Advances in Differential Equations Volume 11, Number 7 (2006), 781–812

ON MULTI-BUMP SEMI-CLASSICAL BOUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS WITH ELECTROMAGNETIC FIELDS

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(Submitted by: Reza Aftabizadeh)

Abstract. We consider the existence and asymptotic behavior of standing wave solutions to nonlinear Schrödinger equations with electromagnetic fields: $ih\frac{\partial\psi}{\partial t} = \left(\frac{h}{i}\nabla - A(x)\right)^2\psi + W(x)\psi - f(|\psi|^2)\psi$ on $\mathbb{R}\times\Omega$. $\Omega\subset\mathbb{R}^N$ is a domain which may be bounded or unbounded. For $h>0$ small we obtain the existence of multi-bump bound states $\psi_h(x,t) = e^{-iEt/h}u_h(x)$ where u_h concentrates simultaneously at possibly degenerate, non-isolated local minima of W as $h\to 0$. We require that $W\geq E$ and allow the possibility that $\{x\in\Omega : W(x)=E\}\neq\emptyset$. Moreover, we describe the asymptotic behavior of u_h as $h\to 0$.

1. Introduction

We are concerned with the nonlinear Schrödinger equation in the presence of an external electromagnetic field:

$$ih\frac{\partial\psi}{\partial t} = \left(\frac{h}{i}\nabla - A(x)\right)^2\psi + W(x)\psi - f(|\psi|^2)\psi, \quad (t,x)\in\mathbb{R}\times\Omega. \tag{1.1}$$

Here h is the Planck constant, i is the imaginary unit, $\Omega\subset\mathbb{R}^N$ is a domain which may be bounded or unbounded, having empty or smooth boundary. $A=(A_1,\dots,A_N):\Omega\to\mathbb{R}^N$ is a vector (magnetic) potential with magnetic field $B=\mathrm{curl}A$, $W:\Omega\to\mathbb{R}$ is a scalar (electric) potential.

Accepted for publication: April 2006.
AMS Subject Classifications: 35J60, 35B33.
[1]Supported by the Alexander von Humboldt foundation and ARC.
[2]Supported by the Alexander von Humboldt foundation and NSFC. No:10571069.

In this paper we are interested in standing wave solutions, i.e. solutions of the form

$$\psi(x,t) = \exp(-iEt/h)u(x), \tag{1.2}$$

where E is a real number, and $u : \Omega \to \mathbb{C}$ is a complex-valued function which satisfies

$$\Big(\frac{h}{i}\nabla - A(x)\Big)^2 u + (W(x) - E)u = f(|u|^2)u, \quad x \in \Omega. \tag{1.3}$$

The transition from quantum mechanics to classical mechanics can be formally performed by letting $h \to 0$. Thus the existence of solutions for h small, semi-classical solutions, has important physical interest. We say that a complex-valued function u defined on Ω is k-bump, if $|u|$ has exactly k local maxima in Ω. In this paper we consider the existence of semi-classical multi-bump solutions to problem (1.3).

In recent years, a lot of work has been devoted to investigating standing wave solutions of problem (1.1) in the case $A(x) \equiv 0$. In this case one is led to looking for positive solutions $u : \mathbb{R}^N \to \mathbb{R}$ of the semilinear elliptic equation

$$-h^2\Delta u + (W(x) - E)u = f(|u|^2)u, \quad x \in \Omega. \tag{1.4}$$

Different approaches have been taken to attack this problem. In [12] the existence of multi-peak bound states for nonlinear Schrödinger equations has been obtained under the condition that $\inf W > E$ and that there are bounded domains $\Lambda_j \subset \Omega = \mathbb{R}^N$, mutually disjoint, such that

$$\inf_{x\in\Lambda_j} W(x) < \inf_{x\in\partial\Lambda_j} W(x) \quad \text{for } j = 1, \dots, k. \tag{1.5}$$

It is proved that for $h > 0$ small, problem (1.4) has a solution u_h which has exactly k maximum points $x_h^j \in \Lambda_j$ satisfying $W(x_h^j) \to \inf_{x\in\Lambda_j} W(x)$ as $h \to 0$. Moreover, $u_h(x)$ is exponentially small for $x \in \mathbb{R}^N \setminus \bigcup_{j=1}^k \Lambda_j$. For more results in that direction we refer to [1, 3, 5, 13, 14, 16, 19, 20, 25, 26, 27, 29, 31, 32] and the references therein. These papers either yield the existence of solutions for arbitrary $h > 0$ or the existence of bump solutions for h sufficiently small.

There are only a few papers dealing with the magnetic case. The first one seems to be [15] where the existence of standing waves to problem (1.1) has been obtained for $h > 0$ fixed and for special classes of magnetic fields. If A and W are periodic functions, the existence of various types of solutions for fixed $h > 0$ has been proved in [2] by applying minimax arguments. Concerning semiclassical bound states, it is proved in [22] that for $h > 0$ small, (1.3) admits a least energy solution which concentrates near the global minimum of W. A multiplicity result for solutions of (1.3) has been obtained in [8] by using a topological argument. There it is also proved that the magnetic potential A only contributes to the phase factor of the solitary solutions of (1.1) for $h > 0$ sufficiently small. In [10] single-bump bound states of (1.3) have been obtained by using perturbation methods. These concentrate near a non-degenerate critical point of W as $h \to 0$.

In this paper, we consider the existence of multi-bump solutions of problem (1.1) which concentrate as $h \to 0$ simultaneously at local, possibly degenerate, possibly non-isolated local minimum points of W. The domain Ω may be quite arbitrary, the nonlinearity f is a general superlinear function, and we only require $W \geq E$. We also consider symmetric equations where the domain and the potentials are invariant under the action of a finite group $G \subset O(N)$ acting on $\mathbb{R}^N$. Related G-invariant problems have been considered in [2, 4, 7, 9], for instance. However in these papers the symmetry was essential for overcoming the lack of compactness of the associated functional. This is not the case here.

In order to obtain the existence of bump solutions we use variational methods and penalization techniques. The penalization argument is rather complicated compared with [6, 12]. We first modify the nonlinearity, and then add a penalty functional $P_h(u)$. This implies the Palais-Smale condition will hold for the modified functional and prevents the concentration outside of $\bigcup_{j=1}^k \Lambda_j$. We have to add yet another penalty functional $Q_h(u)$ in order to control the energy on the zero set of $W - E$ and to obtain the exponential decay estimate of the solution outside $\bigcup_{j=1}^k \Lambda_j$. We want to emphasize that due to the presence of the magnetic vector potential $A(x) \not\equiv 0$, problem (1.3) is a complex-valued equation and we have to find complex-valued solutions, which also makes the problem more complicated.

The paper is organized as follows: in Section 2, we state the conditions imposed on A, W and the nonlinear term f and present the main results of the paper. In Section 3 we construct the penalization functionals and prove that the modified functional admits a nontrivial critical point. In Section 4, we prove that this nontrivial critical point is actually a multi-bump solution of the original problem (1.3) and has the desired properties. An important technical assumption we need is that the least energy of a nontrivial complex-valued solution $u \in H^1(\mathbb{R}^N, \mathbb{C})$ of the limiting equation $-\Delta u + bu + f(|u|^2)u = 0$ is isolated for any $b > 0$. This will be discussed in Section 5 where we show in particular that this isolation condition is equivalent for real and complex least energy solutions.

Throughout the whole paper, we will use the letters C, c and C_j, $j \in \mathbb{N}$, to denote various positive constants, and we will use $O(t)$ and $o(t)$ to mean $|O(t)| \le C|t|$ and $o(t)/t \to 0$, respectively, as $t \to 0$. $o_t(1)$ will be used to denote quantities that tend to 0 as $t \to 0$. $Re\, w$ and $Im\, w$ stand for the real and the imaginary part, $\overline{w}$ for the complex conjugate, of the complex number w.

2. Main results

Setting $V(x) = W(x) - E$, we assume throughout the paper that:

(A) $A : \Omega \to \mathbb{R}^N$ is locally Hölder continuous.

(V_1) $V : \Omega \to \mathbb{R}$ is locally Hölder continuous and

$$\inf_{x\in\partial\Omega} V(x) > \inf_{x\in\Omega} V(x) \ge 0;$$

if Ω is unbounded we require in addition: $\liminf_{x\in\Omega,|x|\to\infty} V(x) > \inf_{x\in\Omega} V(x)$.

(V_2) There are bounded domains $\Lambda_1, \dots \Lambda_k \subset \Omega$, which are mutually disjoint and compactly contained in Ω, such that

$$0 < b_j := \inf_{\Lambda_j} V < \inf_{\partial\Lambda_j} V(x). \tag{2.1}$$

Let E_h be the Hilbert space defined by the completion of $C_0^\infty(\Omega, \mathbb{C})$ under the scalar product

$$(u,v)_h = Re \int_\Omega \Big(\frac{h}{i}\nabla u - A(x)u\Big)\overline{\Big(\frac{h}{i}\nabla v - A(x)v\Big)} + V(x)u\bar{v}\, dx$$

and the associated norm

$$\begin{aligned}\|u\|_{E_h} &= \Big(\int_\Omega \Big|\frac{h}{i}\nabla u - A(x)u\Big|^2 + V(x)|u|^2\, dx\Big)^{1/2} \\ &= \Big(\int_\Omega h^2|\nabla u|^2 + (|A(x)|^2 + V(x))u^2\, dx - 2Re\int_\Omega ihA(x)u\nabla\bar{u}\, dx\Big)^{1/2}.\end{aligned}$$

Setting $D^h u = (D_1^h u, \dots, D_N^h u)$ where $D_j^h = \frac{h}{i}\partial_j - A_j(x)$ we have

$$\|u\|_{E_h}^2 = \int_\Omega |D^h u|^2 + V(x)|u|^2\, dx.$$

Recall the following diamagnetic inequality (see [23] for example):

$$|D^h u(x)| \ge h|\nabla|u|(x)| \quad \text{for all } u \in E_h. \tag{2.2}$$

We want to solve the problem

$$\begin{cases} \Big(\frac{h}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u & x \in \Omega, \\ u(x) \in E_h(\Omega). \end{cases} \tag{2.3}$$

Our assumptions on the nonlinearity f are:

(f_1) $f:(0,\infty)\to\mathbb{R}$ is of class C^1, increasing, and $f(t)=O(t^\mu)$ for some $\mu>0$.

(f_2) $\lim_{t\to\infty}\frac{f(t^2)}{t^s}=0$ for some $s<4/(N-2)$ if $N\geq 3$, some $s>0$ if $N=1,2$.

(f_3) For some $2<\theta<s+2$ we have $0<\frac{\theta}{2}F(t)<f(t)t$ for all $t>0$ where $F(t)=\int_0^t f(\tau)d\tau$.

(f_4) For any $b>0$ the least energy of a nontrivial solution in $H^1(\mathbb{R}^N,\mathbb{C})$ of the limiting equation

$$\Delta u - bu + f(|u|^2)u = 0 \text{ in } \mathbb{R}^N \tag{2.4}$$

is isolated.

The energy functional associated with (2.3) is given by

$$I_h: E_h\to\mathbb{R},\quad I_h(u)=\frac{1}{2}\int_\Omega |D^h u|^2+V|u|^2\,dx-\frac{1}{2}\int_\Omega F(|u|^2). \tag{2.5}$$

It is not difficult to see that under the above assumptions on V and f, the non-trivial critical points of I_h correspond exactly to the classical solutions of problem (2.3) in E_h.

Theorem 2.1. *Suppose that (A), (V_1), (V_2) and (f_1) − (f_4) hold. Then there exists $h_0>0$ such that for every $0<h<h_0$, problem (2.3) admits a non-trivial k-bump solution $u_h\in E_h$. The k local maxima of $|u_h|$ are located at points $x_h^j\in\Lambda_j$, $j=1,\dots,k$, and satisfy $V(x_h^j)\to b_j=\inf_{\Lambda_j}V$ as $h\to 0$, $j=1,\dots,k$. Moreover there exist constants $\omega_j\in\mathbb{R}$ such that*

(i) $I_h(u_h)=\sum_{j=1}^k h^N c_j+o(h^N)$;

(ii) $u_h(x)=\sum_{j=1}^k U_j\left(\frac{x-x_h^j}{h}\right)e^{i(\omega_j+A(x_h^j)\cdot(x-x_h^j)/h)}+\epsilon(x)$ *where U_j is a least energy solution of (2.8) and $\epsilon\in E_h$ satisfies $\|\epsilon\|_{E_h}^2=o(h^N)$;*

(iii) *for each $\delta>0$ there exist $C,c>0$ such that*

$$|u_h(x)|\leq C\exp\left\{-\frac{c}{h}dist\left(x,\bigcup_{j=1}^k B_\delta(x_h^j)\right)\right\}.$$

Remark 2.2. a) In the case $k=1$ we obtain a single-bump solution. In this case, we do not need the isolation condition in (f_4).

b) From Proposition 5.1 and Remark 5.2 in the Appendix, the mountain pass solution of (2.4) is actually a least energy solution and the isolation of the least energy required in (f_4) is satisfied for a large class of nonlinearities f.

c) For $j=1,\dots,k$ let $w_j\in H^1(\mathbb{R}^N,\mathbb{C})$ be a least energy solution of the complex-valued problem

$$\Delta u - b_j u + f(|u|^2)u = 0,\quad u\in H^1(\mathbb{R}^N,\mathbb{C}) \tag{2.6}$$

with energy

$$\begin{aligned} c_j = I_{b_j}(w_j) &:= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_j|^2 + b_j|w_j|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} F(|w_j|^2) \\ &= \inf\{I_{b_j}(u) : I_j'(u)u = 0\}. \end{aligned} \tag{2.7}$$

It is well known [17] that under the conditions $(f_1) - (f_3)$, the least energy solutions of the real-valued problem

$$\begin{cases} \Delta w - b_j w + f(|w|^2)w = 0 & x \in \mathbb{R}^N, \\ w(0) = \max_{x\in\mathbb{R}^N} w(x), & \\ w(x) \to 0, & |x| \to \infty \end{cases} \tag{2.8}$$

are radially symmetric and decay exponentially at infinity. Employing the method in [22], we can show that the least energy associated to (2.8) is also c_j.

One can also obtain multibump solutions using symmetry. For a subgroup $G \subset O(N)$ and $x \in \mathbb{R}^N$ let $Gx := \{gx : g \in G\}$ be the orbit of x, and $G_x := \{g \in G : gx = x\}$ be the isotropy group of x. If $\Omega \subset \mathbb{R}^N$ is invariant under a finite group $G \subset O(N)$; that is, $x \in \Omega$ implies $gx \in \Omega$ for every $g \in G$, a map $\varphi : \Omega \to S$ into any set S is said to be invariant if $\varphi(gx) = \varphi(x)$ for every $g \in G$ and every $x \in \Omega$. Let $E_h^G := \{u \in E_h : u \text{ is invariant}\} \subset E_h$ be the subspace of invariant functions.

Theorem 2.3. *Suppose that $\Omega \subset \mathbb{R}^N$, $A : \Omega \to \mathbb{R}^N$ and $V : \Omega \to \mathbb{R}^N$ are invariant under a finite group $G \subset O(N)$. Suppose moreover that (A), (V_1), (V_2) with $k = 1$ and $(f_1) - (f_3)$ hold. Then there exists $h_0 > 0$ such that for every $0 < h < h_0$, problem* (2.3) *admits a non-trivial multibump solution $u_h \in E_h^G$. $|u_h|$ has a local maximum point $x_h \in \Lambda = \Lambda_1$ and all other maximum points lie on the orbit Gx. Hence $|u_h|$ has precisely $|Gx| = |G|/|G_x|$ local maximum points gx_h, $g \in G$. $V(x_h) = V(gx_h) \to b = \min_\Lambda V$ as $h \to 0$. Moreover there exists a constant $\omega \in \mathbb{R}$ and a least energy solution U of* (2.8) *with $b = b_1$ such that*

(i) *$I_h(u_h) = |Gx|c_j h^N + o(h^N)$;*
(ii) *$u_h(x) = \sum_{Gx_h} U\big(\frac{x-gx_h}{h}\big)e^{i(\omega + A(gx_h)\cdot(x-gx_h)/h)} + \epsilon(x)$ where U is a least energy solution of* (2.8) *and $\epsilon \in E_h^G$ satisfies $\|\epsilon\|^2_{E_h} = o(h^N)$;*
(iii) *for each $\delta > 0$ there exist $C, c > 0$ such that*

$$|u_h(x)| \le C\exp\Big\{-\tfrac{c}{h} dist\Big(x, \textstyle\bigcup_{Gx} B_\delta(gx_h)\Big)\Big\}.$$

Remark 2.4. a) Observe that we do not need hypothesis (f_4) in Theorem 2.3.

b) The number of bumps of the solutions obtained in Theorem 2.2 depends on the interplay between the group action and the potential V. It can be determined precisely if all orbits of minimum points of V in Λ have the same cardinality k: $V(x) = \min_\Lambda V$ implies $|Gx| = k$. This is the case if all such minimum points have the same isotropy group H: $V(x) = \min_\Lambda V$ implies $G_x = H$.

3. Preliminaries

First we introduce some notation. For any given set $M \subset \Omega$ and any $r > 0$ we define $M^r = \{x \in \Omega : dist(x, M) \leq r\}$, $M_h = \{x \in \mathbb{R}^N : hx \in M\}$, and $M_h^r := (M^r)_h$. χ_M denotes the characteristic function of M.

We may assume that there exists $r > 0$ with

$$0 < \inf_{x \in \Lambda_j^{2r}} V(x) < \inf_{x \in \partial\Lambda_j^{2r}} V(x),$$

$\Lambda_j^{2r} \bigcap Z^{2r} = \emptyset$, and $\Lambda_j^{4r} \bigcap \Lambda_m^{4r} = \emptyset$ for $j, m = 1, \ldots, k$, $j \neq m$. We set $\Sigma = \bigcup_{j=1}^k \Lambda_j^{2r}$, $Z := \{x \in \Omega : V(x) = 0\}$, and $\gamma := \inf\{V(x) : x \in \Omega \setminus Z^{2r}\}$. We may also assume that $\liminf_{|x|\to\infty} V(x) \geq 3\gamma$ and $\inf_{x\in\partial\Omega} V(x) \geq 3\gamma$. Finally, we fix

$$0 < \nu < \frac{\gamma}{2}. \tag{3.1}$$

From (f_1) and (f_3) we see that there exists an $a > 0$ such that

$$f(t^2) \leq \nu \quad \text{for } t \in [0, a], \quad \text{and } f(t^2) \geq \nu \quad \text{for } t \geq a.$$

Now we define

$$\tilde{f}(t) = \min\{f(t), \nu\}, \tag{3.2}$$

and

$$g(x, t) = \chi_\Sigma f(t) + (1 - \chi_\Sigma)\tilde{f}(t). \tag{3.3}$$

Setting $G(x, t) = \int_0^t g(x, s)ds$ it is easy to see that

$$G(x, t) \leq g(x, t)t. \tag{3.4}$$

We consider the modified functional

$$L_h : E_h \to \mathbb{R}, \quad L_h(u) = \frac{1}{2}\int_\Omega \left(|D^h u|^2 + V|u|^2\right) - \frac{1}{2}\int_\Omega G(x, |u|^2). \tag{3.5}$$

By (f_4) there exists $\theta_j > 0$ such that for any solution u of problem (2.4), either

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b_j|u|^2\right) - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2) = c_j \tag{3.6}$$

or

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + b_j|u|^2\right) - \frac{1}{2}\int_{\mathbb{R}^N} F(|u|^2) > c_j + \theta_j. \tag{3.7}$$

Here c_j, $j = 1, \ldots, k$, is defined in (2.7).

For $j = 1, \dots, k$ we define the functional

$$L_h^j(u) = \frac{1}{2}\int_{\Lambda_j^{4r}} \Big(|D^h u|^2 + V|u|^2\Big) - \frac{1}{2}\int_{\Lambda_j^{4r}} G(x, |u|^2). \tag{3.8}$$

We also choose $\sigma_j \in (0, \theta_j)$ satisfying

$$\sum_{j=1}^{k} \sigma_j < \frac{1}{2}\min\{c_j : j = 1, \dots, k\}. \tag{3.9}$$

Let M_1 be a positive constant that will be determined later, and define

$$P_h(u) = M_1 \sum_{j=1}^{k} \Big(\Big((L_h^j(u))_+^{1/2} - h^{N/2}(c_j + \sigma_j)^{1/2}\Big)_+\Big)^2, \tag{3.10}$$

and

$$Q_h(u) = M_2\Big(\Big(\Big(\int_{Z^{2r}} |u|^2\Big)^{1/2} - h^{\frac{3}{\mu}+\frac{N}{2}}\Big)_+\Big)^2, \tag{3.11}$$

where $M_2 > 2\nu$ is fixed and $v_+ := \max\{v, 0\}$.

Finally we define the penalized functional $J_h : E_h \to \mathbb{R}$ as follows

$$J_h(u) = L_h(u) + P_h(u) + Q_h(u). \tag{3.12}$$

J_h is of class C^1 because the functionals L_h, P_h and Q_h are of class C^1. The following compactness lemma holds.

Lemma 3.1. *Let $\{u_n\}$ be a sequence in E_h such that $J_h(u_n)$ is bounded and $J_h'(u_n) \to 0$, then u_n has a convergent subsequence.*

Proof. Firstly, we claim that $\{u_n\}$ is bounded in E_h. Indeed, (3.3) implies

$$\begin{aligned}
&L_h(u_n) - \frac{1}{\theta}L_h'(u_n)u_n \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\int_\Omega \big(|D^h u_n|^2 + V|u_n|^2\big) - \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\nu \int_{\Omega\setminus\Sigma} |u_n|^2, \\
&P_h(u_n) - \frac{1}{\theta}P_h'(u_n)u_n \geq -M_1 \sum_{i=1}^{k} h^{N/2}(c_i + \sigma_i)^{1/2} P_h(u_n)^{1/2}, \\
&Q_h(u_n) - \frac{1}{\theta}Q_h'(u_n)u_n = \Big(1 - \frac{2}{\theta}\Big)Q_h(u_n) - \frac{2M_2^{1/2}}{\theta}Q_h^{1/2}(u_n)h^{\frac{3}{\mu}+\frac{N}{2}}.
\end{aligned}$$

Since $P_h(u_n) \leq M_1 \int_\Omega \Big(|D^h u_n|^2 + V|u_n|^2\Big)$ we have

$$C_h o(\|u_n\|_{E_h}) \geq J_h(u_n) - \frac{1}{\theta}J_h'(u_n)u_n \tag{3.13}$$

$$\geq (\frac{1}{2}-\frac{1}{\theta})\int_\Omega \Big(|D^h u_n|^2 + V|u_n|^2\Big) - (\frac{1}{2}-\frac{1}{\theta})\nu\int_{\Omega\setminus\Sigma}|u_n|^2$$
$$- C(h)\Big(\int_\Omega \Big(|D^h u_n|^2 + V|u_n|^2\Big)\Big)^{12} + \Big(1-\frac{2}{\theta}\Big)Q_h(u_n)$$
$$- \frac{2M_2^{1/2}}{q}Q_\varepsilon^{1/2}(u_n)h^{\frac{3}{\mu}+\frac{N}{2}}.$$

By the definition of Q_h, we can easily deduce from (3.13) that

$$C_h o(\|u_n\|_{E_h}) \geq \int_{\mathbb{R}^N}\Big(h^2|\nabla u_n|^2 + V|u_n|^2\Big) + Q_h(u_n), \tag{3.14}$$

which implies that $\{u_n\}$ is bounded in E_h.

Now we choose a subsequence, still denoted by $\{u_n\}$, which converges weakly to u in E_h. In order to complete the proof of the lemma, we only need to prove that this convergence is strong in E_h. As done in [8], it suffices to show that, given $\varepsilon > 0$, there exists an $R > 0$ such that

$$\limsup_{n\to\infty}\int_{\Omega\setminus B_R}\Big(|D^h u_n|^2 + V(x)|u_n|^2\Big) < \varepsilon, \tag{3.15}$$

where B_R denotes the ball with center 0 and radius R.

Without loss of generality we may assume that $\Sigma\cup Z^{2r} \subset B_{R/2}$. Let η_R be a cut-off function such that $\eta_R = 0$ on $B_{R/2}$, $\eta_R = 1$ on $\Omega\setminus B_R$, $0 \leq \eta_R \leq 1$ and $|\nabla\eta_R| \leq c/R$. Taking $\eta_R u_n$ as test functions, we have

$$o(\|u_n\|_{E_h}) = \langle J_h'(u_n), \eta_R u_n\rangle$$
$$\geq Re\Big\{\int_\Omega \Big(|D^h u_n|^2\eta_R + ihD^h u_n\nabla\eta_R\bar{u}_n + V(x)|u_n|^2\eta_R\Big)$$
$$+2M_2\Big(\Big(\int_{\Omega\setminus B_R(0)}|u_n|^2\Big)^{\frac{1}{2}} - h^{\frac{3}{\mu}+\frac{N}{2}}\Big)$$
$$+\Big(\int_{\Omega\setminus B_R(0)}|u_n|^2\Big)^{\frac{1}{2}} - \int_\Omega g(x,|u_n|^2)|u_n|^2\eta_R\Big\}$$

and

$$\int_\Omega |D^h u_n|^2\eta_R + V(x)|u_n|^2\eta_R + Re\int_\Omega ihD^h u_n\nabla\eta_R\bar{u}_n$$
$$\leq \int_\Omega g(x,|u_n|^2)|u_n|^2\eta_R + o(\|u_n\|_{E_h}) \leq \nu\int_\Omega V(x)|u_n|^2\eta_R + o(\|u_n\|_{E_h}).$$

Finally, the Hölder inequality implies

$$\int_{\Omega\setminus B_R} |D^h u_n|^2 + V|u_n|^2 \le \frac{c}{R}\|u_n\|_{L^2(\Omega)}\|D^h u_n\|_{L^2(\Omega)} + o(\|u_n\|_{E_h}),$$

which proves (3.15). □

Now we can apply methods from critical point theory. We consider the class Γ of continuous functions $\gamma : [0,1]^k \to E_h$ such that there are continuous functions $g_j : [0,1] \to E_h$ for $j = 1,\dots,k$ satisfying

(i) $g_j(0) = 0$, $L_h(g_j(1)) < 0$;
(ii) $\mathrm{supp} g_j(t) \subset \Lambda_j^{2r}$ for all $t \in [0,1]$;
(iii) $\gamma(\tau_1,\dots,\tau_k) = \sum_{j=1}^k g_j(\tau_j)$ for all $\tau = (\tau_1,\dots,\tau_k) \in \partial[0,1]^k$;
(iv) $J_h(\gamma(\tau)) \le h^N(\sum_{j=1}^k c_j - \sigma_0)$ for all $\tau \in \partial[0,1]^k$, where $\sigma_0 > 0$ satisfies $\sigma_0 < \frac{1}{2}\min\{c_j \,|\, j = 1,\dots,k\}$.

As we will see in the proof of Lemma 3.3 below, Γ is non-empty. The min-max value associated with the class Γ is given by:

$$C_h = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]^k} J_h(\gamma(\tau)). \tag{3.16}$$

In order to estimate C_h we need to study an auxiliary Neumann problem. We consider the functional L_h^j on $E_h(\Lambda_j^{4r})$. Let Γ_j be the class of all continuous paths $\gamma_j : [0,1] \to E_h(\Lambda_j^{4r})$ such that $\gamma_j(0) = 0$, $L_h^j(\gamma_j(1)) < 0$ and define

$$d_h^j = \inf_{\gamma_j\in\Gamma_j} \sup_{t\in[0,1]^k} L_h^j(\gamma_j(t)). \tag{3.17}$$

We have the following estimate for d_h^j

Lemma 3.2. $d_h^j = h^N(c_j + o(1))$ *as* $h \to 0$.

The proof of the lemma is postponed to the Appendix.

Lemma 3.3. $C_h = h^N\Big(\sum_{j=1}^k c_j + o(1)\Big)$ *as* $h \to 0$.

Proof. Since c_j is the mountain pass value for the limiting functional I^{b_j}, given any $\epsilon > 0$ there exists a path $\gamma_j : [0,1] \to H^1(\mathbb{R}^N,\mathbb{C})$ such that $\gamma_j(0) = 0$, $I^{b_j}(\gamma_j(1)) < 0$ and

$$c_j \le \max_{\tau\in[0,1]} I^{b_j}(\gamma_j(\tau)) \le c_j + \frac{\epsilon}{2k}.$$

Now for $h > 0$ we define the path $\tilde\gamma_j : [0,1] \to E_h$ as

$$\tilde\gamma_j(\tau)(x) = \eta_j(x)\gamma_j\Big(\frac{x-x_j}{h}\Big)e^{i\frac{x-x_j}{h}A(x_j)}, \quad x \in \Omega.$$

Here $x_j \in \Lambda_j$ is chosen such that $V(x_j) = b_j$, and η_j is a C^∞ cut-off function with compact support in Λ_j^{2r}, taking the value 1 except for a small neighborhood of $\partial\Lambda_j^{2r}$. With similar methods as in [12] and [22], it is not difficult to check that

$$L_h(\tilde{\gamma}_j(\tau)) = h^N(I^{b_j}(\gamma_j(\tau)) + o(1)), \quad \text{for all } \tau \in [0,1].$$

Now arguing as in [6] we obtain a path $\gamma \in \Gamma$ such that

$$C_h \le \sup_{\tau\in[0,1]^k} J_h(\gamma(\tau)) \le h^N\Big(\sum_{j=1}^k c_j + o(1)\Big).$$

It remains to prove the lower estimate. Observe that given $\gamma \in \Gamma$ and a continuous map $c : [0,1] \to [0,1]^k$ with $c(0) \in \{0\} \times [0,1]^{k-1}$ and $c(1) \in \{1\} \times [0,1]^{k-1}$, we have $\gamma_1 = \gamma \circ c|_{\Lambda_1^{2r}} \in \Gamma_1$. Moreover, Lemma 3.2 implies

$$\sup_{t\in[0,1]} L_h^1(\gamma_1(t)) \ge h^N(c_1 + o(1)).$$

We have an inequality of this form for every $j = 1, \dots, k$. Thus we can repeat the argument of Coti-Zelati and Rabinowitz [9, Proof of Proposition 3.4] and obtain, for every path $\gamma \in \Gamma$, the existence of a point $\hat{\tau} \in [0,1]^k$ satisfying

$$L_h^j(\gamma(\hat{\tau})) \ge h^N(c_j + o(1)) \quad \text{for } j = 1, \dots, k.$$

Now the definitions of Q_h, g, and the choice of M_2 yield

$$Q_h(u) - \frac{1}{2}\int_{Z^{2r}} G(x, |u|^2) \ge -2M_2 h^{N+6/\mu}. \tag{3.18}$$

Consequently

$$\begin{aligned} \sup_{\tau\in[0,1]^k} J_h(\gamma(\tau)) &\ge \sup_{\tau\in[0,1]^k} \sum_{j=1}^k L_h^j(\gamma(\tau)) - 2M_2 h^{N+6/\mu} \qquad (3.19)\\ &\ge \sum_{j=1}^k L_h^j(\gamma(\hat{\tau})) - 2M_2 h^{N+6/\mu} \ge h^N\Big(\sum_{j=1}^k c_j + o(1)\Big), \end{aligned}$$

which is exactly the required lower estimate. □

Lemma 3.4. *There exists $C > 0$ such that for $h > 0$ sufficiently small $\|u_h\|_{E_h} \le Ch^{N/2}$.*

Proof. The proof is similar to that of the boundedness part of Lemma 3.1. □

4. Proofs of the main results

Since J_h satisfies the Palais-Smale condition by Lemma 3.1 there exists a critical point $u_h \in E_h$ of J_h with $J_h(u_h) = C_h$ and $J_h'(u_h) = 0$.

We define the local weights

$$\rho_h^j = M_1\Big((L_h^j(u))_+^{1/2} - h^{N/2}(c_j+\sigma_j)^{1/2}\Big)_+ (L_h^j(u))_+^{-1/2},$$

and

$$\hat{\psi}_h = 2M_2\bigg(\Big(\int_{Z^{2r}} |u|^2\Big)^{1/2} - h^{\frac{3}{\mu}+\frac{N}{2}}\bigg)_+ \Big(\int_{Z^{2r}} |u|^2\Big)^{-1/2}$$

and then the functions

$$\rho_h = \sum_{j=1}^k \rho_h^j \chi_{\Lambda_j^{4r}}, \qquad \psi_h = \hat{\psi}_h \chi_{Z^{2r}}.$$

The critical point u_h satisfies in the weak sense

$$\begin{aligned} &-\operatorname{div}\Big[(1+\rho_h)\Big(h^2\nabla u_h + \frac{Ah}{i}u_h\Big)\Big] - (1+\rho_h)\Big(\frac{h}{i}A\nabla u_h - |A|^2 u_h\Big) \\ &\quad + \Big((1+\rho_h)V + \psi_h\Big)u_h - (1+\rho_h)g(x,|u_h|)u_h = 0 \end{aligned} \tag{4.1}$$

and

$$\int_\Omega \Big((1+\rho_h)(D^h u_h \overline{D^h\varphi} + V u_h\bar\varphi) + \psi_h u_h \bar\varphi\Big) = \int_\Omega (1+\rho_h) g(x,|u_h|)u_h\bar\varphi$$

for all $\varphi \in E_h$. In order to show that u_h satisfies the original problem (2.3), it suffices to prove $\rho_h \equiv 0$ and $\psi_h \equiv 0$.

Firstly, we prove that u_h cannot concentrate outside the sets Σ and Z^{2r}. We rescale the function u_h as $v_h(x) = u_h(hx)$ for $x \in \Omega_h$. Given $R > 0$, define $N_R(\Lambda_h)$ to be the set $\{x : \operatorname{dist}(x,\Lambda_h) < R\}$ for any set $\Lambda \subset \Omega$.

Lemma 4.1. *There exists $C > 0$ such that for any given $R > 0$,*

$$\int_{\Omega_h\setminus(N_R(\Sigma_h)\cup Z_h^{2r})} \Big(\Big|\frac{1}{i}\nabla v_h - A(hx)v_h\Big|^2 + V(hx)|v_h|^2\Big) \le C\Big(h^{6/\mu} + 1/R\Big) \tag{4.2}$$

for $h \to 0$.

Proof. It follows from (4.1) that

$$\begin{aligned} &-\operatorname{div}\Big[(1+\rho_h(hx))\Big(\nabla v_h + \frac{A(hx)}{i}v_h\Big)\Big] \\ &- (1+\rho_h(hx))\Big(\frac{1}{i}A(hx)\nabla v_h - |A(hx)|^2 v_h\Big) \\ &+ \Big((1+\rho_h(hx))V(hx) + \psi_h(hx)\Big)v_h - (1+\rho_h(hx))g(hx,|v_h|)v_h = 0. \end{aligned} \tag{4.3}$$

For given $R > 0$ and $h > 0$, we choose smooth cut-off functions $0 \le \zeta^j_{R,h}(x) \le 1$ satisfying

$$\zeta^j_{R,h}(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, (\Lambda^{2r}_j)_h) < R/2, \\ 0, & \text{if } \operatorname{dist}(x, (\Lambda^{2r}_j)_h) > R, \end{cases} \tag{4.4}$$

and $|\nabla \zeta^j_R| \le C/R$. We set $\eta_R = 1 - \sum_{j=1}^k \zeta^j_{R,h}$ and use $\overline{\eta_R v_h}$ as a test function in (4.3). This yields

$$\int_{\Omega_h \setminus \Sigma_h} (1 + \rho_h(hx))\Big(\Big|\frac{1}{i}\nabla v_h - A(hx)v_h\Big|^2 + (V(hx) - \tilde{f}(|v_h|^2))|v_h|^2\Big)\eta_R$$
$$+ \int_{\Omega_h \setminus \Sigma_h} \psi_h(hx)|v_h|^2\eta_R - Re\int_{\Omega_h \setminus \Sigma_h}(1 + \rho_h(hx))(\nabla v_h - iA(hx)v_h)\nabla\eta_R\bar{v}_h = 0.$$

If $\int_{Z^{2r}} |u_h|^2 > 4h^{N+6/\mu}$, then $\hat{\psi}_h > M_2$ for h small. As a consequence, $V(x) + \psi_h(x) - \tilde{f}(|v_h|^2)$ has a positive lower bound for $x \in \Omega$. Now by (3.1), (3.2), Lemma 3.4, the uniform boundedness of ρ_h and ψ_h, and by the Hölder inequality,

$$\int_{\Omega_h \setminus N_R(\Sigma_h)} \Big|\frac{1}{i}\nabla v_h - A(hx)v_h\Big|^2 + V(hx)|v_h|^2 \le C/R. \tag{4.5}$$

If $\int_{Z^{2r}} |u_h|^2 \le 4h^{N+6/\mu}$ then the Hölder inequality yields

$$\int_{\Omega_h \setminus (N_R(\Sigma_h) \bigcup Z^{2r}_h)} \Big|\frac{1}{i}\nabla v_h - A(hx)v_h\Big|^2 + V(hx)|v_h|^2 \le Ch^{6/\mu} + C/R.$$

In both cases we have (4.2), so the proof of the lemma is complete. □

Next we prove a lemma which implies $\rho_h \equiv 0$:

Lemma 4.2. *There exists $C > 0$ such that if $M_1 > C$ in (3.10), then*

$$\limsup_{h \to 0} L^j_h(u_h)h^{-N} \le c_j + \frac{1}{2}\sigma_j, \quad \text{for } j = 1, \dots, k.$$

In order to verify this lemma, we need the following preliminary result which will be proved in the Appendix.

Lemma 4.3. *Suppose that $A_0 = (a_1, \dots, a_N) \in \mathbb{R}^N$ is a constant vector and $b > \nu$. Let $v \in H^1(\mathbb{R}^N, \mathbb{C}) \cap C(\mathbb{R}^N, \mathbb{C})$ be a solution of the equation*

$$\Big(\frac{1}{i}\nabla - A_0\Big)^2 v + bv = \chi_{\{x_1 < 0\}} f(|v|^2)v + \chi_{\{x_1 > 0\}}\tilde{f}(|v|^2)v. \tag{4.6}$$

Then $|v| \le a$ for $x_1 > 0$, hence v actually satisfies

$$\Big(\frac{1}{i}\nabla - A_0\Big)^2 v + bv = f(|v|^2)v. \tag{4.7}$$

In the proof of Lemma 4.2 we shall use the fact that for $A_0 \in \mathbb{R}^N$ and $b > 0$ fixed, the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} \Big|\frac{1}{i}\nabla u - A_0 u\Big|^2 + b|u|^2\,dx$$

is equivalent to the usual norm on $H^1(\mathbb{R}^N, \mathbb{C})$. We shall also use the fact that on a bounded domain $K \subset \Omega$ the norm

$$\|u\|^2 := \int_K \Big|\frac{1}{i}\nabla u - A(hx)u\Big|^2 + V(hx)|u|^2\,dx$$

is equivalent to the usual norm on $H^1(K, \mathbb{C})$. A proof can be found in [2, Lemma 2.3].

Proof of Lemma 4.2. We argue by way of contradiction. Suppose that for a sequence $h_n \to 0$ we have critical points u_{h_n} of E_{h_n} such that for certain $j \in \{1, \dots, k\}$

$$\limsup_{n\to\infty} L^j_{h_n}(u_{h_n})h_n^{-N} > c_j + \frac{1}{2}\sigma_j. \tag{4.8}$$

Then we first claim that there exist $S > 0$ and $\alpha > 0$ such that

$$\sup_{y\in(\Lambda_j^{2r})_{h_n}} \int_{B_S(y)} |v_{h_n}|^2 \geq \alpha \quad \text{for } n \geq n_0. \tag{4.9}$$

In fact, we observe from (4.8) that there exists $\beta > 0$ such that

$$\int_{(\Lambda_j^{4r})_{h_n}} \Big(\Big|\frac{1}{i}\nabla v_{h_n} - A(h_n x)v_{h_n}\Big|^2 + V(h_n x)|v_{h_n}|^2\Big) \geq 2\beta.$$

Now Lemma 4.1 implies that for all $R > 0$ large and $h_n > 0$ small

$$\int_{N_R((\Lambda_j^{2r})_{h_n})} \Big(\Big|\frac{1}{i}\nabla v_{h_n} - A(h_n x)v_{h_n}\Big|^2 + V(h_n x)|v_{h_n}|^2\Big) \geq \beta. \tag{4.10}$$

If (4.9) is false, then we may assume that for all $S > 0$ we have

$$\sup_{y\in(\Lambda_j^{2r})_{h_n}} \int_{B_S(x)} |v_{h_n}|^2 \to 0 \qquad \text{as } n \to \infty. \tag{4.11}$$

Let $\zeta^j_{R,h}(x)$ be defined by (4.4) and set $v_n^R = \zeta^j_{2R,h_n}(x)v_{h_n}$. Then (4.11) implies

$$\sup_{x\in\mathbb{R}^N} \int_{B_S(x)} |v_n^R|^2 \to 0 \qquad \text{as } n \to \infty \tag{4.12}$$

for all $S > 0$. The concentration compactness principle and the method used to prove Lemma 2.18 in [9] imply that for each $R > 0$,

$$\int_{\mathbb{R}^N} |v_n^R|^p \to 0 \qquad \text{as } n \to \infty, \text{ for all } p \in (2, 2N/(N-2)).$$

Consequently,

$$\int_{N_R((\Lambda_j^{2r})_{h_n})} |v_{h_n}|^{s+2} \to 0 \qquad \text{as } n \to \infty,$$

where s is as in (f_2). Using $\overline{v_n^R}$ as a test function in (4.3), we obtain

$$\begin{aligned}
&\int_{(\Lambda_j^{4r})_{h_n}} \Big(\Big|\frac{1}{i}\nabla v_{h_n} - A(h_n x)v_{h_n}\Big|^2 + V(h_n x)|v_{h_n}|^2\Big)\zeta_{2R,h_n}^j \\
&\qquad + \int_{(\Lambda_j^{4r})_{h_n}} \psi_{h_n}(h_n x)|v_{h_n}|^2 \zeta_{2R,h_n}^j \\
&= \int_{(\Lambda_j^{4r})_{h_n}} g(h_n x, |v_{h_n}|^2)|v_{h_n}|^2 \zeta_{2R,h_n}^j \\
&\qquad + Re \int_{(\Lambda_j^{4r})_{h_n}} (\nabla v_{h_n} - iA(h_n x)v_{h_n})\nabla \zeta_{2R,h_n}^j \bar{v}_{h_n} \\
&\le \int_{N_R((\Lambda_j^{2r})_{h_n})} |v_{h_n}|^{s+2} + C/R
\end{aligned}$$

which contradicts (4.10) if we choose R and n sufficiently large. Thus we have proved the claim (4.9).

Suppose the sequence $y_n \in (\Lambda_j^{2r})_{h_n}$ satisfies

$$\int_{B_S(y_n)} |v_{h_n}|^2 \ge \alpha > 0 \ \text{ for } \ n \ge n_0. \tag{4.13}$$

Setting $x_n = h_n y_n$ and $v_n = v_{h_n}(y_n + x) = u_{h_n}(x_n + h_n x)$, we have $x_n \in \Lambda_j^{2r}$ and v_n satisfies for $x \in (\Lambda_j'^{4r})_{h_n}$

$$\Big(\frac{1}{i}\nabla - A(x_n + h_n x)\Big)^2 v_n + V(x_n + h_n x)v_n - g(x_n + h_n x, |v_n|^2)v_n = 0, \tag{4.14}$$

where $(\Lambda_j'^{4r})_{h_n} = \{x \in \mathbb{R}^N : x_n + h_n x \in \Lambda_j^{4r}\}$. By Lemma 3.4, the sequence $\{v_n\}$ is bounded in $E(\mathbb{R}^N)$, and the diamagnetic inequality (2.2) immediately yields that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$. Therefore, up to a subsequence, it converges weakly in $H^1(\mathbb{R}^N, \mathbb{R})$ and locally strongly in any $L^p(\mathbb{R}^N, \mathbb{R})$, $1 \le p < 2N/(N-2)$ if $N \ge 3$, and $1 \le p < \infty$ if $N = 1, 2$. Moreover, for each compact subset $K \subset \mathbb{R}^N$ $\{v_n\}$ is bounded in $H^1(K, \mathbb{C})$. We may now use the subsolution estimate [18, Theorem 8.17] in order to get that $\{v_n\}$ is also bounded in $L^\infty_{loc}(\mathbb{R}^N)$ and hence in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ by Schauder's estimate. Thus $\{v_n\}$ converges along a subsequence to $v \in C^2(K)$ and $v \not\equiv 0$ by our assumption. Now the sequence of functions $\chi_\Lambda(x_{h_n} + h_n x)$ can also be assumed to converge weakly in any L^p over K to a function χ with $0 \le \chi \le 1$. Suppose $x_n \to \tilde{x} \in \Lambda_j^{2r}$. Then $v \in H^1(\mathbb{R}^N)$ solves the limit equation

$$\Big(\frac{1}{i}\nabla - A(\tilde{x})\Big)^2 v + V(\tilde{x})v - \hat{g}(x,|v|^2)v = 0 \quad \text{for } x \in \mathbb{R}^N, \tag{4.15}$$

where $\hat{g}(x,t) = \chi(x)f(t) + (1-\chi(x))\tilde{f}(t)$.

If $\mathrm{dist}(y_n, \partial(\Lambda_j^{2r})_{h_n}) \to \infty$, then (4.15) is exactly the equation

$$\Big(\frac{1}{i}\nabla - A(\tilde{x})\Big)^2 v + V(\tilde{x})v - f(|v|^2)v = 0. \tag{4.16}$$

Setting $\tilde{v}(x) = e^{-iA(\tilde{x})x}v(x)$ it follows from (4.13) that $\tilde{v}(x) \not\equiv 0$ and $\tilde{v}(x) \in H^1(\mathbb{R}^N, \mathbb{C})$ solves

$$-\Delta\tilde{v} + V(\tilde{x})\tilde{v} - f(|\tilde{v}|^2)\tilde{v} = 0. \tag{4.17}$$

If $\mathrm{dist}(y_n, \partial(\Lambda_j^{2r})_{h_n}) \le C < \infty$ then (4.15) has the form (4.6) and Lemma 4.3 yields that v solves (4.16) and $\tilde{v}$ solves (4.17). In both cases $\tilde{v}$ is a critical point of the functional $I^{V(\tilde{x})}$.

Now we distinguish two cases:
(i) $\tilde{v}$ is a least energy solution of (4.17);
(ii) $\tilde{v}$ is not a least energy solution of (4.17).
In case (i), by making r and Λ_j smaller if necessary, we may assume

$$c_j \le I^{V(\tilde{x})}(\tilde{v}) \le c_j + \frac{1}{2}\sigma_j.$$

It is not difficult to check as in [12], that along a subsequence, there is a sequence $R_n \to \infty$ such that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2}\int_{B_{R_n}(y_n)} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big)\\
&= \frac{1}{2}\int_{\mathbb{R}^N} \Big(\Big|\frac{1}{i}\nabla v - A(\tilde{x})v\Big|^2 + V(\tilde{x})|v|^2 - F(|v|^2)\Big)\\
&= \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla\tilde{v}|^2 + V(\tilde{x})|\tilde{v}|^2 - F(|\tilde{v}|^2)\Big) = I^{V(\tilde{x})}(\tilde{v}) \le c_j + \frac{1}{3}\sigma_j.
\end{aligned} \tag{4.18}$$

Thus by (4.8) there exists $\delta > 0$ such that for all large n

$$\int_{(\Lambda_j^{4r})_{h_n}\setminus B_{R_n}(y_n)} \Big|\frac{1}{i}\nabla v_{h_n} - A(hx)v_{h_n}\Big|^2 + V(hx)|v_{h_n}|^2 \ge \delta > 0. \tag{4.19}$$

Repeating the procedure to obtain (4.16), we deduce that there exist $S > 0$ and a sequence $\hat{y}_n \in (\Lambda_j^{2r})_{h_n} \setminus B_{R_n}(y_n)$ such that $\int_{B_S(\hat{y}_n)} |v_{h_n}|^2 \geq \alpha > 0$, and, after passing to a subsequence, $v_{h_n}(\hat{y}_n + x)$ converges in $C^2_{loc}(\mathbb{R}^N)$ to a non-zero $\hat{v}$. Furthermore, $\hat{v}$ solves the equation

$$\Big(\frac{1}{i}\nabla - A(\hat{x})\Big)^2 \hat{v} + V(\hat{x})\hat{v} - f(|\hat{v}|^2)\hat{v} = 0,$$

and $\check{v} = e^{-iA(\hat{x})x}\hat{v}(x) \in H^1(\mathbb{R}^N, \mathbb{C})$ solves

$$-\Delta\check{v} + V(\hat{x})\check{v} - f(|\check{v}|^2)\check{v} = 0.$$

where $\hat{x} = \lim_{n\to\infty} h_n\hat{y}_n \in \Lambda_j^{2r}$. Hence, $I^{V(\hat{x})}(\check{v}) \geq c_j$.

Now we claim

$$\lim_{n\to\infty} L^j_{h_n}(u_{h_n})h_n^{-N} \geq I^{V(\tilde{x})}(\tilde{v}) + I^{V(\hat{x})}(\check{v}) \geq 2c_j. \tag{4.20}$$

To prove (4.20), firstly, we show that

$$\max_{x\in\partial(\Lambda_j^{4r})_{h_n}} v_{h_n}(x) \to 0 \quad \text{as } n \to \infty. \tag{4.21}$$

It is sufficient to show that $\max_{x\in\partial\Lambda_j^{4r}} u_{h_n}(x) \to 0$ as $n \to \infty$. Suppose to the contrary that there exist subsequences, still denoted by $\{h_n\}$, and $\{y_n\} \subset \Lambda_j^{4r}$, such that $h_n \to 0$, $y_n \to y_0 \in \partial\Lambda_j^{4r}$ as $n \to \infty$, and $u_{h_n}(y_n) \geq \delta > 0$. Choose $\rho > 0$ such that $B_\rho(y_0) \subset \Omega \setminus (\Lambda_1^{2r} \cup \cdots \cup \Lambda_k^{2r} \cup Z^{2r})$. We may assume $\{y_n\} \subset B_\rho(y_0)$. Using the above scaling technique on $B_\rho(y_0)$, it is easy to prove that $w_n(x) := u_{h_n}(y_n + h_nx)$ converges in C^2 on any compact set to some function w. Moreover w satisfies

$$-\Delta w - \frac{2}{i}A(y_0)\nabla w + |A(y_0)|^2 w + V(y_0)w - \tilde{f}(|w|^2)w = 0.$$

By Lemma 3.4 and Fatou's lemma, $w \in H^1(\mathbb{R}^N, \mathbb{C})$, hence

$$\int_{\mathbb{R}^N} \Big|\frac{1}{i}\nabla w - A(y_0)w\Big|^2 + V(y_0)|w|^2 = \int_{\mathbb{R}^N} \tilde{f}(|w|^2)|w|^2.$$

But $\tilde{f}(|w|^2) \leq \nu$, so

$$\int_{\mathbb{R}^N} \Big|\frac{1}{i}\nabla w - A(y_0)w\Big|^2 + \nu|w|^2 \leq 0,$$

which implies $w \equiv 0$. This contradicts the fact that $\max w(x) \geq \delta$, and therefore (4.21) holds.

Now we verify (4.20). Define

$$\varphi(x) = v_{h_n}(x)\eta\Big(\frac{|x - y_{h_n}|}{R}\Big) + v_{h_n}(x)\eta\Big(\frac{|x - \hat{y}_{h_n}|}{R}\Big),$$

where η is a C^∞ function with $\eta(t) = 0$ for $t \le 1$ and $\eta(t) = 1$ for $t \ge 2$. Using $\overline{\varphi}$ in equation (4.14) as a test function and employing (4.21), we derive

$$\begin{aligned}
&\int_{(\Lambda_j^{4r})_{h_n}\setminus(B_{2R}(y_{h_n})\cup B_{2R}(\hat{y}_{h_n}))} \Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 \\
&\ge \int_{(\Lambda_j^{4r})_{h_n}\setminus(B_{R}(y_{h_n})\cup B_{R}(\hat{y}_{h_n}))} g(h_n x, |v_n|^2)|v_n|^2 + O(1/R) + o_{h_n}(1) \\
&\ge \int_{(\Lambda_j^{4r})_{h_n}\setminus(B_{R}(y_{h_n})\cup B_{R}(\hat{y}_{h_n}))} G(h_n x, |v_n|^2) + O(1/R) + o_{h_n}(1).
\end{aligned}$$

As a consequence we get

$$\begin{aligned}
2L_{h_n}^j(u_{h_n})h_n^{-N} &= \int_{(\Lambda_j^{4r})_{h_n}\setminus(B_{R}(y_{h_n})\cup B_{R}(\hat{y}_{h_n}))} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 \\
&\qquad + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big) \\
&+ \int_{B_R(y_{h_n})} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big) \\
&+ \int_{B_R(\hat{y}_{h_n})} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big) \\
&\ge \int_{B_R(y_{h_n})} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big) + O(1/R) \\
&+ \int_{B_R(\hat{y}_{h_n})} \Big(\Big|\frac{1}{i}\nabla v_n - A(h_n x)v_n\Big|^2 + V(h_n x)|v_n|^2 - G(h_n x, |v_n|^2)\Big) + o_{h_n}(1) \\
&\ge \int_{B_R(0)} \Big(\Big|\frac{1}{i}\nabla v - A(\tilde{x})v\Big|^2 + V(\tilde{x})|v|^2 - F(|v|^2)\Big) \\
&+ \int_{B_R(0)} \Big(\Big|\frac{1}{i}\nabla \hat{v} - A(\hat{x})\hat{v}\Big|^2 + V(\hat{x})|\hat{v}|^2 - F(|\hat{v}|^2)\Big) + O(1/R).
\end{aligned}$$

(4.20) follows because R is arbitrary.

Using the same argument we can check that $L_{h_n}^j(u_{h_n})h_n^{-N} \ge 0$ for $j = 1, \dots, k$. Now as a consequence of (3.18) and the method to obtain (3.19), it follows that

$$\sum_{j=1}^k c_j \ge \liminf_{n\to\infty} J_{h_n}(u_{h_n})h_n^{-N} \ge 2c_j + M_1\Big((2c_j)^{1/2} - (c_j + \sigma_j)^{1/2}\Big)_+^2. \quad (4.22)$$

Now we choose

$$C_1 = \frac{\sum_{j=1}^k c_j}{\min\{((2c_j)^{1/2} - (c_j + \sigma_j)^{1/2})^2 : j = 1, \dots, k\}},$$

so that (4.22) is impossible if $M_1 > C_1$.

In case (ii) we deduce from (3.7) that

$$I^{V(\tilde{x})}(\tilde{v}) > c_j + \theta_j.$$

Proceeding exactly as was done in the proof of (4.22) we get

$$\sum_{j=1}^k c_j \geq \liminf_{n\to\infty} J_{h_n}(u_{h_n}) h_n^{-N} \geq (c_j + \theta_j) + M_1\{(c_j + \theta_j)^{1/2} - (c_j + \sigma_j)^{1/2}\}_+^2. \tag{4.23}$$

Now setting

$$C_2 := \frac{\sum_{j=1}^k c_j}{\min\{((c_j + \theta_j)^{1/2} - (c_j + \sigma_j)^{1/2})^2 : j = 1, \dots, k\}},$$

we also obtain that (4.23) is impossible if $M_1 > C_2$.

Therefore, in both cases (i) and (ii) we arrive at a contradiction if we choose $M_1 > C = \max\{C_1, C_2\}$. It follows that the assumption (4.8) does not hold if $M_1 > C$. □

The following lemma implies $\psi_h \equiv 0$ for $h > 0$ sufficiently small.

Lemma 4.4. *For any $\delta > 0$ there exist constants $C > 0$ and $c > 0$ such that, for $h > 0$ small enough,*

$$|u_h(x)| \leq C \exp\Big(-\frac{c}{\hbar} \operatorname{dist}\Big(x, \bigcup_{j=1}^k \Lambda_j^{2r+2\delta}\Big)\Big), \quad \textit{for } x \in \Omega.$$

Proof. By Lemma 4.2 and (4.3), v_h satisfies

$$\Big(\frac{\nabla}{i} - A(hx)\Big)^2 v_h + (V(hx) + \psi_h(hx))v_h = g(hx, |v_h|^2)v_h, \quad \text{for } x \in \Omega_h.$$

Exploiting Kato's inequality (see [30, Theorem X.33])

$$\Delta|v_h| \geq -Re\Big(\frac{\overline{v}_h}{|v_h|}\Big(\frac{\nabla}{i} - A(hx)\Big)^2 v_h\Big),$$

we obtain

$$\Delta|v_h| - (V(hx) + \psi_h(hx))|v_h| + g(hx, |v_h|^2)|v_h| \geq 0 \quad \text{for } x \in \Omega_h. \tag{4.24}$$

As a consequence of (g_1) and (g_2) there exist $p \in (1, (N+2)/(N-2))$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 1, 2$ and C_1, $C_2 > 0$ such that $g(x,t)t \leq C_1 t + C_2 t^p$

for $x \in \Omega_h$, $t \geq 0$. Now Moser iteration [18] yields $\|v_h\|_{L^\infty(\Omega_h)} \leq C$. Hence, from Lemma 4.1 we see that for any $q \geq 2$

$$\int_{\Omega_h \setminus \left(\bigcup_{j=1}^k (\Lambda_j^{2r+\delta})_h \cup Z_h^{2r} \right)} |v_h|^q \leq C \int_{\Omega_h \setminus \left(\bigcup_{j=1}^k (\Lambda_j^{2r+\delta})_h \cup Z_h^{2r} \right)} |v_h|^2 \leq o_h(1).$$

This implies

$$\|v_h\|_{L^\infty\left(\Omega_h \setminus \left(\bigcup_{j=1}^k (\Lambda_j^{2r+\delta})_h \cup Z_h^{2r} \right)\right)} \to 0 \quad \text{as } h \to 0. \tag{4.25}$$

By the same argument as used in [3], we can find $C > 0$ and $c > 0$ such that

$$|u_h(x)| \leq C \exp\Big(-\frac{c}{\hbar} \operatorname{dist}\Big(x, \Big(\bigcup_{j=1}^k \Lambda_j^{2r+2\delta} \Big) \cup Z^{2r} \Big) \Big) \quad \text{for } x \in \Omega. \tag{4.26}$$

Since V is 0 on Z, we can not directly apply the maximum principle to get exponential decay estimates on a neighborhood of Z. Instead we will follow an idea from [6]. Firstly we need the following estimate:

$$\int_{Z^{2r}} |u_h|^2 \leq 4h^{N+6/\mu} \quad \text{for } h > 0 \text{ small.} \tag{4.27}$$

We prove this indirectly. If (4.27) is false then there exists a sequence $h_n \to 0$ such that $\int_{Z^{2r}} |u_{h_n}|^2 > 4h_n^{N+6/\mu}$, which implies $\hat{\psi}_{h_n} > M_2$ for n large. Consequently, $V(x)+\psi_{h_n}(x)$ has a positive lower bound for $x \in \Omega$. Moreover, with the same argument as in the proof of (4.5), we see

$$\int_{\Omega_{h_n} \setminus N_R(\Sigma_{h_n})} \Big(\Big| \frac{1}{i} \nabla v_{h_n} - A(h_n x) v_{h_n} \Big|^2 + \nu |v_{h_n}|^2 \Big) \leq C/R. \tag{4.28}$$

As in the proof of (4.25) we obtain

$$|u_{h_n}(x)| \leq C \exp\Big(-\frac{c}{h_n} \operatorname{dist}\Big(x, \bigcup_{j=1}^k \Lambda_j^{2r+2\delta} \Big) \Big) \quad \text{for } x \in \Omega,$$

which implies $\int_{Z^{2r}} |u_{h_n}|^2 < 4h_n^{N+6/\mu}$ for n large, a contradiction.

Therefore, (4.27) holds. Also by Moser iteration and [18, Theorem 9.26], there exists some $C_1 > 0$ such that

$$\|v_h\|_{L^\infty(Z_h^{2r})} \leq C_1 h^{3/\mu}. \tag{4.29}$$

(4.25) and (4.29) yield, for $h > 0$ sufficiently small,

$$g(x, |u_h|^2) = f(|u_h|^2), \quad \text{for } x \in \Omega \setminus \bigcup_{j=1}^k \Lambda_j^{2r+2\delta}.$$

Now (f_1) implies $f(|v_h|) \le C_2 h^3$ for some $C_2 > 0$. Hence for some constants C_3, C_4, $C_5 > 0$, it follows from (4.24) and (4.26) that

$$\begin{cases} \Delta|v_h| + C_4 h^3 |v_h| \ge 0, & x \in \operatorname{int}(Z^{2r+4\delta}), \\ |v_h| \le C_2 \exp(-\frac{C_5}{\hbar}), & x \in \partial Z^{2r+3\delta}. \end{cases} \tag{4.30}$$

Let Φ be the first eigenfunction and λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \operatorname{int}(Z^{2r+4\delta}), \\ \varphi = 0, & x \in \partial Z^{2r+4\delta}. \end{cases}$$

We can assume that $\Phi(x) \ge 1$ for $x \in Z^{2r+3\delta}$ and $\max\{\Phi(x) : x \in Z^{2r+4\delta}\} < \infty$. For $a, b > 0$, define $\Phi_h(x) := a\exp(-b/\hbar)\Phi(hx)$. Then

$$\begin{cases} \Delta\Phi_h + h^2\lambda_1\Phi_h = 0, & x \in \operatorname{int}(Z_h^{2r+4\delta}), \\ \Phi_h \ge a\exp(-\frac{b}{\hbar}), & x \in \partial Z_h^{2r+3\delta}. \end{cases}$$

Therefore the comparison principle yields that for some constants $C, c > 0$,

$$\|v_h\|_{L^\infty(Z_h^{2r+3\delta})} \le C\exp(-c/h).$$

Combined with (4.26) this proves Lemma 4.4. □

Lemma 4.5. $\lim_{h\to 0} L_h^j(u_h)h^{-N} = c_j$ *for all* $j = 1, \dots, k$.

Proof. This lemma can be proved by using Lemma 3.3, Lemma 4.2 and Lemma 4.4. See also [6] or [12]. □

Proof of Theorem 2.1. Firstly, we claim that there exists some $\delta > 0$ independent of h such that for each $j = 1, \dots, k$, $\max_{x\in\Lambda_j^{4r}} |u_h(x)| \ge \delta$. In fact, if there exist $j_0 \in \{1, \dots, k\}$ and a subsequence h_n with $|u_{h_n}(x)| \to 0$ uniformly in $\Lambda_{j_0}^{4r}$, then Lemma 4.1 and Lemma 4.4 imply

$$\begin{aligned} \int_{\Lambda_{j_0}^{4r}} \left(|D^h u_{h_n}|^2 + V(x)|u_{h_n}|^2\right) &= \int_{\Lambda_{j_0}^{4r}} g(x, |u_{h_n}|^2)|u_{h_n}|^2 + o(\|u_{h_n}\|_{E_h}) \\ &= o(\|u_{h_n}\|_{E_h}), \end{aligned}$$

contradicting Lemma 4.5.

Now we prove that u_h solves problem (2.3) for h small. We choose $x_h^j \in \Lambda_j^{4r}$ such that $|u_h(x_h^j)| = \max_{x\in\Lambda_j^{4r}} |u_h(x)|$ and $|u_h(x_h^j)| \ge \delta$. Set $v_h^j(x) = u_h(x_h^j + hx)$ and denote $\Theta_j = \{x \in \Lambda_j : V(x) = b_j\}$. Proceeding as in the proof of (4.21), we see that $\|u_h\|_{L^\infty(\Lambda_j^{4r}\setminus\Lambda_j^{2r})} \to 0$. Therefore, using the scaling argument from the proof of Lemma 4.2 and the fact $\lim_{h\to 0} L_h^j(u_h)h^{-N} = c_j$, we see that after passing to a subsequence, $x_h^j \to x_0^j \in \Theta_j$, $v_h^j(x) \to v^j$ in $C^2_{loc}(\mathbb{R}^N)$, where v^j solves

$$-\Delta v^j - \frac{2}{i}A(x_0^j)\nabla v^j + |A(x_0^j)|^2 v^j + V(x_0^j)v^j - f(|v^j|^2)v^j = 0 \quad \text{for } x \in \mathbb{R}^N. \tag{4.31}$$

Hence, for any $\delta > 0$, we have $\|u_h\|_{L^\infty(\Lambda^{4r}\setminus\Theta_j^\delta)} \to 0$. Using once more the argument in [3] and Lemma 4.4, we can easily check that there exist constants $C, c > 0$ such that for $h > 0$ small

$$|u_h(x)| \le C \exp\Big(-\frac{c}{h}\operatorname{dist}\Big(x, \bigcup_{j=1}^k \Theta_j^\delta\Big)\Big) \quad \text{for } x \in \Omega. \tag{4.32}$$

This implies that u_h is a solution of the original problem (2.3) for $h > 0$ sufficiently small. By Lemma 4.5 and (4.32), Part (i) of Theorem 2.1 holds.

Next we prove Part (ii). We firstly claim that there exists $\omega_j \in \mathbb{R}$, such that $v^j = e^{i\omega_j + iA(x_0^j)x}U_j$, where U_j is a real-valued least energy solution of the real-valued problem (2.8) with $m = j$. In fact, $U_j^c := v^j e^{-iA(x_0^j)x}$, solves the complex-valued problem

$$-\Delta U_j^c + V(x_0^j)U_j^c = f(|U_j^c|^2)U_j^c, \quad \text{for } x \in \mathbb{R}^N.$$

Next we prove

$$\int_{\mathbb{R}^N}\Big(|\nabla|U_j^c||^2 + V(x_0^j)|U_j^c|^2\Big) = \int_{\mathbb{R}^N} f(|U_j^c|^2)|U_j^c|^2.$$

Suppose that $\theta \in \mathbb{R}$ satisfies

$$\int_{\mathbb{R}^N}\Big(|\nabla|\theta U_j^c||^2 + V(x_0^j)|\theta U_j^c|^2\Big) = \int_{\mathbb{R}^N} f(|\theta U_j^c|^2)|\theta U_j^c|^2.$$

If $\theta \ne 1$, then $\theta \in (0,1)$ because $|\nabla|U_j^c|| \le |\nabla U_j^c|$ a.e. The function

$$q(t) := \frac{1}{2}\int_{\mathbb{R}^N} f(|tU_j^c|^2)|tU_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|tU_j^c|^2)$$

satisfies $q'(t) = \int_{\mathbb{R}^N} t^3 f'(t|U_j^c|)|U_j^c|^4 > 0$ for $t > 0$, so $q(t)$ is strictly increasing in $[0,\infty)$. Hence,

$$\begin{aligned} c_j &= \frac{1}{2}\int_{\mathbb{R}^N} f(|U_j^c|^2)|U_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|U_j^c|^2) \\ &> \frac{1}{2}\int_{\mathbb{R}^N} f(|\theta U_j^c|^2)|\theta U_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|\theta U_j^c|^2) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla|\theta U_j^c||^2 + V_0|\theta U_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|\theta U_j^c|^2) \ge c_j, \end{aligned}$$

which is impossible. Therefore $\theta = 1$ and Lemma 4.5 yields for $h \to 0$

$$\begin{aligned} c_j + o(1) &= h^{-N} L_h^j(u_h) + o(1) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_j^c|^2 + V(x_0^j)|U_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|U_j^c|^2) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla |U_j^c||^2 + V(x_0^j)|U_j^c|^2 - \frac{1}{2}\int_{\mathbb{R}^N} F(|U_j^c|^2) \geq c_j. \end{aligned}$$

Therefore, $|U_j^c|$ is a least energy solution of (2.8) which we denote by U_j, and $|\nabla U_j^c| = |\nabla |U_j^c|| = |\nabla U_j|$ for almost every x. According to [24], there exists a constant $\omega_j \in \mathbb{R}$ such that $U_j^c = e^{i\omega_j} U_j$. Thus, the claim is true.

We set $\varphi^j(x) = U_j^c(x) e^{iA(x_0^j)x}$ and $\psi_h^j = v_h^j - \varphi^j$. Then $|\psi_h^j| \to 0$ weakly in $H^1(K,\mathbb{R})$ on any bounded subset $K \subseteq \mathbb{R}^N$. Indeed, by the diamagnetic inequality (2.2), $|\psi_h^j|$ is bounded in $H^1(K,\mathbb{R})$, so after passing to a subsequence, $|\psi_h^j| \to \phi^j$ weakly in $H^1(K,\mathbb{R})$ and $|\psi_h^j| \to \phi^j$ almost everywhere. However, as seen in the proof of Lemma 4.2, $v_h^j \to \varphi^j$ almost everywhere, so $|\psi_h^j| \to 0$ weakly in $H^1(K,\mathbb{R})$. By (4.32) and the exponential decay of $|\varphi^j|$, in order to prove Part (ii) of Theorem 1.1 it suffices to prove that

$$\int_{\Lambda'_{j,h}} \frac{1}{i}\Big(\nabla \psi_h^j - A(x_h^j + hx)|\psi_h^j|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big) \to 0, \quad \text{as } h \to 0 \tag{4.33}$$

where $\Lambda'_{j,h} := \{x : x_h^j + hx \in \Lambda_j\}$.

It is easy to check that

$$\Big(\frac{1}{i}\nabla - A(x_h^j + hx)\Big)^2 v_h^j + V(x_h^j + hx) v_h^j = f(|v_h^j|^2) v_h^j \quad \text{for } x \in \Lambda'_{j,h}$$

and

$$\Big(\frac{1}{i}\nabla - A(x_h^j + hx)\Big)^2 \varphi^j + V(x_h^j + hx)\varphi^j = f(U_j^2)\varphi^j + I(x) \quad \text{for } x \in \Lambda'_{j,h},$$

where

$$\begin{aligned} I(x) = {} & \frac{2}{i}\Big(A(x_h^j + hx) - A(x_h^j)\Big)\varphi^j - \Big(|A(x_h^j + hx)|^2 - |A(x_h^j)|^2\Big)\varphi^j \\ & + \frac{1}{i}\operatorname{div}(A(x_h^j + hx) - A(x_h^j))\varphi^j + (V(x_h^j + hx) - V(x_h^j))\varphi^j. \end{aligned}$$

A direct computation yields

$$\Big(\frac{1}{i}\nabla - A(x_h^j + hx)\Big)^2 \psi_h^j + V(x_h^j + hx)\psi_h^j = f(|v_h^j|^2) v_h^j - f(U_j^2)\varphi^j - I(x)$$

for $x \in \Lambda'_{j,h}$ and $\psi_h^j = v_h^j - \varphi^j$ for $x \in \partial\Lambda'_{j,h}$. Multiplying the above differential equation by $\overline{\psi^j}_h$ and integrating by parts yields for $h \to 0$

$$\begin{aligned}
&\int_{\Lambda'_{j,h}} \Big|\frac{1}{i}\nabla\psi_h - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2 \qquad (4.34)\\
&= \int_{\Lambda'_{j,h}} \Big(f(|v_h^j|^2)v_h^j - f(U_j^2)\varphi^j - I(x)\Big)\overline{\psi^j}_h - \int_{\partial\Lambda'_{j,h}} \frac{\partial\psi_h^j}{\partial n}\overline{\psi_h^j}\\
&= \int_{\Lambda'_{j,h}} f(|\psi_h^j|^2)|\psi_h^j|^2 + O\Big(\int_{\Lambda'_{j,h}} |\psi_h^j|^{(s+4)/2}U_j^{s/2}\Big)\\
&+ O\Big(\int_{\Lambda'_{j,h}} |\psi_h^j|^{(s+2)/2}U_j^{(s+2)/2}\Big)\\
&+ O\Big(\int_{\Lambda'_{j,h}} |\psi_h^j|^{(\mu+4)/2}U_j^{\mu/2}\Big) + O\Big(\int_{\Lambda'_{j,h}} |\psi_h^j|^{(\mu+2)/2}U_j^{(\mu+2)/2}\Big)\\
&+ \int_{\Lambda'_{j,h}} f(|\psi_h^j|^2)\varphi^j\overline{\psi^j}_h + \int_{\Lambda'_{j,h}} f(U_j^2)(\psi_h^j)^2 - \int_{\Lambda'_{j,h}} I(x)\overline{\psi^j}_h - \int_{\partial\Lambda'_{j,h}} \frac{\partial\psi_h^j}{\partial n}\overline{\psi^j}_h\\
&= \int_{\Lambda'_{j,h}} f(|\psi_h^j|^2)|\psi_h^j|^2 + o_h(1) - \int_{\Lambda'_{j,h}} I(x)\overline{\psi^j}_h.
\end{aligned}$$

The last equality follows from the weak convergence of $|\psi_h^j|$ in $H^1(K,\mathbb{R})$ and the exponential decay of U_j.

On the other hand, from the Hölder continuity of $V(x)$ and $A(x)$ and the exponential decay of U_j, it follows that $\int_{\Lambda'_{j,h}} I(x)\overline{\psi^j}_h = o_h(1)$. Hence,

$$\int_{\Lambda'_{j,h}} \Big(\Big|\frac{1}{i}\nabla\psi_h^j - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big) = \int_{\Lambda'_{j,h}} f(|\psi_h^j|^2)|\psi_h^j|^2 + o_h(1).$$

Suppose there exists some $j \in \{1, \dots, k\}$ with

$$\int_{\Lambda'_{j,h}} \Big(\Big|\frac{1}{i}\nabla\psi_h^j - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big) \to \rho > 0 \quad \text{for } h \to 0.$$

Then $(f_1) - (f_3)$ imply for $h \to 0$

$$\begin{aligned}
&\frac{1}{2}\int_{\Lambda'_{j,h}} \Big(\Big|\frac{1}{i}\nabla\psi_h^j - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big) - \frac{1}{2}\int_{\Lambda'_{j,h}} F(|\psi_h^j|^2)\\
&\qquad \geq \frac{1}{2}\int_{\Lambda'_{j,h}} \Big(\Big|\frac{1}{i}\nabla\psi_h^j - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big)
\end{aligned}$$

$$- \frac{1}{\theta} \int_{\Lambda'_{j,h}} f(|\psi_h^j|^2)|\psi_h^j|^2$$
$$\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\rho + o_h(1).$$

Proceeding exactly as in the proof of (4.34) we obtain

$$h^{-N}\Big(\frac{1}{2}\int_{\Lambda_j}\Big(\Big|\frac{h}{i}\nabla uh - A(x)u_h\Big|^2 + V(x)|u_h|^2\Big) - \frac{1}{2}\int_{\Lambda_j} F(|u_h|^2)\Big)$$
$$= \frac{1}{2}\int_{\Lambda'_{j,h}}\Big(\Big|\frac{1}{i}\nabla\varphi^j - A(x_h^j + hx)\varphi^j\Big|^2 + V(x_h^j + hx)|\varphi^j|^2\Big) - \frac{1}{2}\int_{\Lambda'_{j,h}} F(|\varphi^j|^2)$$
$$+ \frac{1}{2}\int_{\Lambda'_{j,h}}\Big(\Big|\frac{1}{i}\nabla\psi_h^j - A(x_h^j + hx)\psi_h^j\Big|^2 + V(x_h^j + hx)|\psi_h^j|^2\Big)$$
$$- \frac{1}{2}\int_{\Lambda'_{j,h}} F(|\psi_h^j|^2) + o_h(1)$$
$$\geq c_j + (\tfrac{1}{2} - \tfrac{1}{\theta})\rho + o_h(1) > c_j.$$

Consequently,

$$h^{-N}J_h(u_h)$$
$$= h^{-N}\sum_{j=1}^{k}\frac{1}{2}\int_{\Lambda_j}\Big(\Big|\frac{h}{i}\nabla u_h - A(x)u_h\Big|^2 + V(x)|u_h|^2 - F(|u_h|^2)\Big) + o_h(1)$$
$$\geq \sum_{j=1}^{k} c_j + k(\frac{1}{2} - \frac{1}{\theta})\rho + o_h(1) > \sum_{j=1}^{k} c_j,$$

which contradicts Lemma 3.3. Now (4.33) and therefore Part (ii) of Theorem 2.1 follow.

Next we show that $|u_h|$ has exactly k local maxima in Ω. (4.32) implies that the local maxima should be in $\bigcup_{j=1}^{k}\Lambda_j$, so we only need to show that $|u_h|$ has a unique local maximum in Λ_j for every $j \in \{1, \dots, k\}$. Suppose that this is not true, then there exist some $j_0 \in \{1, \dots, k\}$ and a sequence $h_n \to 0$ such that $|u_{h_n}|$ possesses at least two local maxima $y_n^1, y_n^2 \in \Lambda_{j_0}$. We distinguish between two cases:

(1) $(|y_n^1 - y_n^2|)/h_n \to c < \infty$;

(2) $(|y_n^1 - y_n^2|)/h_n \to \infty$.

Set $v_n(x) = u_{h_n}(y_n^1 + h_n x)$. Proceeding as in the proof of Lemma 4.2, we can prove that v_n converges along a subsequence in the C^2-sense over compact sets to a function v and $|v| \in H^1(\mathbb{R}^N, \mathbb{R})$ is a least energy solution of (2.8) with $b_m = b_{j_0}$. It is well known that the family of least energy solutions of (2.8) has a local maximum and is radially symmetric and radially decreasing. So Case (1) cannot happen.

Now we consider Case (2). By Theorem 2.1 (ii) there exist $\omega_1, \omega_2 \in \mathbb{R}$ and two least energy solutions V_1 and V_2 of (2.8) with $b_m = b_{j_0}$ such that as $n \to \infty$

$$\int_{\Lambda_{j_0}} \Big|\frac{h_n}{i}\nabla(W_{1,n}-W_{2,n})-A(x)(W_{1,n}-W_{2,n})\Big|^2+V(x)|W_{1,n}-W_{2,n}|^2 = o(h_n^N),$$

where $W_{q,n} = V_q((x-y_n^q)/h_n)e^{i(\omega_q+A(y_n^q)(x-y_n^q)/h_n)}$, $q = 1,2$. We set $D_n = \{y : y_n^1+h_ny \in \Lambda_{j_0}\}$ and $y_n = (y_n^1-y_n^2)/h_n$, so $|y_n| \to \infty$. By the exponential decay of V_q and $|\nabla V_q|$, we have

$$\begin{aligned}
0 = &\lim_{n\to\infty} h_n^{-N}\Big(\int_{\Lambda_{j_0}} \Big|\frac{h_n}{i}\nabla W_{1,n} - A(x)W_{1,n}\Big|^2 + V(x)|W_{1,n}|^2\Big)\\
&+ \lim_{n\to\infty} h_n^{-N}\Big(\int_{\Lambda_{j_0}} \Big|\frac{h_n}{i}\nabla W_{2,n} - A(x)W_{2,n}\Big|^2 + V(x)|W_{2,n}|^2\Big)\\
&- 2\lim_{n\to\infty}\int_{D_n}\Big\{|\nabla V_1|\cdot|\nabla V_2(y-y_n)| + |A(y_n^1)||\nabla V_2(y-y_n)|V_1\\
&+ |A(y_n^2)||\nabla V_1|V_2(y-y_n) + |A(y_n^1)||A(y_n^2)|V_1V_2(y-y_n)\\
&+ (|A(y_n^1+h_ny)|^2 + V(y_n^1+h_ny))V_1V_2(y-y_n)\\
&+ |A(y_n^1+h_ny)|(V_1|\nabla V_2(y-y_n)| + V_2(y-y_n)|\nabla V_1|)\Big\}\\
&\geq c > 0,
\end{aligned}$$

which is a contradiction. Hence Case (2) is impossible also. Therefore u_h has exactly k local maxima in Ω.

In order to complete our proof of Theorem 2.1 it remains to prove Part (iii). In fact, by the same argument as in the proof of (4.32), this can be easily derived from the maximum principle and the fact that u_h has only one local maxima; see also [14]. □

Remark 4.6. We can prove the existence of multi-bump solutions and the exponential decay property even when f is not increasing as required in (f_1). The proof of Theorem 2.1 shows that in this case we only need to assume that the mountain pass value in $H^1(\mathbb{R}^N,\mathbb{C})$ of the energy functional associated to (2.4) is the least among all the nontrivial critical values. Such a result has been proved in [21] under very weak hypotheses.

Proof of Theorem 2.3. Since the main procedure of the proof is similar to that of Theorem 2.1, we give here only a sketch. For convenience, we use similar notation as above, in particular $b \equiv b_j$, $c \equiv c_j$.
Step 1: Here we set $\Sigma := \bigcup_{g\in G} g\Lambda$ and consider the functional

$$J_h := L_h + Q_h : E_h^G \to \mathbb{R}$$

with

$$L_h(u) = \frac{1}{2}\int_\Omega \Big(|D^h u|^2 + V|u|^2\Big) - \frac{1}{2}\int_\Omega G(x,|u|^2)$$

and

$$Q_h(u) = M_2\Big(\Big(\int_{Z^{2r}} u^2\Big)^{1/2} - h^{\frac{3}{\mu}+\frac{N}{2}}\Big)_+^2.$$

It is easy to prove that J_h has the mountain pass geometry and satisfies the Palais-Smale condition in E_h^G; see the proof of Lemma 3.1. Therefore J_h has a nontrivial critical point u_h in E_h^G. Moreover, taking the test function

$$u_\eta(x) = \sum_{j=1}^{k} \eta(|x - g^j(x_0)|) U_j((x - g^j(x_0))/h) e^{i\frac{x-g^j(x_0)}{h}A(g^j(x_0))} \in E_h^G,$$

where $x_0 \in \Lambda_m$ satisfies $V(x_0) = b$ and $\eta \in C_0^\infty(B_{2\rho}(0))$ satisfies supp$\eta(|x - x_0|) \subset \Lambda$, $\eta(|x|) = 1$ on $B_\rho(0)$, it is not difficult to prove that $J_h(u_h) \le h^N c(k + o(1))$.
Step 2: Applying the scaling argument used in the proof of Lemma 4.2, one can show that if $x_h \in \Lambda$ is such that $u_h(x_h) \ge \zeta > 0$, then after passing to a subsequence, $V(x_h) \to b$, which implies $\|u_h\|_{L^\infty(\partial\Sigma)} \to 0$. Moreover, $g(x,|u|^2) \le f(|u|^2)$ implies $J_h(u_h) = h^N c(k + o(1))$. Hence, $\|u_h\|_{L^\infty(\Omega\setminus\Sigma)} \to 0$. The same arguments used to prove Lemma 4.4 yield that u_h decays exponentially outside Σ. As a consequence, u_h solves the original problem (2.3).
Step 3: Using the symmetry of u_h and the arguments to prove (i),(ii) and (iii) of Theorem 2.1, one can complete the proof of Theorem 2.3. □

5. Appendix

First we consider the isolation of the least energy associated to problem (2.4).

Proposition 5.1. *The isolation of the least energy c is equivalent for the problem (2.4) on $\mathbb{C}$ or on $\mathbb{R}$.*

Proof. The assumptions (f_1), (f_2) and (f_3) imply that the mountain pass solution of problem (2.4) is a least energy solution on the corresponding Nehari manifold

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 + b|u|^2 = \int_{\mathbb{R}^N} f(|u|^2)|u|^2 \right\}.$$

Suppose that $u = (u_1, u_2) := u_1 + iu_2$ is a least energy solution, where $u_1, u_2 \in H^1(\mathbb{R}^N, \mathbb{R})$. Then $(|u_1|, |u_2|)$ has on $\mathcal{N}$ the same energy c as (u_1, u_2) and thus minimizes the energy on $\mathcal{N}$. Therefore it is a solution. Thus $|u_1|, |u_2|$ are smooth and u_1, u_2 must have fixed sign.

Now u_1, u_2 are both principal eigenvectors of the operator $-\Delta - f(|u|^2) + b$ on $L^2(\mathbb{R}^N)$. It is easy to prove that the principal eigenvalue is simple, hence $u_2 = Bu_1$. Thus u is a rotation of a real-valued positive solution.

Suppose that $u_n + iv_n$ are solutions with energy $c_n \to c$ as $n \to \infty$. Then (f_3) implies that u_n and v_n are bounded in $H^1(\mathbb{R}^N)$. Using Kato's inequality, the diamagnetic inequality and Moser iteration, we can deduce $\|u_n + iv_n\|_{L^\infty(\mathbb{R}^N)} < C$. Now we claim that $|u_n + iv_n| \to 0$ uniformly in n as $|x| \to \infty$. Indeed, from the fact that $c > 0$ we get $\|u_n + iv_n\|_{L^\infty(\mathbb{R}^N)} > \alpha$ for some $\alpha > 0$. If the claim does not hold, then there exist $x_n, y_n \in \mathbb{R}^N$ and $a > 0$ satisfying

$$|x_n - y_n| \to \infty, \ \ |u_n(x_n) + iv_n(x_n)| > a, \ \ |u_n(y_n) + iv_n(y_n)| > a.$$

With arguments similar to those in the proof of Lemma 4.2, we can prove that $u_n(\cdot + x_n) + iv_n(\cdot + x_n) \to w_1$ weakly in $L^2(\mathbb{R}^N)$ and $u_n(\cdot + y_n) + iv_n(\cdot + y_n) \to w_2$ weakly in $L^2(\mathbb{R}^N)$, where $w_1, w_2 \in H^1(\mathbb{R}^N, \mathbb{C})$ are nontrivial solutions of (2.4). The convergence also holds in C^2 on any compact sets. Hence, as in the proof (4.20), it follows that $c + o(1) = c_n > 2c - \delta > c$ for some small $\delta > 0$ and large n, which is impossible. Hence $|u_n + iv_n| \to 0$ uniformly on n as $|x| \to \infty$. Now using a standard argument, it follows that there exist constants $K > 0, \gamma > 0$ such that

$$|u_n(x + x_n) + iv_n(x + x_n)| \le Ke^{-\gamma|x|} \quad \text{uniformly in } n.$$

From the above exponential decay one can deduce that $u_n(\cdot + x_n) + iv_n(\cdot + x_n)$ converges in $L^2(\mathbb{R}^N)$ to $\tilde{u} + i\tilde{v}$, a least energy solution of (2.4). Moreover, $u_n(\cdot + x_n) + iv_n(\cdot + x_n)$ converges in an exponentially weighted L^∞-norm to $\tilde{u} + i\tilde{v}$. By using a rotation, we can assume that $\tilde{u} \not\equiv 0$ and $\tilde{v} \not\equiv 0$. Thus $\tilde{u}$ has fixed sign. Now, $-\Delta u_n = (f(|u_n|^2 + |v_n|^2) - b)u_n$, while $-\Delta \tilde{u} = (f(|\tilde{u}|^2 + |\tilde{v}|^2) - b)\tilde{u}$. Since $\tilde{u}$ has fixed sign, 0 is the principal eigenvalue of $-\Delta - (f(|\tilde{u}|^2 + |\tilde{v}|^2) - b)$, which is simple. Hence, for large n, $-\Delta - (f(|u_n|^2 + |v_n|^2) - b)$ has a unique eigenvalue near zero and this will be the smallest eigenvalue. Since 0 is an eigenvalue, it must be the principal eigenvalue and hence u_n has fixed sign. Similarly, v_n has fixed sign. As before, we can then deduce that $v_n = B_n u_n$, where B_n are constants. Thus after a rotation, $v_n \equiv 0$. □

Remark 5.2. Thus we only need to consider the isolation of the least energy c associated to problem (2.4) on $\mathbb{R}$. Moreover, we only need to worry about positive solutions and in fact radial solutions as a consequence of the moving plane method. Next, if f is analytic, the isolation of c always holds. This follows by an argument similar to one from [6] (where the method was applied in $C^0(\mathbb{R}^N)$). In fact, one can work in the radial subspace of $C^0(\mathbb{R}^N)$. Using the theory of finite-dimensional real analytic sets one can show that any solution $\tilde{u}$ close to a solution u_0 can be connected to u_0 by a continuous piecewise differentiable path of solutions. Thus $\tilde{u}$ has the same energy as u_0. Finally, concerning the property of real analyticity, it can be satisfied by a much larger class of nonlinearities including typically terms like $g(y) = \sum_{j=1}^{k} \alpha_j |y|^{p_j-1} y + k(y)$, where $p_j > 1$ need not be an integer and $k(y)$ is real analytic in a neighborhood of $[0,\infty)$. Indeed, one can work in a space $H := \{u \in L^\infty_{rad}(\mathbb{R}^N) : \sup(1+r)^{(N-1)/2} e^r |u(r)| < \infty\}$ with the norm $\|\cdot\|_H := \sup_{r\geq 0}(1+r)^{(N-1)/2} e^r |\cdot|$. One finds that the positive solutions are in the interior of the cone $\mathcal{C} := \{u \in H : u \geq 0\}$, and shows as in the paper [11] that real analyticity holds in H near the positive solutions. This is enough to use the same argument as in the appendix to [6] in H. (Note that by the above arguments the set of solutions of least energy is also locally compact in H.)

Proof of Lemma 3.2. Using the test path constructed in the beginning of the proof of Lemma 3.3, we obtain the following upper estimate

$$d_h^j \leq h^N(c_j + o(1)). \tag{5.1}$$

In order to complete the proof, it suffices to prove

$$d_h^j \geq h^N(c_j + o(1)). \tag{5.2}$$

Recall that for any $h > 0$ the norm

$$\|u\|^2 := \int_{\Lambda_j^{4r}} \left|\frac{1}{i}\nabla u - A(hx)u\right|^2 + V(hx)|u|^2\,dx$$

is equivalent to the usual H^1-norm because Λ_j^{4r} is bounded. It follows that the functional L_h^j satisfies the Palais-Smale condition. The mountain pass lemma yields that d_h^j is a critical value of L_h^j. Let w_h be an associated critical point, so $w_h \not\equiv 0$ solves the following equation

$$\begin{cases} (\frac{h}{i}\nabla - A(x))^2 w_h + V(x)w_h = g(x,|w_h|^2)w_h, & x \in \Lambda_j^{4r}, \\ \frac{\partial w_h}{\partial n} = 0, & x \in \partial\Lambda_j^{4r}. \end{cases} \tag{5.3}$$

Employing the arguments used to prove (4.21), we see that

$$\|w_h\|_{L^\infty(\Lambda_j^{4r}\setminus\Lambda_j^{2r})} \to 0 \quad \text{as} \quad h \to 0.$$

Consider a maximum point $x_h \in \Lambda_j^{2r}$ of w_h. There exists $\delta > 0$ such that $w_h(x_h) \geq \delta$ uniformly in $h > 0$. We define $v_h(x) := w_h(x_h + hx)$ and take a sequence $h_n \to 0$. Using (5.1) and the fact that $w_\hbar$ satisfies (5.3), we obtain a subsequence denoted still by $\{h_n\}$ such that $x_{h_n} \to x_0 \in \Lambda_j^{2r}$ and $v_{h_n} \to v \in H^1(\mathbb{R}^N, \mathbb{C})$ in the C^1-norm on compact sets, where v satisfies an equation of the form (4.6). It follows now from Lemma 4.3 that v actually solves

$$\Big(\frac{1}{i}\nabla - A(x_0)\Big)^2 v + V(x_0)v = f(|v|^2)v.$$

Setting $w = e^{-iA(x_0)x}v$, the arguments from the proof of Lemma 4.2 show that

$$\lim_{n\to\infty} L^j_{h_n}(w_{h_n})h_n^{-N} \geq I^{V(x_0)}(w) \geq c_j. \tag{5.4}$$

Since (5.4) holds for a subsequence of every given sequence h_n, we conclude that (5.2) is true. □

Proof of Lemma 4.3. We first show that $|v| \leq a$ on $\{x_1 = 0\}$. Obviously v also satisfies

$$\overline{\Big(\frac{1}{i}\nabla - A_0\Big)^2 v} + b\overline{v} = \chi_{\{x_1<0\}} f(|v|^2)\overline{v} + \chi_{\{x_1>0\}}\tilde{f}(|v|^2)\overline{v}. \tag{5.5}$$

Multiplying (4.6) by $\partial\overline{v}/\partial x_1$ and (5.5) by $\partial v/\partial x_1$ and then integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^{N-1}} dx' \int_{-\infty}^{+\infty} \frac{\partial}{\partial x_1}\Big(\Big|\frac{1}{i}\nabla v - A_0 v\Big|^2 + b|v|^2\Big) + \int_{\mathbb{R}^{N-1}} (F(|v(0,x')|^2) - \tilde{F}(|v(0,x')|^2))dx' = 0.$$

The first summand above is zero. Since $F(s) \geq \tilde{F}(s)$, and since equality only holds in the case $s \leq a$, we deduce $|v(0,x')| \leq a$.

Next we prove $|v(x_1,x')| \leq a$ for $x_1 \geq 0$. From (4.6) and Kato's inequality (see Theorem X.33 in [30])

$$\Delta|v| \geq -Re\Big(\frac{\overline{v}}{|v|}\Big(\frac{\nabla}{i} - A_0\Big)^2 v\Big),$$

we get

$$-\Delta|v| + b|v| \le \chi_{\{x_1<0\}} f(|v|^2)|v| + \chi_{\{x_1>0\}} \tilde{f}(|v|^2)|v|. \tag{5.6}$$

Taking $\eta = (|v| - a)_+ \chi_{x_1>0}$ as a test function in (5.6), we obtain

$$\int_{\{x_1>0\}} \Big(|\nabla(|v| - a)_+|^2 + (b - \tilde{f}(|v|^2))(|v| - a)_+^2 \Big) \le 0.$$

This implies $|v| \le a$ in $\{x_1 > 0\}$ since $b > \nu \ge \tilde{f}(|v|^2)$. □

Acknowledgement. The second and the third authors would like to thank the Humboldt foundation for financial support and the mathematics institute at the University of Giessen for their hospitality during our visit.

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