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# POINTWISE ASYMPTOTIC BEHAVIOR OF PERTURBED VISCOUS SHOCK PROFILES

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**Abstract.** We consider the asymptotic behavior of perturbations of Lax and overcompressive-type viscous shock profiles arising in systems of regularized conservation laws with strictly parabolic viscosity, and also in systems of conservation laws with partially parabolic regularizations such as arise in the case of the compressible Navier–Stokes equations and in the equations of magnetohydrodynamics. Under the necessary conditions of spectral and hyperbolic stability, together with transversality of the connecting profile, we establish detailed pointwise estimates on perturbations from a sum of the viscous shock profile under consideration and a family of diffusion waves which propagate perturbation signals along outgoing characteristics. Our approach combines the recent $L^p$-space analysis of Raoofi [33] with a straightforward bootstrapping argument that relies on a refined description of nonlinear signal interactions, which we develop through convolution estimates involving Green's functions for the linear evolutionary PDE that arises upon linearization of the regularized conservation law about the distinguished profile. Our estimates are similar to, though slightly weaker than, those developed by Liu in his landmark result on the case of weak Lax-type profiles arising in the case of identity viscosity [21].

## 1. Introduction

Consider a "viscous shock profile", or traveling-wave solution

$$u(x,t) = \bar{u}(x-st), \qquad \lim_{z\to\pm\infty} \bar{u}(z) = u_\pm, \tag{1.1}$$

of a system of conservation laws

$$u_t + F(u)_x = (B(u)u_x)_x, \quad x \in \mathbb{R}; \;\; u,\, F \in \mathbb{R}^n;\, B \in \mathbb{R}^{n\times n}, \tag{1.2}$$

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for which we take

$$u = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{1.3}$$

$u^I \in \mathbb{R}^{n-r}$, $u^{II} \in \mathbb{R}^r$, and

$$Re\sigma b_2 \geq \theta, \tag{1.4}$$

with $\theta > 0$ and where $\sigma$ denotes spectrum.

Equations of the form (1.2) arise in a wide variety of applications, including such well-known examples as the compressible Navier–Stokes equations, the equations of magnetohydrodynamics (see [40]), and the equations of viscoelastic materials. The long-time behavior of solutions of (1.2) is often characterized by a family of viscous shock profiles (1.1), which serves as a regularization for the shock pattern solution of the associated hyperbolic problem

$$u_t + F(u)_x = 0. \tag{1.5}$$

In this context, we expect that a viscous profile will persist in the pattern only if it is individually stable to small fluctuations in initial conditions, and consequently the stability of viscous shock profiles has long been a subject of considerable interest and research effort (see [42] and the references therein).

An important advance in the study of such waves and their stability was the identification in [42] of a spectral criterion based on the *Evans function* (see particularly [1, 6, 9, 14, 18, 29, 36, 39, 42]). Briefly, the Evans function, typically denoted $D(\lambda)$, serves as a characteristic function for the linear operator $L$ that arises upon linearization of (1.2) about the stationary profile $\bar{u}(x)$. More precisely, away from essential spectrum, zeros of the Evans function correspond in location and multiplicity with eigenvalues of $L$ [1, 9, 42]. It was shown in [42] and [29], respectively for the strictly parabolic and real viscosity cases, that $L^1 \cap L^p \to L^p$ linearized orbital stability of the profile, $p > 1$, is equivalent to the Evans function condition,

($\mathcal{D}$) There exist precisely $\ell$ zeroes of $D(\cdot)$ in the nonstable half-plane $\Re\lambda \geq 0$, necessarily at the origin $\lambda = 0$.

Here, $\ell$ is the dimension of the manifold connecting $u_-$ and $u_+$.

Stability criterion ($\mathcal{D}$) has been shown to hold in all cases for small amplitude Lax shocks arising in strictly parabolic systems [8, 10, 13, 16, 17, 31, 32], as well as for large-amplitude shocks in such cases as Lax-type waves arising in isentropic Navier–Stokes equations for the gamma-law gas as $\gamma \to 1$ [31], and undercompressive shocks arising in Slemrod's model for van der Waal gas dynamics [40] (see [34] for Slemrod's model). More generally, condition ($\mathcal{D}$) can be verified by numerical calculation [2, 3, 5, 4]. In the case of strictly parabolic systems such *spectral stability*, along with standard technical hypotheses on $F$ and $B$ (see $(\mathcal{H}0)$–$(\mathcal{H}4)$, in Remark 1.7 below), has been shown sufficient for establishing nonlinear stability of Lax, under-, over-, and mixed under–overcompressive shock profiles [12], while for mixed hyperbolic–parabolic regularization, condition $(\mathcal{D})$ (along with (H0)–(H3) and (A1)–(A3) below) has been shown sufficient for establishing nonlinear stability for Lax and overcompressive shock profiles [33, 30, 40]. Except in the case of [33], these nonlinear analyses are carried out through consideration of the perturbation

$$v(x,t) = u(x,t) - \bar{u}^{\delta(t)}(x),$$

where $\delta(t)$ is introduced as a local phase chosen in such a way that $u$ is compared with the nearby manifold of connections between shock endstates $u_\pm$ rather than with the particular connection $\bar{u}$ (in the Lax case, $\delta(t)$ is simply a shift in position of the wave, and this choice insures that the shapes of $u$ and $\bar{u}$ are compared, rather than their positions). In this way, orbital stability is considered. In this context, and under the assumption of stability criterion $(\mathcal{D})$, it can be shown that for initial perturbations

$$|v(x,0)| \le E_0(1+|x|)^{-3/2},$$

for some $E_0$ sufficiently small, there holds

$$\|v(x,t)\|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \tag{1.6}$$

from which we observe asymptotic decay in time for all $p > 1$.

A natural refinement of this type of analysis regards the consideration of *diffusion waves*, which are defined as exact solutions to a family of convecting Burgers' equations (convection along outgoing characteristics of the underlying hyperbolic problem), and which carry precisely the $L^1$ mass in (1.6) that does not decay asymptotically in time [20, 21]. Considering, then, the perturbation

$$v(x,t) = u(x,t) - \bar{u}^{\delta(t)}(x) - \varphi(x,t),$$

where $\varphi(t,x)$ represents the sum of diffusion waves, we have

$$\int_{-\infty}^{+\infty} \varphi(x,0)dx = \int_{-\infty}^{+\infty} (u(x,0) - \bar{u}^{\delta(t)}(x))dx,$$

and consequently,

$$\int_{-\infty}^{+\infty} v(x,0)dx = 0.$$

In this way, we reduce to the case of zero-mass initial data, for which perturbations decay equally from above and below the profile, and the asymptotic rate of decay is doubled (at least at the linear level, and for an initial perturbation with sufficiently fast spatial decay). Working in this context, Raoofi has shown that stability criterion $(\mathcal{D})$, along with (H0)–(H3) and (A1)–(A3), are sufficient for establishing that for initial perturbations

$$v(x,t) = u(x,t) - \bar{u}^{\delta_*}(x) - \varphi(x,0) - \frac{\partial \bar{u}^{\delta}}{\partial \delta}\delta(t),$$

where $\delta_*$ is the asymptotic shape/location of the shock, and with $|u(\cdot,0) - \bar{u}|$ sufficiently small, there holds [33]

$$\|v(\cdot,t)\|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}.$$

The goal of the current analysis is both to refine the analysis of Raoofi to the case of pointwise (rather than $L^p$) estimates, and to introduce a simplified bootstrapping argument through which estimates on the perturbation $v(t,x)$ emerge in straightforward fashion. Our estimates are similar to those of Liu, developed for Lax-type profiles in the case of identity viscosity and under the assumption of weak shock strength. Our analysis has no such limitations, though we note that the form of our viscosity and the shock strength under consideration are encoded in our spectral assumptions.

Throughout the analysis, we will work in a coordinate system moving along with the shock, so that, without loss of generality, we consider a standing profile $\bar{u}(x)$, which satisfies the traveling-wave ordinary differential equation (ODE)

$$B(\bar{u})\bar{u}' = F(\bar{u}) - F(u_-). \tag{1.7}$$

Considering the block structure of $B$, this can be written as:

$$F^I(u^I, u^{II}) \equiv F^I(u_-^I, u_-^{II}) \tag{1.8}$$

and

$$b_1(u^I)' + b_2(u^{II})' = F^{II}(u^I, u^{II}) - F^{II}(u_-^I, u_-^{II}). \tag{1.9}$$

We are interested in the asymptotic and pointwise behavior of $\tilde{u}$, a solution of (1.2) and a perturbation of $\bar{u}$.

We assume that, by some invertible change of coordinates $u \to w(u)$, possibly but not necessarily connected with a global convex entropy, followed if necessary by multiplication on the left by a nonsingular matrix function $S(w)$, equations (1.2) may be written in the *quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde{A}^0 w_t + \tilde{A} w_x = (\tilde{B} w_x)_x + G, \quad w = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix}, \tag{1.10}$$

$w^I \in \mathbb{R}^{n-r}$, $w^{II} \in \mathbb{R}^r$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, where, defining $w_\pm := w(u_\pm)$, we have:

(A1) $\tilde{A}(w_\pm)$, $\tilde{A}_{11}$, $\tilde{A}^0$ are symmetric, $\tilde{A}^0 > 0$.

(A2) Dissipativity: no eigenvector of $dF(u_\pm)$ lies in the kernel of $B(u_\pm)$. ((A2) is equivalent to the assertion that no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(w_\pm)$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(w_\pm)$.)

(A3) $\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}$, $\tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$, with $Re\tilde{b}(w) \geq \theta$ for some $\theta > 0$, for all $W$, and $\tilde{g}(w_x, w_x) = \mathbf{O}(|w_x|^2)$.

Here, the coefficients of (1.10) may be expressed in terms of the original equation (1.2), the coordinate change $u \to w(u)$, and the approximate symmetrizer $S(w)$, as

$$\begin{aligned} \tilde{A}^0 &:= S(w)(\partial u/\partial w), \quad \tilde{A} := S(w)dF(u(w))(\partial u/\partial w), \\ \tilde{B} &:= S(w)B(u(w))(\partial u/\partial w), \quad G = -(dSw_x)B(u(w))(\partial u/\partial w)w_x. \end{aligned} \tag{1.11}$$

Along with the above structural assumptions, we make the technical hypotheses:

(H0) $F$, $B$, $w$, $S \in C^8$.

(H1) The eigenvalues of $\tilde{A}_* := \tilde{A}_{11}(\tilde{A}^0_{11})^{-1}$ are (i) distinct from 0; (ii) of common sign; and (iii) of constant multiplicity with respect to $u$.

(H2) The eigenvalues of $dF(u_\pm)$ are real, distinct, and nonzero.

(H3) Nearby $\bar{u}$, the set of all solutions of (1.1)–(1.2) connecting the same values $u_\pm$ forms a smooth manifold $\{\bar{u}^\delta\}$, $\delta \in \mathcal{U} \subset \mathbb{R}^\ell$, $\bar{u}^0 = \bar{u}$.

We note that structural assumptions (A1)–(A3) and technical hypotheses (H0)–(H2) are broad enough to include such cases as the compressible Navier–Stokes equations, the equations of magnetohydrodymics, and Slemrod's model for van der Waal gas dynamics [40]. Moreover, the existence of waves $\bar{u}$ satisfying (H3) has been established in each of these cases.

**Definition 1.1.** *An ideal shock*

$$u(x,t) = \begin{cases} u_- & x < st, \\ u_+ & x > st, \end{cases} \tag{1.12}$$

*is classified as undercompressive, Lax, or overcompressive type according as* $i - n$ *is less than, equal to, or greater than* 1*, where* $i$*, denoting the sum of the dimensions* $i_-$ *and* $i_+$ *of the center–unstable subspace of* $df(u_-)$ *and the center–stable subspace of* $df(u_+)$*, represents the total number of characteristics incoming to the shock.*

*A viscous profile* (1.1) *is classified as pure undercompressive type if the associated ideal shock is undercompressive and* $\ell = 1$*, pure Lax type if the corresponding ideal shock is Lax type and* $\ell = i - n$*, and pure overcompressive* *type if the corresponding ideal shock is overcompressive and $\ell = i - n$, $\ell$ as in (H3). Otherwise it is classified as mixed under–overcompressive type; see* [22, 42].

Pure Lax type profiles are the most common type, and the only type arising in standard gas dynamics, while pure over- and undercompressive type profiles arise in magnetohydrodynamics (MHD) and phase-transitional models.

Under assumptions (A0)–(A3) and (H0)–(H3), or their analogs in the real viscosity case, condition $(\mathcal{D})$ is equivalent to (i) *strong spectral stability*, $\sigma(L) \subset \{\Re\lambda \le 0\} \cup \{0\}$, (ii) *hyperbolic stability* of the associated ideal shock, and (iii) *transversality* of $\bar{u}$ as a solution of the connection problem in the associated traveling-wave ODE, where hyperbolic stability is defined for Lax and undercompressive shocks by the Lopatinski condition of [23, 24, 25, 7] and for overcompressive shocks by an analogous long-wave stability condition (see $(\mathcal{D}$ii) below); see [42, 29, 43, 38, 39, 40] for further explanation. From now on, we assume $(\mathcal{D})$ to hold along with (A0)–(A3) and (H0)–(H4). We also assume that the the shock is a pure Lax or overcompressive one.

Setting $A_\pm := df(u_\pm)$, $\Gamma_\pm := d^2 f(u_\pm)$, and $B_\pm := B(u_\pm)$, denote by

$$a_1^- < a_2^- < \cdots < a_n^- \quad \text{and } a_1^+ < a_2^+ < \cdots < a_n^+ \tag{1.13}$$

the eigenvalues of $A_-$ and $A_+$, and $l_j^\pm$, $r_j^\pm$ left and right eigenvectors associated with each $a_j^\pm$, normalized so that $(l_j^T r_k)_\pm = \delta_k^j$, where $\delta_k^j$ is the Kronecker delta function, returning 1 for $j = k$ and 0 for $j \neq k$. Under this notation, hyperbolic stability of $\bar{u}$, a Lax or overcompressive shock profile, is the condition:

$(\mathcal{D}$ii) The set $\{r^\pm; a^\pm \gtrless 0\} \cup \{\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_i} dx; i = 1, \cdots, \ell\}$ forms a basis for $\mathbb{R}^n$, with $\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_i} dx$ computed at $\delta = 0$.

As said before, $(\mathcal{D}$ii) is satisfied whenever $(\mathcal{D})$ holds.

Define scalar diffusion coefficients

$$\beta_j^\pm := (l_j^T B r_j)_\pm \tag{1.14}$$

and scalar coupling coefficients

$$\gamma_j^\pm := (l_j^T \Gamma(r_j, r_j))_\pm. \tag{1.15}$$

Following [19, 20, 21], define for a given mass $m_j^-$ the scalar diffusion waves $\varphi_j^-(x, t; m_j^-)$ as (self-similar) solutions of the Burgers equations

$$\varphi_{j,t}^- + a_j^- \varphi_{j,x}^- - \beta_j^- \varphi_{j,xx}^- = -\gamma_j^- ((\varphi_j^-)^2)_x \tag{1.16}$$

with point-source initial data

$$\varphi_j^-(x,-1) = m_j^- \delta_0(x), \tag{1.17}$$

and similarly for $\varphi_j^+(x,t;m_j^+)$. Given a collection of masses $m_j^\pm$ prescribed on outgoing characteristic modes $a_j^- < 0$ and $a_j^+ > 0$, define

$$\varphi(x,t) = \sum_{a_j^-<0} \varphi_j^-(x,t;m_j^-) r_j^- + \sum_{a_j^+>0} \varphi_j^+(x,t;m_j^+) r_j^+. \tag{1.18}$$

Also define

$$\begin{aligned} \psi_1(x,t) := \chi(x,t) \sum_{a_j^-<0} (1+t)^{-1/2}(1+|x-a_j^- t| + t^{\frac{1}{3}})^{-3/4} \\ + \chi(x,t) \sum_{a_j^+>0} (1+t)^{-1/2}(1+|x-a_j^+ t| + t^{\frac{1}{3}})^{-3/4}, \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} \bar\psi_1(x,t) := \chi(x,t) \sum_{a_j^-<0} (1+t)^{-1/2}(1+|x-a_j^- t|)^{-3/4} \\ + \chi(x,t) \sum_{a_j^+>0} (1+t)^{-1/2}(1+|x-a_j^+ t|)^{-3/4}, \end{aligned} \tag{1.20}$$

where $\chi(x,t) = 1$ for $x \in [a_1^- t, a_n^+ t]$ and zero otherwise. Also,

$$\begin{aligned} \psi_2(x,t) := \sum_{a_j^-<0} (1+|x-a_j^- t| + t^{1/2})^{-3/2} \\ + \sum_{a_j^+>0} (1+|x-a_j^+ t| + t^{1/2})^{-3/2}, \end{aligned} \tag{1.21}$$

and

$$\alpha(x,t) := \chi(x,t)(1+t)^{-3/4}(1+|x|)^{-1/2}. \tag{1.22}$$

Theorem 1.2 and Corollaries 1.4 and 1.5 are the main results of this paper.

**Theorem 1.2.** *Assume* (A1)–(A3), (H0)–(H3) *and* ($\mathcal{D}$) *hold, and $\bar u$ is a pure Lax or overcompressive shock profile. Assume also that $\tilde u$ solves (1.2) with initial data $\tilde u_0$ and that, for initial perturbation $u_0 := \tilde u_0 - \bar u$, we have $|u_0|_{L^1\cap H^4} \le E_0$, $|u_0(x)| \le E_0(1+|x|)^{-\frac{3}{2}}$, and $|\partial_x u_0(x)| \le E_0(1+|x|)^{-\frac{1}{2}}$, for*

*$E_0$ sufficiently small. Then there is an $\ell$-array function $\delta(t)$, and a small constant $\delta_*$, such that if $v := \tilde{u} - \bar{u}^{\delta_*} - \varphi - \frac{\partial \bar{u}^\delta}{\partial \delta}\delta$, then*

$$|v(x,t)| \le CE_0(\psi_1 + \psi_2 + \alpha), \tag{1.23}$$

*and*

$$|v_x(x,t)| \le CE_0 t^{-\frac{1}{2}}(1+t)^{\frac{1}{2}}(\bar{\psi}_1 + \psi_2 + \alpha); \tag{1.24}$$

*furthermore,*

$$|\delta(t)| \le CE_0(1+t)^{-\frac{1}{2}}, \tag{1.25}$$

*and*

$$|\dot{\delta}(t)| \le CE_0(1+t)^{-1}, \tag{1.26}$$

*for some constant $C$ (independent of $x, t$ and $E_0$).*

The proof of Theorem 1.2 uses a straightforward bootstrapping argument, combined with $L^p$ estimates proved in [33] and restated here in the following proposition.

**Proposition 1.3.** *Under the conditions of Theorem 1.2,*

$$|v(\cdot,t)|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}, \tag{1.27}$$

*for any $p$, $1 \le p \le \infty$; also*

$$|v(\cdot,t)|_{H^s} \le CE_0(1+t)^{-\frac{1}{2}}, \tag{1.28}$$

*for any $s \le 4$, and for some constant $C$.*

**Proof of Proposition 1.3.** See [33]. In [33] the bound (1.28) was established for $s \le 3$; basically, to prove (1.28), it was shown, using energy estimates, that the $H^3$-norm of $u$ is controlled by the $H^3$-norm of the initial data and the $L^2$-norm of $u$. However, the same proof (with a slight modification) can be used for $s = 4$; we need only to assume that the initial data is small in $H^4$ and the coefficients are in $C^8$ (rather than $C^6$ as in [33]). The reason we need two times differentiability of the coefficients is in the fact that we need the exponential decay of $\bar{u}$ and its derivatives to their endstates up to $2s$ derivatives; see (5.56) in [33] and Lemma 2.1 in the present work. □

A Taylor expansion gives us

$$\bar{u}^{\delta_* + \delta(t)} - \bar{u}^{\delta_*} = \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t) + \mathbf{O}(|\delta(t)|^2 e^{-k|x|}), \tag{1.29}$$

since $\bar{u}$ approaches its endstates at an exponential rate. (see Lemma 2.1). But $|\delta(t)|^2 e^{-k|x|}$ is then smaller than the right-hand side of (1.23). Hence we have the following.

**Corollary 1.4.** *Under the assumption of Theorem* 1.2,

$$\tilde{u}(x,t) - \bar{u}^{\delta_* + \delta(t)}(x) = \varphi(x,t) + \mathbf{O}(\psi_1 + \psi_2 + \alpha). \tag{1.30}$$

*Also,*

$$|\tilde{u}(\cdot,t) - \bar{u}^{\delta_* + \delta(t)} - \varphi(\cdot,t)|_{L^p} = \mathbf{O}((1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}). \tag{1.31}$$

Without "instantaneous shock tracking" $\delta(t)$, however, we obtain the following, which Liu proved for the *artificial viscosity* case [21].

**Corollary 1.5.** *Under the assumption of Theorem* 1.2,

$$|\tilde{u}(\cdot,t) - \bar{u}^{\delta_*} - \varphi(\cdot,t)|_{L^p} = \begin{cases} \mathbf{O}((1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}) & \text{for } 1 \leq p \leq 2 \\ \mathbf{O}((1+t)^{-\frac{1}{2}}) & \text{for } 2 \leq p \leq \infty. \end{cases} \tag{1.32}$$

The picture of asymptotic behavior described in Theorem 1.2 and Corollary 1.4 was introduced on heuristic grounds by Liu [20] in the context of small-amplitude Lax-type shock waves and artificial (identity) viscosity $B = I$, and, along with the accompanying analysis of [19] described below, played an important role in the subsequent analysis by Szepessy and Xin in [35] establishing for the first time stability (with no rate) of small-amplitude Lax-type shock profiles with $B = I$. Bounds (1.27) validating this picture were established for large-amplitude Lax or overcompressive profiles and general viscosity by Raoofi [33]. See also earlier pointwise arguments sketched (but not completed) in [21, 42]. In the proof of Theorem 1.2 we will assume and use the bounds (1.27) and (1.28), already established in [33].

**Remark 1.6.** The estimates of Theorem 1.2 are similar to, though weaker than, those developed by Liu in the case of identity viscosity and weak shock strength [21]. In particular, Liu's estimates provide slightly sharper results in terms of the pointwise bounds, and also give more information along the characteristic modes. In order to provide results as sharp as those of Liu, we would have to treat the critical nonlinearity $\varphi v$ similarly to the way we treat the nonlinearity $\varphi^2$, which we analyze by a clever approach of Liu's in which he integrates the nonlinear interaction integral by parts in $t$, making use of the observation that $(\partial_t + a_j^- \partial_x)\varphi_j^-$ decays similarly to the $t$-derivative of a heat kernel [19]. (For the argument in our context, see (4.37) and the surrounding discussion, for which under a shift of coordinates, differentiation with respect to $\tau$ replaces the operator $(\partial_t + a_j^- \partial_x)$. The resulting estimate is stated in (3.20) of Lemma 3.2.) In order to apply this approach to the term $\varphi v$, we would additionally have to carry an estimate on $(\partial_t + a_j^- \partial_x)v$ through our argument. We leave the full details of this calculation to a separate paper

[11]. We remark also that the estimates of Liu are uniform in shock strength $\epsilon := |u_+ - u_-|$ as $\epsilon \to 0$. More specifically, $\epsilon$ appears explicitly in Liu's estimates, and the estimates remain largely unchanged as $\epsilon \to 0$, though one of the estimates (denoted $\chi_i$) increases slightly due to the loss of a term $\epsilon^{-1}\psi_i(x,t)$ as a possibility in an estimate that takes the minimum of three quantities (the other two quantities remain uniform in $\epsilon$) [21]. In our case, shock strength is assumed fixed, and our estimates are not uniform as $\epsilon \to 0$. In particular, coefficients in the Green's function estimates we use blow up as $\epsilon \to 0$, and we must counter this by reducing $E_0$, the small constant multiplying our initial perturbation. We regard the extension of our analysis to the uniform case in (small) shock strength as an important future project. One of the advantages of our approach, on the other hand, is the use of the instantaneous shock tracking $\delta(t)$, as well as using the previously established $L^p$ norms, which makes our proof both more straightforward and easier to generalize. We also remark that since we proceed from Green's function estimates obtained for possibly large-amplitude shock profiles (information about the amplitude is encoded in the spectrum of the linearized operator), our analysis applies to this case as well. Finally, the $L^p$ estimates of Corollary 1.5 recover the estimates of [33], with a slight improvement on the shift location estimates $|\delta(t)|$ and $\dot{\delta}(t)$. As mentioned in [33], these estimates, in the case $p > 2$, are a slight improvement of Liu's.

**Remark 1.7** (Remarks on the Strictly Parabolic case)**.** The case of the strictly parabolic systems can be treated in a very similar way, the proof being almost identical. However, we need fewer assumptions for the equations, or for the initial perturbation, in this case. Basically, instead of (A1)–(A3) and (H0)–(H3), we assume the following assumptions in the strictly parabolic case.

*($\mathcal{H}0$)* $f, B \in C^3$.

*($\mathcal{H}1$)* $Re\,\sigma(B) > 0$.

*($\mathcal{H}2$)* The eigenvalues of $df(u_\pm)$ are real, distinct, and nonzero.

*($\mathcal{H}3$)* For some $\theta > 0$, and all real $k$, we have

$$Re\,\sigma(-ikdf(u_\pm) - k^2B(u_\pm)) < -\theta k^2.$$

*($\mathcal{H}4$)* The set of all stationary solutions of (1.1)-(1.2) near $\bar{u}$, connecting the same values $u_\pm$, forms a smooth manifold $\{\bar{u}^\delta\}, \delta \in \mathbb{R}^\ell, \bar{u}^0 = \bar{u}$.

We also assume that ($\mathcal{D}$) holds. For the initial perturbation we only need $u_0 \in C^{1+\alpha}$ for some $0 < \alpha < 1$. In Theorem 1.2 concerning the real viscosity (partially parabolic) case, we require the pointwise bound on the derivative of the initial perturbation—i.e. the bound $|\partial_x u_0(x)| \leq E_0(1+|x|)^{-\frac{1}{2}}$—only in order to control the derivative in the hyperbolic modes, which are absent in the strictly parabolic case. The necessary bounds on the derivatives in the strictly parabolic case can be achieved through short time estimates similar to ones carried out in [33, 12, 42]. On the other hand, as we are assuming less on the initial data, (1.28) does not necessarily hold. See [33] for more details.

## 2. Linearized equations and Green's function bounds

Before stating the Green's function bounds for the linearized equation, we need to do some preparation. First we need the exponential decay of $\bar{u}$ to its endstates. The following lemma proved in [28] provides us with that.

**Lemma 2.1.** *Given (H1)–(H3), the endstates $u_\pm$ are hyperbolic rest points of the ODE determined by* (1.9) *on the $r$-dimensional manifold* (1.8); *i.e., the coefficients of the linearized equations about $u_\pm$, written in local coordinates, have no center subspace. In particular, under regularity (H0),*

$$D_x^j D_\delta^i(\bar{u}^\delta(x) - u_\pm) = \mathbf{O}(e^{-\alpha|x|}), \quad \alpha > 0,\ 0 \leq j \leq 8,\ i = 0, 1, \tag{2.1}$$

*as $x \to \pm\infty$.*

Instead of linearizing about $\bar{u}(\cdot)$, we linearize about $\bar{u}^{\delta_*}(\cdot)$, where $\delta_*$, determined *a priori* by the mass of the perturbation, would be the asymptotic location or shape of the shock. Linearizing around $\bar{u}^{\delta_*}(\cdot)$ gives us

$$v_t = Lv := -(Av)_x + (Bv_x)_x, \tag{2.2}$$

with

$$B(x) := B(\bar{u}^{\delta_*}(x)), \quad A(x)v := df(\bar{u}^{\delta_*}(x))v - dB(\bar{u}^{\delta_*}(x))v\bar{u}_x^{\delta_*}, \tag{2.3}$$

and let

$$G(x,t;y) := e^{Lt}\delta_y(x). \tag{2.4}$$

Denoting $A^\pm := A(\pm\infty)$, $B^\pm := B(\pm\infty)$, and considering Lemma 2.1, it follows that

$$|A(x) - A^-| = \mathbf{O}(e^{-\eta|x|}), \quad |B(x) - B^-| = \mathbf{O}(e^{-\eta|x|}) \tag{2.5}$$

as $x \to -\infty$, for some positive $\eta$, similarly for $A^+$ and $B^+$, as $x \to +\infty$. Also $|A(x) - A^\pm|$ and $|B(x) - B^\pm|$ are bounded for all $x$.

Define the *(scalar) characteristic speeds* $a_1^\pm < \cdots < a_n^\pm$ (as above) to be the eigenvalues of $A^\pm$, and the left and right *(scalar) characteristic modes* $l_j^\pm$, $r_j^\pm$ to be corresponding left and right eigenvectors, respectively (i.e., $A^\pm r_j^\pm = a_j^\pm r_j^\pm$, etc.), normalized so that $l_j^+ \cdot r_k^+ = \delta_k^j$ and $l_j^- \cdot r_k^- = \delta_k^j$.

Following Kawashima [15], define associated *effective scalar diffusion rates* $\beta_j^{\pm} : j = 1, \cdots, n$ by the relation

$$\begin{pmatrix} \beta_1^{\pm} & & 0 \\ & \vdots & \\ 0 & & \beta_n^{\pm} \end{pmatrix} = \text{diag } L^{\pm} B^{\pm} R^{\pm}, \tag{2.6}$$

where $L^{\pm} := (l_1^{\pm}, \ldots, l_n^{\pm})^t$, $R^{\pm} := (r_1^{\pm}, \ldots, r_n^{\pm})$ diagonalize $A^{\pm}$.

Assume for $A$ and $B$ the block structures:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Also, let $a_j^*(x)$, $j = 1, \ldots, (n-r)$ denote the eigenvalues of

$$A_* := A_{11} - A_{12} B_{22}^{-1} B_{21},$$

with $l_j^*(x)$, $r_j^*(x) \in \mathbb{R}^{n-r}$ associated left and right eigenvectors, normalized so that $l_j^{*t} r_j \equiv \delta_k^j$. More generally, for an $m_j^*$-fold eigenvalue, we choose $(n-r) \times m_j^*$ blocks $L_j^*$ and $R_j^*$ of eigenvectors satisfying the *dynamical normalization*

$$L_j^{*t} \partial_x R_j^* \equiv 0,$$

along with the usual static normalization $L_j^{*t} R_j \equiv \delta_k^j I_{m_j^*}$; as shown in Lemma 4.9, [27], this may always be achieved with bounded $L_j^*$, $R_j^*$. Associated with $L_j^*$, $R_j^*$, define extended $n \times m_j^*$ blocks

$$\mathcal{L}_j^* := \begin{pmatrix} L_j^* \\ 0 \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -B_{22}^{-1} B_{21} R_j^* \end{pmatrix}.$$

The eigenvalues $a_j^*$ and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of the degenerate system (1.2).

Define local, $m_j \times m_j$ *dissipation coefficients*

$$\eta_j^*(x) := -L_j^{*t} D_* R_j^*(x), \quad j = 1, \ldots, J \leq n-r,$$

where

$$D_*(x) := A_{12} B_{22}^{-1} \Big[ A_{21} - A_{22} B_{22}^{-1} B_{21} + A_* B_{22}^{-1} B_{21} + B_{22} \partial_x (B_{22}^{-1} B_{21}) \Big]$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

At $x = \pm\infty$, these reduce to the corresponding quantities identified by Zeng [37, 26] in her study by Fourier transform techniques of decay to *constant solutions* $\bar{U} \equiv u_\pm$ of hyperbolic–parabolic systems, i.e., of limiting equations

$$U_t = L_\pm U := -A_\pm U_x + B_\pm U_{xx}.$$

As a consequence of dissipativity, (A2), we obtain (see, e.g., [15, 26, 29])

$$\beta_j^\pm > 0, \quad Re\sigma(\eta_j^{*\pm}) > 0 \quad \text{for all } j. \tag{2.7}$$

However, note that the dynamical dissipation coefficient $D_*(x)$ *does not* agree with its static counterpart, possessing an additional term $B_{22}\partial_x(B_{22}^{-1}B_{21})$, and so we cannot conclude that (2.7) holds everywhere along the profile, but only at the endpoints. This is an important difference in the variable-coefficient case; see Remarks 1.11-1.12 of [29] for further discussion.

**Proposition 2.2.** [29] *Under assumptions (A1)–(A3), (H0)–(H3), and ($\mathcal{D}$), the Green's distribution $G(x,t;y)$ associated with the linearized evolution equations may be decomposed as*

$$G(x,t;y) = H + E + S + R, \tag{2.8}$$

*where, for $y \le 0$:*

$$\begin{aligned} H(x,t;y) &:= \sum_{j=1}^{J} a_j^{*-1}(x)a_j^*(y)\mathcal{R}_j^*(x)\zeta_j^*(y,t)\delta_{x-\bar{a}_j^*t}(-y)\mathcal{L}_j^{*t}(y) \\ &= \sum_{j=1}^{J} \mathcal{R}_j^*(x)\mathcal{O}(e^{-\eta_0 t})\delta_{x-\bar{a}_j^*t}(-y)\mathcal{L}_j^{*t}(y), \end{aligned} \tag{2.9}$$

*where the averaged convection rates $\bar{a}_j^* = \bar{a}_j^*(x,t)$ in (2.9) denote the time-averages over $[0,t]$ of $a_j^*(x)$ along backward characteristic paths $z_j^* = z_j^*(x,t)$ defined by*

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(t) = x,$$

*and the dissipation matrix $\zeta_j^* = \zeta_j^*(x,t) \in \mathbb{R}^{m_j^* \times m_j^*}$ is defined by the dissipative flow*

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*)\zeta_j^*, \quad \zeta_j^*(0) = I_{m_j};$$

*and $\delta_{x-\bar{a}_j^*t}$ denotes the Dirac distribution centered at $x - \bar{a}_j^*t$.*

$$E(x,t;y) := \sum_{j=1}^{\ell} \frac{\partial \bar{u}^\delta(x)}{\partial \delta_j}_{|\delta=\delta_*} e_j(y,t), \tag{2.10}$$

$$e_j(y,t) := \sum_{a_k^->0} \Big( errfn\,\Big(\frac{y+a_k^-t}{\sqrt{4\beta_k^-t}}\Big) - errfn\,\Big(\frac{y-a_k^-t}{\sqrt{4\beta_k^-t}}\Big)\Big) l_{jk}^- \tag{2.11}$$

$$S(x,t;y) := \chi_{\{t\geq 1\}} \sum_{a_k^-<0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^-t)^2/4\beta_k^-t} \tag{2.12}$$

$$+ \chi_{\{t\geq 1\}} \sum_{a_k^->0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^-t)^2/4\beta_k^-t} \Big(\frac{e^{-x}}{e^x+e^{-x}}\Big)$$

$$+ \chi_{\{t\geq 1\}} \sum_{a_k^->0,\, a_j^-<0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} (4\pi\bar\beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^-t} \Big(\frac{e^{-x}}{e^x+e^{-x}}\Big),$$

$$+ \chi_{\{t\geq 1\}} \sum_{a_k^->0,\, a_j^+>0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} (4\pi\bar\beta_{jk}^+ t)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar\beta_{jk}^+t} \Big(\frac{e^{x}}{e^x+e^{-x}}\Big),$$

*with*

$$z_{jk}^{\pm}(y,t) := a_j^{\pm}\Big(t - \frac{|y|}{|a_k^-|}\Big) \tag{2.13}$$

*and*

$$\bar\beta_{jk}^{\pm}(x,t;y) := \frac{x^{\pm}}{a_j^{\pm}t}\beta_j^{\pm} + \frac{|y|}{|a_k^-t|}\Big(\frac{a_j^{\pm}}{a_k^-}\Big)^2 \beta_k^-. \tag{2.14}$$

*The remainder $R$ and its derivatives have the following bounds.*

$$R(x,t;y) = \mathbf{O}(e^{-\eta(|x-y|+t)}) \tag{2.15}$$

$$+ \sum_{k=1}^{n} \mathbf{O}\Big((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\Big) t^{-1/2} e^{-(x-y-a_k^-t)^2/Mt}$$

$$+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^-t|\geq|y|\}} \mathbf{O}((t+1)^{-1/2}t^{-1/2}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}$$

$$+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^-t|\geq|y|\}} \mathbf{O}((t+1)^{-1/2}t^{-1/2}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-},$$

$$R_y(x,t;y) = \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t}) \delta_{x-\bar a_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \tag{2.16}$$

$$+ \sum_{k=1}^{n} \mathbf{O}\Big((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|} + e^{-\eta|y|}\Big) t^{-1} e^{-(x-y-a_k^-t)^2/Mt}$$

$$+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}$$

$$+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-},$$

$$R_x(x,t;y) = \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t}) \delta_{x-\bar a_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \tag{2.17}$$

$$+ \sum_{k=1}^{n} \mathbf{O}\left( (t+1)^{-1} e^{-\eta x^+} + e^{-\eta|x|} \right) t^{-1} (t+1)^{1/2} e^{-(x-y-a_k^- t)^2/Mt}$$

$$+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}$$

$$+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}.$$

*Moreover, for $|x-y|/t$ sufficiently large, $|G| \le Ce^{-\eta t} e^{-|x-y|^2/Mt)}$ as in the strictly parabolic case.*

Setting $\tilde G := S + R$, so that $G = H + E + \tilde G$, we have the following alternative bounds for $\tilde G$.

**Proposition 2.3** ([42], [29, 12]). *Under the assumptions of Proposition 2.2, $\tilde G$ has the following bounds.*

$$|\partial_{x,y}^{\alpha} \tilde G(x,t;y)| \le C(t^{-|\alpha|/2} + |\alpha_x| e^{-\eta|x|}) \Big( \sum_{k=1}^{n} t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} e^{-\eta x^+}$$

$$+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} t^{-1/2} e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}, \tag{2.18}$$

$$+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-} \Big),$$

*for $y \le 0$, and symmetrically for $y \ge 0$, for some $\eta$, $C$, $M > 0$, where $a_j^{\pm}$ are as in Proposition 2.2, $\beta_k^{\pm} > 0$, $x^{\pm}$ denotes the positive/negative part of $x$, and the indicator function $\chi_{\{|a_k^- t|\ge|y|\}}$ is $1$ for $|a_k^- t| \ge |y|$ and $0$ otherwise. Moreover, all estimates are uniform in the supressed parameter $\delta_*$.*

**Remark 2.4.** We will refer to the three differently scaled diffusion kernels in (2.18) respectively as the *convection* kernel, the *reflection* kernel, and the *transmission* kernel. We recall the notation

$$\text{errfn}(z) := \frac{1}{2\pi}\int_{-\infty}^{z} e^{-\xi^2} d\xi.$$

**Remark 2.5.** The Green's function bounds for the strictly parabolic case (see Remark 1.7) are very similar. The main difference is that, in the strictly parabolic case, the hyperbolic part $H$ is absent in the decomposition of $G$ as in (2.8).

**Remark 2.6.** From (2.11), we obtain by straightforward calculation (see [29]) the bounds

$$\begin{aligned}
|e_j(y,t)| &\le C \sum_{a_k^->0} \Big(\text{errfn}\Big(\frac{y+a_k^- t}{\sqrt{4\beta_k^- t}}\Big) - \text{errfn}\Big(\frac{y-a_k^- t}{\sqrt{4\beta_k^- t}}\Big)\Big),\\
|\partial_t e_j(y,t)| &\le C t^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt},\\
|\partial_y e_j(y,t)| &\le C t^{-1/2} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt}\\
|\partial_{yt} e_j(y,t)| &\le C t^{-1} \sum_{a_k^->0} e^{-|y+a_k^- t|^2/Mt}
\end{aligned} \tag{2.19}$$

for $y \le 0$, and symmetrically for $y \ge 0$.

## 3. Nonlinear analysis

Let $\tilde{u}$ solve (1.2), and, using ($\mathcal{D}$ii), assume that

$$\int_{-\infty}^{+\infty} \tilde{u}(x,0) - \bar{u}(x) = \sum_{a_j^-<0} m_j r_j^- + \sum_{a_j^+>0} m_j r_j^+ + \sum_{i=1}^{\ell} \int c_i \frac{\partial \bar{u}^\delta}{\partial \delta_i}_{|\delta=0}$$

with the $m_i$'s and $c_i$'s small enough. Using the implicit function theorem, we can find $\delta_*$ such that

$$\int_{-\infty}^{+\infty} \tilde{u}(x,0) - \bar{u}^{\delta_*}(x) = \sum_{a_j^-<0} m'_j r_j^- + \sum_{a_j^+>0} m'_j r_j^+$$

where each $m_i'$ is just "slightly" different from $m_i$. Notice that this way we have no "mass" in any $\int \frac{\partial \bar{u}^\delta}{\partial \delta_i}$ direction anymore. With a slight abuse of notation we drop the $'$ sign from each $m_i'$ and denote it simply by $m_i$.

**Remark 3.1.** In the case of Lax–type shock waves, $\bar{u}^\delta(x) = \bar{u}(x+\delta)$, hence $\frac{\partial \bar{u}^\delta}{\partial \delta} = u'(x)$, and $\delta_*$ can be explicitly computed: $\delta_* = c_1$.

Let $u(x,t) = \tilde{u}(x,t) - \bar{u}^{\delta_*}(x)$ and use a Taylor's expansion around $\bar{u}^{\delta_*}(x)$ to find

$$u_t + (A(x)u)_x - (B(x)u_x)_x = -(\Gamma(x)(u,u))_x + Q(u,u_x)_x, \tag{3.1}$$

where $\Gamma(x)(u,u) = d^2 f(\bar{u}^{\delta_*})(u,u) - d^2 B(\bar{u}^{\delta_*})(u,u)\bar{u}_x^{\delta_*}$ and

$$Q(u,u_x) = \mathbf{O}(|u||u_x| + |u|^3).$$

Denote $\Gamma^\pm = \Gamma(\pm\infty)$. Define constant coefficients $b_{ij}^\pm$ and $\Gamma_{ijk}^\pm$ to satisfy

$$\Gamma^\pm(r_j^\pm, r_k^\pm) = \sum_{i=1}^n \Gamma_{ijk}^\pm r_i^\pm, \quad B^\pm r_j^\pm = \sum_{i=1}^n b_{ij}^\pm r_i^\pm. \tag{3.2}$$

Then, of course, $\beta_i^\pm = b_{ii}^\pm$ and $\gamma_i^\pm := \Gamma_{iii}^\pm$.

Now define $\varphi_i^-$ by (1.16) and (1.17), and likewise for $\varphi_i^+$. Finally $\varphi$ is defined by (1.18). Then set $v := u - \varphi - \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$, with $\delta(t)$ to be defined later, and assuming $\delta(0) = 0$. Notice that, by our choice of $\delta_*$ and diffusion waves $\varphi_i^\pm$'s, we have zero initial mass of $v$, i.e.,

$$\int_{-\infty}^{+\infty} v(x,0)dx = 0. \tag{3.3}$$

Replacing $u$ with $v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$ in (3.1) ( $\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ computed at $\delta = \delta_*$), and using the fact that $\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ satisfies the linear time independent equation $Lv = 0$, we will have

$$v_t - Lv = \Phi(x,t) + \mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t))_x + \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta}(t), \tag{3.4}$$

where

$$\begin{aligned}\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) = \mathbf{O}(&|v|^2 + |\varphi||v| + |v||\frac{\partial \bar{u}^\delta}{\partial \delta}\delta| + |\varphi||\frac{\partial \bar{u}^\delta}{\partial \delta}\delta| + |\frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^2 \\ &+ |(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_x| + |\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^3).\end{aligned} \tag{3.5}$$

Furthermore

$$\mathcal{F}(v,\varphi,\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta(t))_x=\mathbf{O}\big(\mathcal{F}(v,\varphi,\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta) + |(v+\varphi+\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta)_x||(v^{II}+\varphi+\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta)_x| + |v+\varphi+\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta||(v^{II}+\varphi+\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta)_{xx}|\big), \tag{3.6}$$

and $\Phi(x,t):=-\varphi_t-(A(x)\varphi)_x+(B(x)\varphi_x)_x-(\Gamma(x)(\varphi,\varphi))_x$. For $\Phi$ we write

$$\begin{aligned}\Phi(x,t)=&-(\varphi_t+A\varphi_x-B\varphi_{xx}+\Gamma(\varphi,\varphi)_x)\\ =&-\sum_{a_i^-<0}\varphi_t^i r_i^- +(A(x)\varphi^i r_i^-)_x-(B(x)\varphi_x^i r_i^-)_x+(\Gamma(x)(\varphi^i r_i^-,\varphi^i r_i^-))_x\\ &-\sum_{a_i^+>0}\varphi_t^i r_i^+ +(A(x)\varphi^i r_i^+)_x-(B(x)\varphi_x^i r_i^+)_x+(\Gamma(x)(\varphi^i r_i^+,\varphi^i r_i^+))_x\\ &-\sum_{i\neq j}(\varphi_i\varphi_j\Gamma(x)(r_i^{\pm},r_j^{\pm}))_x.\end{aligned} \tag{3.7}$$

Let us write a typical term of the first summation ($a_i^-<0$) in the following form:

$$\begin{aligned}&\varphi_t^i r_i^- +(A(x)\varphi^i r_i^-)_x-(B(x)\varphi_x^i r_i^-)_x+(\Gamma(x)(\varphi^i r_i^-,\varphi^i r_i^-))_x\\ &\quad=\big[(A(x)-A^-)\varphi^i r_i^- -(B(x)-B^-)\varphi_x^i r_i^- +(\Gamma(x)-\Gamma^-)(\varphi^i r_i^-,\varphi^i r_i^-)\big]_x\\ &\quad+\varphi_t^i r_i^- +(\varphi_x^i A^- r_i^-)-(\varphi_{xx}^i B^- r_i^-)+((\varphi^i)_x^2\Gamma^-(r_i^-,r_i^-)).\end{aligned} \tag{3.8}$$

Now we use the definition of $\varphi^i$ in (1.16) and the definition of the coefficients $b_{ij}$ and $\Gamma_{ijk}$ in (3.2) to write the last part of (3.8) in the following form:

$$\begin{aligned}&\varphi_t^i r_i^- +(\varphi_x^i A^- r_i^-)-(\varphi_{xx}^i B^- r_i^-)+((\varphi^i)_x^2\Gamma^-(r_i^-,r_i^-))\\ &\qquad=-\varphi_{xx}^i\sum_{j\neq i}b_{ij}^- r_j^- -(\varphi^i)_x^2\sum_{j\neq i}\Gamma_{jii}^- r_j^-.\end{aligned} \tag{3.9}$$

Similar statements hold for $a_i^+>0$ with minus signs replaced with plus signs.

Now we employ Duhamel's principle to obtain from (3.4)

$$\begin{aligned}v(x,t)=&\int_{-\infty}^{+\infty}G(x,t;y)v_0(y)dy\\ &+\int_0^t\int_{-\infty}^{+\infty}G(x,t-s;y)\mathcal{F}(\varphi,v,\frac{\partial\bar{u}^{\delta}}{\partial\delta}\delta)_y(y,s)dy\,ds\end{aligned}$$

$$+\int_0^t\int_{-\infty}^{+\infty}G(x,t-s;y)\Phi(y,s)dy\,ds+\frac{\partial\bar u^\delta}{\partial\delta}\delta(t). \tag{3.10}$$

To get the above we used

$$\int_{-\infty}^{+\infty}G(x,t;y)\frac{\partial\bar u^\delta}{\partial\delta_i}(y)dy=e^{Lt}\frac{\partial\bar u^\delta}{\partial\delta_i}=\frac{\partial\bar u^\delta}{\partial\delta_i},$$

and $\delta(0)=0$.

Assuming

$$\begin{aligned}\delta_i(t)=&-\int_{-\infty}^{\infty}e_i(y,t)v_0(y)dy\\&-\int_0^t\int_{-\infty}^{+\infty}e_i(y,t-s)\mathcal{F}(\varphi,v,\frac{\partial\bar u^\delta}{\partial\delta}\delta)_y(y,s)dyds\\&-\int_0^t\int_{-\infty}^{+\infty}e_i(x,t-s;y)\Phi(y,s)dy\,ds,\end{aligned} \tag{3.11}$$

and using (3.10), (3.11) and $G=H+E+\tilde G$ we obtain:

$$\begin{aligned}v(x,t)=&\int_{-\infty}^{+\infty}(H+\tilde G)(x,t;y)v_0(y)dy\\&+\int_0^t\int_{-\infty}^{+\infty}(H+\tilde G)(x,t-s;y)\mathcal{F}(\varphi,v,\frac{\partial\bar u^\delta}{\partial\delta}\delta)_y(y,s)dy\,ds\\&+\int_0^t\int_{-\infty}^{+\infty}(H+\tilde G)(x,t-s;y)\Phi(y,s)dy\,ds.\end{aligned} \tag{3.12}$$

In addition to $v(x,t)$ and $\delta(t)$, we will keep track in our argument of $v_x(x,t)$ and $\dot\delta(t)$, the latter of which satisfies

$$\begin{aligned}\dot\delta_i(t)=&-\int_{-\infty}^{\infty}\partial_te_i(y,t)v_0(y)dy\\&-\int_0^t\int_{-\infty}^{+\infty}\partial_te_i(y,t-s)\mathcal{F}(\varphi,v,\frac{\partial\bar u^\delta}{\partial\delta}\delta)_y(y,s)dyds\\&-\int_0^t\int_{-\infty}^{+\infty}\partial_te_i(y,t-s)\Phi(y,s)dy\,ds,\end{aligned} \tag{3.13}$$

where we have taken advantage of the observation, apparent from (2.19), that $e(y,0)=0$.

The following lemmas are the main ingredients for the proof of Theorem 1.2. Their proofs, however, are postponed to Section 4.

**Lemma 3.2** (Estimates for linear part)**.** *Suppose that for some $E_0 > 0$, we have that $v_0(y)$ satisfies the conditions*

$$|v_0(y)| \le E_0(1+|y|)^{-3/2}, \qquad \int_{-\infty}^{+\infty} v_0(y)dy = 0.$$

*Then there holds*

$$\Big|\int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy\Big| \le CE_0\psi_2(x,t), \tag{3.14}$$

$$\Big|\int_{-\infty}^{+\infty} e_i(y,t)v_0(y)dy\Big| \le CE_0(1+t)^{-1/2}, \tag{3.15}$$

$$\Big|\int_{-\infty}^{+\infty} \partial_t e_i(y,t)v_0(y)dy\Big| \le CE_0(1+t)^{-3/2}, \tag{3.16}$$

*where $C$ does not depend on $E_0$.*

**Lemma 3.3.** *If $|v_0(x)| \le E_0(1+|x|)^{-\frac{3}{2}}$, then*

$$\int_{-\infty}^{+\infty} H(x,t;y)v_0(y)dy \le CE_0e^{-\theta t}(1+|x|)^{-\frac{3}{2}}. \tag{3.17}$$

*If $|\partial_x v_0(x)| \le E_0(1+|x|)^{-\frac{1}{2}}$, then*

$$\int_{-\infty}^{+\infty} H_x(x,t;y)v_0(y)dy \le CE_0e^{-\theta t}(1+|x|)^{-\frac{1}{2}} \tag{3.18}$$

*for some $\theta > 0$. The right-hand sides of (3.17) and (3.18) are obviously of order $\alpha(x,t)$.*

**Lemma 3.4** (Estimates for nonlinear part)**.** *For $G(x,t;y)$ as in Propositions 2.2 and 2.3, we have*

$$\int_0^t \int_{-\infty}^{+\infty} |\tilde{G}_y(x,t-s;y)|\Psi(y,s)dyds \le C(\bar{\psi}_1+\psi_2+\alpha)(x,t), \tag{3.19}$$

$$\Big|\int_0^t \int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\Phi(y,s)dyds\Big| \le CE_0(\bar{\psi}_1+\psi_2)(x,t), \tag{3.20}$$

$$\int_0^t \int_{-\infty}^{+\infty} |\partial_y e_i(y,t-s)|\Psi(y,s)dyds \le C(1+t)^{-3/4}, \tag{3.21}$$

$$\Big|\int_0^t \int_{-\infty}^{+\infty} e_i(y,t-s)\Phi(y,s)dyds\Big| \le CE_0(1+t)^{-1/2}, \tag{3.22}$$

$$\Big|\int_0^t \int_{-\infty}^{+\infty} \partial_t e_i(y,t-s)\Phi(y,s)dyds\Big| \le CE_0(1+t)^{-1}, \tag{3.23}$$

$$\int_0^t \int_{-\infty}^{+\infty} |\partial_{yt} e_i(y, t-s)| \Psi(y, s) dy ds \le C(1+t)^{-1}, \tag{3.24}$$

*for $\Phi(y, s)$ as in (3.7) and*

$$\begin{aligned} \Psi(y, s) &= (1+s)^{-1/4} s^{-1/2} (\bar{\psi}_1 + \psi_2 + \alpha + \varphi)(y, s) \\ &\quad + (1+s)^{-1/2} s^{-1/2} e^{-\eta|y|}. \end{aligned} \tag{3.25}$$

**Lemma 3.5.** *If $|\Upsilon(y, s)| \le s^{-1/2} (\bar{\psi}_1 + \psi_2 + \alpha + \varphi)(y, s) + s^{-1/2} e^{-\eta|y|}$, then*

$$\Big| \int_0^t \int_{-\infty}^{+\infty} H(x, t-s; y) \Upsilon(y, s) dy ds \Big| \le C(\bar{\psi}_1 + \psi_2 + \alpha)(x, t). \tag{3.26}$$

*If $|\partial_y \Upsilon(y, s)| \le s^{-1/2} (\bar{\psi}_1 + \psi_2 + \alpha + \varphi + e^{-\eta|y|})(y, s)$, then*

$$\Big| \int_0^t \int_{-\infty}^{+\infty} H_x(x, t-s; y) \Upsilon(y, s) dy ds \Big| \le C(\bar{\psi}_1 + \psi_2 + \alpha)(x, t). \tag{3.27}$$

*Also,*

$$\Big| \int_0^t \int_{-\infty}^{+\infty} H(x, t-s; y) \Phi(y, s) dy ds \Big| \le C E_0 (\bar{\psi}_1 + \psi_2 + \alpha)(x, t)(x, t), \tag{3.28}$$

*and similarly with $H$ replaced by $H_x$.*

**Proof of Theorem 1.2.** We use estimates (1.27) and (1.28) in order to obtain a supremum norm on $v$ and its first, second and third derivatives in $x$. We prove (1.23) with $\psi_1$ replaced by $\bar{\psi}_1$, i.e. we prove

$$|v(x, t)| \le C E_0 (\bar{\psi}_1 + \psi_2 + \alpha); \tag{3.29}$$

but then, as we have $|v(x, t)| \le C E_0 (1+t)^{-\frac{3}{4}}$ (this is (1.27) for $p = \infty$), and as

$$\min\{(1+t)^{-\frac{3}{4}}, \bar{\psi}_1(x, t)\} \sim \psi_1(x, t),$$

the proof of (1.23) would be immediate.

Let

$$\begin{aligned} \zeta(t) := &\sup_{y, 0 \le s \le t} \frac{|v(y, s)|}{(\bar{\psi}_1 + \psi_2 + \alpha)(y, s)} + \sup_{y, 0 \le s \le t} \frac{|v_x(y, s)|}{t^{-\frac{1}{2}} (1+t)^{\frac{1}{2}} (\bar{\psi}_1 + \psi_2 + \alpha)(y, s)} \\ &+ \sup_{0 \le s \le t} |\delta(s)| (1+s)^{\frac{1}{2}} + \sup_{0 \le s \le t} |\dot{\delta}(s)| (1+s). \end{aligned} \tag{3.30}$$

Our aim is to show that

$$\zeta(t) \le C(E_0 + E_0 \zeta(t)), \tag{3.31}$$

from which we conclude $\zeta(t) \leq \frac{CE_0}{1-CE_0} \leq \frac{1}{2}$, if $E_0 \leq \frac{1}{2C}$. Equivalent to (3.31) is

$$|v(x,t)| \leq C(E_0 + E_0\zeta(t))(\bar{\psi}_1 + \psi_2 + \alpha)(y,s), \tag{3.32}$$

$$|\delta(t)| \leq C(E_0 + E_0\zeta(t))(1+t)^{-\frac{1}{2}}, \tag{3.33}$$

and similar statements for $v_x$ and $\dot{\delta}$.

*Estimates of* $v(x,t)$. Looking at (3.12), there are three parts that should be estimated. The first part, the linear part, is carried out using (3.17) and (3.14). For the second part, we notice that, by (3.5), (3.30), (1.27), (1.25), (1.26), Lemma 2.1, and the definition and bounds of $\varphi$,

$$|\mathcal{F}(y,s)| \leq C(E_0 + \zeta(t))\Psi(y,s)$$

with $\mathcal{F}(y,s)$ as in (3.5), and $\Psi$ as in (3.25). Hence, by integration by parts,

$$\begin{aligned}\Big|\int_0^t \int_{-\infty}^{+\infty} &\tilde{G}(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_y(y,s)dy\,ds\Big| \\ &= \Big|\int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds\Big| \\ &\leq C(E_0 + E_0\zeta(t))h(x,t),\end{aligned}$$

by (3.19).

As we do not have good estimates for $H_y$ we cannot do the same with the part containing $H$, so, instead, we notice that

$$|\mathcal{F}_y(y,s)| \leq C(E_0 + \zeta(t))\Upsilon(y,s),$$

where $\mathcal{F}_y(y,s)$ is as in (3.6), and $\Upsilon(y,s)$ is as in Lemma 3.5. (The subtle fact is that, as we have $v_{xx}$ in $\mathcal{F}_y(y,s)$, we need to use (1.28), hence having only $(1+t)^{-\frac{1}{2}}$, instead of $(1+t)^{-\frac{3}{4}}$ which appears in $\Psi$.) Therefore we can use (3.26) to obtain the desired estimate. The third integral of (3.12) can be estimated similarly using (3.20) and (3.28).

*Estimates of* $\delta(t)$ *and* $\dot{\delta}(t)$. This is done similarly using (3.11) and (3.21)–(3.24).

*Estimates of* $v_x(x,t)$. By (3.12),

$$\begin{aligned}v_x(x,t) = &\int_{-\infty}^{+\infty} (H_x + \tilde{G}_x)(x,t;y)v_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} (H_x + \tilde{G}_x)(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_y(y,s)dy\,ds, \\ &+ \int_0^t \int_{-\infty}^{+\infty} (H_x + \tilde{G}_x)(x,t-s;y)\Phi(y,s)dy\,ds.\end{aligned} \tag{3.34}$$

The parts involving $H_x$ are treated using (3.18), (3.27) and the similar statement for $\Phi$ in Lemma 3.5; this because, due to (1.28), $\mathcal{F}_{yy}(y,s)$ has bounds similar to those for $\mathcal{F}_y(y,s)$. For $\int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y)v_0(y)dy$, we notice that $\tilde{G}_x$ is at least as good as $\tilde{G}$ away from 0. Therefore, for $t \geq 1$, a proof identical to that of (3.14) can also prove

$$\int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y)v_0(y)dy \leq CE_0\psi_2(x,t).$$

On the other hand, for $t \leq 1$ we use the fact that $\tilde{G}_x(x,t;y \sim t^{-\frac{1}{2}}\tilde{G}(x,mt;y)$ for $t$ near zero and some positive $m$. This, together with (1.28) provides us with the necessary bounds for the linear part of the calculations.

Finally,

$$\begin{aligned}\int_0^t \int_{-\infty}^{+\infty} &\tilde{G}_x(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_y(y,s)dy\,ds \\ &= \int_0^{t-1} \int_{-\infty}^{+\infty} \tilde{G}_{xy}(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds \\ &+ \int_{t-1}^{t} \int_{-\infty}^{+\infty} \tilde{G}_x(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_y(y,s)dy\,ds.\end{aligned}$$

The first integral can be computed exactly the same way (3.19) is proved, for away from $t-s=0$, $\tilde{G}_{xy}(x,t-s;y)$ is as well behaved as $\tilde{G}_{xy}(x,t-s;y)$. For the second integral, a close investigation of the proof of (3.19) shows us that, if we limit the time integration to $t-1 \leq s \leq t$, then the same proof works to prove that

$$\int_{t-1}^{t} \int_{-\infty}^{+\infty} |\tilde{G}_x(x,t-s;y)||\Upsilon(y,s)|dyds \leq C(\bar{\psi}_1 + \psi_2 + \alpha)(x,t),$$

with $\Upsilon$ as in Lemma 3.5. This fact plus the bound for $\mathcal{F}_y$ provides us with the estimates needed for the second integral.

This concludes the proof of Theorem 1.2. □

**Remark 3.6.** We observe that our argument in the proof of Theorem 1.2 assumes that $\zeta(t)$ does not have a discontinuous step from a finite value to an infinite value. Though Theorem 1.2 regards behavior as $t \to \infty$, the possibility of such a jump at each given time is a short-time phenomenon, in which case we can apply the parametrix development of [42], augmented in the case of mixed hyperbolic–parabolic regularization by the energy estimate approach of [33]. More precisely, in the case of strictly parabolic regularization, we write

$$u_t + F(u)_x = (B(u)u_x)_x$$

in the form

$$u_t = \hat{L}u := (\hat{A}(x,t)u)_x + (\hat{B}(x,t)u_x)_x,$$

where the coefficients $\hat{A}$ and $\hat{B}$ depend on $x$ and $t$ through $u(x,t)$, and hence their behavior is governed by the regularity of $u(x,t)$ (precise forms are given in [42] in the proof of Corollary 11.4). Regarding $\hat{L}$ as a linear operator with associated Green's function $\hat{G}(x,t;y,s)$, we fix a reference time $T$ and propogate $u(x,t)$ forward in time as

$$u(x,t) = \mathcal{T}u := \int_{-\infty}^{+\infty} \hat{G}(x,t;y,T)u(y,T)dy,$$

where $t$ is taken sufficiently close to $T$. Following the analysis of [42], it can be shown that for $u$ sufficiently smooth—in particular, $u \in H^s$, $s \geq 4$ suffices—the operator $\mathcal{T}$ is a contraction on $L^\infty$, hence bounding instantaneous jumps in the supremum norm as $t$ increases.

In the case of mixed hyperbolic–parabolic regularization, we employ our assumption that there exists an invertible change of coordinates $u \to w(u)$ so that (1.2) can be written in the quasilinear, partially symmetric, hyperbolic–parabolic form,

$$\tilde{A}^0 w_t + \tilde{A}w_x = (\tilde{B}w_x)_x + G, \quad w = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix}, \tag{3.35}$$

where $w^I \in \mathbb{R}^{n-r}$, $w^{II} \in \mathbb{R}^r$, $x \in \mathbb{R}$, $t > 0$, where (A1)–(A3) hold (see (1.10) and the surrounding discussion). Letting then $\bar{G}(x,t;y,s)$ denote the Green's function associated with (3.35), we again fix a reference time $T$ and propagate $w(t,x)$ forward in time as

$$w(x,t) = \int_{-\infty}^{+\infty} \bar{G}(x,t;y,T)w(y,T)dy + \int_T^t \int_{-\infty}^{+\infty} \bar{G}(x,t;y;s)G(y,s)dyds.$$

In this case, Raoofi has shown through energy estimates that the operator (redefined from above)

$$\mathcal{T}w := \int_{-\infty}^{+\infty} \bar{G}(x,t;y,T)w(y,T)dy + \int_T^t \int_{-\infty}^{+\infty} \bar{G}(x,t;y;s)G(y,s)dyds$$

is bounded for $w \in H^s$, $s = 3$, and contractive in the class $L^2$, giving convergence in $L^2$ to an $H^s$ solution. Similarly as in the case of strictly parabolic regularization, this argument can be extended locally to estimates on $\zeta(t)$, again limiting possible discontinuities to finite steps.

## 4. Integral estimates

**Proof of Lemma 3.2.** We begin by establishing that for $\tilde{G}$ satisfying

$$\|\tilde{G}\|_{L^1_x(\mathbb{R})} \le C_1, \quad \|\tilde{G}_y\|_{L^\infty_x(\mathbb{R})} \le C_2 t^{-1}, \quad \|\tilde{G}_y\|_{L^1_x(\mathbb{R})} \le C_3 t^{-1/2},$$

for some positive constants $C_1$, $C_2$, and $C_3$, we have

$$\Big| \int_{-\infty}^{+\infty} \tilde{G}(x,t;y) v_0(y) dy \Big| \le C E_0 (1+t)^{-3/4}. \tag{4.1}$$

First, in the event that $t$ is bounded, we have

$$\Big| \int_{-\infty}^{+\infty} \tilde{G}(x,t;y) v_0(y) dy \Big| \le \|\tilde{G}\|_{L^1_x(\mathbb{R})} \|v_0\|_{L^\infty} \le C_1 E_0.$$

In the alternative case, for which we take $t$ bounded away from 0, we integrate by parts to obtain

$$\int_{-\infty}^{+\infty} \tilde{G}(x,t;y) v_0(y) dy = \int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y) V_0(y) dy,$$

where

$$V_0(x) := \int_{-\infty}^{x} v_0(x) dx, \qquad |V_0(y)| \le C_4 E_0 (1+|y|)^{-1/2}.$$

We have, then

$$\begin{aligned}
&\Big| \int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y) V_0(y) dy \Big| \\
&\le \Big| \int_{\{|y| \le \sqrt{t}\}} \tilde{G}_y(x,t;y) V_0(y) dy \Big| + \Big| \int_{\{|y| \ge \sqrt{t}\}} \tilde{G}_y(x,t;y) V_0(y) dy \Big| \\
&\le C_2 t^{-1} \int_{\{|y| \le \sqrt{t}\}} |V_0(y)| dy + C_4 E_0 (1+\sqrt{t})^{-1/2} \int_{\{|y| \ge \sqrt{t}\}} |\tilde{G}_y(x,t;y)| dy \\
&\le C_5 E_0 t^{-3/4},
\end{aligned}$$

establishing (4.1).

It now follows from (4.1) that if there exists some $a_j^\pm \gtrless 0$ so that $|x - a_j^\pm t| \le \sqrt{t}$, then $t^{-3/4} \le C|x - a_j^\pm t|^{-3/2}$, and the first estimate is apparent. In the event that $|x - a_j^\pm t| \ge \sqrt{t}$ for all $a_j^\pm$, we consider the case in which there exists some $a_j^\pm \gtrless 0$ so that $|x - a_j^\pm t| \le \epsilon t$, where $\epsilon > 0$ is sufficiently small so that for all $j \ne k$, $|x - a_k^\pm t| \ge \eta t$, for some $\eta > 0$. In this case, we compute

$$\int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy \tag{4.2}$$
$$= \int_{\{|y|\leq \frac{|x-a_j^{\pm}t|}{N}\}} \tilde{G}(x,t;y)v_0(y)dy + \int_{\{|y|\geq \frac{|x-a_j^{\pm}t|}{N}\}} \tilde{G}(x,t;y)v_0(y)dy,$$

where $N$ will be chosen sufficiently large in the analysis. For the second of these last two integrals, integration of $\tilde{G}$ immediately gives an estimate by

$$CE_0(1+|x-a_j^{\pm}t|)^{-3/2},$$

which is sufficient for our first estimate since we are in the case $|x-a_j^{\pm}t| \geq \sqrt{t}$. Alternatively, for the first integral in (4.2), we integrate by parts to obtain

$$\Big|\int_{\{|y|\leq \frac{|x-a_j^{\pm}t|}{L}\}} \tilde{G}(x,t;y)v_0(y)dy\Big| \tag{4.3}$$
$$\leq \Big|\tilde{G}(x,t,\frac{x-a_j^{\pm}t}{N})V_0(\frac{x-a_j^{\pm}t}{N})\Big| + \Big|\tilde{G}(x,t,-\frac{x-a_j^{\pm}t}{N})V_0(-\frac{x-a_j^{\pm}t}{N})\Big|$$
$$+ \int_{\{|y|\leq \frac{|x-a_j^{\pm}t|}{N}\}} |\tilde{G}_y(x,t;y)||V_0(y)|dy.$$

It remains to estimate each of these last three terms for each summand in the estimates on $|\tilde{G}|$ and $|\tilde{G}_y|$ from Lemma 2.3. As each case is similar, we proceed only with estimates on the convection terms.

In the case $y<0$ and for the convection Green's function estimate, we have

$$\Big|\tilde{G}(x,t;-\frac{|x-a_j^-t|}{N})V_0(-\frac{|x-a_j^-t|}{N})\Big|$$
$$\leq CE_0\sum_{k=1}^{n} t^{-\frac{1}{2}} e^{-\frac{(x-\frac{|x-a_j^-t|}{N}-a_k^-t)^2}{Mt}} e^{-\eta x^+}(1+|x-a_j^-t|)^{-\frac{1}{2}}.$$

In the event that $j=k$, we have

$$|x-a_k^-t| - (\frac{x-a_k^-t}{N}) \geq (x-a_k^-t)(1-\frac{1}{N}),$$

which for $N$ sufficiently large leads immediately to an estimate by

$$CE_0 t^{-1/2} e^{-\frac{(x-a_k^-t)^2}{Lt}}(1+|x-a_k^-t|)^{-1/2},$$

for some $L$ sufficiently large. According to the boundedness of

$$\frac{|x-a_k^- t|}{t^{1/2}} e^{-\frac{(x-a_k^- t)^2}{Lt}}, \tag{4.4}$$

we obtain the claimed estimate. On the other hand, if $j \neq k$, then $|x-a_j^- t| \leq \epsilon t$ and $|x-a_k^- t| \geq \eta t$, so that, for $N$ sufficiently large, we have exponential decay in $t$.

We next consider the integral in (4.3), for which in the case of the convection estimate and for $y \leq 0$, we have

$$\int_{\{y \leq -\frac{|x-a_j^- t|}{N}\}} \sum_{k=1}^{n} t^{-1} e^{-\frac{(x-\frac{|x-a_j^- t|}{N}-a_k^- t)^2}{Mt}} e^{-\eta x^+} |V_0(y)| dy.$$

Proceeding as in the boundary case, we see that in the case $j \neq k$, the kernel decays at exponential rate in $t$, while for $j = k$, upon integration of $V_0(y)$, we have an estimate,

$$C E_0 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} (1+|x-a_j^- t|)^{1/2} \leq \bar{C} E_0 |x-a_j^- t|^{-3/2}.$$

For the second estimate in Lemma 3.2, we observe from Remark 2.6 the estimate

$$|\partial_y e_i(y,t)| \leq C t^{-1/2} \sum_{a_k^- > 0} e^{-\frac{(y+a_k^- t)^2}{Mt}}.$$

Integrating by parts, then, we have

$$\begin{aligned} \Big| \int_{-\infty}^{+\infty} e_i(y,t) v_0(y) dy \Big| &= \Big| \int_{-\infty}^{+\infty} \partial_y e_i(y,t) V_0(y) dy \Big| \\ &\leq C t^{-1/2} \sum_{a_k^- > 0} \int_{-\infty}^{+\infty} e^{-\frac{(y+a_k^- t)^2}{Mt}} E_0 (1+|y|)^{-1/2} dy. \end{aligned}$$

In this last integrand, we observe the inequality

$$e^{-\frac{(y+a_k^- t)^2}{Mt}} E_0 (1+|y|)^{-1/2} \leq C e^{-\frac{(y+a_k^- t)^2}{Mt}} E_0 (1+t)^{-1/2},$$

from which the claimed estimate is immediate.

The final estimate in Lemma 3.2 is proven similarly to the second estimate in Lemma 2 of [12]. □

**Proof of Lemma 3.3.** Looking at (2.9), we notice that in order to estimate $\int_{-\infty}^{+\infty} H(x,t;y)v_0(y)dy$ it suffices to estimate

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathcal{R}_j^*(x)\mathcal{O}(e^{-\eta_0 t})\delta_{x-\bar{a}_j^* t}(-y)\mathcal{L}_j^{*t}(y)v_0(y)dy &\leq CE_0 e^{-\eta_0 t} v_0(\bar{a}_j^* t - x) \\ \leq CE_0 e^{-\eta_0 t}(1+|\bar{a}_j^* t - x|)^{-\frac{3}{2}} &\leq CE_0 e^{-\eta_0 t}(1+|x|)^{-\frac{3}{2}}(1+|\bar{a}_j^* t|)^{\frac{3}{2}} \\ &\leq CE_0 e^{-\frac{\eta_0 t}{2}}(1+|x|)^{-\frac{3}{2}}.\end{aligned}$$

Here we used the crude inequality

$$\frac{1}{1+|a+b|} \leq \frac{1+|b|}{1+|a|} \tag{4.5}$$

and the fact that $\bar{a}_j^*$ $\mathcal{R}_j^*$ and $\mathcal{L}_j^{*t}$ are bounded. This gives us (3.17). Estimate (3.18) is obtained similarly. □

**Proof of Lemma 3.4.** For Lemma 3.4, the proof of each estimate requires the analysis of several cases. We proceed by carrying out detailed calculations in the most delicate cases and sufficing to indicate the appropriate arguments in the others. In particular, we will always consider the case $x, y \leq 0$. The case $y \leq 0 \leq x$ is similar (though certainly not identical) to the reflection case for $x, y \leq 0$, and the estimates for $y \geq 0$ are entirely symmetric. The analysis of each type of kernel—convection, transmission, and scattering—is similar, and we carry out details only in the case of convection. (This terminology is reviewed in Remark 2.4.)

*Nonlinearity* $(1+s)^{-1/4}s^{-1/2}\bar{\psi}_1$: We begin by estimating integrals

$$\int_0^t \int_{-|a_1^-|s}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s|)^{-3/4} s^{-1/2}(1+s)^{-3/4} dyds. \tag{4.6}$$

In the event that $|x| \geq |a_1^-|t$, we write

$$x - y - a_j^-(t-s) = (x - a_1^- t) - (y - a_1^- s) + (a_1^- - a_j^-)(t-s), \tag{4.7}$$

and observe that in the current setting ($x \leq 0$, $y \in [-|a_1^-|s, 0]$, $a_1^- \leq a_k^-$), there is no cancellation between these three summands. Integrating $(1+|y - a_k^- s|)^{-3/4}$ for $s \in [0, t/2]$ and integrating the kernel for $s \in [t/2, t]$, we obtain an estimate by

$$\begin{aligned}&C_1 t^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_0^{t/2} s^{-1/2}(1+s)^{-1/2} ds \\ &\quad + C_2 t^{-1/2}(1+t)^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds\end{aligned} \tag{4.8}$$

$$\leq Ct^{-1}\ln(e+t)e^{-\frac{(x-a_1^- t)^2}{Lt}},$$

which is sufficient by the argument of (4.4). We observe that the seeming blowup as $t \to 0$ can be eliminated. Integrating the kernel in (4.6), we have an estimate by

$$\begin{aligned} &C\int_0^t (t-s)^{-1/2}s^{-1/2}ds \\ &C_1 t^{-1/2}\int_0^{t/2} s^{-1/2}ds + C_2 t^{-1/2}\int_{t/2}^t (t-s)^{-1/2}ds \leq C. \end{aligned} \tag{4.9}$$

For $|x| \leq |a_1^-|t$, we write

$$x - y - a_j^-(t-s) = (x - a_j^-(t-s) - a_k^- s) - (y - a_k^- s), \tag{4.10}$$

from which we observe the inequality

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y-a_k^- s|)^{-3/4} \\ &\leq C\Big[e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}(1+|y-a_k^- s|)^{-3/4}e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} \\ &+ e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|)^{-3/4}\Big], \end{aligned} \tag{4.11}$$

for some $\bar{M}$ sufficiently large and $\epsilon > 0$ sufficiently small. The analysis can now be divided into three cases: (i) $a_k^- < 0 < a_j^-$, (ii) $a_k^- < a_j^- < 0$, and (iii) $a_j^- \leq a_k^- < 0$. Each critical argument appears in Case (ii), and so we consider only it in detail. For the first estimate in (4.11), we have

$$\begin{aligned} &\int_0^t \int_{-|a_1^-|s}^0 (t-s)^{-1}e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} \\ &\quad\times(1+|y-a_k^- s|)^{-3/4}s^{-1/2}(1+s)^{-3/4}dyds. \end{aligned}$$

In the event that $|x| \geq |a_k^-|t$, we write

$$x - a_j^-(t-s) - a_k^- s = (x - a_k^- t) - (a_j^- - a_k^-)(t-s), \tag{4.12}$$

for which there is no cancellation between the summands, and we can proceed exactly as in (4.8). On the other hand, for $|x| \leq |a_j^-|t$, we write

$$x - a_j^-(t-s) - a_k^- s = (x - a_j^- t) - (a_k^- - a_s^-)s, \tag{4.13}$$

for which again there is no cancellation between summands, and we can proceed exactly as in (4.8). For the critical case $|a_j^-|t \leq |x| \leq |a_k^-|t$, we subdivide the analysis into cases: $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (4.13) the estimate

$$\begin{aligned} & e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}}(1+s)^{-3/4} \\ & \le C\Big[e^{-\frac{(x-a_j^- t)^2}{\bar{M}(t-s)}}(1+s)^{-3/4} + e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}(1+|x-a_j^- t|)^{-3/4}\Big]. \end{aligned} \tag{4.14}$$

For the first, we can proceed exactly as in (4.8), while for the second, we integrate $(1+|y-a_k^- s|)^{-3/4}$ to obtain the estimate

$$\begin{aligned} & C_1 t^{-1}(1+|x-a_j^- t|)^{-3/4}\int_0^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} s^{-1/2}(1+s)^{1/4} ds \\ & \quad \le C t^{-1/2}(1+|x-a_j t|)^{-3/4}, \end{aligned}$$

where in establishing this last inequality we have integrated $e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}$ in $s$. For $s \in [t/2, t]$, we observe through (4.12) the estimate

$$\begin{aligned} & (t-s)^{-3/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} \\ & \qquad \le C\Big[(t-s)^{-3/4} e^{-\frac{(x-a_k^- t)^2}{\bar{M}(t-s)}} + |x-a_k^- t|^{-3/4} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}\Big]. \end{aligned} \tag{4.15}$$

For the first of these estimates, we proceed exactly as in (4.8), while for the second, we integrate the kernel in $y$ to obtain an estimate by

$$\begin{aligned} & C_2|x-a_k^- t|^{-\frac{3}{4}} t^{-\frac{1}{2}}(1+t)^{-\frac{3}{4}}\int_{t/2}^t (t-s)^{\frac{1}{4}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} ds \\ & \quad \le C|x-a_k^- t|^{-3/4}(1+t)^{-1/2}. \end{aligned}$$

The apparent blowup as $x$ approaches $a_k^- t$ can be removed similarly as in (4.9).

For the second estimate in (4.11), we have

$$\begin{aligned} & \int_0^t \int_{-|a_1^-|s}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} \\ & \quad \times (1+|y-a_k^- s| + |x-a_j^-(t-s)-a_k^- s|)^{-3/4} s^{-1/2}(1+s)^{-3/4} dy ds. \end{aligned}$$

Again, we proceed in detail only in Case (ii), $a_k^- < a_j^- < 0$. In the event that $|x| \ge |a_k^-|t$ (though keeping in mind that we have already considered the case

$|x| \geq |a_1^- |t)$, we use (4.12), for which there is no cancellation between the summands, and upon integration of the kernel, we have an estimate by

$$\begin{aligned} &C(1+|x-a_k^- t|)^{-3/4} \int_0^t (t-s)^{-1/2} s^{-1/2} (1+s)^{-3/4} ds \\ &\quad \leq C_1 (1+|x-a_k^- t|)^{-3/4} t^{-1/2} \int_0^{t/2} s^{-1/2} (1+s)^{-3/4} ds \\ &+C_2 (1+|x-a_k^- t|)^{-3/4} t^{-1/2} (1+t)^{-3/4} \int_{t/2}^t (t-s)^{-1/2} ds, \end{aligned} \tag{4.16}$$

from which we obtain an estimate by $\bar{\psi}_1$. Likewise, in the case $|x| \leq |a_j^-|t$, we use (4.13), for which there is no cancellation between summands, and we can proceed similarly as in (4.16). For the critical case $|a_j^-|t \leq |x| \leq |a_k^-|t$, we subdivide the analysis into cases: $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (4.13) the estimate

$$\begin{aligned} &(1+|y-a_k^- s| + |x-a_j^-(t-s) - a_k^- s|)^{-3/4} (1+s)^{-3/4} \\ &\leq C\Big[(1+|x-a_j^- t|)^{-3/4} (1+s)^{-3/4} \\ &+ (1+|y-a_k^- s| + |x-a_j^-(t-s) - a_k^- s|)^{-3/4} (1+s+|x-a_j^- t|)^{-3/4}\Big]. \end{aligned}$$

For the first of these estimates, we can proceed as in (4.16), while for the second, we integrate the kernel in $y$ to obtain an estimate by

$$\begin{aligned} &C_1 (1+|x-a_j^- t|)^{-3/4} t^{-1/2} \int_0^{t/2} (1+|x-a_j^-(t-s) - a_k^- s|)^{-3/4} s^{-1/2} ds \\ &\quad \leq C(1+t)^{-1/2} (1+|x-a_j t|)^{-3/4}. \end{aligned}$$

For $s \in [t/2, t]$, we observe through (4.12) the estimate

$$\begin{aligned} &(t-s)^{-3/4} (1+|y-a_k^- s| + |x-a_j^-(t-s) - a_k^- s|)^{-3/4} \\ &\leq (t-s)^{-3/4} \Big[(1+|x-a_k^- t|)^{-3/4} \\ &+ |x-a_k^- t|^{-3/4} (1+|y-a_k^- s| + |x-a_j^-(t-s) - a_k^- s|)^{-3/4}\Big]. \end{aligned}$$

For the first of these estimates, we proceed exactly as in (4.8), while for the second, we integrate the kernel in $y$ to obtain an estimate by

$$\begin{aligned} &C_2 |x-a_k^- t|^{-\frac{3}{4}} t^{-\frac{1}{2}} (1+t)^{-\frac{3}{4}} \int_{t/2}^t (t-s)^{\frac{1}{4}} (1+|x-a_j^-(t-s) - a_k^- s|)^{-\frac{3}{4}} ds \\ &\quad \leq C|x-a_k^- t|^{-3/4} (1+t)^{-3/4}. \end{aligned}$$

*Nonlinearity* $(1+s)^{-1/4}s^{-1/2}\psi_2$: We next consider integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s|+s^{1/2})^{-3/2} s^{-1/2}(1+s)^{-1/4} dyds. \tag{4.17}$$

In this case, we again observe (4.10), from which we obtain the estimate

$$\begin{aligned} & e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s|+s^{1/2})^{-3/2} \qquad (4.18)\\ & \le C\Big[e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s|+s^{1/2})^{-3/2} \\ & + e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\Big]. \end{aligned}$$

As in our analysis of the nonlinearity $(1+s)^{-1/4}s^{-1/2}\bar{\psi}_1$, we focus on the case $a_k^- < a_j^- < 0$, compared to which the other cases are either similar or more straightforward. For the first estimate in (4.18), we have

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\epsilon\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} \times (1+|y-a_k^- s|+s^{1/2})^{-3/2} s^{-1/2}(1+s)^{-1/4} dyds.$$

In the event that $|x| \ge |a_k^-|t$, we employ (4.12), for which there is no cancellation between the summands. In this case, we have an estimate by

$$\begin{aligned} & C_1 t^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_0^{t/2} (1+s^{1/2})^{-1/2} s^{-1/2}(1+s)^{-1/4} ds \\ & \quad + C_2(1+t^{1/2})^{-3/2} t^{-1/2}(1+t)^{-1/4} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds \\ & \quad \le C t^{-1}\ln(e+t) e^{-\frac{(x-a_k^- t)^2}{Lt}}, \end{aligned} \tag{4.19}$$

where the seeming blowup as $t \to 0$ can be eliminated as in (4.9). In the case $|x| \le |a_j^-|t$, we employ (4.13) for which again there is no cancellation between the summands and we can proceed as in (4.19).

We turn next to the critical case $|a_j^-|t \le |x| \le |a_k^-|t$, for which we divide the analysis into cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe through (4.13) the inequality

$$e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{M(t-s)}} s^{-1/2}(1+s)^{-1/4} \le C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}} s^{-1/2}(1+s)^{-1/4}$$

$$+ e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}|x-a_j^- t|^{-1/2}(1+|x-a_j^- t|)^{-1/4}\Big].$$

For the first of these last two estimates, we can proceed as in (4.19), while for the second, we have an estimate of the form

$$C_1 t^{-1}|x-a_j^- t|^{-\frac{1}{2}}(1+|x-a_j^- t|)^{-\frac{1}{4}}\int_0^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}(1+s^{\frac{1}{2}})^{-\frac{1}{2}}ds$$
$$\leq Ct^{-1/2}|x-a_j t|^{-1/2}(1+|x-a_j t|)^{-1/4},$$

where as usual the singular behavior can be removed. For $s\in[t/2,t]$, we observe through (4.12) the inequality

$$(t-s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}$$
$$\leq C\Big[|x-a_k^- t|^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}} + (t-s)^{-3/4}e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

For the second of these last two estimates, we can proceed as in (4.19), while for the first we have an estimate of the form

$$C_1|x-a_k^- t|^{-\frac{3}{4}}t^{-\frac{1}{2}}(1+t)^{-\frac{1}{4}}(1+t^{\frac{1}{2}})^{-\frac{3}{2}}\int_{t/2}^{t}(t-s)^{\frac{1}{4}}e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{\bar{M}(t-s)}}ds$$
$$\leq C|x-a_k^- t|^{-3/4}(1+t)^{-3/4}.$$

For the second estimate in (4.18), we have

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}$$
$$\times(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{\frac{1}{2}})^{-\frac{3}{2}}s^{-\frac{1}{2}}(1+s)^{-\frac{1}{4}}dyds.$$

Focusing on the case $a_k^- < a_j^- < 0$, we observe as before that,

for $|x|\geq |a_k^-|t$, we have no cancellation between the summands in (4.12), and consequently there holds

$$(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}$$
$$\leq C(1+|y-a_k^- s|+|x-a_k^- t|+(t-s)+s^{1/2})^{-3/2},$$

from which, upon integration of the kernel, we have an estimate by

$$C_1(1+|x-a_k^- t|+t^{1/2})^{-3/2}\int_0^t (t-s)^{-1/2}s^{-1/2}(1+s)^{-1/4}$$
$$\leq C(1+t)^{-1/4}(1+|x-a_k^- t|+t^{1/2})^{-3/2}. \tag{4.20}$$

In the event that $|x| \le |a_j^-|t$, we have no cancellation between the summands in (4.13), and consequently there holds

$$\begin{aligned}&(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\ &\le C(1+|y-a_k^- s|+|x-a_k^- t|+s)^{-3/2},\end{aligned}$$

from which we have an estimate by

$$\begin{aligned}&C_1 t^{-1}(1+|x-a_j^- t|)^{-1/2}\int_0^{t/2} s^{-1/2}(1+s)^{-1/4}ds\\ &\quad + C_2 t^{-1/2}(1+t)^{-1/4}(1+|x-a_j^- t|+t)^{-3/2}\int_{t/2}^t (t-s)^{-1/2}ds\\ &\quad \le Ct^{-3/4}(1+|x-a_j^- t|)^{-1/2}.\end{aligned} \tag{4.21}$$

In the critical case $|a_j^-|t \le |x| \le |a_k^-|t$, we subdivide the analysis further into cases $s \in [0, t/2]$ and $s \in [t/2, t]$. In the case $s \in [0, t/2]$, we observe through (4.13) the estimate

$$\begin{aligned}&(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}s^{-1/2}(1+s)^{-1/4}\\ &\le C\Big[(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+|x-a_j^- t|+s^{1/2})^{-3/2}\\ &\quad \times s^{-1/2}(1+s)^{-1/4}\\ &+(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\ &\times (s+|x-a_j^- t|)^{-1/2}(1+s+|x-a_j^- t|)^{-1/4}\Big].\end{aligned}$$

For the first of these last two estimates, we can proceed as in (4.21), while for the second, we obtain an estimate by

$$\begin{aligned}&C_1 t^{-1}|x-a_j^- t|^{-1/2}(1+|x-a_j^- t|)^{-1/4}\\ &\quad\times\int_0^{t/2}(1+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-1/2}ds\\ &\quad \le Ct^{-1/2}|x-a_j^- t|^{-1/2}(1+|x-a_j t|)^{-1/4},\end{aligned}$$

where the apparent singularities can be eliminated as in (4.9). In the case $s \in [t/2, t]$, we observe through (4.12) the estimate

$$\begin{aligned}&(t-s)^{-3/4}(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\ &\le C\Big[|x-a_k^- t|^{-3/4}(1+|y-a_k^- s|+|x-a_j^-(t-s)-a_k^- s|+s^{1/2})^{-3/2}\\ &+(t-s)^{-3/4}(1+|y-a_k^- s|+|x-a_k^- t+s^{1/2})^{-3/2}\Big].\end{aligned}$$

For the second of the last two estimates, keeping in mind that here $s \in [t/2, t]$, we can proceed as in (4.20), while for the first we integrate the kernel to obtain an estimate by

$$\begin{aligned} &C_1|x-a_k^- t|^{-3/4} t^{-1/2}(1+t)^{-1/4} \\ &\quad \times \int_{t/2}^t (t-s)^{1/4}(1+|x-a_j^-(t-s)-a_k^- s| + |x-a_j^- t| + s^{1/2})^{-3/2} ds \\ &\quad \le C|x-a_k^- t|^{-3/4} t^{-1/2}(1+t)^{-1/4}, \end{aligned}$$

where the apparent singularities can be eliminated as in (4.9).

*Nonlinearity* $(1+s)^{-1/4}s^{-1/2}\alpha$: We next consider integrals of the form

$$\int_0^t \int_{-|a_1^-|s}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} s^{-1/2} (1+|y|)^{-1/2} dy ds. \tag{4.22}$$

In the case $|x| \ge |a_1^-|t$, we observe that there is no cancellation between summands in decomposition (4.7), and consequently, we have an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_0^{t/2} (1+s)^{-1/2} s^{-1/2} ds \\ &\quad + C_2(1+t)^{-1} t^{-1/2} e^{-\frac{(x-a_1^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds \\ &\quad \le C t^{-1} \ln(e+t) e^{-\frac{(x-a_1 t)^2}{Lt}}, \end{aligned} \tag{4.23}$$

where the apparent blowup as $t \to 0$ can be removed as in (4.9). For the remainder of the analysis, we restrict our attention to the case $|x| \le |a_1^-|t$ and $a_j^- < 0$ (the case $a_j^- > 0$ is similar and more direct). Observing that, for $y \in [-|a_1^-|s, 0]$, we have

$$(1+s)^{-1} \le C(1+s+|y|)^{-1},$$

we have the inequality

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} s^{-1/2} (1+|y|)^{-1/2} \Big|_{\{y \in [-|a_1^-|s, 0]\}} \\ &\le C\Big[ e^{-\epsilon \frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} s^{-1/2} (1+|y|)^{-1/2} \\ &+ e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s+|y|+|x-a_j^-(t-s)|)^{-1} \\ &\times (s+|y|+|x-a_j^-(t-s)|)^{-1/2} (1+|y|+|x-a_j^-(t-s)|)^{-1/2} \Big]. \end{aligned} \tag{4.24}$$

For the first estimate in (4.24), we observe that for $|x| \geq |a_j^-|t$, there is no cancellation between $x - a_j^- t$ and $a_j^- s$, and consequently we have an estimate by

$$C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_0^{t/2} (1+s)^{-1/2} s^{-1/2} ds \tag{4.25}$$
$$+C_2(1+t)^{-1} t^{-1/2} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t/2}^t (t-s)^{-1/2} ds \leq C t^{-1} \ln(e+t) e^{-\frac{(x-a_j^- t)^2}{Lt}},$$

where the apparent blowup as $t \to 0$ can be removed as in (4.9). In the case $|x| \leq |a_j^-|t$, we divide the analysis into the cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we have the inequality

$$e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} (1+s)^{-1} s^{-1/2} \leq C \Big[ e^{-\frac{(x-a_j^- t)^2}{Lt}} e^{-\epsilon \frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} (1+s)^{-1} s^{-1/2}$$
$$+ e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} (1+s+|x-a_j^- t|)^{-1} (s+|x-a_j^- t|)^{-1/2} \Big].$$

For the first of these last two estimates, we proceed as in (4.25), while for the second, upon integration of $(1+|y|)^{-1/2}$, we have an estimate on (4.22) by

$$\begin{aligned} &C_1 t^{-1} (1+|x-a_j^- t|)^{-3/4} \int_0^{t/2} e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} \\ &\quad \times (1+s+|x-a_j^- t|)^{-1/4} (s+|x-a_j^- t|)^{-1/2} (1+s)^{1/2} ds \\ &\quad \leq C t^{-1/2} (1+|x-a_j^- t|)^{-3/4}. \end{aligned}$$

In the case $s \in [t/2, t]$, we observe the inequality

$$(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} \leq C|x|^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}},$$

from which, upon integration of the kernel, we obtain an estimate on (4.22) by

$$C_2 (1+t)^{-1} t^{-1/2} |x|^{-1/2} \int_{t/2}^t e^{-\frac{(x-a_j(t-s))^2}{\bar{M}(t-s)}} ds \leq C t^{-1} |x|^{-1/2}.$$

For the second estimate in (4.24), we observe that in the case $|x| \geq |a_j^-|t$, we have no cancellation between $(x - a_j^- t)$ and $a_j^- s$, and consequently can estimate

$$\begin{aligned} &C_1 t^{-1/2}(1+|x-a_j^- t|)^{-3/2}\int_0^{t/2} s^{-1/2}ds \\ &+C_2 t^{-1/2}(1+|x-a_j^- t|)^{-3/2}\int_{t/2}^{t}(t-s)^{-1/2} \\ &\le C(1+|x-a_j^- t|)^{-3/2}, \end{aligned} \tag{4.26}$$

which suffices since the case $|x-a_j^- t| \le C\sqrt{t}$ requires only $t^{-3/4}$ decay. In the case $|x| \le |a_j^-|t$, we subdivide the analysis further into the cases $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, we observe the inequality

$$(1+s+|x-a_j^-(t-s)|)^{-1} \le C(1+s+|x-a_j^- t|)^{-1},$$

from which, upon integration of the kernel, we obtain an estimate by

$$\begin{aligned} &C_1 t^{-1/2}(1+|x-a_j^- t|)^{-1}\int_0^{t/2} s^{-1/2}(1+|x-a_j^-(t-s)|)ds \\ &\le C t^{-1/2}(1+|x-a_j^- t|)^{-1}. \end{aligned}$$

For $s \in [t/2, t]$, we observe the inequality

$$\begin{aligned} &(t-s)^{-1/2}(1+|y|+|x-a_j^-(t-s)|)^{-1/2} \\ &\le \Big[|x|^{-1/2}(1+|y|+|x-a_j^-(t-s)|)^{-1/2}+(t-s)^{-1/2}(1+|y|+|x|)^{-1/2}\Big]. \end{aligned}$$

For the first of these last two estimates, we obtain an estimate on (4.22) by

$$C_2(1+t)^{-1}t^{-1/2}|x|^{-1/2}\int_{t/2}^{t}(1+|x-a_j^-(t-s)|)^{-1/2}ds \le C(1+t)^{-1}|x|^{-1/2},$$

while for the second, we have an estimate by

$$C_2(1+t)^{-1}t^{-1/2}(1+|x|)^{-1/2}\int_{t/2}^{t}(t-s)^{-1/2}ds \le C(1+t)^{-1}|x|^{-1/2}.$$

*Nonlinearity* $s^{-1/2}(1+s)^{-1/4}\varphi$: We next consider integrals of the form

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}(1+s)^{-3/4}s^{-1/2}e^{-\frac{(y-a_k^- s)^2}{Ms}}dyds. \tag{4.27}$$

In this case we observe from Lemma 6 of [12] the equality

$$\begin{aligned} &e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\frac{(y-a_k^- s)^2}{Ms}} \\ &= e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}e^{-\frac{t}{Ms(t-s)}(y-\frac{xs-(a_j^-+a_k^-)(t-s)s}{t})^2}, \end{aligned} \tag{4.28}$$

from which direct integration over $y$ leads to an estimate by

$$Ct^{-1/2}\int_0^t (t-s)^{-1/2}(1+s)^{-3/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}.$$

As in the previous analyses, we focus on the case $a_k^- < a_j^- < 0$, and note that analysis of the remaining case $a_j^- \le a_k^- < 0$ is similar. In the event that $|x| \ge |a_k^-|t$, we observe that there is no cancellation between the summands of (4.12) and we have an estimate by

$$C_1 t^{-1}e^{-\frac{(x-a_k^-t)^2}{Lt}}\int_0^{t/2}(1+s)^{-3/4}ds + C_2 t^{-1/2}(1+t)^{-3/4}e^{-\frac{(x-a_k^-t)^2}{Lt}}\int_{t/2}^t (t-s)^{-1/2}ds \le Ct^{-3/4}e^{-\frac{(x-a_k^-t)^2}{Lt}}. \tag{4.29}$$

For $|x| \le |a_k^-|t$, we divide the analysis into cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, use (4.13), for which the summands cancel and we have the estimate (4.14). For the first estimate in (4.14), we can proceed exactly as in (4.29), while for the second, we have an estimate by

$$C_1 t^{-1}(1+|x-a_j^-t|)^{-3/4}\int_0^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds \le Ct^{-1/2}(1+|x-a_jt|)^{-3/4},$$

where the apparent blowup as $t \to 0$ can be removed as in (4.9). For $s \in [t/2, t]$, we observe through (4.12) the estimate (4.15). For the first estimate in (4.15), we proceed as in (4.29), while for the second we have an estimate by

$$C_2 t^{-1/2}|x-a_k^-t|^{-3/4}(1+t)^{-3/4}\int_{t/2}^t (t-s)^{1/4}e^{-\frac{(x-a_j^-(t-s)-a_k^-s)^2}{Mt}}ds \le Ct^{-1/2}|x-a_k^-t|^{-3/4}.$$

*Nonlinearity* $(1+s)^{-1}e^{-\eta|y|}$: We next consider integrals of the form

$$\int_0^t\int_{-\infty}^0 (t-s)^{-1}e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}}e^{-\eta|y|}(1+s)^{-1}dyds. \tag{4.30}$$

First, we observe that in the event $|y| \ge |a_1^-|s$, we have exponential decay in both $y$ and $s$, from which the claimed estimate readily follows. For what remains, then, we focus on the case $|y| \le |a_1^-|s$. In the event that $|x| \ge |a_1^-|t$, we write

$$x-y-a_j^-(t-s) = (x-a_1^-t)-(y-a_1^-s)-(a_j^- - a_1^-)(t-s),$$

for which there is no cancellation between the summands (in the case $y \in [a_1^- s, 0]$), and we immediately arrive at an estimate by

$$Ct^{-1} e^{-\frac{(x-a_1^- t)^2}{Lt}}.$$

In the case $|x| \le |a_1^-| t$, we focus on the critical case $a_j^- < 0$. Here, we observe the estimate

$$e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} e^{-\eta|y|} \le C\Big[e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} e^{-\eta|y|} + e^{-\eta_1|x-a_j^-(t-s)|} e^{-\eta_2|y|}\Big], \tag{4.31}$$

for some constants $\bar{M} > 0$, $\eta_1 > 0$, and $\eta_2 > 0$. For the first estimate in (4.31), we have

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} e^{-\eta|y|} (1+s)^{-1} dy ds.$$

In the event that $|x| \ge |a_j^-| t$, we write

$$x - a_j^-(t-s) = (x - a_j^- t) + a_j^- s, \tag{4.32}$$

for which there is no cancellation between the summands, and we arrive at an estimate by

$$\begin{aligned} &C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_0^{t/2} (1+s)^{-1} ds + C_2 (1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t/2}^{t-1} (t-s)^{-1} ds \\ &\quad + C_3 (1+t)^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_{t-1}^{t} (t-s)^{-1/2} ds \le C t^{-1} \ln(e+t) e^{-\frac{(x-a_j^- t)^2}{Lt}}. \end{aligned} \tag{4.33}$$

For $|x| \le |a_j^-| t$, we divide the analysis into cases, $s \in [0, t/2]$ and $s \in [t/2, t]$. In the case $s \in [0, t/2]$, we observe the cancellation between summands in (4.32), which leads to the estimate

$$e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} \le C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}} (1+s)^{-1} + e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} (1+|x-a_j^- t|)^{-1}\Big].$$

For the first of these last two estimates, we proceed exactly as in (4.33), while for the second, we have an estimate by

$$C_1 t^{-1} (1+|x-a_j^- t|)^{-1} \int_0^{t/2} e^{-\frac{(x-a_j^-(t-s))^2}{\bar{M}(t-s)}} ds \le C t^{-1/2} (1+|x-a_j^- t|)^{-1}.$$

For $s \in [t/2, t]$, we observe that for $|a_j^-|(t-s) \le (1/2)|x|$, we have the estimate

$$(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} \le C(t-s)^{-1/2} e^{-\frac{x^2}{L|x|}} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}},$$

while for $|a_j^-|(t-s) \ge (1/2)|x|$, we have the estimate

$$(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} \le C|x|^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}}.$$

For the second of these last two estimates, we obtain an estimate

$$C_2(1+t)^{-1}|x|^{-1/2} \int_{t/2}^{t} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s))^2}{M(t-s)}} ds \le C(1+t)^{-3/4}|x|^{-1/2}.$$

For the first, we have the same decay in $t$, with exponential decay in $|x|$. For the second estimate in (4.31), we have

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\eta_1|x-a_j^-(t-s)|} e^{-\eta_2|y|} (1+s)^{-1} dyds.$$

In the event that $|x| \ge |a_j^-|t$, we observe that there is no cancellation between the summands of (4.32), so that

$$e^{-\eta_1|x-a_j^-(t-s)|} \le Ce^{-\eta_3|x-a_j^- t|} e^{-\eta_4|a_j^-|s},$$

which leads immediately to an estimate better than $\psi_2$. For $|x| \le |a_j^-|t$, we proceed similarly as with the first estimate in (4.31).

This concludes our analysis of the main case, the first estimate in Lemma 3.4.

*Second estimate of Lemma* 3.4. For the second estimate in Lemma 3.4, we consider in detail the case of nonlinearity $\partial_y(\varphi^i)^2$, for which the other cases are similar. Following the analysis of [33], we divide the integration over $s$ as

$$\begin{aligned} \int_0^t \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y) \partial_y (\varphi^k)^2 dyds &= \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y) \partial_y (\varphi^k)^2 dyds \\ &+ \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y) \partial_y (\varphi^k)^2 dyds \\ &+ \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y) \partial_y (\varphi^k)^2 dyds. \end{aligned} \tag{4.34}$$

For the first integral in (4.34), we focus on the case $y < 0$ and on the convection term of the Green's function, for which we must estimate integrals of the form

$$\int_0^{\sqrt{t}} \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_k^- s)^2}{Ms}} dyds.$$

According to Lemma 6 from [12], we can write (4.28), from which direct integration over $y$ leads to an estimate by

$$Ct^{-1/2} \int_0^{\sqrt{t}} (t-s)^{-1/2} (1+s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds. \qquad (4.35)$$

We have three cases to consider: (1) $a_k^- < 0 < a_j^-$, (2) $a_k^- < a_j^- < 0$, and (3) $a_j^- < a_k^- < 0$; here we point out that the case $a_k^- = a_j^-$ does not arise: the reason is that, according to (3.9), the $(\varphi^k)^2$ term in $\Phi$ occurs only in the $r_j^-$ direction for $j \neq k$. Hence if we consider the detailed description of $S$ in Green's function bounds (2.12), we can see that the case $a_k^- = a_j^-$ does not arise (since $l_j^- r_k^- = 0$). We will focus on the case $a_k^- < a_j^- < 0$, to which the remaining cases are similar. First, in the event that $|x - a_j^- t| \leq C\sqrt{t}$ we can conclude decay of the form $e^{-\frac{(x-a_j^- t)^2}{Mt}}$ by boundedness. In the case $|x - a_j^- t| \geq C\sqrt{t}$, for $C$ sufficiently large, and for $s \in [0, \sqrt{t}]$, we have

$$|x - a_j^-(t-s) - a_k^- s| = |(x - a_j^- t) - (a_k^- - a_j^-)s| \geq c_1 |x - a_j^- t|,$$

for which we can estimate (4.35) for $t > 1$ by

$$C_1 t^{-1} e^{-\frac{(x-a_j^- t)^2}{Lt}} \int_0^{\sqrt{t}} (1+s)^{-1/2} \leq Ct^{-1}(1+\sqrt{t})^{1/2} e^{-\frac{(x-a_j^- t)^2}{Lt}}. \qquad (4.36)$$

For the third integral in (4.34), we can proceed similarly to obtain an estimate by

$$Ct^{-1/4}(1+t)^{-1/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}.$$

In the case of the second integral in (4.34), we closely follow the approach of [19], using the framework of [33]. Defining for some fixed $k$ $\phi(x,t)$ as a Burgers kernel, we have

$$\begin{cases} \phi_t - \beta_k^- \phi_{xx} = -\gamma_k^- (\phi^2)_x & \text{for } t > -1, \\ \phi(x + a_k^-, 1) = m_k \delta_0 & t = -1, \end{cases}$$

for which we have $\varphi^k(x,t) = \phi(x - a_k^- t, t)$, where the $\varphi^k(x,t)$ are as in (1.16). In this notation, the second integral in (4.34) becomes

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} \tilde{G}(x, t-s; y)(\phi(y - a_k^- s, s)^2)_y dy ds. \tag{4.37}$$

The idea of Liu is to take advantage of the observation that time derivatives of the heat kernel decay at the same algebraic rate as two space derivatives, $t^{-3/2}$, and also of the relation between time and space derivatives of $\tilde{G}$ and $\phi$. Intuitively, we can think that we would like to integrate (4.37) by parts in $s$ to put time derivatives on $\tilde{G}$, but in order to facilitate this, we would like the derivatives on $\phi^2$ to be with respect to $s$. Though convecting kernels such as $\tilde{G}$ do not enjoy this property of fast-decaying time derivatives, we observe that under a change of integration variable, the leading-order term in $\tilde{G}$ can be converted into a heat kernel. More precisely, according to Proposition 2.2, the leading-order contribution to the scattering piece of the Green's function is given by $g(x - a_j^- t, t)$, where $g(x,t)$ is the heat kernel

$$g(x,t;y) := ct^{-1/2} e^{-\frac{(x-y)^2}{4\beta_j t}},$$

for a constant $c = r_j^- (l_j^-)^{\mathrm{tr}} / \sqrt{4\pi\beta_j^-}$. Under the change of variables $\xi = y + a_j^-(t-s)$, the second integral in (4.34) becomes, for this leading-order term,

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)_\xi d\xi ds.$$

Following [33], we write for $g = g(x,\tau,\xi)$ and $\phi = \phi(\xi,\tau)$,

$$\begin{aligned}
&\Big(g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)\Big)_s \\
&= -g_\tau(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2) \\
&\quad + (a_j^- - a_k^-) g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)_\xi \\
&\quad\quad + g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)_\tau,
\end{aligned}$$

from which we have

$$\begin{aligned}
&(a_j^- - a_k^-) g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s)^2)_\xi \\
&\quad = \Big(g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)\Big)_s \\
&\quad\quad + g_\tau(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2) \\
&\quad\quad - g(x, t-s; \xi)(\phi(\xi - a_j^-(t-s) - a_k^- s, s)^2)_\tau.
\end{aligned} \tag{4.38}$$

We proceed now by analyzing (under integration) each term on the right-hand side of (4.38). For the first, we have

$$\begin{aligned}&\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}\Big(g(x,t-s;\xi)(\phi(\xi-a_j^-(t-s)-a_k^-s,s)^2)\Big)_s d\xi ds\\&=\int_{-\infty}^{+\infty}g(x,\sqrt{t};\xi)(\phi(\xi-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}),t-\sqrt{t})^2)d\xi\\&+\int_{-\infty}^{+\infty}g(x,t-\sqrt{t};\xi)(\phi(\xi-a_j^-(t-\sqrt{t})-a_k^-\sqrt{t},\sqrt{t})^2)d\xi.\end{aligned} \tag{4.39}$$

For the first integral in (4.39), we have an estimate by

$$C\int_{-\infty}^{+\infty}(\sqrt{t})^{-1/2}e^{-\frac{(x-\xi)^2}{M\sqrt{t}}}(1+(t-\sqrt{t}))^{-1}e^{-\frac{(\xi-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}d\xi.$$

Writing now $\xi=z+a_j^-\sqrt{t}$, we can rewrite this last integral as

$$C\int_{-\infty}^{+\infty}(\sqrt{t})^{-1/2}e^{-\frac{(x-z-a_j^-\sqrt{t})^2}{M\sqrt{t}}}(1+(t-\sqrt{t}))^{-1}e^{-\frac{(z-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}dz.$$

We have, then, according to Lemma 6 of [12], with $t-s$ replaced by $\sqrt{t}$,

$$\begin{aligned}&e^{-\frac{(x-z-a_j^-\sqrt{t})^2}{M\sqrt{t}}}e^{-\frac{(z-a_k^-(t-\sqrt{t}))^2}{M(t-\sqrt{t})}}\\&=e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}}e^{-\frac{t}{M\sqrt{t}(t-\sqrt{t})}(\xi-\frac{xs-(a_j^-+a_k^-)\sqrt{t}(t-\sqrt{t})^2}{t})}.\end{aligned}$$

Upon integration in $\xi$, then, we have an estimate by

$$Ct^{-1/2}(1+(t-\sqrt{t}))^{-1}(t-\sqrt{t})^{1/2}e^{-\frac{(x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t}))^2}{Mt}}.$$

In the event that $|x-a_k^-t|\le C\sqrt{t}$, we have exponential decay $e^{-\frac{(x-a_k^-t)^2}{Mt}}$ from boundedness, while for $|x-a_k^-t|\ge C\sqrt{t}$, $C$ sufficiently large, we have

$$|x-a_j^-\sqrt{t}-a_k^-(t-\sqrt{t})|=|(x-a_k^-t)-(a_j^--a_k^-)\sqrt{t}|\ge c|x-a_k^-t|.$$

In either case, we conclude for $t$ sufficiently large an estimate $Ct^{-1}e^{-\frac{(x-a_k^-t)^2}{Lt}}$. The second integral in (4.39) can be analyzed similarly, and we obtain an estimate of the same form. For the second integral arising from (4.38), we have

$$\int_{\sqrt{t}}^{t-\sqrt{t}}\int_{-\infty}^{+\infty}(t-s)^{-3/2}e^{-\frac{(x-\xi)^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(\xi-a_j^-(t-s)-a_k^-s)^2}{Ms}}d\xi ds.$$

Returning to our original coordinates $\xi = y + a_j^-(t-s)$, we have

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-3/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_k^- s)^2}{Ms}} d\xi ds.$$

Proceeding now exactly as in the derivation of (4.35), we arrive at an estimate by

$$Ct^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-1}(1+s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds. \tag{4.40}$$

As in previous arguments, we will focus on the case $a_k^- < a_j^- < 0$, for which the remaining cases are similar. First, in the event that $|x| \geq |a_k^-|t$, we observe that there is no cancellation between summands in (4.12), and consequently that we can estimate (4.40) for $t$ large enough so that for $\sqrt{t} \leq t/2$

$$\begin{aligned} &C_1 t^{-3/2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{\sqrt{t}}^{t/2} (1+s)^{-1/2} ds \\ &\quad + C_2 t^{-1/2}(1+t)^{-1/2} e^{-\frac{(x-a_k^- t)^2}{Lt}} \int_{t/2}^{t-\sqrt{t}} (t-s)^{-1} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ &\quad \leq Ct^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}}. \end{aligned} \tag{4.41}$$

For $|x| \leq |a_j^-|t$, we observe that there is no cancellation between summands in (4.13), and consequently we can estimate (4.40) similarly as in (4.41) to obtain an estimate by $Ct^{-1}e^{-\frac{(x-a_j^- t)^2}{Lt}}$. For the critical case $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into the cases $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe through (4.13) the inequality

$$\begin{aligned} &e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} (1+s)^{-1/2} \\ &\quad \leq C\Big[ e^{-\frac{(x-a_j^- t)^2}{Lt}} (1+s)^{-1/2} + e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} (1+|x-a_j^- t|)^{-1/2} \Big]. \end{aligned}$$

For the first of these last two estimates, we proceed as in (4.41), while for the second, we have an estimate by

$$\begin{aligned} C_1 t^{-3/2}(1+|x-a_j^- t|)^{-1/2} &\int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ &\leq Ct^{-3/2}(1+t)^{1/2}(1+|x-a_j^- t|)^{-1/2}. \end{aligned}$$

For $s \in [t/2, t-\sqrt{t}]$, we observe through (4.12) the estimate

$$(t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} \leq C\Big[|x-a_k^- t|^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} + (t-s)^{-1/2} e^{-\frac{(x-a_k^- t)^2}{Lt}}\Big].$$

For the second of these last two estimates we can proceed as in (4.41) to determine an estimate by $Ct^{-1} e^{-\frac{(x-a_k^- t)^2}{Lt}}$, while for the first, we have an estimate by

$$C_1 t^{-1/2}(1+t)^{-1/2}|x-a_k^- t|^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (t-s)^{-1/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \leq C(1+t)^{-1/2}|x-a_k^- t|^{-1/2}(\sqrt{t})^{-1/2}.$$

For the final expression on the right-hand side of (4.38), we have integrals of the form

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-\xi)^2}{M(t-s)}} (1+s)^{-2} e^{-\frac{(\xi-a_j^-(t-s)-a_k^- s)^2}{Ms}} d\xi ds.$$

Re-writing in our original variable $\xi = y + a_j^-(t-s)$, we have

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-1/2} e^{-\frac{(x-y-a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-2} e^{-\frac{(y-a_k^- s)^2}{Ms}} d\xi ds.$$

Proceeding now exactly as in the derivation of (4.40), we arrive at an estimate by

$$Ct^{-1/2} \int_{\sqrt{t}}^{t-\sqrt{t}} (1+s)^{-3/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds. \tag{4.42}$$

For the cases $|x| \geq |a_k^-|t$ and $|x| \leq |a_j^-|t$, we can proceed as in (4.41) to arrive at estimates of the form $t^{-1/4}\varphi$, while in the case $|a_j^-|t \leq |x| \leq |a_k^-|t$, we divide the analysis into cases, $s \in [\sqrt{t}, t/2]$ and $s \in [t/2, t-\sqrt{t}]$. For $s \in [\sqrt{t}, t/2]$, we observe through (4.13) the estimate

$$e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} (1+s)^{-3/2} \leq C\Big[e^{-\frac{(x-a_j^- t)^2}{Lt}} (1+s)^{-3/2} + (1+|x-a_j^- t| + t^{1/2})^{-3/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}}\Big].$$

For the first of these last two estimates, we proceed as in (4.41), while for the second we have an estimate by

$$\begin{aligned} Ct^{-1/2}(1+|x-a_j^- t|+t^{1/2})^{-3/2}\int_{\sqrt{t}}^{t/2} e^{-\frac{(x-a_j^-(t-s)-a_k^- s)^2}{Mt}} ds \\ \le C(1+|x-a_j^- t|+t^{1/2})^{-3/2}. \end{aligned}$$

For the case $s \in [t/2, t-\sqrt{t}]$, we similarly arrive at an estimate by $(1+t)^{-1}|x-a_k^- t|^{-1/2}$.

*Excited estimates.* We turn now to the estimates in Lemma (3.4) that involve the *excited* terms $e_i(y,t)$. Observing that in the Lax and overcompressive cases, we have the estimate

$$|\partial_y e_i(y,t)| \le Ct^{-1/2} \sum_{a_k^->0} e^{-\frac{(y+a_k^- t)^2}{Mt}},$$

integrals in the third estimate of Lemma 3.4 take the form

$$\int_0^t \int_{-|a_1^-|s}^0 (t-s)^{-1/2} e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}} \Psi(s,y) dy ds,$$

which differ from those of the previous analysis only by the factor $(t-s)^{-1/2}$. Proceeding almost exactly as in our analysis of the first estimate of Lemma 3.4, we determine an estimate on these integrals of $C(1+t)^{-3/4}$.

*Excited diffusion wave estimates.* The third and fourth estimates in Lemma 3.4 are similar and straightforward to prove, and we consider only the third. As each summand in $\Phi(y,s)$ can be dealt with similarly (see (3.7)–(3.9)), we focus on the nonlinearity $\partial_y(\varphi^k)^2$, for which integration by parts in $y$ and the estimates of Remark 2.6 yield integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1/2} e^{-\frac{(y+a_j^-(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_k^- s)^2}{Ms}} dy ds,$$

with $a_j^- > 0$ (the excited terms correspond to mass convecting into the shock layer) and $a_k < 0$ (diffusion waves carry mass away from the shock layer). According to Lemma 6 of [12], we have the relation

$$e^{-\frac{(y+a_j^-(t-s))^2}{M(t-s)}} e^{-\frac{(y-a_k^- s)^2}{Ms}} = e^{-\frac{(a_j^-(t-s)+a_k^- s)^2}{Mt}} e^{-\frac{t}{Ms(t-s)}(y+\frac{(a_j^-+a_k^-)(t-s)s}{t})^2},$$

from which integration over $y$ leads to an estimate of the form

$$Ct^{-1/2}\int_0^t (1+s)^{-1/2} e^{-\frac{(a_j^-(t-s)+a_k^- s)^2}{Mt}} ds.$$

In either the case $s \in [0, t/\gamma]$ or $s \in [t - t/\gamma, t]$, for $\gamma$ sufficiently large, we have

$$|a_j^-(t-s) + a_k^- s| \geq \eta t,$$

for some $\eta > 0$, through which we have exponential decay in $t$. In the case $s \in [t/\gamma, t - t/\gamma]$, we integrate the kernel in $s$ to obtain an estimate by $C(1+t)^{-1/2}$.

For the final estimate in Lemma 3.4, we have for $y < 0$ integrals of the form

$$\int_0^t \int_{-\infty}^0 (t-s)^{-1} e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}} \Psi(y,s) dy ds,$$

where $a_k^- < 0$. In the case of nonlinearities $(1+s)^{-1/4} s^{-1/2}(\bar{\psi}_1 + \psi_2 + \varphi)$, we can proceed by setting $x = 0$ in the scattering estimates to obtain an estimate by $(1+t)^{-5/4}$. In the case of nonlinearity $(1+s)^{-1} s^{-1/2}(1+|y|)^{-1/2}$, we have the estimate

$$e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}} (1+|y|)^{-1/2} \leq C e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}} (1+|y|+(t-s))^{-1/2},$$

from which we immediately obtain an estimate by

$$C_1 t^{-1/2}(1+t)^{-1/2} \int_0^{t/2} (1+s)^{-1} s^{-1/2} ds$$
$$+ C_2(1+t)^{-1} t^{-1/2} \int_{t/2}^t (t-s)^{-1/2}(1+(t-s))^{-1/2} ds \leq C(1+t)^{-1}.$$

Finally, in the case of nonlinarity $(1+s)^{-1} e^{-\eta|y|}$, we have the estimate

$$e^{-\frac{(y+a_k^-(t-s))^2}{M(t-s)}} e^{-\eta|y|} \leq C e^{-\eta_1(t-s)} e^{-\eta_2|y|},$$

from which we immediately obtain an estimate by

$$C_1 e^{-\frac{\eta_1}{4}t} \int_0^{t/2} e^{-\frac{\eta_1}{2}(t-s)} ds + C_2(1+t)^{-1} \int_{t/2}^t e^{-\eta_1(t-s)} ds \leq C(1+t)^{-1}.$$

This concludes the proof of Lemma 3.4 □

**Proof of Lemma 3.5.** To show (3.26) we need to estimate

$$\int_0^t \int_{-\infty}^{+\infty} \mathcal{R}_j^*(x) \mathcal{O}(e^{-\eta_0(t-s)}) \delta_{x-\bar{a}_j^*(t-s)}(-y) \mathcal{L}_j^{*t}(y) \Upsilon(y,s) dy ds$$
$$\leq C \int_0^t (e^{-\eta_0(t-s)}) |f(-x + \bar{a}_j^*(t-s), s)| ds$$

by the boundedness of $\mathcal{R}_j^*$ and $\mathcal{L}_j^{*t}$. A typical term of $\bar{\psi}_1$ is a term of the form $(1+t)^{-\frac{1}{2}}(1+|x-a_i^- t|)^{-\frac{3}{4}}$ (or with $a_i^-$ replaced by $a_i^+$). Hence for the terms coming from $\bar{\psi}_1$ the above is of the order

$$\int_0^t e^{-\eta_0(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}(1+|x-\bar{a}_j^*(t-s)-a_i^- s|)^{-\frac{3}{4}}ds$$

$$=\int_0^t e^{-\eta_0(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}(1+|x-a_i^- t-(\bar{a}_j^*-a_i^-)(t-s)|)^{-\frac{3}{4}}ds;$$

now, using (4.5), this is smaller than

$$\int_0^t e^{-\eta_0(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}(1+|x-a_i^- t|)^{-\frac{3}{4}}(|1+(\bar{a}_j^*-a_i^-)(t-s)|)^{\frac{3}{4}}ds.$$

Notice that $(\bar{a}_j^*-a_i^-)\le C$. Now we use half of $\eta$ to neutralize $t-s$, and to get

$$(1+|x-a_i^- t|)^{-\frac{3}{4}}\int_0^t e^{-\frac{\eta_0}{2}(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds\le(1+|x-a_i^- t|)^{-\frac{3}{4}}(1+t)^{-1},$$

obviously absorbable in $\bar{\psi}_1+\psi_2+\alpha$. For the terms coming from $\psi_2$, we notice that, by the inequality $a^2+b^2\ge 2ab$,

$$(1+|y-a_i^- s|+\sqrt{s})^{-\frac{3}{2}}\le C(1+s)^{-\frac{3}{8}}(1+|y-a_i^- s|)^{-\frac{3}{4}}.$$

Hence applying the same calculations as for $\bar{\psi}_1$ gives us an estimate of $(1+|x-a_i^- t|)^{-\frac{3}{4}}(1+t)^{-\frac{7}{8}}$. For $\alpha$, we apply the same procedure, and using again (4.5), we have an estimate by

$$C\int_0^t e^{-\eta_0(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{3}{4}}(1+|x-\bar{a}_j^*(t-s)|)^{-\frac{1}{2}}ds$$

$$\le C\int_0^t e^{-\eta_0(t-s)}s^{-\frac{1}{2}}(1+s)^{-\frac{3}{4}}(1+|x|)^{-\frac{1}{2}}(1+|\bar{a}_j^*(t-s)|)^{\frac{1}{2}}ds$$

$$\le C(1+|x|)^{-\frac{1}{2}}\int_0^t e^{-\frac{\eta_0}{2}(t-s)}(1+s)^{-\frac{5}{4}}ds\le C(1+|x|)^{-\frac{1}{2}}(1+t)^{-1}.$$

This finishes the proof of (3.26) and (3.27). The proof of (3.28) is similar, since all the terms in $\Phi$ are smaller than terms looking like $C(1+t)^{-\frac{3}{2}}e^{\frac{-(x-a_i^- t)^2}{Mt}}$, which is bounded by $C(1+t)^{-\frac{3}{4}}(1+|x-a_i^- t|+\sqrt{t})^{-\frac{3}{2}}$.

This finishes the proof of Lemma 3.5. □

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## REFERENCES

[1] J. Alexander, R. Gardner, and C.K.R.T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math., 410 (1990), 167–212.

[2] L. Brin, "Numerical testing of the stability of viscous shock waves," Ph.D. thesis under the direction of K. Zumbrun, Indiana University (1998).

[3] L. Brin, *Numerical testing of the stability of viscous shock waves*, Mat. Contemp., 70 (2001), 1071–1088.

[4] T. Bridges, G. Derks, and G. Gottwald, *Stability and instability of solitary waves of the fifth-order KdV equation: a numerical framework*, Physica D, 172 (2002), 190–216.

[5] L. Brin and K. Zumbrun, *Analytically varying eigenvectors and the stability of viscous shock waves*, Mat. Contemp., 22 (2002), 19–32

[6] J.W. Evans, *Nerve Axon Equations I–IV*, Indiana U. Math. J., 21 (1972), 877–885; 22 (1972), 75–90; 22 (1972), 577–594; 24 (1975), 1169–1190.

[7] H. Freistühler, *Some results on the stability of non-classical shock waves,* J. Partial Diff. Eqs., 11 (1998), 23–38.

[8] H. Freistühler and P. Szmolyan, *Spectral stability of small shock waves,* Arch. Ration. Mech. Anal., 164 (2002), 287–309.

[9] R. Gardner and K. Zumbrun, *The Gap Lemma and geometric criteria for instability of viscous shock profiles*, Comm. Pure Appl. Math., 51 (1998), 797–855.

[10] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws,* Arch. Rational Mech. Anal., 95 (1986), 325–344.

[11] P. Howard, M. Raoofi, and K. Zumbrun, *Asymptotic behavior of perturbed viscous shock profiles*, to appear in J. Hyperbolic Differential Equations, 3 (2006).

[12] P. Howard and K. Zumbrun, *Stability of undercompressive shock profiles,* to appear in J. Differential Equations.

[13] J. Humpherys and K. Zumbrun, *Spectral stability of small amplitude shock profiles for dissipative symmetric hyperbolic–parabolic systems,* Z. Angew. Math. Phys., 53 (2002), 20–34.

[14] C.K.R.T. Jones, *Stability of the traveling wave solution of the FitzHugh–Nagumo system*, Trans. Amer. Math. Soc., 286 (1984), 431–469.

[15] S. Kawashima, "Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics," Thesis, Kyoto University, (1983).

[16] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion,* Comm. Math. Phys., 101 (1985), 97–127.

[17] S. Kawashima, A. Matsumura, and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas,* Proc. Japan Acad. Ser. A Math. Sci., 62 (1986), 249–252.

[18] T. Kapitula and B. Sandstede, *Stability of bright solitary-wave solutions to perturbed nonlinear Schrodinger equations*, Physica D, 124 (1998), 58–103.

[19] T.P. Liu, *Interaction of nonlinear hyperbolic waves*, Nonlinear analysis (Taipei, 1989), 171–183.

[20] T.P. Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Memoirs of American Math. Soc., 56 (1985).

[21] T.P. Liu, *Pointwise convergence to shock waves for viscous conservation laws*, Comm. Pure Appl. Math., 50 (1997), 1113–1182.

[22] T.P. Liu and K. Zumbrun, *On nonlinear stability of general undercompressive viscous shock waves,* Comm. Math. Phys., 174 (1995), 319–345.

[23] A. Majda, *The stability of multi-dimensional shock fronts – a new problem for linear hyperbolic equations,* Mem. Amer. Math. Soc., 275 (1983).

[24] A. Majda, *The existence of multi-dimensional shock fronts,* Mem. Amer. Math. Soc., 281 (1983).

[25] A. Majda, "Compressible fluid flow and systems of conservation laws in several space variables," Springer-Verlag, New York (1984), viii+ 159 pp.

[26] T.-P. Liu and Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic–parabolic systems of conservation laws*, American Math. Soc. memoirs, 599 (1997).

[27] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks,* Indiana Univ. Math. J., 51 (2002), 773–904.

[28] C. Mascia and K. Zumbrun, *Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems*, Arch. Ration. Mech. Anal., 172 (2004), 93–131.

[29] C. Mascia and K. Zumbrun, *Pointwise Green function bounds for shock profiles with degenerate viscosity,* Arch. Ration. Mech. Anal., 169 (2003), 177–263.

[30] C. Mascia and K. Zumbrun, *Stability of large-amplitude viscous shock profiles of hyperbolic–parabolic systems.* Arch. Rational Mech. Anal., 172 (2004), 93–131.

[31] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math., 2 (1985), 17–25.

[32] R. Plaza and K. Zumbrun, *An Evans function approach to spectral stability of small-amplitude viscous shock profiles,* Discrete Contin. Dyn. Syst.. 10 (2004), 885–924.

[33] M. Raoofi, $L^p$ *Asymptotic behavior of perturbed viscous shock profiles,* J. Hyperbolic Differ. Equ., 2 (2005), 595–644.

[34] M. Slemrod, *Admissibility criteria for propagating phase boundaries in a van der Waals fluid*, Arch. Rational Mech. Anal., 81 (1983), 301–315.

[35] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves,* Arch. Rat. Mech. Anal., 122 (1993), 53–103.

[36] D. Serre and K. Zumbrun *Viscous and inviscid stability of multidimensional planar shock fronts,* Indiana Univ. Math. J., 48 (1999), 937–992.

[37] Y. Zeng, $L^1$ *asymptotic behavior of compressible, isentropic, viscous* 1*-d flow,* Comm. Pure Appl. Math., 47 (1994), 1053–1092.

[38] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhuser Boston, Boston, MA, 2001.

[39] K. Zumbrun, *Planar stability criteria for viscous shock waves of systems with real viscosity,* CIME summer school notes, preprint (2004).

[40] K. Zumbrun, *Stability of large amplitude shock waves of compressible Navier–Stokes equation,* to appear in *Handbook of Fluid Mechanics, Volume IV*, (2004).

[41] K. Zumbrun, *Dynamical stability of phase transitions in the p-system with viscosity-capillarity,* SIAM J. Appl. Math. 60 (2000), 1913–1924.

[42] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*. Indiana Mathematics Journal, 47 (1998), 741–871.

[43] K. Zumbrun and D. Serre, *Viscous and inviscid stability of multidimensional planar shock fronts,* Indiana Univ. Math. J., 48 (1999), 937–992.