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AN INTEGRODIFFERENTIAL WAVE EQUATION

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Abstract. This paper is devoted to the study of the integrodifferential equation

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1 u(s)ds + f(t), \quad t \geq 0,$$

where A is a Hille-Yosida operator in a Banach space X, $A_1 \in \mathcal{L}(D(A); X)$ and a has bounded variation. Existence, uniqueness and estimates of strict and weak solutions are proved by extrapolation methods and the Miller scheme. Applications are given to the Cauchy-Dirichlet problem for the integrodifferential wave equation

$$w_{tt}(t,x) = w_{xx}(t,x) + \int_0^t a(t-s)w_{xx}(s,x)ds + f(t,x), \quad t \geq 0, \quad x \in [0,\ell].$$

0. Introduction and notation

The abstract integrodifferential Volterra equation

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1 u(s)ds + f(t), \quad t \geq; \quad u(0) = u_0, \tag{0.1}$$

where $A : D(A) \subset X \to X$ is a linear operator in a Banach space X and $A_1 : D(A) \to X$ is linear and continuous has been intensively studied under the assumption that A is the generator of a semigroup (in particular when $D(A)$ is dense in X). See e.g. [4]. The motivation for our study of $(0,1)$ is its application to the integrodifferential wave equation

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + \int_0^t a(t-s)w_{xx}(s,x)ds + f(t,x), & t \geq 0,\ x \in [0,\ell] \\ w(t,0) = w(t,\ell) = 0 & t \geq 0 \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x) & x \in [0,\ell]. \end{cases} \tag{0.2}$$

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If we are interested in classical solutions we must choose X as a Banach space of continuous functions: this choice yields also pointwise estimates but problem (0.2) is reduced to an abstract problem (0.1) where $D(A)$ is not dense in X (this is due to the boundary condition $(0,2)_2$). In fact we will study (0.1) under the assumption that A is a Hille-Yosida operator i.e., satisfies all the conditions required to be the generator of a semigroup except the density of its domain.

The solution of (0.1) is obtained by using the so-called Miller scheme (see [10], [12]) which reduces the integrodifferent equation in X to a homogeneous differential equation in a suitable product space. This procedure requires results from extrapolation spaces and Favard classes and was used in [11] to solve (0.1) when A has a continuous inverse and $A_1 = A$: here we avoid these restrictions and consider also the weak solutions obtainable under minimal conditions on the data. Some of the results about the strict solutions were proved in [15] which considers a delay-problem for the wave equation.

The only property we assume on the kernel of (0.1) or (0.2) is the boundedness of its variation. Under this assumption it was studied in [11] and in [13] with integrated semigroup methods; for different assumptions see e.g. [8] and [14].

In addition to the customary symbols, when I is an interval of $\mathbb{R}$ and X a Banach space we set

$$W^{I,p}(I;X) = \left\{ u : I \to X;\, \exists v \in \mathcal{L}^b(I : X),\, u(t) - u(s) = \int_s^t v(\sigma)d\sigma;\, t, s \in I \right\}.$$

If $A : D(A) \subset X \to X$ is a linear operator, $\rho(A)$ denotes the resolvent set and $R(\lambda, A)$ the resolvent function. $D(A)$ will be given the graph norm.

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1. Hille-Yosida operators and Favard classes

Let $X = (x, \|\cdot\|)$ denote a Banach space and $A : D(A) \subset X \to X$ a linear operator. As mentioned before we want to use a sup-norm which forces us to deal with operators with nondense domain.

Definition 1.1. A is a Hille-Yosida operator of type (M, ω) if there exist $M, \omega \in \mathbb{R}$ such that $(\omega, +\infty) \subseteq \rho(A)$ and $\|(\lambda - \omega)^n(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leq M$ for each $\lambda > \omega$ and $n \in \mathbb{N}$. We set

$$\omega' = \sup(0, \omega).$$

If in addition $\overline{D(A)} = X$, then A is the generator of a semigroup in X. In any case we can associate to A a generator of a semigroup by virtue of the following theorem (see [7]; Theorem 12.2.4):

Theorem 1.2. *Let* $X_0 = (\overline{D(A)}, \|\cdot\|_0)$, *where* $\|\cdot\|_0 = \|\cdot\|$ *and* A_0 *be the part of* A *in* X_0 *i.e.,*

$$\begin{cases} D(A_0) = \{x \in D(A);\ Ax \in \overline{D(A)}\} \\ A_0 x = Ax, \end{cases} \tag{1.1}$$

then A_0 *is the generator of a semigroup* $T_0(t)$ *in* X_0 *of type* (M, ω).

We will use a real interpolation space between $D(A_0)$ and X_0 called the Favard class of the semigroup $T_0(t)$ (see [2], Chapter 3).

Definition 1.3. Let A_0 be the generator of a semigroup $T_0(t)$ in the Banach space X_0. The Favard class of $T_0(t)$ is the Banach space

$$F(A_0) := \left\{ x \in X_0; [x] = \sup_{0<t\le 1} \left\| \frac{T_0(t)x - x}{t} \right\|_0 < \infty \right\}$$

with norm $\|x\|_{F(A_0)} := \|x\|_0 + [x]$.

Theorem 1.4. *Let* A_0 *be the generator associated to the Hille-Yosida operator* A *and* $F(A_0)$ *its Favard class. Then we have*

$$D(A) \hookrightarrow F(A_0). \tag{1.2}$$

Proof. As $D(A_0)$ is dense in X_0 we have $\lim_{\lambda\to+\infty} \lambda(\lambda - A_0)^{-1}x = x$ for each $x \in X_0$; hence for $x \in D(A)$ and $0 < t < 1$

$$\lim_{\lambda\to+\infty} \|(T_0(t) - I)\lambda(\lambda - A_0)^{-1}x\|_0 = \|T_0(t)x - x\|_0. \tag{1.3}$$

If $\lambda > \omega$ and $0 < t \le 1$, we have

$$\begin{aligned} \|(T_0(t) - I)\lambda(\lambda - A_0)^{-1}x\| &= \left\| A_0 \int_0^t T_0(s)\lambda(\lambda - A_0)^{-1}x ds \right\| \\ &= \left\| \int_0^t T_0(s)\lambda A(\lambda - A)^{-1}x ds \right\| \le M^2 t e^{\omega'} \frac{\lambda}{\lambda - \omega} \|Ax\|. \end{aligned}$$

From (1.3) we deduce $\frac{1}{t}\|T_0(t)x - x\| \le M^2 t e^{\omega'} \|Ax\|$ and (1.2) follows. $\square$

2. Extrapolation spaces

In this section we collect the main definitions and results about the extrapolation spaces. For their proofs we refer to [11] where it is also supposed that $A^{-1} \in \mathcal{L}(x)$. We will write explicitly only the nonobvious generalizations to our situation. In this section we assume that $A_0 : D(A_0) \subset X_0 \to X_0$ is the generator of a semigroup in a Banach space $X_0 = (\mathcal{X}_0, \|\cdot\|_0)$ and that $\lambda \in \rho(A_0)$ is chosen.

Definition 2.1. The extrapolation space X_{-1} of X_0 (with respect to A_0) is the completion of $\mathcal{X}_0$ endowed with the norm

$$\|x\|_{-1} := \|R(\lambda, A_0)x\|_0, \quad x \in X_0, \tag{2.1}$$

i.e.,

$$X_{-1} = (\mathcal{X}_0, \|\cdot\|_{-1})^{\sim}. \tag{2.2}$$

We will identify $\mathcal{X}_0$ with its canonical image in X_{-1} and so $\bar{X}_0^{\|\cdot\|_{-1}} = X_{-1}$ and

$$X_0 \hookrightarrow X_{-1}. \tag{2.3}$$

Theorem 2.2. *$A_0 : D(A_0) \subset X_0 \to X_{-1}$ can be continuously extended into an operator $A_{-1} \in \mathcal{L}(X_0, X_{-1})$.*

Definition 2.3. $A_{-1} : D(A_{-1}) \subset X_{-1} \to X_{-1}$ with $D(A_{-1}) = X_0$ is called the operator extrapolated from A_0.

Theorem 2.4. *$\lambda - A_{-1}$ is a bijective isometry from X_0 to $X_{-1}, \lambda \in \rho(A_{-1})$ and*

$$\|(\lambda - A_{-1})x\|_{-1} = \|x\|_0, \quad x \in X_0. \tag{2.4}$$

Moreover, $R(\lambda, A_{-1})|_{X_0} = R(\lambda, A_0)$ and

$$\lambda x - A_{-1}x \in X_0 \Rightarrow x \in D(A_0). \tag{2.5}$$

Theorem 2.5. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator and $\lambda \in \rho(A)$. Let A_0 be the generator associated to A and A_{-1} its extrapolated operator. If we define in $\mathcal{X}$ the norm $\|x\|_{-1} = \|R(\lambda, A)x\|$, $x \in \mathcal{X}$, then we have $X_{-1} = (\mathcal{X}, \|\cdot\|_{-1})^{\sim}$. By identifying $\mathcal{X}$ with its canonical image in $(\mathcal{X}, \|\cdot\|_{-1})^{\sim}$, we have*

$$X \hookrightarrow X_{-1}. \tag{2.6}$$

Theorem 2.6. *$A_{-1} : X_0 \subset X_{-1} \to X_{-1}$ is an extension of $A : D(A) \subset X \to X$. We have $R(\lambda, A_{-1})|_X = R(\lambda, A)$ and*

$$\lambda x - A_{-1}x \in X \Rightarrow x \in D(A). \tag{2.7}$$

We define now a semigroup in X_{-1}, extrapolated from $T_0(t)$, in an equivalent but simpler way than that used in [11].

Theorem 2.7. *Setting for each* $t > 0 : T_{-1}(t) = (\lambda - A_{-1})T_0(t)(\lambda - A_{-1})^{-1}$ *we obtain a semigroup in* X_{-1} *with generator* $A_{-1} : X_0 \subset X_{-1} \to X_{-1}$. *Moreover* $T_{-1}(t)$ *is an extension of* $T_0(t)$ *and* $\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{\mathcal{L}(X_0)}$.

Proof. As $\lambda - A_{-1}$ is a bijective isometry of X_0 into X_{-1} the result is a consequence of the properties of similar semigroups (see e.g. [5] page 59).

Theorem 2.8. *For the Favard classes* $F_0 := F(A_0)$ *and* $F_{-1} := F(A_{-1})$ *we have the following properties*:

$$(\lambda - A_{-1})(F_0) = F_{-1} \quad \text{and} \quad \|(\lambda - A_{-1})x\|_{F_{-1}} = \|x\|_{F_0}, \quad x \in F_0 \tag{2.8}$$

$$D(A) \subset F_0 \subset X_0 \subset X \subset F_{-1} \subset X_{-1}. \tag{2.9}$$

3. The Miller scheme

In this section we recall a method due to R. K. Miller (see [10] and [12]) to reduce a nonhomogeneous Cauchy problem to a homogeneous one in a suitable product space.

Let us first state a known result which will be used later.

Theorem 3.1. *In the Banach space* $L^1(\mathbb{R}_+; X_{-1})$, *the translation semigroup defined for* $f \in L^1(\mathbb{R}_+; X)$ *by*

$$(S(t)f)(s) = f(t+s), \quad s \geq 0 \quad a.e.$$

is strongly continuous and its generator $B : D(B) \subset L^1(\mathbb{R}_+; X) \to L^1(\mathbb{R}_+, X)$ *is given by*

$$\begin{cases} D(B) = W^{1,1}(\mathbb{R}_+, X_{-1}) \\ (Bf)(s) = f'(s), \quad s \geq 0 \quad a.e. \end{cases}$$

From this theorem we can deduce the following

Theorem 3.2. *In the Banach space*

$$Z_{-1} := X_{-1} \oplus L^1(\mathbb{R}_+; X_{-1})$$

with norm

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_{Z_{-1}} := \|x\|_{-1} + \|f\|_{L^1(\mathbb{R}_+; X_{-1})}$$

the semigroup

$$G_{-1}(t) \begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} T_{-1}(t)x + (T_{-1} * f)(t) \\ S(t)f \end{pmatrix},$$

where

$$(T_{-1} * f)(t) = \int_0^t T_{-1}(t-s)f(s)ds, \quad t \geq 0$$

has generator $\mathcal{A}_{-1} : D(\mathcal{A}_{-1}) \subset Z_{-1} \to Z_{-1}$ *given by*

$$\begin{cases} D(\mathcal{A}_{-1}) = X_0 \oplus W^{1,1}(\mathbb{R}_+; X_{-1}) \\ \mathcal{A}_{-1}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} A_{-1}x + f(0) \\ f' \end{pmatrix}. \end{cases}$$

We will need to find a subspace of Z_{-1} on which $G_{-1}(t)$ is still a semigroup. To this end we use the following lemma (which was proved in [11] under the additional assumption that $A^{-1} \in \mathcal{L}(x)$).

Lemma 3.3. *Given* $f \in L^1(\mathbb{R}_+; X)$ *we have for each* $t > 0$ *:*

$$(T_{-1} * f)(t) \in X_0, \tag{3.1}$$

$$\|(T_{-1} * f)(t)\|_0 \leq c_1 e^{2\omega' t}\|f\|_{L^1(0,t;X)} \text{ where } c_1 \text{ is independent of } f \text{ and } t, \tag{3.2}$$

$$\lim_{t\to 0} \|(T_{-1} * f)(t)\|_0 = 0. \tag{3.3}$$

Proof. Given $x \in X$ and $t > 0$, set

$$V(t)x = \int_0^t T_{-1}(s)x\, ds.$$

As X_0 is the domain of the generator of $T_{-1}(t)$ we have

$$V(t)x \in X_0. \tag{3.4}$$

For each $\mu > \omega$ we have $\|(\mu-\omega)R(\mu, A)\| \leq M$; hence, setting $y := R(\lambda, A)x$ we get

$$\begin{aligned} \|(\mu - \omega)(T_0(t) - I)R(\mu, A_0)y\|_0 &= \left\|(\mu - \omega)A_0\int_0^t T_0(s)R(\mu, A_0)y\, ds\right\|_0 \\ &\leq M^2 te^{\omega' t}\|Ay\|. \end{aligned} \tag{3.5}$$

As $y \in \overline{D(A_0)}$ we have

$$\lim_{\mu\to+\infty} \|(\mu - \omega)R(\mu, A_0)y - y_0\|_0 = 0$$

hence,

$$\lim_{\mu\to+\infty} \|(\mu - \omega)(T_0(t) - I)R(\mu, A_0)y\|_0 = \|(T_0(t) - I)y\|_0$$

and so by virtue of (3.5)

$$\|(T_0(t) - I)y\|_0 \leq M^2 te^{\omega' t}\|Ay\|. \tag{3.6}$$

As

$$V(t)x = \int_0^t (\lambda - A_{-1})T_{-1}(s)y\,ds = \lambda\int_0^t T_{-1}(s)y\,ds - T_0(t)y + y$$

we get from (3.6)

$$\|V(t)x\|_0 \le c_1 t e^{\omega' t}\|x\|, \tag{3.7}$$

where c_1 is independent of x and t.

Let $\mathcal{S}$ denote the set of functions $\varphi : \mathbb{R}_+ \to X$ such that $\varphi(t) = \sum_{i=0}^n x_i \cdot \alpha_i(t)$, $t \ge 0$ where $0 = t_0 < t_1 < \ldots < t_{n+1}$, $x_i \in X$ and α_i is the characteristic function of $[t_i, t_{i+1}] (i = 0, 1, \ldots, n)$. Let us prove that for each $\varphi \in \mathcal{S}$ and $t > 0$

$$(T_{-1} * \varphi)(t) \in X_0 \tag{3.8}$$

$$\|(T_{-1} * \varphi)(t)\|_0 \le c_1 M e^{2\omega' t}\|\varphi\|_{L^1(0,t;X)}. \tag{3.9}$$

If $t \ge t_{n+1}$ we have $0 \le t_i < t_{i+1} \le t$ $(i = 0, 1, \ldots, n)$ and so

$$\begin{aligned}(T_{-1} * \varphi)(t) &= \sum_{i=0}^n \int_{t_i}^{t_{i+1}} T_{-1}(t-s)x_i ds \\ &= \sum_{i=0}^n T_{-1}(t - t_{i+1}) \int_0^{t_{i+1}-t_i} T_{-1}(s)x_i ds \\ &= \sum_{i=0}^n T_{-1}(t - t_{i+1}) V(t_{i+1} - t_i)x_i\end{aligned}$$

hence, by virtue of (3.4) we deduce (3.8). Moreover from (3.7) we get

$$\|(T_{-1} * \varphi)(t)\|_0 \le M e^{\omega' t}\sum_{i=0}^n c_1(t_{i+1} - t_i)e^{\omega' t}\|x_i\| = c_1 M e^{2\omega' t}\|\varphi\|_{L^1(0,t;X)},$$

i.e., (3.9).

If $0 < t < t_{n+1}$, let α be the characteristic function of $[0, t]$. Setting $\varphi_1 = \varphi \cdot \alpha$ we have $\varphi_1 \in \mathcal{S}$ and $\mathrm{spt}\varphi_1 \subseteq [0, t]$, so we can apply to φ_1 the result just proved and deduce from (3.8)

$$(T_{-1} * \varphi)(t) = (T_{-1} * \varphi_1)(t) \in X_0,$$

and from (3.9)

$$\begin{aligned}\|(T_{-1} * \varphi)(t)\|_0 &= \|(T_{-1} * \varphi_1)(t)\|_0 \le c_1 M e^{2\omega' t}\|\varphi_1\|_{L^1(0,t;X)} \\ &= c_1 M e^{2\omega' t}\|\varphi\|_{L^1(0,t;X)},\end{aligned}$$

and so (3.8) and (3.9) are proved also in this case.

Given $f \in L^1(\mathbb{R}_+;X)$ let $\{\varphi_n\} \subset \mathcal{S}$ be such that

$$\lim_{n\to\infty} \|f - \varphi_n\|_{L^1(\mathbb{R};X)} = 0$$

hence for $t \geq 0$

$$\lim_{n\to\infty} \|(T_{-1} * f)(t) - (T_{-1} * \varphi_n)(t)\|_{-1} = 0. \tag{3.10}$$

Fix $t > 0$: from (3.9) we deduce that there exists $\lim_{n\to\infty}(T_{-1} * \varphi_n)(t)$ in X_0 and so (from (3.10)) we have $(T_{-1} * f)(t) \in X_0$ (i.e., (3.1)) and

$$\lim_{n\to\infty} \|(T_{-1} * f)(t) - (T_{-1} * \varphi_n)(t)\|_0 = 0.$$

Setting $\varphi = \varphi_n$ in (3.9) and letting $n \to \infty$ we deduce (3.2). From this (3.3) follows. □

We can prove now a restriction theorem for the semigroup $G_{-1}(t)$.

Theorem 3.4. *If $G(t)$ is the restriction of $G_{-1}(t)$ to the Banach space*

$$Z = X_0 \oplus L^1(\mathbb{R}_+;X)$$

with norm

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|_Z = \|x\|_0 + \|f\|_{L^1(\mathbb{R}_+,X)},$$

then $G(t)$ is a strongly continuous semigroup in Z with generator $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ given by

$$\begin{cases} D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \oplus W^{1,1}(\mathbb{R}_+;X); Ax + f(0) \in X_0 \right\} \\ \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} Ax + f(0) \\ f' \end{pmatrix}. \end{cases} \tag{3.11}$$

Moreover, the graph-norm of $\mathcal{A}$ on $D(\mathcal{A})$ is equivalent to the norm of $D(A) \oplus W^{1,1}(\mathbb{R}_+;X)$.

Proof. We have $Z \hookrightarrow Z_{-1}$ and Z is invariant for $G_{-1}(t)$ by virtue of (3.1). For each $t > 0$ we have $G_{-1}(t) \in \mathcal{L}(Z)$ because from (3.2) we deduce that for $f \in L^1(\mathbb{R}_+;X)$

$$\|(T_{-1} * f)(t)\|_0 \leq c_1 e^{2\omega' t} \|f\|_{L^1(\mathbb{R}_+;X)}.$$

In addition we have for each $\binom{x}{f} \in Z$ that

$$\lim_{t\to 0} \left\| G_{-1}(t) \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} x \\ f \end{pmatrix} \right\|_Z = 0$$

by virtue of (3.3). In conclusion $G(t) := G_{-1}(t)|_Z$ is a strongly continuous semigroup and so its generator $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ is the part of $\mathcal{A}_1$ in Z, i.e.,

$$\begin{cases} D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_{-1}) \cap Z; \mathcal{A}_{-1}\begin{pmatrix} x \\ f \end{pmatrix} \in Z \right\} \\ \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{A}_{-1}\begin{pmatrix} x \\ f \end{pmatrix} \end{cases}$$

hence $\binom{x}{f} \in D(\mathcal{A})$ if and only if $x \in X_0$, $f \in W^{1,1}(\mathbb{R}_+; X_{-1}) \cap L^1(\mathbb{R}_+; X)$, $f' \in L^1(\mathbb{R}_+; X)$ and $A_{-1}x + f(0) \in X_0$. We deduce $f \in W^{1,1}(\mathbb{R}_+; X)$ and $\lambda x - A_{-1}x \in X$; this implies (see (2.7)) $x \in D(A)$ and so (3.11) is proved.

For the last part of the theorem it is sufficient to observe that if $f \in W^{1,1}(\mathbb{R}_+; X)$, then $f(0) = -\int_0^{+\infty} f'(s)ds$ hence for $x \in D(A)$

$$\|Ax\| \leq \|Ax + f(0)\| + \|f'\|_{L^1(\mathbb{R}_+;X)}$$

and

$$\|Ax + f(0)\| \leq \|Ax\| + \|f'\|_{L^1(\mathbb{R}_+;X)}. \qquad \square$$

4. STRICT SOLUTION OF THE ABSTRACT INTEGRODIFFERENTIAL EQUATION

To apply the Miller scheme to the abstract integrodifferential equation (0.1) we will show that the integral term corresponds to a suitable additive perturbation of the operator $\mathcal{A}$ which preserves the semigroup generator property. To show this we state more precisely the assumption on the kernel.

Definition 4.1. We say that $a : \mathbb{R}_+ \to \mathbb{R}$ belongs to $BV(\mathbb{R}_+)$ if for each $[t_0, t_1] \subseteq \mathbb{R}_+$ the total variation $V_{t_0}^{t_1}(a)$ of a on $[t_0, t_1]$ is bounded and

$$V_0^\infty(a) := \lim_{t\to+\infty} V_0^t(a) < +\infty. \tag{4.1}$$

In this case we have (see [1] Lemma A1)

$$|a(t)| \leq |a(0)| + V_0^\infty(a), \quad t \geq 0 \tag{4.2}$$

and if in addition $a \in \mathcal{L}^1(\mathbb{R}_+)$ also

$$\int_0^{+\infty} |a(t+s) - a(s)|ds \leq t \cdot V_0^\infty(a), \quad t \geq 0. \tag{4.3}$$

Theorem 4.2. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$. If we define $\mathcal{B} : D(\mathcal{A}) \to Z$ as*

$$\mathcal{B}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ a(\cdot)A_1 x \end{pmatrix}, \tag{4.4}$$

then there exists $c > 0$ such that for each $x \in D(A)$ we have

$$\left\| \mathcal{B}\begin{pmatrix} x \\ f \end{pmatrix} \right\|_{F(\mathcal{A})} \le c_2(\|x\| + \|Ax\|) \tag{4.5}$$

where $F(\mathcal{A})$ is the Favard class of the semigroup $G(t)$ (defined in Theorem 3.4)

Moreover, $\mathcal{B} \in \mathcal{L}(D(\mathcal{A}), F(\mathcal{A}))$ and $\mathcal{A} + \mathcal{B} : D(A) \subset Z \to Z$ generates a semigroup in Z.

Proof. Given $x \in D(A)$ and $t \in (0,1]$ we have by virtue of (3.2) and (4.3)

$$\begin{aligned}
&\left\| \begin{pmatrix} 0 \\ a(\cdot)A_1x \end{pmatrix} \right\|_Z + \frac{1}{t}\left\| (e^{t\mathcal{A}} - I)\begin{pmatrix} 0 \\ a(\cdot)A_1x \end{pmatrix} \right\|_Z \\
&= \int_0^{+\infty} |a(s)| \|A_1x\| ds + \frac{1}{t}\left\| \begin{pmatrix} (T_{-1} * a(\cdot)A_1x)(t) \\ [a(t+\cdot) - a(\cdot)]A_1x \end{pmatrix} \right\|_Z \\
&= \int_0^{+\infty} |a(s)| \|A_1x\| ds + \frac{1}{t}\left\| \int_0^t T_{-1}(t-s)a(s)A_1x ds \right\| \\
&\quad + \frac{1}{t}\int_0^{+\infty} |a(t+s) - a(s)| \|A_1x\| ds \le \|a\|_{L^1(\mathbb{R}_+)} \|A_1x\| \\
&\quad + c_1 e^{2\omega' t} \|A_1x\| \frac{1}{t}\int_0^t |a(s)| ds + V_0^\infty(a)\|A_1x\| \le c_2(\|x\| + \|Ax\|),
\end{aligned}$$

where c_2 is independent of x and t. We deduce that $\mathcal{B}\binom{x}{f} \in F(\mathcal{A})$ for each $\binom{x}{f} \in D(\mathcal{A})$. As $f(0) = -\int_0^{+\infty} f'(s)ds$ we have

$$\|x\| + \|Ax\| \le \left\| \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} \right\|_Z$$

hence $\mathcal{B} \in \mathcal{L}(D(\mathcal{A}), F(\mathcal{A}))$. This implies, by virtue of a Desch-Schappacher theorem (see [4], Example 3), the last part of the theorem. □

We will use the following result of semigroup theory.

Theorem 4.3. *Let $B_1 : D \subset X \to X$ and $B_2 : D \subset X \to X$ be generators of semigroups $T_1(t)$ and $T_2(t)$ in a Banach space X. For each $x \in D$ and $t > 0$ we have*

$$T_1(t)x = T_2(t)x + \int_0^t T_2(t-s)(B_1 - B_2)T_1(s)x\, ds. \tag{4.6}$$

Proof. Setting, for $x \in D$ and $t \geq 0$, $u_1(t) = T_1(t)x$ and $u_2(t) = T_2(t)x$, we have that $u(t) := u_1(t) - u_2(t)$ is a solution of the problem

$$\begin{cases} u'(t) = B_2 u(t) + B_1 u_1(t) - B_2 u_1(t), & t \geq 0 \\ u(0) = 0 \end{cases}$$

and from this (4.6) follows. □

The next theorem provides the existence, uniqueness and estimate of a strict solution in $\mathbb{R}_+$ of the abstract problem (0.1).

Theorem 4.4. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. Given*

$$u_0 \in D(A) \quad \text{and} \quad f \in W^{1,1}(\mathbb{R}_+, X) \quad \text{such that} \quad Au_0 + f(0) \in \overline{D(A)} \tag{4.7}$$

there exists a unique $u \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ solution of

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1 u(s)ds + f(t), \quad t \geq 0; \quad u(0) = u_0 \tag{4.8}$$

and a continuous and increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, independent of u_0 and f, such that

$$\|u'(t)\| + \|u(t)\|_{D(A)} \leq \rho(t)(\|u_0\|_{D(A)} + \|f\|_{W^{1,1}(\mathbb{R}_+,X)}), \quad t \geq 0. \tag{4.9}$$

Proof. Let us first prove the uniqueness of a solution $u \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ of (4.8).

To this end let us suppose that u is such a solution when $u_0 = f = 0$ and prove that $u = 0$. Defining $H : \mathbb{R}_+ \to Z$ as

$$H(t) = \begin{pmatrix} 0 \\ a(\cdot)A_1 u(t) \end{pmatrix}, \quad t \geq 0$$

we deduce from Theorem 4.2 that $H \in C(\mathbb{R}_+ : F(\mathcal{A}))$. Setting

$$W(t) = \int_0^t e^{(t-s)\mathcal{A}} H(s)ds, \quad t \geq 0 \tag{4.10}$$

we get from Theorem 7.3 of [12] that $W \in C^1(\mathbb{R}_+, Z) \cap C(\mathbb{R}_+, D(\mathcal{A}))$ and

$$W'(t) = \mathcal{A}W(t) + H(t), \quad t \geq 0; \quad W(0) = 0. \tag{4.11}$$

From (4.8) with $f = 0$ we deduce that the function $t \to \varphi(t) := \int_0^t a(t-s)A_1 u(s)ds$ belongs to $C(\mathbb{R}_+, X)$; hence u is a mild solution of the problem

$$u'(t) = A_{-1}u(t) + \varphi(t), \quad t \geq 0, \quad u(0) = 0,$$

i.e.,

$$\begin{aligned} u(t) &= \int_0^t T_{-1}(t-\sigma)d\sigma \int_0^\sigma a(\sigma-s)A_1u(s)ds \\ &= \int_0^t ds \int_s^t T_{-1}(t-\sigma)a(\sigma-s)A_1u(s)d\sigma \qquad (4.12) \\ &= \int_0^t ds \int_0^{t-s} T_{-1}(t-s-\sigma)a(\sigma)A_1u(s)d\sigma. \end{aligned}$$

We deduce from (4.10) and (4.12) that

$$\begin{aligned} W(t) &= \begin{pmatrix} \int_0^t ds \int_0^{t-s} T_{-1}(t-s-\sigma)a(\sigma)A_1u(s)d\sigma \\ \int_0^t a(t-s+\cdot)A_1u(s)ds \end{pmatrix} \\ &= \begin{pmatrix} u(t) \\ \int_0^t a(t-s+\cdot)A_1u(s)ds \end{pmatrix} \qquad (4.13) \end{aligned}$$

hence

$$BW(t) = \begin{pmatrix} 0 \\ a(\cdot)A_1u(t) \end{pmatrix} = H(t)$$

which implies by virtue of (4.11)

$$W'(t) = (\mathcal{A}+\mathcal{B})W(t), \quad t \geq 0, \quad W(0) = 0.$$

As $\mathcal{A}+\mathcal{B}$ generates a semigroup in Z (see Theorem 4.2) we deduce $W=0$; hence from (4.13) $u=0$ and the uniqueness is proved.

To prove the existence let us observe that setting $U_0 := \binom{u_0}{f}$ we have $U_0 \in D(\mathcal{A}+\mathcal{B}) = D(\mathcal{A})$ hence $U(t) := e^{t(\mathcal{A}+\mathcal{B})}U_0$ is the unique solution of

$$U'(t) = (\mathcal{A}+\mathcal{B})U(t), \quad t \geq 0; \quad U(0) = U_0$$

and $U \in C^1(\mathbb{R}_+, Z) \cap C(\mathbb{R}_+; D(\mathcal{A}))$. As the graph norm of $\mathcal{A}$ on $D(\mathcal{A})$ is equivalent to the norm of $D(A) \oplus W^{1,1}(\mathbb{R}_+, X)$ (see Theorem 3.4), setting for $t \geq 0$

$$\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} := e^{t(\mathcal{A}+\mathcal{B})}U_0$$

we get $u \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+; D(A)), u(0) = u_0$. Moreover from

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} = (\mathcal{A}+\mathcal{B})\begin{pmatrix} u(t) \\ F(t) \end{pmatrix}, \quad t \geq 0$$

we deduce

$$u'(t) = Au(t) + F(t)(0), \quad t \geq 0. \tag{4.14}$$

If we apply Theorem 4.3 to $B_1 = \mathcal{A} + \mathcal{B}$, $B_2 = \mathcal{A}$ and $D = D(\mathcal{A})$ we deduce

$$e^{t(\mathcal{A}+\mathcal{B})}U_0 = e^{t\mathcal{A}}U_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}e^{s(\mathcal{A}+\mathcal{B})}U_0 ds, \quad t \geq 0 \tag{4.15}$$

and from this

$$F(t)(\cdot) = f(t+\cdot) + \int_0^t a(t-s+\cdot)A_1 u(s)ds, \quad t \geq 0.$$

Hence from (4.14) we conclude that u is a solution of (4.8).

To prove estimate (4.9) let us suppose that $\|e^{t\mathcal{A}}\|_{\mathcal{L}(Z)} \leq \tilde{M}e^{\tilde{\omega}t}$, $t \geq 0$. Equation (4.15) can be written as

$$U(t) = e^{t\mathcal{A}}U_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}U(s)ds, \quad t \geq 0. \tag{4.16}$$

As $U \in C(\mathbb{R}_+, D(\mathcal{A}))$ we have $\mathcal{B}U \in C(\mathbb{R}_+; F(\mathcal{A}))$: hence from Theorem 6.1 of [12] we get

$$\left\| \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}U(s)ds \right\|_{D(\mathcal{A})} \leq \tilde{M} \int_0^t e^{\tilde{\omega}(t-s)} \|\mathcal{B}U(s)\|_{F(\mathcal{A})} ds.$$

As $U_0 \in D(\mathcal{A})$ we deduce from (4.16), setting $\|B\| = \|B\|_{\mathcal{L}(D(\mathcal{A}),F(\mathcal{A}))}$

$$\|U(t)\|_{D(\mathcal{A})} \leq \tilde{M}e^{\tilde{\omega}t}\|U_0\|_{D(\mathcal{A})} + \tilde{M}\|B\| \int_0^t e^{\tilde{\omega}(t-s)}\|U(s)\|_{D(\mathcal{A})} ds$$

and so from Gronwall's lemma

$$\|U(t)\|_{D(\mathcal{A})} \leq \tilde{M}\|U_0\|_{D(\mathcal{A})} \exp[(\tilde{M}\|B\| + \tilde{\omega})t].$$

As the graph norm of $\mathcal{A}$ on $D(\mathcal{A})$ is equivalent to the norm of $D(A) \oplus W^{1,1}(\mathbb{R}_+, X)$ (see Theorem 3.4) we deduce the existence of a continuous and increasing function $\tilde{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$, independent of u_0 and f, such that

$$\|u(t)\|_{D(A)} \leq \tilde{\rho}(t)(\|u_0\|_{D(A)} + \|f\|_{W^{1,1}(\mathbb{R}_+,X)}). \tag{4.17}$$

From equation (4.8) we deduce

$$\|u'(t)\| \leq \|Au(t)\| + \|A_1\|_{\mathcal{L}(D(A),X)}\|a\|_{L^1(\mathbb{R}_+)} \sup_{0\leq s\leq t} \|u(s)\|_{(D(A)} + \|f(t)\|, \ t \geq 0$$

and so from (4.17) we obtain (4.9). □

We will consider now problem (4.8) in a compact interval $[0, T]$. While the existence of the solution is an easy consequence of the existence in $\mathbb{R}_+$, the uniqueness result is less obvious. To this end we need the following result.

Lemma 4.5. *A function* $f \in W^{1,1}(0,T;X)$ *has an extension* $\hat{f} \in W^{1,1}(\mathbb{R}_+, X)$ *such that*

$$\|\hat{f}\|_{W^{1,1}(\mathbb{R}_+,X)} \le \hat{c}\|f\|_{W^{1,1}(0,T;X)}, \tag{4.18}$$

where $\hat{c}$ *is independent of* f. *A function* $a \in BV(0,T)$ *has an extension* $\hat{a} \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$ *such that*

$$V_0^\infty(\hat{a}) \le |a(T)| + V_0^T(a). \tag{4.19}$$

Proof. For the first part see Theorem 2.2 of [9]. To prove the second part it is sufficient to consider the null extension of a. ☐

Theorem 4.6. *Let* $A : D(A) \subset X \to X$ *be a Hille-Yosida operator,* $A_1 \in \mathcal{L}(D(A), X)$ *and* $a \in BV(0,T)$. *Given*

$$u_0 \in D(A) \quad \text{and} \quad f \in W^{1,1}(0,T;X) \quad \text{such that} \quad Au_0 + f(0) \in \overline{D(A)} \tag{4.20}$$

there exists a unique solution $u \in C^1(0,T;X) \cap C(0,T;D(A))$ *of*

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1u(s)ds + f(t), \quad t \in [0,T]; \quad u(0) = u_0 \tag{4.21}$$

and a continuous and increasing function $\hat{\rho} : [0,T] \to \mathbb{R}_+$, *independent of* u_0 *and* f, *such that*

$$\|u'(t)\| + \|u(t)\|_{D(A)} \le \hat{\rho}(t)(\|u_0\|_{D(A)} + \|f\|_{W^{1,1}(0,T;X)}), \quad t \in [0,T]. \tag{4.22}$$

Proof. Let $\hat{f}$ and $\hat{a}$ be the be the extensions of f and a given by Lemma 4.5. Theorem 4.4 gives a solution of (4.8) with $a = \hat{a}$ and $f = \hat{f}$; its restriction to $[0,T]$ is a solution of (4.21) and by virtue of (4.18) estimate (4.19) implies (4.22).

To prove the uniqueness of the solution in $[0,T]$ it is sufficient to assume that $u \in C^1(0,T;X) \cap C(0,T;D(A))$ is a solution of

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1u(s)ds, \quad t \in [0,T]; \quad u(0) = 0 \tag{4.23}$$

and to prove that u vanishes in $[0,T]$. If we show that u can be extended to a solution $\hat{u} \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ of

$$\hat{u}'(t) = A\hat{u}(t) + \int_0^t a(t-s)A_1\hat{u}(s)ds, \quad t \ge 0; \quad \hat{u}(0) = 0, \tag{4.24}$$

then the uniqueness result of Theorem 4.4 implies $\hat{u} = 0$ and so $u = 0$.

Let us observe that setting

$$g(t) = \int_0^T \hat{a}(T+t-s)A_1u(s)ds, \quad t \in [0,T]$$

we have $g \in W^{1,1}(0,T;X)$ (see Theorem 7.1 of the Appendix). Moreover, $u(T) \in D(A)$ and

$$Au(T) + g(0) = u'(T) \in \overline{D(A)}$$

hence from the first part of the proof, the problem

$$v'(t) = Av(t) + \int_0^t a(t-s)A_1 v(s)ds + g(t), \quad t \in [0,T]; \quad v(0) = u(T) \quad (4.25)$$

has a solution $v \in C^1(0,T;X) \cap (0,T;D(A))$. Let us define $\hat{u} : [0,2T] \to X$ as

$$\hat{u}(t) = \begin{cases} u(t), & 0 \le t \le T \\ v(t-T), & T < t \le 2T. \end{cases} \quad (4.26)$$

As $u(T) = v(0)$ and $\hat{u}'_-(T) = \hat{u}'_+(T)$ we have

$$\hat{u} \in C^1(0,2T;X) \cap C(0,2T;D(A)).$$

From (4.23) we deduce

$$\hat{u}'(t) = A\hat{u}(t) + \int_0^t \hat{a}(t-s)A_1\hat{u}(s)ds, \quad t \in [0,T]; \quad \hat{u}(0) = 0. \quad (4.27)$$

If $t \in [T,2T]$, by replacing t with $t-T$ in (4.25) we get

$$\begin{aligned} v'(t-T) &= Av(t-T) + \int_0^{t-T} a(t-T-s)A_1 v(s)ds + \int_0^T \hat{a}(t-s)A_1 u(s)ds \\ &= Av(t-T) + \int_T^t a(t-s)A_1 v(s-T)ds + \int_0^T \hat{a}(t-s)A_1 u(s)ds; \end{aligned}$$

i.e., by virtue of (4.26)

$$\hat{u}'(t) = A\hat{u}(t) + \int_0^t \hat{a}(t-s)A_1\hat{u}(s)da \quad t \in (T,2T],$$

which together with (4.27) shows that $\hat{u}$ is a solution on $[0,2T]$ of problem (4.24). As the procedure can be iterated we get the existence of an extension $\hat{u}$ of u to a solution of problem (4.24) in $\mathbb{R}_+$ and the conclusion follows. □

Remark 4.7. The compatibility condition on the data $Au_0 + f(0) \in \overline{D(A)}$ given in (4.7) and (4.20) is necessary to obtain a solution $u \in C^1(0,T;X) \cap C(0,T;D(A))$ because from (4.8) and (4.20) $Au_0 + f(0) = u'(0) \in \overline{D(A)}$.

Remark 4.8. From the uniqueness result of Theorem 4.6 it follows that if u is the strict solution of (4.8) in $\mathbb{R}_+$ given by Theorem 4.4 for each $T > 0$ estimate (4.22) holds.

5. Weak solutions of the abstract integrodifferential equation

In this section we want to relax the regularity on the data u_0 and f and investigate the existence of a solution in a weak sense of the integrodifferential equation.

As we deal with a Hille-Yosida operator A we can use a variant of the definition of F-solution (or solution in the sense of Friedrichs) given in [3] for the case $a = 0$. This type of solution is the limit of strict solutions of an approximating problem. More precisely:

Definition 5.1. Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$. Given

$$f \in L^1(\mathbb{R}_+, X) \quad \text{and} \quad u_0 \in \overline{D(A)} \tag{5.1}$$

a function $u \in C(\mathbb{R}_+, X)$ is called an F-solution of the problem

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1u(s)ds + f(t), \quad t \geq 0; \quad u(0) = u_0 \tag{5.2}$$

if there exists $\{u_n\} \subset C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ such that, setting

$$\begin{cases} u_n'(t) - Au_n(t) - \displaystyle\int_0^t a(t-s)A_1u_n(s)ds =: f_n(t), \quad t \geq 0 \\ u_n(0) =: u_{0n} \end{cases} \tag{5.3}$$

we have $f_n \in L^1(\mathbb{R}_+, X)$ and

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R}_+,X)} = 0, \quad \lim_{n\to\infty} \|u_{0n} - u_0\| = 0 \tag{5.4}$$

$$\text{for each } T > 0, \lim_{n\to\infty} \|u_n(t) - u(t)\| = 0 \text{ uniformly for } t \in [0,T]. \tag{5.5}$$

Therefore, $u(0) = u_0$ and $u_0 = \lim_{n\to\infty} u_n(0) \in \overline{D(A)}$; hence (5.1) is necessary for the existence of an F-solution.

Before proving an a-priori estimate for an F-solution we need an estimate for the strict solution with norms different from those used in Section 4.

Theorem 5.2. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(0,T)$. Given $u_0 \in D(A), f \in L^1(0,T;X)$, let $u \in C^1(0,T;X) \cap C(0,T;D(A))$ be the strict solution of*

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1u(s)ds + f(t), \quad t \in [0,T]; \quad u(0) = u_0. \tag{5.6}$$

Then there exists a continuous and increasing function $\rho^* : [0,T] \to \mathbb{R}_+$, independent of u_0 and f, such that

$$\|u(t)\| \le \rho^*(T)(\|u_0\| + \|f\|_{L^1(0,T;X)}), \quad t \in [0,T]. \tag{5.7}$$

Proof. Set, for $t \in [0,T]$, $v(t) := \int_0^t u(s)ds$ and $b(t) := \int_0^t a(s)ds$. By integrating (5.6) we get

$$u(t)-u_0 = A\int_0^t u(s)ds + \int_0^t ds \int_0^s a(s-\sigma)A_1u(\sigma)ds + \int_0^t f(s)ds, \ \ t \in [0,T]. \tag{5.8}$$

As

$$\int_0^t ds \int_0^s a(s-\sigma)A_1 u(\sigma)d\sigma = \int_0^t A_1u(\sigma)d\sigma \int_\sigma^s a(s-\sigma)ds$$
$$= \int_0^t b(t-\sigma)A_1u(\sigma)d\sigma = A_1\int_0^t b(t-\sigma)u(\sigma)d\sigma$$

and $v' = u$, integrating by parts, we deduce from (5.8)

$$u(t)-u_0 = A\int_0^t u(s)ds + \int_0^t a(t-\sigma)A_1v(\sigma)d\sigma + \int_0^t f(s)ds, \ \ t \in [0,T], \tag{5.9}$$

hence v is a strict solution of the equation

$$v'(t) = Av(t) + \int_0^t a(t-\sigma)A_1v(\sigma)d\sigma + g(t), \quad t \in [0,T]; \quad v(0) = 0, \tag{5.10}$$

where

$$g(t) := u_0 + \int_0^t f(s)ds.$$

As $g \in W^{1,1}(0,T;X)$ and

$$\|g\|_{W^{1,1}(0,T;X)} \le (1+T)(\|u_0\| + \|f\|_{L^1(0,T;X)})$$

by virtue of estimate (4.22) applied to the solution of problem (5.10) we deduce

$$\|v'(t)\| \le \hat{\rho}(t)\|g\|_{W^{1,1}(0,T;X)}, \quad t \in [0,T]$$

from which (5.7) follows. □

We can prove now an a priori estimate of the F-solution of problem (5.2).

Theorem 5.3. *If u is an F-solution of problem* (5.2)*, then for each $T > 0$ we have*

$$\|u(t)\| \le \rho^*(T)(\|u_0\| + \|f\|_{L^1(0,T;X)}), \quad t \in [0,T], \tag{5.11}$$

where ρ^* is given by Theorem 5.2

Proof. By definition there exist $\{u_n\}$ satisfying (5.3)-(5.5); and so for each $n \in \mathbb{N}$ u_n is the strict solution of (5.3); hence by virtue of Theorem 5.2 we have for each $T > 0$

$$\|u_n(t)\| \le \rho^*(T)(\|u_{0n}\| + \|f_n\|_{L^1(0,T;X)}), \quad t \in [0,T],$$

and so for $n \to \infty$ we deduce (5.10). □

We can prove now the existence, uniqueness and estimate of the F-solution of the abstract integrodifferential equation.

Theorem 5.4. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$. Given $u_0 \in \overline{D(A)}$ and $f \in L^1(\mathbb{R}_+, X)$ there exists a unique F-solution of problem* (5.2) and

$$\|u(t)\| \le \overline{M} e^{\overline{\omega} t}(\|u_0\| + \|f\|_{L^1(\mathbb{R}_+;X)}), \quad t \ge 0, \tag{5.12}$$

where $\overline{M}$ and $\overline{\omega}$ are independent of u_0 and f.

Proof. The uniqueness is a consequence of estimate (5.10). To prove the existence let us use the notation of Section 4 and observe that $\binom{u_0}{f} \in Z = \overline{D(\mathcal{A})}$; hence there exists $\left\{\binom{u_{0n}}{f_n}\right\} \subset D(\mathcal{A})$ such that

$$\lim_{n\to\infty} \left\| \begin{pmatrix} u_{0n} \\ f_n \end{pmatrix} - \begin{pmatrix} u_0 \\ f \end{pmatrix} \right\|_Z = 0. \tag{5.13}$$

Moreover, from the proof of Theorem 4.4 we deduce that

$$u_n(t) := \mathrm{pr}_1 e^{t(\mathcal{A}+\mathcal{B})} \begin{pmatrix} u_{0n} \\ f_n \end{pmatrix}, \quad t \ge 0$$

is the strict solution of (5.3); moreover (5.4) holds. If the semigroup generated by $\mathcal{A} + \mathcal{B}$ is of type $(\overline{M}, \overline{\omega})$ we have for $n \in \mathbb{N}$, $t \ge 0$

$$\|u_n(t)\| \le \overline{M} e^{\overline{\omega} t} \left\| \begin{pmatrix} u_{0n} \\ f_n \end{pmatrix} \right\|_Z. \tag{5.14}$$

From (5.13) we deduce that for each $T > 0$ there exists $u(t) := \lim_{n\to\infty} u_n(t)$ uniformly for $t \in [0,T]$ and (5.11). In conclusion u is an F-solution of (5.2). □

In an obvious way it is possible to define an F-solution of the abstract integrodifferential equation in a compact interval $[0,T]$.

Definition 5.5. Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A), X)$ and $a \in BV(0,T)$. Given

$$f \in L^1(0,T;X) \quad \text{and} \quad u_0 \in \overline{D(A)} \tag{5.15}$$

a function $u \in C(0,T;X)$ is called an F-solution of problem

$$u'(t) = Au(t) + \int_0^t a(t-s)A_1u(s)ds + f(t), \quad t \in [0,T]; \quad u(0) = u_0 \tag{5.16}$$

if there exists $\{u_n\} \subset C^1(0,T;X) \cap C(0,T;D(A))$ such that setting

$$\begin{cases} u_n'(t) - Au_n(t) - \displaystyle\int_0^t a(t-s)A_1u_n(s)ds =: f_n(t), \quad t \in [0,T] \\ u_n(0) =: u_{0n} \end{cases} \tag{5.17}$$

we have

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(0,T;X)} = 0, \quad \lim_{n\to\infty} \|u_{0n} - u_o\| = 0 \tag{5.18}$$

$$\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0 \quad \text{uniformly for} \quad t \in [0,T]. \tag{5.19}$$

Observe that (5.17) implies by virtue of Corollary 7.2 of the Appendix that $f_n \in L^1(0,T;X)$.

Theorem 5.6. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator, $A_1 \in \mathcal{L}(D(A),X)$ and $a \in BV(0,T)$. Given $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$ there exists a unique F-solution of problem* (5.16) *and*

$$\|u(t)\| \le \overline{M}e^{\overline{\omega}t}(\|u_0\| + \|f\|_{L^1(0,T;X)}), \quad t \in [0,T], \tag{5.20}$$

where $\overline{M}$ and $\overline{\omega}$ are independent of u_0 and f. □

Proof. If we extend by zero a and f to $\hat{a} \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$ and $\hat{f} \in L^1(\mathbb{R}_+,X)$ we can obtain from Theorem 5.4 an F-solution $\hat{u}$ to problem (5.2) with $a = \hat{a}$ and $f = \hat{f}$; its restriction to $[0,T]$ is an F-solution of (5.16) and satisfies (5.20). To prove its uniqueness we can use estimate (5.10) which holds also for an F-solution on $[0,T]$ of problem (5.16). □

6. An integrodifferential wave equation

In this section we want to apply the abstract results obtained in Sections 4 and 5 to find strict and weak solutions of problem (0.2).

Given $\ell, T > 0$ we set

$$J := [0,\ell], \quad Q := \mathbb{R}_+ \times J, \quad Q_T := [0,T] \times J, \quad Q' := [0,T] \times \{0,\ell\}.$$

The first step is the reduction of our problem

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + \displaystyle\int_0^t a(t-s)w_{xx}(s,x)ds + f(t,x), & (t,x) \in Q \\ w(t,x') = 0 & (t,x) \in Q' \\ w(0,x) = w_0(x) & x \in J \\ w_t(0,x) = w_1(x) & x \in J \end{cases} \tag{6.1}$$

to a first-order system; after this we will write it as an abstract integrodifferential equation in a Banach space of continuous functions; this forces the use of operators with nondense domain (because of the homogeneous boundary condition $(6.1)_2$) but yields solutions in the classical sense and pointwise estimates of all the derivatives.

Let us start with the reduction to a first-order problem, proved in [15] and recalled here for the reader's convenience.

Theorem 6.1. *Let $a : \mathbb{R}_+ \to \mathbb{R}$, $f : Q \to \mathbb{R}$, $w_0 \in C^2(J)$ and $w_1 \in C^1(J)$ be given. If $w \in C^2(Q)$ is a solution of problem* (6.1)*, then setting*

$$u = \tfrac{1}{2}(w_t + w_x), \quad v = \tfrac{1}{2}(w_t - w_x) \tag{6.2}$$

we have that u, $v \in C^1(Q)$ and

$$\begin{cases} u_t(t,x) = u_x(t,x) + \displaystyle\int_0^t a(t-s)\frac{u_x(s,x) - v_x(s,x)}{2}ds + \frac{1}{2}f(t,x), & (t,x) \in Q \\ v_t(t,x) = -v_x(t,x) + \displaystyle\int_0^t a(t-s)\frac{u_x(s,x) - v_x(s,x)}{2}ds + \frac{1}{2}f(t,x), & (t,x) \in Q \\ u(0,x) = \dfrac{1}{2}(w_1(x) + w_0'(x));\ v(0,x) = \dfrac{1}{2}(w_1(x) - w_0'(x)), & x \in J \\ u(t,x') + v(t,x') = 0 & (t,x) \in Q'. \end{cases} \tag{6.3}$$

Conversely, if

$$w_0(x') = 0, \quad x' = 0, \ell \tag{6.4}$$

and problem (6.3) *has a solution $u, v \in C^1(Q)$, then setting*

$$w(t,x) = \int_0^x [u(t,y) - v(t,y)]dy, \quad (t,x) \in Q \tag{6.5}$$

we have that $w \in C^2(Q)$ and w is a solution of problem (6.1)*.*

Proof. If $w \in C^2(Q)$ is a solution of (6.1) and u, v are defined by (6.2) it can be checked that

$$w_{tt} - w_{xx} = u_t + v_t - u_x + v_x.$$

As $w_{xt} = w_{tx}$ we have $u_t - u_x = v_t + v_x$ hence

$$w_{tt} - w_{xx} = 2(u_t - u_x) = 2(v_t + v_x).$$

As $w_{xx} = u_x - v_x$ we deduce $(6.3)_{1,2}$ from $(6.1)_1$; $(6.3)_3$ is a consequence of $(6.1)_{3,4}$. From $(6.1)_2$ we obtain $(6.3)_4$ because for $(t, x') \in Q'$ we have

$$0 = w_t(t, x') = u(t, x') + v(t, x')$$

and so the first part of the theorem is proved.

Conversely let $u, v \in C^1(Q)$ satisfy system (6.3) and let (6.4) hold. If we define w as in (6.5) we deduce from $(6.3)_{1,2,4}$

$$w_t(t, x) = u(t, x) + v(t, x), \quad (t, x) \in Q \tag{6.6}$$

which implies by virtue of $(6.3)_{1,2}$ and (6.5)

$$w_{tt}(t, x) = w_{xx}(t, x) + \int_0^t a(t - s) w_{xx}(s, x) ds + f(t, x), \quad (t, x) \in Q,$$

i.e., $(6.1)_1$. From $(6.3)_3$ and (6.4) we get $(6.1)_3$ because

$$w(0, x) = \int_0^x [u(0, y) - v(0, y)] dy = \int_0^x w_0'(y) dy = w_0(x).$$

By using (6.6) and $(6.3)_4$ we obtain

$$w_t(0, x) = u(0, x) + v(0, x) = w_1(x),$$

i.e., $(6.1)_4$. Finally we have $w(t, 0) = 0$ and by virtue of (6.6), $(6.3)_4$ and (6.4)

$$w(t, \ell) = \int_0^t w_s(s, \ell) ds + w(0, \ell) = \int_0^t [u(s, \ell) + v(s, \ell)] ds + w_0(\ell) = 0$$

and also $(6.1)_2$ is proved. □

To write in an abstract form system (6.3) we will choose X, A and A_1 as follows

$$X = C(J) \oplus C(J) \tag{6.7}$$

with norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \|u\|_{C(J)} + \|v\|_{C(J)}. \tag{6.8}$$

$A : D(A) \subset X \to X$ is given by

$$\begin{cases} D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X;\ u, v \in C^1(J), u(x') + v(x') = 0,\ x' = 0, \ell \right\} \\ A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ -v' \end{pmatrix} \end{cases} \tag{6.9}$$

and $A_1 : D(A) \to X$ is defined as

$$A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u' - v' \\ u' - v' \end{pmatrix}. \tag{6.10}$$

Let us observe that

$$X_0 := \overline{D(A)} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X;\ u(x') + v(x') = 0;\ x' = 0, \ell \right\}. \tag{6.11}$$

Theorem 6.2. *A is a Hille-Yosida operator of type $(0,1)$ and*

$$A_1 \in \mathcal{L}(D(A), X).$$

Proof. See Theorem 2.1 of [6]. □

Now we can apply to problem (6.1) the abstract results of Section 4 about the strict solutions of the integrodifferential equation.

Theorem 6.3. *Let $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$, $f \in W^{1,1}(\mathbb{R}_+, C(J))$, $w_0 \in C^2(J)$ and $w_1 \in C^1(J)$ be such that for $x' = 0, \ell$ we have*

$$\begin{cases} w_0(x') = 0 \\ w_1(x') = 0 \\ w_0''(x') + f(0, x') = 0, \end{cases} \tag{6.12}$$

then problem (6.1) has a unique solution $w \in C^2(Q)$ and for each $T > 0$

$$\|w\|_{C^2(Q_T)} \le \rho(T)(\|w_0\|_{C^2(J)} + \|w_1\|_{C'(J)} + \|f\|_{W^{1,1}(0,T;C(J))}) \tag{6.13}$$

where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and increasing function independent of w_0, w_1 and f. Conditions (6.12) are necessary for the existence of a C^2-solution of problem (6.1).

Proof. We want to apply Theorem 4.4 to the problem

$$U'(t) = AU(t) + \int_0^t a(t-s) A_1 U(s) ds + F(t), \quad t \ge 0; \quad U(0) = U_0, \tag{6.14}$$

where A and A_1 are defined as above and

$$F(t) := \frac{1}{2} \begin{pmatrix} f(t, \cdot) \\ f(t, \cdot) \end{pmatrix}, \quad U_0 := \frac{1}{2} \begin{pmatrix} w_1 + w_0' \\ w_1 - w_0' \end{pmatrix}. \tag{6.15}$$

From our assumptions we deduce that

$$U_0 \in D(A), \quad F \in W^{1,1}(\mathbb{R}_+, X) \quad \text{and} \quad AU_0 + F(0) \in \overline{D(A)}$$

hence Theorem 4.4 yields a solution $U \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, D(A))$ of (6.14) such that for each $T > 0$

$$\|U\|_{C^1(0,T;X)} + \|U\|_{C(0,T;D(A))} \le \hat{\rho}(T)(\|U_0\|_{D(A)} + \|F\|_{W^{1,1}(0,T;X)}) \tag{6.16}$$

with $\hat{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, increasing and independent of U_0 and F (see Remark 4.8).

Setting

$$U(t) := \begin{pmatrix} u(t,\cdot) \\ v(t,\cdot) \end{pmatrix}$$

we have that $u, v \in C^1(Q)$ and (6.14) can be written as system (6.3) which therefore has a C^1-solution. From Theorem 6.1 we obtain a solution $w \in C^2(Q)$ of (6.1). Moreover (6.16) implies (6.13). The uniqueness of the solution of problem (6.1) follows from the uniqueness of the strict solution of problem (6.14). Finally note that if there exists a solution $w \in C^2(Q)$ of (6.1) from $(6.1)_2$ we get for $x' = 0, \ell$:

$$w(0,x') = w_t(0,x') = w_{tt}(0,x') = 0$$

hence from $(6.1)_{1,3,4}$ we deduce the compatibility conditions on w_0, w_1 and f given by (6.12). □

We pass now to the study of the solutions of problem (6.1) in the case in which the data w_0, w_1 and f are less regular. To this purpose we give a suitable definition of weak solution of problem (6.1).

Definition 6.4. Given $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$ and

$$w_0 \in C^1(J) \quad \text{with} \quad w_0(x') = 0, \quad x' = 0, \ell \tag{6.17}$$

$$w_1 \in C(J) \quad \text{with} \quad w_1(x') = 0, \quad x' = 0, \ell \tag{6.18}$$

$$f \in L^1(\mathbb{R}_+, C(J)) \tag{6.19}$$

a function $w \in C^1(Q)$ is called an F-solution of problem (6.1) if for each $n \in \mathbb{N}$ there exists $w_n \in C^2(Q)$, a strict solution of the problem

$$\begin{cases} D_{tt}w_n(t,x) = D_{xx}w_n(t,x) \\ \quad + \displaystyle\int_0^t a(t-s)D_{xx}w_n(s,x)ds + f_n(t,x), & (t,x) \in Q \\ w_n(t,x') = 0, & (t,x) \in Q' \\ w_n(0,x) = w_{0,n}(x), & x \in J \\ D_t w_n(0,x) = w_{1,n}(x), & x \in J \end{cases} \tag{6.20}$$

with $f_n \in L^1(\mathbb{R}_+, C(J))$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R}_+,C(J)} = 0 \tag{6.21}$$

$$\lim_{n\to\infty} \|w_{0n} - w_0\|_{C^1(J)} = 0 \tag{6.22}$$

$$\lim_{n\to\infty} \|w_{1n} - w_1\|_{C(J)} = 0 \tag{6.23}$$

and for each $T > 0$

$$\lim_{n\to\infty} \|w_n - w\|_{C^1(Q_T)} = 0. \tag{6.24}$$

From these it follows that $w(0,x) = w_0(x)$ and $D_t w(0,x) = w_1(x)$; $x \in J$; hence (6.17) and (6.18) are necessary for the existence of an F-solution.

To prove the existence of an F-solution of (6.1) we need an approximation result.

Lemma 6.5. *Given w_0, w_1 and f satisfying* (6.17)-(6.19) *there exist $\{w_{0n}\} \subseteq C^2(J)$; $\{w_{1n}\} \subseteq C^1(J)$ and $\{f_n\} \subseteq W^{1,1}(\mathbb{R}_+, C(J))$ such that*

$$w_{0n}(x') = 0, \qquad x' = 0, \ell \tag{6.25}$$

$$w_{1n}(x') = 0, \qquad x' = 0, \ell \tag{6.26}$$

$$w''_{0n}(x') + f_n(0, x') = 0, \qquad x' = 0, \ell \tag{6.27}$$

$$\lim_{n\to\infty} \|w_{0n} - w_0\|_{C^1(J)} = 0 \tag{6.28}$$

$$\lim_{n\to\infty} \|w_{1n} - w_1\|_{C(J)} = 0 \tag{6.29}$$

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R}_+, C(J))} = 0. \tag{6.30}$$

Proof. Let $\{\tilde{w}_{1n}\} \subseteq C^2(J)$ be such that

$$\lim_{n\to\infty} \|\tilde{w}_{1n} - w_1\|_{C(J)} = 0.$$

Setting

$$w_{1n}(x) = \tilde{w}_{1n}(x) - \left(1 - \frac{x}{\ell}\right) \tilde{w}_{1n}(0) - \frac{x}{\ell} \tilde{w}_{1n}(\ell), \quad x \in J$$

we have $\{w_{1n}\} \subset C^2(J)$ and (6.26), (6.29) hold.

Similarly, we obtain $\{\alpha_n\} \subset C^2(J)$ such that

$$\begin{cases} \lim_{n\to\infty} \|\alpha_n - w_0\|_{C^1(J)} = 0 \\ \alpha_n(x') = 0, \quad n \in \mathbb{N}, \quad x' = 0, \ell. \end{cases}$$

Let us choose $\{f_n\} \subset W^{1,1}(\mathbb{R}_+; C(J))$ such that (6.30) holds and set for each $n \in \mathbb{N}$

$$a_n := -f_n(0,0) - \alpha''_n(0) \qquad b_n := -f_n(0,\ell) - \alpha''_n(\ell)$$

and

$$\beta_n(x) = \frac{1}{n}\int_0^x \operatorname{sen}(na_n y)dy, \quad \gamma_n(x) = \frac{1}{n}\int_0^{\ell-x} \operatorname{sen}(nb_n y)dy, \quad x \in J.$$

Choosing $\varphi \in C^2(J)$ such that $\varphi(x) = 1$ for $x \in [0, \frac{\ell}{3}]$ and $\varphi(x) = 0$ for $x \in [\frac{2}{3}\ell, \ell]$ and setting

$$\delta_n(x) := \varphi(x)\beta_n(x) + (1-\varphi(x))\gamma_n(x), \quad x \in J$$

we have

$$\delta_n(x') = 0, \quad x' = 0, \ell; \quad \delta_n''(0) = a_n, \quad \delta_n''(\ell) = b_n$$

$$\lim_{n\to\infty} \|\delta_n\|_{C^1(J)} = 0.$$

From this it follows that $w_{0n}(x) := \alpha_n(x) + \delta_n(x)$, $x \in J$ satisfy (6.25), (6.27) and (6.28).

Theorem 6.6. *Let $a \in BV(\mathbb{R}_+) \cap \mathcal{L}^1(\mathbb{R}_+)$ and*

$$w_0 \in C^1(J) \quad \text{with} \quad w_0(x') = 0, \quad x' = 0, \ell \tag{6.31}$$

$$w_1 \in C(J) \quad \text{with} \quad w_1(x') = 0, \quad x' = 0, \ell \tag{6.32}$$

$$f \in L^1(\mathbb{R}_+, C(J)). \tag{6.33}$$

There exists a unique F-solution w of problem (6.1) *and for each $T > 0$ we have*

$$\|w\|_{C^1(Q_T)} \le \hat{\rho}(T)(\|w_0\|_{C^1(J)} + \|w_1\|_{C(J)} + \|f\|_{L^1(0,T;C(J))}), \tag{6.34}$$

where $\hat{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and increasing function, independent of w_0, w_1 and f.

Proof. To prove the uniqueness, let w be an F-solution of (6.1) corresponding to $w_0 = w_1 = f = 0$ and let $\{w_n\}\{f_n\}\{w_{0n}\}$ and $\{w_{1n}\}$ satisfy the properties (6.20)-(6.24). Setting

$$\begin{aligned} u_n(t,x) &= \tfrac{1}{2}(D_t w_n(t,x) + D_x w_n(t,x)), \\ v_n(t,x) &= \tfrac{1}{2}(D_t w_n(t,x) - D_x w_n(t,x)), \quad (t,x) \in Q \end{aligned}$$

we know from Theorem 6.1 that $\{u_n, v_n\}$ is a solution of problem (6.3) with $f = f_n$, $w_0 = w_{0n}$ and $w_1 = w_{1n}$. Hence

$$U_n(t) := \begin{pmatrix} u_n(t,\cdot) \\ v_n(t,\cdot) \end{pmatrix}, \quad t \ge 0$$

is a strict solution of

$$U_n'(t) = AU_n(t) + \int_0^t a(t-s)A_1U_n(s)ds + F_n(t), \ \ t \geq 0, \ \ U_n(0) = U_{0n}, \quad (6.35)$$

where

$$F_n(t) = \frac{1}{2}\begin{pmatrix} f_n(t,\cdot) \\ f_n(t,\cdot) \end{pmatrix} \quad \text{and} \quad U_{0n} = \frac{1}{2}\begin{pmatrix} w_{n1} + w_{n0}' \\ w_{n1} - w_{n0}' \end{pmatrix}.$$

Setting

$$U(t) = \frac{1}{2}\begin{pmatrix} w_t(t,\cdot) + w_x(t,\cdot) \\ w_t(t,\cdot) - w_x(t,\cdot) \end{pmatrix}, \quad t \geq 0$$

we deduce from (6.21)-(6.24) that for each $T > 0$

$$\lim_{n\to\infty} \|U_n(t) - U(t)\|_X = 0 \quad \text{uniformly for} \quad t \in [0,T]$$

$$\lim_{n\to\infty} \|F_n\|_{L^1(\mathbb{R}_+,C(J))} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|U_{0n}\|_X = 0.$$

We conclude that U is an F-solution of the abstract problem

$$U'(t) = AU(t) + \int_0^t a(t-s)A_1U(s)ds, \quad t \geq 0, \quad U(0) = 0.$$

From the uniqueness proved in Theorem 5.4 we deduce that $U = 0$ hence $w(t,x)$ is constant for $(t,x) \in Q$; but from (6.20) and (6.24) we get $w(0,0) = \lim_{n\to\infty} w_n(0,0) = 0$ therefore $w \equiv 0$ and the uniqueness is proved.

To prove the existence of an F-solution of (6.1) let w_0, w_1 and f satisfy (6.31)-(6.33) and choose $\{w_{0n}\}, \{w_{1n}\}$ and $\{f_n\}$ given by Lemma 6.5. Setting for $t \geq 0$

$$F_n(t) = \frac{1}{2}\begin{pmatrix} f_n(t,\cdot) \\ f_n(t,\cdot) \end{pmatrix}, \quad F(t) = \frac{1}{2}\begin{pmatrix} f(t,\cdot) \\ f(t,\cdot) \end{pmatrix}$$

and

$$U_{0n} = \frac{1}{2}\begin{pmatrix} w_{1n} + w_{0n}' \\ w_{1n} - w_{0n}' \end{pmatrix}, \quad U_0 = \frac{1}{2}\begin{pmatrix} w_1 + w_0' \\ w_1 - w_0' \end{pmatrix}$$

we see that

$$U_{0n} \in D(A), \quad F_n \in W^{1,1}(\mathbb{R}_+, X), \quad AU_{0n} + F_n(0) \in \overline{D(A)}$$

hence (from Theorem 4.4) there exists a strict solution U_n of

$$U_n'(t) = AU_n(t) + \int_0^t a(t-s)A_1U_n(s)ds + F_n(t), \ \ t \geq 0; \ \ U_n(0) = U_{0n} \quad (6.36)$$

such that for each $T > 0$ (see Remark 4.7)

$$\|U_n(t)\|_X \leq \hat{\rho}(T)(\|U_{0n}\|_X + \|F_n\|_{L^1(0,T;X)}), \quad t \in [0,T]. \quad (6.37)$$

From (6.28)-(6.30) we have

$$\lim_{n\to\infty} \|U_{0n} - U_0\|_X = 0, \quad \lim_{n\to\infty} \|F_n - F\|_{L^1(\mathbb{R}_+,X)} = 0$$

hence we deduce from (6.37) the existence of $U \in C(\mathbb{R}_+, X)$ such that for each $T > 0$

$$\lim_{n\to\infty} \|U_n(t) - U(t)\|_X = 0, \quad \text{uniformly for} \quad t \in [0,T]$$

and

$$\|U(t)\|_X \le \hat{\rho}(T)(\|U_0\|_X + \|F\|_{L^1(0,T;X)}), \quad t \in [0,T]. \tag{6.38}$$

Set

$$U(t) =: \begin{pmatrix} u(t,\cdot) \\ v(t,\cdot) \end{pmatrix}, \quad t \ge 0$$

and

$$w(t,x) = \int_0^x [u(t,y) - v(t,y)]dy, \quad (t,x) \in Q.$$

As U_n is a strict solution of (6.36), setting

$$U(t) =: \begin{pmatrix} u_n(t,\cdot) \\ v_n(t,\cdot) \end{pmatrix}, \quad t \ge 0$$

and

$$w_n(t,x) = \int_0^x [u_n(t,y) - v_n(t,y)]dy, \quad (t,x) \in Q$$

we deduce from Theorem 6.1 that w_n is a strict solution of (6.20) and also that

$$D_x w_n(t,x) = u_n(t,x) - v_n(t,x), \quad D_t w_n(t,x) = u_n(t,x) + v_n(t,x).$$

Hence $w_n \in C^2(Q)$ and for each $T > 0$

$$\lim_{n\to\infty} \|w_n - w\|_{C^1(Q_T)} = 0.$$

In conclusion w is an F-solution of problem (6.1). To prove estimate (6.34) let us observe that (6.37) implies that for each $T > 0$

$$\|w_n\|_{C^1(Q_T)} \le \hat{\rho}(T)(\|w_{0n}\|_{C^1(J)} + \|w_{1n}\|_{C(J)} + \|f_n\|_{L^1(0,T;C(J)})$$

and so for $n \to \infty$ we get (6.34). □

7. APPENDIX

For the reader's convenience we prove in this section a useful result which was used in the previous sections.

Theorem 7.1. *Let X be a Banach space, $K \in BV(c,d)$ and $f \in C(a,b;X)$ with*

$$a' := b + c < b' := a + d.$$

Setting

$$(\hat{S}f)(t) = \int_a^b \hat{K}(t-s) f(s) ds, \quad t \in [a', b'] \tag{7.1}$$

we have $\hat{S}f \in W^{1,\infty}(a', b'; X)$ and

$$\|(\hat{S}f)'\|_{L^\infty(a',b';X)} \le \mathrm{var}\hat{K} \cdot \|f\|_{C(a,b;X)}. \tag{7.2}$$

Proof. Let $c_i \in X$ $(i = 1, \dots n)$, $a = s_1 < s_2 < \dots < s_{n-1} = b$ and X_i be the characteristic function of $[s_i, s_{i+1}]$ $(i = 1, \dots n)$. Setting $\varphi(s) = \sum_{i=1}^n c_i X_i(s)$, $s \in [a,b]$ we have for $t \in [a', b']$

$$(\hat{S}\varphi)(t) = \sum_{i=1}^n c_i \int_{s_i}^{s_{i+1}} \hat{K}(t-s) ds = \sum_{i=1}^n c_i \int_{t-s_{i+1}}^{t-s_i} \hat{K}(s) ds$$

hence

$$\|(\hat{S}\varphi)(t)\| \le \|\varphi\|_{L^1(a,b;X)} \|\hat{K}\|_{L^\infty(c,d)}. \tag{7.3}$$

Moreover, $(\hat{S}\varphi) \in W^{1,\infty}(a', b'; X)$ and for $t \in (a', b')$ almost everywhere

$$(\hat{S}\varphi)'(t) = \sum_{i=1}^n c_i [\hat{K}(t-s_i) - \hat{K}(t-s_{i+1})]$$

from which

$$\|(\hat{S}\varphi)'(t)\| \le \|\varphi\|_{L^\infty(a,b;X)} \cdot \mathrm{var}\hat{K}, \quad t \in (a', b'). \tag{7.4}$$

Let $\{\varphi_n\}$ be a sequence of step functions from $[a,b]$ to X such that

$$\lim_{n\to\infty} \|\varphi_n - f\|_{L^\infty(a,b;X)} = 0 \tag{7.5}$$

hence

$$\lim_{n\to\infty} \|\hat{S}\varphi_n - \hat{S}f\|_{L^\infty(a',b';X)} = 0.$$

From (7.4) and (7.5) we get the existence in $L^\infty(a', b'; X)$ of $\lim_{n\to\infty} (\hat{S}\varphi_n)'$: hence $\hat{S}f \in W^{1,\infty}(a', b'; X)$ and

$$\lim_{n\to\infty} \|(\hat{S}\varphi_n)' - (\hat{S}f)'\|_{L^\infty(a',b';X)}.$$

Setting $\varphi = \varphi_n$ in (7.4) and letting $n \to \infty$ we deduce (7.2). □

Corollary 7.2. *Let X be a Banach space, $K \in BV(0,T)$ and $f \in C(0,T;X)$. Setting*

$$(Sf)(t) = \int_0^t K(t-s)f(s)ds, \quad t \in [0,T] \tag{7.6}$$

we have $Sf \in W^{1,\infty}(0,T;X)$ and

$$\|(Sf)'\|_{L^\infty(0,T;X)} \le (|K(0)| + \mathrm{var}K)\|f\|_{C(0,T;X)}. \tag{7.7}$$

Proof. Extending K to zero we obtain $\hat{K} \in BV(-T,T)$ and $\mathrm{var}\hat{K} = |K(0)| + \mathrm{var}K$; then we can apply Theorem 7.1 to get the conclusion. □

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