xi)(t+1)^{\alpha+1}\right)^\beta \tag{3.81}$$

$$\Big(a(\xi)(t+1)^{\alpha+1} + \big(a(\xi)(t+1)^{\alpha+1}\big)^2\Big)$$
$$\times \exp\big(-c_1 a(\xi)(t+1)^{\alpha+1}\big)\,(|u_0(\xi)| + |u_1(\xi)|)$$
$$+ C\exp\Big(-\frac{b_0}{1-\alpha}(t+1)^{1-\alpha}\Big)|u_1(\xi)|$$

for every $(t,\xi) \in \mathcal{G}_-$, where C is a positive constant depending only on α, β, b_0, b_1 and b_2. Since

$$\sup_{s\geq 0} s^p \exp(-c_1 s) < \infty, \tag{3.82}$$

for every fixed $p \in \mathbb{R}$, and since $\exp(-c_2(t+1)^{1-\alpha}) = O(t^{-m})$ for every $m \geq 0$, (3.81) implies (3.79).

In the same way, we obtain (3.80) from (3.71). □

By the above corollary, we obtain the decay estimate of the difference between the solution of the wave equation and the solution of the corresponding parabolic equation for low frequency:

Corollary 7. *Let β be an arbitrary non-negative constant. There exists a positive constant C depending only on α, β, b_0, b_1 and b_2 such that*

$$a(\xi)^\beta |u(t,\xi) - v(t,\xi)| \leq C(t+1)^{\alpha-1-\beta(\alpha+1)}(|u_0(\xi)| + |u_1(\xi)|), \tag{3.83}$$

for every $(t,\xi) \in \mathcal{G}_-$, where $u(t,\xi)$ is the solution of (2.1) *and $v(t,\xi)$ is the solution of* (2.2).

Proof. By (3.6) and the formula

$$v(t,\xi) = \exp\Big(-\int_0^t \frac{a(\xi)}{b(s)}ds\Big)v_0(\xi),$$

the sum of (3.79) and (3.80) implies the assertion. □

For the difference between the derivatives of the solutions, we have the following estimate.

Lemma 4. *Let β be an arbitrary non-negative constant. There exists a positive constant C depending only on α, β, b_0, b_1 and b_2 such that*

$$a(\xi)^\beta \big|u'(t,\xi) - v'(t,\xi)\big| \leq C(t+1)^{\alpha-2-\beta(\alpha+1)}(|u_0(\xi)| + |u_1(\xi)|), \tag{3.84}$$

for every $(t,\xi) \in \mathcal{G}_-$, where $u(t,\xi)$ is the solution of (2.1) *and $v(t,\xi)$ is the solution of* (2.2).

Proof. In the proof, C denotes various positive constants depending only on the operator α, β, b_0, b_1 and b_2.

Since $v(t,x)$ is the solution of (2.2), we have

$$v'(t,\xi) = -\frac{a(\xi)}{b(t)}v(t,\xi). \tag{3.85}$$

From (3.7) and (3.85), we have

$$\begin{aligned}|u'(t,\xi)-v'(t,\xi)| &\le \Big|\Big(\frac{b(t)}{2\sqrt{b^2(t)-4a(\xi)}}+\frac{1}{2}\Big)w_-(t,\xi)\Big| \\ &+\Big|\frac{2a(\xi)}{\sqrt{b^2(t)-4a(\xi)}(b(t)+\sqrt{b^2(t)-4a(\xi)})}w_+(t,\xi)-\frac{a(\xi)}{b(t)}v(t,\xi)\Big| \\ &:= K_1+K_2.\end{aligned} \tag{3.86}$$

By (3.71), (3.2), (0.2), (3.82) and the boundedness of $a(\xi)(\le \lambda_0)$, we have

$$\begin{aligned}a(\xi)^\beta K_1 &\le 2a(\xi)^\beta|w_-(t,\xi)| \le C\Big(a(\xi)^\beta\exp(-c_2(t+1)^{1-\alpha}) \\ &\quad + a(\xi)^{\beta+1}\exp\left(-c_1a(\xi)(t+1)^{\alpha+1}\right)(t+1)^{2\alpha-1}\Big)\times(|u_0(\xi)|+|u_1(\xi)|) \\ &\le C(t+1)^{\alpha-2-(\alpha+1)\beta}\left(|u_0(\xi)|+|u_1(\xi)|\right).\end{aligned} \tag{3.87}$$

We estimate K_2 by dividing it into two terms.

$$\begin{aligned}K_2 &\le \Big|\frac{a(\xi)}{\sqrt{b^2(t)-4a(\xi)}}\Big(\frac{2}{b(t)+\sqrt{b^2(t)-4a(\xi)}}-\frac{1}{b(t)}\Big)w_+(t,\xi)\Big| \\ &\quad + \frac{a(\xi)}{b(t)}\Big|\frac{1}{\sqrt{b^2(t)-4a(\xi)}}w_+(t,\xi)-v(t,\xi)\Big| := K_{2,1}+K_{2,2}.\end{aligned} \tag{3.88}$$

By (3.3), (3.2) and (0.2), we have

$$\begin{aligned}K_{2,1} &= \frac{4a(\xi)^2}{(\sqrt{b(t)^2-4a(\xi)}+b(t))^2 b(t)\sqrt{b(t)^2-4a(\xi)}}|w_+(t,\xi)| \\ &\le \frac{8a(\xi)^2}{b(t)^4}|w_+(t,\xi)| \le Ca(\xi)^2(t+1)^{4\alpha}|w_+(t,\xi)|.\end{aligned} \tag{3.89}$$

From (3.23), (3.25), (3.22) and (3.53), it follows that

$$\begin{aligned}|w_+(t,\xi)| &= \frac{h(t,a(\xi))}{h(0,a(\xi))}\exp(-B_-(t,\xi))W_+(t,\xi) \\ &\le Cb(t)\exp(-c_1a(\xi)(t+1)^{\alpha+1})(|u_0(\xi)|+|u_1(\xi)|) \\ &\le C(t+1)^{-\alpha}\exp(-c_1a(\xi)(t+1)^{\alpha+1})(|u_0(\xi)|+|u_1(\xi)|)\end{aligned} \tag{3.90}$$

for every $(t,\xi) \in \mathcal{G}_-$. Substituting this inequality into (3.89), we obtain

$$K_{2,1} \le Ca(\xi)^2(t+1)^{3\alpha} \exp(-c_1 a(\xi)(t+1)^{\alpha+1})(|u_0(\xi)| + |u_1(\xi)|). \quad (3.91)$$

Thus we obtain

$$\begin{aligned} a(\xi)^\beta K_{2,1} &\le (t+1)^{\alpha-2-\beta(\alpha+1)} \\ &\quad \times (a(\xi)(t+1)^{\alpha+1})^{2+\beta} \exp\left(-c_1 a(\xi)(t+1)^{\alpha+1}\right) (|u_0(\xi)| + |u_1(\xi)|) \\ &\le C(t+1)^{\alpha-2-\beta(\alpha+1)}(|u_0(\xi)| + |u_1(\xi)|). \end{aligned} \quad (3.92)$$

By (3.50) and (0.2), we have

$$a(\xi)^\beta K_{2,2} \le C(t+1)^{\alpha-2-(\alpha+1)\beta} (|u_0(\xi)| + |u_1(\xi)|). \quad (3.93)$$

Inequalities (3.86), (3.87), (3.88), (3.92) and (3.93) imply (3.84). □

4. Estimate for high frequency

In this section, we prove that each solution of (2.1) and (2.2) itself has a better decay estimate than any polynomial order on $\mathcal{G}_+ \cup [T_0, t_0] \times [\lambda_0, \infty)$.

Lemma 5. *There exists a positive constant C and c_5 depending only on α, b_0, b_1 and b_2 such that*

$$(1 + a(\xi))|u(t,\xi)|^2 + \left|u'(t,\xi)\right|^2 \le C \exp\left(-c_5(t+1)^{1-\alpha}\right) e_0(0,\xi), \quad (4.1)$$

for every $(t,\xi) \in \mathcal{G}_+ \cup \mathcal{G}_0$, where

$$e_0(t,\xi) := \left|u'(t,\xi)\right|^2 + a(\xi)|u(t,\xi)|^2.$$

Proof. In the proof C denotes various positive constants depending only on α, b_0, b_1 and b_2.

Let (t,ξ) be an arbitrary fixed element of $\mathcal{G}_+$. Put

$$e(\tau,\xi) := \frac{1}{2}\left|u'(\tau)\right|^2 + \frac{1}{2}a(\xi)|u(\tau,\xi)|^2 + b_3 b(\tau) u(\tau)\overline{u'(\tau)},$$

for $\tau \ge t/2$, where

$$b_3 := \frac{3b_0^2}{256} \min\{\frac{1}{b_1^2}, \frac{b_0^2}{b_2^2}\}.$$

Here we note that

$$b_3 \le \frac{1}{4}. \quad (4.2)$$

From the definition of $\Lambda(t)$ (see (2.3)) and the assumption that $(t,\xi) \in \mathcal{G}_+$, it follows that

$$(\tau+1)^{-2\alpha} \le (\frac{t}{2}+1)^{-2\alpha} \le 2^{2\alpha}(t+1)^{-2\alpha} \le \frac{64}{3b_0^2}\Lambda(t) \le \frac{64}{3b_0^2}a(\xi) \quad (4.3)$$

for $\tau \geq t/2$. Thus together with using (0.2) and the definitions of b_3, we have

$$b_3 b(\tau)^2 \leq b_3 b_1^2 (\tau+1)^{-2\alpha} \leq \frac{1}{4} a(\xi), \tag{4.4}$$

for $\tau \geq t/2$. Inequalities (4.2) and (4.4) yield

$$\begin{aligned} b_3 b(\tau) \Big| u(\tau,\xi)\overline{u'(\tau,\xi)} \Big| &\leq \frac{1}{4} \big| u'(\tau,\xi) \big|^2 + b_3^2 b(\tau)^2 |u(\tau,\xi)|^2 \\ &\leq \frac{1}{4} \big| u'(\tau,\xi) \big|^2 + \frac{1}{4} b_3 a(\xi) |u(\tau,\xi)|^2 \leq \frac{1}{4} e_0(\tau,\xi) \end{aligned} \tag{4.5}$$

for every $\tau \geq t/2$. Thus we have

$$\frac{1}{4} e_0(\tau,\xi) \leq e(\tau,\xi) \leq e_0(\tau,\xi) \tag{4.6}$$

for every $\tau \geq t/2$. Since $u(\tau,\xi)$ is the solution of (2.1), we have

$$\begin{aligned} \frac{d}{d\tau} e(\tau,\xi) = &-b(\tau)(1-b_3) \big| u'(\tau,\xi) \big|^2 - b_3 b(\tau) a(\xi) |u(\tau,\xi)|^2 \\ &+ b_3(-b(\tau)^2 + b'(\tau)) \Re(u(\tau,\xi)\overline{u'(\tau,\xi)}). \end{aligned} \tag{4.7}$$

By the assumptions (0.2) and (0.3), we have

$$\begin{aligned} b_3 |b'(\tau) \Re(u(\tau,\xi)\overline{u'(\tau,\xi)})| &\leq \frac{b_2 b_3}{b_0} (\tau+1)^{-1} b(\tau) |u(\tau,\xi)| |u'(\tau,\xi)| \\ &\leq \Big(\frac{1}{4} |u'(\tau,\xi)|^2 + \Big(\frac{b_2 b_3}{b_0} (\tau+1)^{-1} \Big)^2 |u(\tau,\xi)|^2 \Big) b(\tau). \end{aligned} \tag{4.8}$$

By (4.3) and the definition of b_3, we have

$$\Big(\frac{b_2 b_3}{b_0} (\tau+1)^{-1} \Big)^2 \leq \Big(\frac{b_2 b_3}{b_0} \Big)^2 (\tau+1)^{-2\alpha} \leq \frac{1}{4} b_3 a(\xi)$$

for $\tau \geq t/2$. Hence, we obtain

$$b_3 |b'(\tau)| \Re(u(\tau,\xi)\overline{u'(\tau,\xi)}) \leq \Big(\frac{1}{4} |u'(\tau,\xi)|^2 + \frac{b_3}{4} a(\xi) |u(\tau,\xi)|^2 \Big) b(\tau). \tag{4.9}$$

From (4.7), (4.5) and (4.9), together with using inequalities (4.2) and (4.6), we obtain

$$\begin{aligned} \frac{d}{d\tau} e(\tau,\xi) \leq &-b(\tau)(1-b_3) \big| u'(\tau,\xi) \big|^2 - b_3 b(\tau) a(\xi) |u(\tau,\xi)|^2 \\ &+ \frac{b(\tau)}{2} \Big(\big| u'(\tau,\xi) \big|^2 + b_3 a(\xi) |u(\tau,\xi)|^2 \Big) \\ = &-b(\tau) \Big((\frac{1}{2} - b_3) \big| u'(\tau,\xi) \big|^2 + \frac{b_3}{2} a(\xi) |u(\tau,\xi)|^2 \Big) \end{aligned}$$

$$\le -\frac{b_3}{2} b(\tau) e_0(\tau,\xi) \le -\frac{b_3}{2} b(\tau) e(\tau,\xi)$$

for $\tau \ge t/2$. Thus, it follows that

$$e(t,\xi) \le \exp\Big(-\frac{b_3}{2}\int_{t/2}^{t} b(s)ds\Big) e(\frac{t}{2},\xi). \tag{4.10}$$

By assumption (0.2), we have

$$\int_{t/2}^{t} b(s)ds \ge \frac{b_0}{1-\alpha}\Big((t+1)^{1-\alpha} - (\frac{t}{2}+1)^{1-\alpha}\Big) \ge 2c_6(t+1)^{1-\alpha}, \tag{4.11}$$

where $2c_6 = b_0(1-2^{\alpha-1})/(1-\alpha)$. Inequalities (4.10) and (4.11) yield

$$e(t,\xi) \le \exp\left(-2c_5(t+1)^{1-\alpha}\right) e(\frac{t}{2},\xi), \tag{4.12}$$

where $c_5 = \frac{1}{2}c_6 b_3$. From the fact that $u(\tau,\xi)$ is the solution of the equation (2.1), it follows that $\frac{d}{ds}e_0(s,\xi) \le 0$ for every $s \ge 0$, and therefore

$$e_0(s,\xi) \le e_0(0,\xi), \tag{4.13}$$

for every $s \ge 0$. From (4.12), (4.6) and (4.13) with $s = t/2$, we obtain

$$e_0(t,\xi) \le 4e(t,\xi) \le 4\exp\left(-2c_5(t+1)^{1-\alpha}\right) e_0(0,\xi). \tag{4.14}$$

The assumption $(t,\xi) \in \mathcal{G}_+$ means $a(\xi) \ge \Lambda(t)$. Thus we have

$$\begin{aligned}(1+a(\xi))|u(t,\xi)|^2 + \left|u'(t,\xi)\right|^2 &\le (\frac{1}{a(\xi)}+1)e_0(t,\xi)\\ &\le \Big(\frac{16}{3b_0^2}(t+1)^{2\alpha}+1\Big)e_0(t,\xi).\end{aligned} \tag{4.15}$$

Substituting (4.14) into (4.15), we have

$$\begin{aligned}&(1+a(\xi))|u(t,\xi)|^2 + \left|u'(t,\xi)\right|^2\\ &\le 4\Big(\frac{16}{3b_0^2}(t+1)^{2\alpha}+1\Big)\exp\Big(-2c_5(t+1)^{1-\alpha}\Big)e_0(0,\xi)\\ &= 4\Big(\frac{16}{3b_0^2}(t+1)^{2\alpha}+1\Big)\exp\left(-c_5(t+1)^{1-\alpha}\right)\exp\left(-c_5(t+1)^{1-\alpha}\right)e_0(0,\xi)\\ &\le C\exp\left(-c_5(t+1)^{1-\alpha}\right)e_0(0,\xi),\end{aligned}$$

since $\sup_{t\ge 0}(t+1)^{2\alpha}\exp\left(-c_5(t+1)^{1-\alpha}\right) < \infty$. Thus (4.1) holds for $(t,\xi) \in \mathcal{G}_+$. Inequality (4.1) for $(t,\xi) \in \mathcal{G}_0$ is trivial by (4.13). □

Lemma 6. *Let T_0 be an arbitrary positive number. Let β be an arbitrary non-negative constant. There exists a positive constant C and c_7 depending only on α, β, b_0, b_1, b_2 and T_0 such that*

$$a(\xi)^{\beta}(1+a(\xi))^{\frac{1}{2}}|v(t,\xi)| \le C\exp\left(-c_7(t+1)^{1-\alpha}\right)\left(|u_0(\xi)|^2+|u_1(\xi)|^2\right)^{\frac{1}{2}}, \tag{4.16}$$

for every $(t,\xi) \in \mathcal{G}_+ \cup \{(t,\xi); (t,a(\xi)) \in [T_0,\infty)\times[\lambda_0,\infty)\}$.

Proof. In the proof C denotes various positive constants depending only on α, b_0, b_1 and T_0. By (0.2), we have

$$\begin{aligned}|v(t,\xi)| &= |v(0,\xi)|\exp\Big(-\int_0^t \frac{a(\xi)}{b(s)}ds\Big)\\ &\le |v(0,\xi)|\exp\Big(-\frac{a(\xi)}{b_1(\alpha+1)}((t+1)^{\alpha+1}-1)\Big)\end{aligned} \tag{4.17}$$

for every $t \ge 0$. There exists $\theta_0 > 0$ such that

$$(t+1)^{\alpha+1} - 1 > \theta_0(t+1)^{\alpha+1}$$

for every $t > T_0$. Thus, for every $\gamma \ge 0$, we have by using (3.82) that

$$\begin{aligned}a(\xi)^{\gamma}|v(t,\xi)| &\le a(\xi)^{\gamma}\exp\Big(-\frac{\theta_0 a(\xi)}{b_1(\alpha+1)}(t+1)^{\alpha+1}\Big)|v(0,\xi)|\\ &\le a(\xi)^{\gamma}\exp\Big(-\frac{1}{2}\frac{\theta_0 a(\xi)}{b_1(\alpha+1)}(T_0+1)^{\alpha+1}\Big)\\ &\qquad\times\exp\Big(-\frac{1}{2}\frac{\theta_0 a(\xi)}{b_1(\alpha+1)}(t+1)^{\alpha+1}\Big)|v(0,\xi)|\\ &\le C\exp\Big(-\frac{1}{2}\frac{\theta_0 a(\xi)}{b_1(\alpha+1)}(t+1)^{\alpha+1}\Big)(|u_0(\xi)|+|v_0(\xi)|)\end{aligned} \tag{4.18}$$

for every $t \ge T_0$. Since

$$\begin{aligned}&a(\xi)(t+1)^{\alpha+1}\\ &\quad\ge \begin{cases}\dfrac{3b_0}{16}(t+1)^{1-\alpha} & \text{for } (t,\xi)\in\mathcal{G}_+,\\ \lambda_0(t+1)^{\alpha+1} \ge \lambda_0(t+1)^{1-\alpha} & \text{for } \big(t,a(\xi)\big)\in[T_0,t_0]\times[\lambda_0,\infty),\end{cases}\end{aligned}$$

(4.18) with $\gamma = \beta$ and $\beta + 1/2$ implies (4.16). □

5. Proof of Theorem 1

Now we have prepared to prove Theorem 1.

Proof of Theorem 1. In the proof C denotes various positive constants depending only on α, β, γ, b_0, b_1, b_2 and T_0.

Let $t \geq T_0$ be an arbitrary fixed number.

(i) We first prove the inequality (1.1) for $\gamma = 0$. Integrating the square of (3.83) in Corollary 7 on $\mathcal{G}_-(t)$, we obtain

$$\begin{aligned} &\Big(\int_{\xi \in \mathcal{G}_-(t)} a(\xi)^{2\beta} \left| u(t,\xi) - v(t,\xi) \right|^2 d\mu_\xi \Big)^{1/2} \\ &\leq C(t+1)^{\alpha-1-(\alpha+1)\beta} \Big(\int_{\xi \in \mathcal{G}_-(t)} \left(|u_0(\xi)|^2 + |u_1(\xi)|^2 \right) d\mu_\xi \Big)^{1/2} \\ &\leq C(t+1)^{\alpha-1-(\alpha+1)\beta} \left(\|u_0\| + \|u_1\| \right). \end{aligned} \tag{5.1}$$

Integrating (4.1) in Lemma 5 multiplied by $a(\xi)^{2\beta}$ on $\mathcal{G}_+(t)$, we obtain

$$\begin{aligned} &\Big(\int_{\xi \in \mathcal{G}_+(t)} a(\xi)^{2\beta} \left((1+a(\xi))|u(t,\xi)|^2 + \left|u'(t,\xi)\right|^2 \right) d\mu_\xi \Big)^{1/2} \\ &\leq C \exp\left(-c_5(t+1)^{1-\alpha}\right) \Big(\int_{\xi \in \mathcal{G}_+(t)} a(\xi)^{2\beta} e_0(0,\xi) d\mu_\xi \Big)^{1/2} \\ &\leq C \exp\left(-c_5(t+1)^{1-\alpha}\right) \left(\|u_0\|_{\beta+\frac{1}{2}} + \|u_1\|_\beta \right). \end{aligned} \tag{5.2}$$

Integrating the square of (4.16) on $\mathcal{G}_+(t)$, we obtain

$$\begin{aligned} &\Big(\int_{\xi \in \mathcal{G}_+(t)} a(\xi)^{2\beta} (1+a(\xi)) |v(t,\xi)|^2 d\mu_\xi \Big)^{1/2} \\ &\leq C \exp\left(-c_7(t+1)^{1-\alpha}\right) \Big(\int_{\xi \in \mathcal{G}_+(t)} (|u_0(\xi)|^2 + |u_1(\xi)|^2) d\mu_\xi \Big)^{1/2} \\ &\leq C \exp\left(-c_7(t+1)^{1-\alpha}\right) (\|u_0\| + \|u_1\|). \end{aligned} \tag{5.3}$$

Summing (5.1), (5.2) and (5.3) up, we obtain (1.1) with $\gamma = 0$.

(ii) Next, we prove (1.2) with $\gamma = 0$. Integrating the square of (3.84) in Lemma 4 on $\mathcal{G}_-(t)$, we obtain

$$\begin{aligned} &\Big(\int_{\xi \in \mathcal{G}_-(t)} a(\xi)^{2\beta} \left| u'(t,\xi) - v'(t,\xi) \right|^2 d\mu_\xi \Big)^{1/2} \\ &\leq C(t+1)^{\alpha-2-\beta(\alpha+1)} \Big(\int_{\xi \in \mathcal{G}_-(t)} \left(|u_0(\xi)|^2 + |u_1(\xi)|^2 \right) d\mu_\xi \Big)^{1/2} \\ &\leq C(t+1)^{\alpha-2-(\alpha+1)\beta} \left(\|u_0\| + \|u_1\| \right). \end{aligned} \tag{5.4}$$

Since $v'(t,\xi) = -\frac{1}{b(t)}a(\xi)v(t,\xi)$, inequality (5.3) with β replaced by $\beta + 1/2$ yields

$$\begin{aligned}&\Big(\int_{\xi\in\mathcal{G}_+(t)} a(\xi)^{2\beta}\left|v'(t,\xi)\right|^2 d\mu_\xi\Big)^{1/2}\\ &\le C\exp\left(-c_7(t+1)^{1-\alpha}\right)(\|u_0\|+\|u_1\|).\end{aligned}\tag{5.5}$$

Summing (5.4), (5.2) and (5.5) up, we obtain (1.2) with $\gamma = 0$.

(iii) We consider the case $\gamma > 0$. We apply the result (1.1) to initial data $\tilde{u}_0$ and $\tilde{u}_1$. Let $\tilde{u}(t)$ and $\tilde{v}(t)$ be solutions of (0.1) and (0.11)–(0.12) with u_0 and u_1 replaced by $\tilde{u}_0$ and $\tilde{u}_1$, respectively. Inequality (1.1) with β and γ replaced by $\beta+\gamma$ and 0 yields

$$\begin{aligned}&\left\|A^{\beta+\gamma}(\tilde{u}(t)-\tilde{v}(t))\right\|_{1/2}\\ &\le C(t+1)^{\alpha-1-(\alpha+1)(\beta+\gamma)}\left(\|\tilde{u}_0\|_{\beta+\gamma+1/2}+\|\tilde{u}_1\|_{\beta+\gamma}\right)\\ &\le C(t+1)^{\alpha-1-(\alpha+1)(\beta+\gamma)}\left(\|u_0\|_{\beta+1/2}+\|\tilde{u}_0\|+\|u_1\|_{\beta}+\|\tilde{u}_1\|\right).\end{aligned}\tag{5.6}$$

Since $A^\gamma\tilde{u}(t) = u(t)$ and $A^\gamma\tilde{v}(t) = v(t)$, the inequality (5.6) implies (1.1). In the same way we obtain (1.2) for $\gamma > 0$.

6. Proof of Theorem 2

Proof of Theorem 2. In the proof C denotes various positive constants depending only on α, β, γ, b_0, b_1, b_2.

(i) We first prove the inequality (1.5) for $\gamma = 0$. In the same way as in the proof of (3.79), we have by (0.2), (3.6), (3.71) and (3.90) together with (3.82) that

$$\begin{aligned}&|a(\xi)^\beta u(t,\xi)| \le C(a(\xi)^\beta\exp(-c_1a(\xi)(t+1)^{\alpha+1}) + e^{-c_2(t+1)^{1-\alpha}}\\ &+ Ca(\xi)^{\beta+1}\exp\left(-c_1a(\xi)(t+1)^{\alpha+1}\right)(t+1)^{3\alpha-1}(|u_0(\xi)|+|u_1(\xi)|)\\ &\le C(t+1)^{-\beta(\alpha+1)}(|u_0(\xi)|+|u_1(\xi)|)\end{aligned}\tag{6.1}$$

for every $(t,\xi)\in\mathcal{G}_-$. By (4.1) in Lemma 5, we have

$$\begin{aligned}&a(\xi)^\beta|u(t,\xi)| \le C\exp\left(-\frac{c_5}{2}(t+1)^{1-\alpha}\right)a(\xi)^\beta(1+a(\xi))^{-1/2}e_0(0,\xi)^{\frac{1}{2}}\\ &\le C\exp\left(-\frac{c_5}{2}(t+1)^{1-\alpha}\right)\left(a(\xi)^\beta|u_0(\xi)|+a(\xi)^\beta(1+a(\xi))^{-1/2}|u_1(\xi)|\right)\end{aligned}\tag{6.2}$$

for every $(t,\xi)\in\mathcal{G}_+\cup\mathcal{G}_0$. Using inequalities (6.1) and (6.2), we obtain (1.5) for $\gamma = 0$ in the same way as in the proof of Theorem 1 (i).

(ii) Next we prove that the inequality (1.5) also holds for $\gamma > 0$, in the same way as in the proof of Theorem 1 (iii). We apply the result (1.5) for initial data $\tilde{u}_0$ and $\tilde{u}_1$. Let $\tilde{u}(t)$ be a solution of (0.1) with u_0 and u_1 replaced by $\tilde{u}_0$ and $\tilde{u}_1$, respectively. Inequality (1.5) with β and γ replaced by $\beta+\gamma$ and 0 yields

$$\begin{aligned} \|A^\beta u(t)\| &= \|A^{\beta+\gamma}\tilde{u}(t)\| \\ &\le C(t+1)^{-(\alpha+1)(\beta+\gamma)}\left(\|\tilde{u}_0\|_{\beta+\gamma} + \|\tilde{u}_1\|_{\max\{\beta+\gamma-\frac{1}{2},0\}}\right) \\ &\le C(t+1)^{-(\alpha+1)(\beta+\gamma)}\left(\|u_0\|_\beta + \|\tilde{u}_0\| + \|\tilde{u}_1\|_{\max\{\beta+\gamma-\frac{1}{2},0\}}\right), \end{aligned} \tag{6.3}$$

which means (1.5). Since

$$\|\tilde{u}_1\|_{\max\{\beta+\gamma-\frac{1}{2},0\}} \le \|u_1\|_{\max\{\beta-\frac{1}{2},0\}} + \|\tilde{u}_1\|,$$

(1.6) holds for $\gamma > 0$.

(iii) In the same way, we have by (3.7), (3.71) and (3.90) together with (3.82) that

$$\begin{aligned} &|a(\xi)^\beta u'(t,\xi)| \le Ca(\xi)^\beta(|w_-(t,\xi)| + a(\xi)(t+1)^{2\alpha}|w_+(t,\xi)|) \\ &\le C\Big(a(\xi)^{\beta+1}e^{-c_2(t+1)^{1-\alpha}} + Ca(\xi)^{\beta+1}((t+1)^{2\alpha-1} + (t+1)^\alpha) \\ &\qquad \times \exp\left(-c_1 a(\xi)(t+1)^{\alpha+1}\right)\Big)(|u_0(\xi)| + |u_1(\xi)|) \\ &\le C(t+1)^{-1-\beta(\alpha+1)}(|u_0(\xi)| + |u_1(\xi)|) \end{aligned} \tag{6.4}$$

for every $(t,\xi) \in \mathcal{G}_-$. Using inequalities (6.4) and (4.1) in Lemma 5, we obtain (1.7) with $\gamma = 0$ in the same way as in (i), and obtain it for general γ in the same way as in (ii).

Acknowledgements. The author expresses her sincere gratitude to the referees for careful readings, valuable suggestions and helpful comments.

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