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A FAST BLOWUP SOLUTION TO AN ELLIPTIC-PARABOLIC SYSTEM RELATED TO CHEMOTAXIS

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Abstract. We consider radial blowup solutions to an elliptic-parabolic system in N-dimensional Euclidean space. The system is introduced to describe several phenomena, for example, motion of bacteria by chemotaxis and equilibrium of self-attracting clusters. In the case where $N \geq 3$, we can find positive and radial backward self-similar solutions which blow up in finite time. In the present paper, in the case where $N \geq 11$, we show the existence of a radial blowup solution whose blowup speed is faster than the one of backward self-similar solutions, by using so-called asymptotic matched expansion techniques.

1. Introduction

In this paper, we consider the system

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \mathbf{R}^N \times (0,T), \\ 0 = \Delta v + u & \text{in } \mathbf{R}^N \times (0,T), \end{cases} \tag{1.1}$$

where $N = 1, 2, 3, \cdots$.

The system (1.1) was introduced as a simplified system of

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v), \\ v_t = \Delta v - v + u \end{cases} \tag{1.2}$$

(see [10]). The system (1.2) has been introduced as a model for several biological problems (see [12]) and physical problems (see [2]).

In a biological problem, the system (1.2) describes the aggregation of cellular slime molds, owing to the motion of the cells moving towards higher concentrations of a chemical substance produced by themselves. In the model, $v(x,t)$ represents the concentration of the chemical substance, and $u(x,t)$

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represents the density of cells. We think that the blowup of solutions corresponds to the aggregation of cells. The systems (1.1) and (1.2) have blowup solutions (see [7, 18]).

Here and henceforth, we say that a solution (u, v) blows up at $x = q$ and $t = T$, if there exists a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T)$ such that $\lim_{n\to\infty}(x_n, t_n) = (q, T)$ and $\lim_{n\to\infty} |u(x_n, t_n)| = \infty$. Then, we say that the point q and the time T are a blowup point and blowup time, respectively.

The system (1.1) has positive and radial backward self-similar solutions (see [10, 3, 18]). Here, we say that (u, v) is a backward self-similar solution to (1.1), if (u, v) is a solution to (1.1) and if u has the form $u(x, t) = (T - t)^{-1}\overline{u}(x/\sqrt{T - t})$ for any $T > 0$ with a function $\overline{u}$. We call the function $\overline{u}$ a profile function of the solution (u, v). Since each profile function found in [10, 3, 18] has a limit $\overline{c} = \lim_{|x|\to\infty} |x|^2\overline{u}(x) \in (0, \infty)$, the corresponding self-similar solution (u, v) blows up at $x = 0$ and $t = T$ and satisfies

$$\lim_{t\to T}(T - t)\text{ess.sup}_{x\in\mathbf{R}^N}|u(x, t)| = \sup_{x\in\mathbf{R}^N} \overline{u}(x) \in (0, \infty) \tag{1.3}$$

and

$$u(\cdot, t) \to \frac{\overline{c}}{|x|^2} \quad \text{as } t \to T \quad \text{in the sense of measures.} \tag{1.4}$$

We say that the blowup is Type I, if the blowup solution (u, v) to (1.1), whose blowup time is T, satisfies

$$\limsup_{t\to T}(T - t)\text{ess.sup}_{x\in\mathbf{R}^N}|u(x, t)| < \infty.$$

In the case where the domain Ω is $\{x \in \mathbf{R}^2 : |x| < R\}$ $(0 < R < \infty)$, there exists a radial blowup solution (u, v) to (1.2) in Ω, which has a delta function singularity at $x = 0$ and $t = T$ $(< \infty)$, and satisfies

$$\lim_{t\to T}(T - t)\text{ess.sup}_{x\in\Omega}|u(x, t)| = \infty \tag{1.5}$$

(see [7]). We say that the blowup is Type II, if the blowup solution (u, v) to (1.1), whose blowup time is T, satisfies

$$\limsup_{t\to T}(T - t)\text{ess.sup}_{x\in\mathbf{R}^N}|u(x, t)| = \infty.$$

In this paper, we show that the system (1.1) has a radial blowup solution satisfying (1.4) and (1.5) for $N \geq 11$. More precisely, we show the following. For $N \geq 11$, let us put

$$\nu = \frac{1}{4}\left[-(N - 2) + \sqrt{(N - 10)(N - 2)}\right] \in [-3/2, -1),$$

$$\alpha = 2\nu + \frac{N-2}{2} = \frac{1}{2}\sqrt{(N-10)(N-2)}$$

and $\eta = (2+\nu)/(-2\nu+1)$.

Theorem 1. *For any $0 < \varepsilon \ll 1$ and $K \gg 1$, there exists a radial solution (u, v) to (1.1) satisfying the following.*

(i) It holds that

$$C_1(T-t)^{-1+((\nu+2)/\nu)} \le \max_{0 \le r \le K(T-t)^{(1+2\eta)/2}} u(r,t) \le C_2(T-t)^{-(1+3\eta)}$$

for $0 < t < T$ with some positive constants C_1 and C_2.

In particular, it holds that

$$C_1(T-t)^{-1+((\nu+2)/\nu)} \le u(0,t) \le C_3(T-t)^{-(1+3\eta)} \quad \text{for } 0 < t < T$$

with a positive constant C_3.

(ii) For any positive constant C_4, it holds that

$$\left| u(r,t) - \frac{2(N-2)}{r^2} \right| \le \frac{C_5}{r^2}(T-t)^{\eta}$$
$$\text{for } K(T-t)^{(2\eta+1)/2} \le r \le C_4(T-t)^{1/2} \text{ and } 0 < T-t \ll 1$$

with a positive constant C_5.

(iii) It holds that

$$\left| u(r,t) - \frac{2(N-2)}{r^2} \right| \le \frac{C_6\varepsilon}{r^2} \quad \text{for } K(T-t)^{(2\eta+1)/2} < r \quad \text{and } 0 < T-t \ll 1$$

with a positive constant C_6, which is independent of ε.

In order to find that solution, we use an argument similar to the one in [8, 14]. In [8], Herrero and Velázquez showed the existence of a radial solution u to

$$u_t = \Delta u + u^p \quad \text{in } \mathbf{R}^N \times (0,T) \tag{1.6}$$

exhibiting Type II blowup for $p > (N - 2\sqrt{N-1})/(N-4-2\sqrt{N-1})$ and $N \ge 11$. In [14], Mizoguchi showed a result, which is essentially the same as that in [8], by using a method which is a slight modification of the one in [8].

That is to say, the global strategy of [8, 14] and the one of this paper are the same. That global strategy is as follows.

In the present paper, we introduce self-similar variables

$$\Psi(y,\tau) = M(r,t), \tag{1.7}$$

in order to find the solution (u,v) to (1.1) blowing up at $(x,t)=(0,T)$, where

$$M(r,t)=\frac{1}{\omega_N r^{N-2}}\int_{|x|<r}u(x,t)dx, \tag{1.8}$$

$$\tau=-\log(T-t),\quad y=x/\sqrt{T-t},\quad \tau_0=-\log T$$

and ω_N is the area of a unit sphere in $\mathbf{R}^N$. Then, Ψ satisfies

$$\begin{aligned}\mathcal{M}(\Psi)=\frac{\partial\Psi}{\partial\tau}-\frac{\partial^2\Psi}{\partial y^2}-\Big(\frac{N-3}{y}-\frac{y}{2}\Big)\frac{\partial\Psi}{\partial y}+\frac{2(N-2)}{y^2}\Psi\\ -\frac{\Psi}{y^2}\Big\{y\frac{\partial\Psi}{\partial y}+(N-2)\Psi\Big\}=0\quad\text{in }(0,\infty)\times[\tau_0,\infty)\end{aligned} \tag{1.9}$$

and

$$\Psi(0,\cdot)=0\quad\text{in }[\tau_0,\infty). \tag{1.10}$$

In order to get a blowup solution to (1.1), we consider the solution Ψ to (1.9) and (1.10) satisfying $\lim_{\tau\to\infty}\Psi(\cdot,\tau)=\Psi_\infty\equiv 2$. Here, Ψ_∞ is the singular stationary solution to (1.9) and (1.10). Then, we consider $\phi(y,\tau)=\Psi(y,\tau)-\Psi_\infty(y)$ satisfying

$$\begin{aligned}\mathcal{N}(\phi)&=\mathcal{M}(\Psi)=\mathcal{M}(\phi+\Psi_\infty)\\ &=\frac{\partial\phi}{\partial\tau}+A\phi-F(\cdot,\phi)=0\quad\text{in }(0,\infty)\times[\tau_0,\infty)\end{aligned} \tag{1.11}$$

and

$$\phi(0,\cdot)=-2\quad\text{in }[\tau_0,\infty), \tag{1.12}$$

where

$$A\phi=-\frac{\partial^2\phi}{\partial y^2}-\Big(\frac{N-1}{y}-\frac{y}{2}\Big)\frac{\partial\phi}{\partial y}-\frac{2(N-2)}{y^2}\phi \tag{1.13}$$

and

$$F(y,\phi)=\frac{(N-2)}{y^2}\phi^2+\frac{\phi}{y}\frac{\partial\phi}{\partial y}.$$

The Friedrichs' extension of A is a self-adjoint operator in a suitable function space. Letting the eigenvalues and the eigenfunctions be $\{\lambda_j\}_{j\geq 0}$ and $\{\varphi_j\}_{j\geq 0}$, respectively, the solution ϕ to (1.11) and (1.12) has a representation

$$\phi(y,\tau)=\sum_{j=0}^{\infty}a_j(\tau)\varphi_j(y).$$

Since we have $\lambda_j>0$ for $j\geq 2$ (see Lemma 2.1), we can find a solution ϕ satisfying $\phi(\cdot,\tau)\to 0$ as $\tau\to\infty$ in the region $y\geq Ke^{-\eta\tau}$. On the other hand, we construct the solution ϕ in the region $y\leq Ke^{-\eta\tau}$, by using a rescaled stationary solution, a rescaled self-similar solution and a comparison theorem. Then, we get the solution ϕ in $(0,\infty)$ by showing that one can match the solution in the region $y \le Ke^{-\eta\tau}$ with the solution in the region $y \ge Ke^{-\eta\tau}$.

That is the global strategy of the present paper, which is the same as the one of [8, 14].

However, in [8, 14], $w(y,\tau) = (T-t)^{1/(p-1)}u(x,t)$ and $q(y,\tau) = w(y,\tau) - w_\infty(y)$ are investigated, where u is a solution to (1.6) and w_∞ is the singular stationary solution to (1.6). Then, q satisfies

$$q_\tau + Bq - h(q) = 0, \tag{1.14}$$

where

$$Bq = -\Delta q + \frac{y}{2}\cdot\nabla q - pw_\infty^{p-1}q + \frac{1}{p-1}q$$

and

$$h(q) = (q+w_\infty)^p - w_\infty^p - pw_\infty^{p-1}q.$$

Estimates of the solution q to (1.14) follow from the representation

$$q(\cdot,\tau) = e^{-(\tau-\tau_0)B}q(\cdot,\tau_0) + \int_{\tau_0}^{\tau} e^{-(\tau-\tilde\tau)B}h(q(\cdot,\tilde\tau))d\tilde\tau$$

and estimates of the kernel of $e^{-\tau B}$. On the other hand, estimates of the solution ϕ to (1.11) follow from the representation

$$\phi(\cdot,\tau) = e^{-(\tau-\tau_0)A}\phi(\cdot,\tau_0) + \int_{\tau_0}^{\tau} e^{-(\tau-\tilde\tau)A}F(\cdot,\phi(\cdot,\tilde\tau))d\tilde\tau$$

and estimates of the kernel of $e^{-\tau A}$ and its differentiation, since F includes the term ϕ_y. Moreover, we must estimate the differentiation of ϕ in order to get estimates of the solution (u,v) to (1.1). Then, our proof is more complicated than the one in [8, 14].

We conclude this introduction by describing the plan of this paper.

In Section 2, we will investigate some properties of functional spaces, the operator A in (1.13), a stationary solution and a self-similar solution. In Section 3, we will show Theorem 1 except the proof of a key proposition. Section 4 will be devoted to showing the key proposition. Section 5 will be devoted to the proofs of some technical lemmas.

2. Preliminary

Henceforth, we consider only radial solutions to (1.1) with $N \ge 11$.

First, we shall explain the definition, existence and uniqueness of the solution (u,v) to (1.1) in the space $L^\infty(\mathbf{R}^N;1)\times L^\infty(\mathbf{R}^N;-1/2)$, where

$$L^\infty(\mathbf{R}^N;p)=\left\{f\in L^\infty(\mathbf{R}^N):\mathrm{ess.sup}_{x\in\mathbf{R}^N}(1+|x|^2)^p|f(x)|<\infty\right\}$$

for $p\in\mathbf{R}$.

For any radial function $u_0\in L^\infty(\mathbf{R}^N;1)$, there exists the unique function $u\in C([0,T);L^\infty(\mathbf{R}^N;1))$, which is radial in x for each $t\in[0,T)$, satisfying

$$\begin{aligned}u(x,t)=&\int_{\mathbf{R}^N}\mathcal{G}(x-\tilde{x},t)u_0(\tilde{x})d\tilde{x} \\ &-\int_0^t\int_{\mathbf{R}^N}\Big[\nabla_{\tilde{x}}\mathcal{G}(x-\tilde{x},t-\tilde{t})\cdot\frac{\tilde{x}}{\omega_N|\tilde{x}|^N}\int_{|\hat{x}|\le|\tilde{x}|}u(\hat{x},\tilde{t})d\hat{x}\Big]u(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t} \\ &\text{in } \mathbf{R}^N\times[0,T)\end{aligned}\tag{2.1}$$

with a positive constant T, where $\mathcal{G}$ is the Gauss kernel of $\partial_t-\Delta$ in $\mathbf{R}^N$.

Then, we define $v\in C([0,T);L^\infty(\mathbf{R}^N;-1/2))$ as

$$v(x,t)=C-\int_0^{|x|}\frac{1}{\omega_N r^{N-1}}\Big(\int_{|\tilde{x}|\le r}u(\tilde{x},t)d\tilde{x}\Big)dr,\tag{2.2}$$

where C is an arbitrary constant.

Therefore, for each radial function $u_0\in L^\infty(\mathbf{R}^N;1)$ we define the solution to (1.1) with $u(\cdot,0)=u_0$ as $(u,v)\in C([0,T);L^\infty(\mathbf{R}^N;1)\times L^\infty(\mathbf{R}^N;-1/2))$ satisfying (2.1) and (2.2). Moreover, it holds that $u\ge 0$ in $\mathbf{R}^N\times[0,T)$, if $u_0\ge 0$ in $\mathbf{R}^N$.

Let (u,v) be a radial solution to (1.1) in $\mathbf{R}^N\times(0,T)$ with $T\in(0,\infty)$.

The function M in (1.8) satisfies

$$\begin{aligned}\frac{\partial M}{\partial t} =& \frac{\partial^2 M}{\partial r^2}+\frac{(N-3)}{r}\frac{\partial M}{\partial r}-\frac{2(N-2)}{r^2}M \\ &+\frac{M}{r^2}\Big\{r\frac{\partial M}{\partial r}+(N-2)M\Big\} \quad \text{in } (0,\infty)\times[0,T)\end{aligned}\tag{2.3}$$

and $M(0,\cdot)=0$ in $[0,T)$. Let

$$L^2_w=\Big\{f\in L^2_{loc}((0,\infty)):\int_0^\infty|f(y)|^2y^{N-1}e^{-y^2/4}dy<\infty\Big\}.$$

For any f, $g\in L^2_w$, put

$$\langle f,g\rangle=\int_0^\infty f(y)g(y)y^{N-1}e^{-y^2/4}dy,\quad \|f\|=\sqrt{\langle f,f\rangle}$$

and

$$H_w^1 = \left\{ f \in H_{loc}^1((0,\infty)) : \|f\|_1 = \sqrt{\|f\|^2 + \|f_y\|^2} < \infty \right\},$$

and let H_w^{-1} be the dual space of H_w^1 with respect to $\langle \cdot, \cdot \rangle$. Then, the norm is defined as $\| \cdot \|_{-1} = \sup\{\langle \cdot, f \rangle : \|f\|_1 \leq 1\}$.

We define the domain of A in (1.13) as $D(A) = \{f \in H_w^1 : \ Af \in L_w^2\}$.

Here and henceforth, we denote the Friedrichs' extension of A as A. Then, the operator A is a self-adjoint operator in L_w^2.

Let $\Gamma(\cdot)$ be the gamma function, and let $L_j^\alpha(y)$ be the associated Laguerre polynomials satisfying $L_j^\alpha(0) > 0$ (see [1]). Putting

$$\varphi_j(y) = \sqrt{\frac{j!}{2^{N-1}\Gamma(j+\alpha+1)}} \left(\frac{y^2}{4}\right)^\nu L_j^\alpha(y^2/4) \quad \text{for } j = 0, 1, 2, \cdots, \tag{2.4}$$

it holds that

$$\frac{d^2\varphi_j}{dy^2} + \left(\frac{N-1}{y} - \frac{y}{2}\right)\frac{d\varphi_j}{dy} + \frac{2(N-2)}{y^2}\varphi_j + (j+\nu)\varphi_j = 0$$

and that there exist $\varphi_{0j} = \lim_{y\to 0} y^{-2\nu}\varphi_j(y) > 0$ and $\lim_{y\to\infty} y^{-2(j+\nu)}\varphi_j(y)$.

The following lemma is an immediate conclusion of properties of the associated Laguerre polynomials. Thus, we omit the proof.

Lemma 2.1. *$\{\varphi_j\}_{j=0,1,2,\cdots}$ is a complete orthonormal basis in L_w^2. For each $j = 0, 1, 2, \cdots$, φ_j satisfies $A\varphi_j = \lambda_j \varphi_j$ with $\lambda_j = j + \nu$.*

The following lemma is an estimate similar to the one in [8, 14].

Lemma 2.2. *For $g \in H_w^1$, it holds that*

$$\int_0^\infty \frac{1}{y^2}|g(y)|^2 y^{N-1} e^{-y^2/4} dy \quad \leq \quad \frac{4\|g_y\|^2}{(N-2)^2} + \frac{\|g\|^2}{(N-2)}.$$

The proof of this lemma is found in Section 5.1.

Let $\tilde{A} : H_w^{-1} \to H_w^{-1}$ be the operator with domain $D(\tilde{A}) = H_w^1$ such that $\tilde{A}\varphi_j = \lambda_j \varphi_j$, where φ_j is the function in (2.4).

The following lemma can be shown, by using an argument similar to the proof of [14, Lemma 3.1]. Thus, we omit the proof.

Lemma 2.3. *Let $0 < \beta < (N+2)/2$. If $g \in L_{loc}^1((0,\infty))$ satisfies*

$$g(y) \leq C(y^{-\beta} + 1) \quad \text{for } y > 0$$

with some constant $C > 0$, then g can be regarded as an element of H_w^{-1}.

For $g,\ f \in H^1_w$, it holds that

$$\tilde{A}g = -\frac{d^2 g}{dy^2} - \Big(\frac{N-1}{y} - \frac{y}{2}\Big)\frac{dg}{dy} - \frac{2(N-2)}{y^2} g \qquad \text{in } H^{-1}_w$$

and that

$$\langle \tilde{A}g, f\rangle = \int_0^\infty \Big(\frac{dg}{dy}\cdot\frac{df}{dy} - \frac{2(N-2)}{y^2} g\cdot f\Big) y^{N-1} e^{-y^2/4} dy. \tag{2.5}$$

It follows from $N \geq 11$, Lemmas 2.2 and 2.3 that

$$\langle \tilde{A}g, g\rangle + \Big(3 - \frac{8}{N-2}\Big)\|g\|^2 \geq \Big(\frac{N-10}{N-2}\Big)\|g\|_1^2 \qquad \text{for } g \in H^1_w. \tag{2.6}$$

We can find a stationary solution Ψ_{ST} to (2.3) satisfying

$$\Psi_{ST}(0) = \Psi'_{ST}(0) = 0, \qquad \lim_{r\to\infty} \Psi_{ST}(r) = 2,$$
$$0 < \Psi_{ST}(r) < 2 \qquad \text{and} \qquad \Psi'_{ST}(r) > 0$$

(see [2]).

Here and henceforth, we fix a stationary solution. The stationary solution satisfies the following property.

Lemma 2.4. *It holds that*

$$\Psi_{ST}(r) = 2 - c_{ST}(1+o(1))r^{2\nu} \qquad \text{as } r \to \infty$$

and that

$$\Psi'_{ST}(r) = -2\nu c_{ST}(1+o(1))r^{2\nu-1} \qquad \text{as } r\to\infty$$

with a positive constant c_{ST}.

The proof of this lemma is found in Section 5.2.

For $K > 0$, let us put

$$\Psi_1(y,\tau) = \left(1 + \tilde{c}_{ST}K^{-2}e^{-\eta\tau}\right)\Psi_{ST}\left(K^{-(\nu+1)/\nu}e^{-\lambda_2\tau/(2\nu)}y\right),$$

where $\tilde{c}_{ST} = \min(c_{ST}, 1)/8$ and c_{ST} is the constant in Lemma 2.4.

Putting

$$\Psi_{SE}(y) = \frac{4y^2}{2(N-2)+y^2},$$

Ψ_{SE} satisfies (1.9). For $K > 0$, let us put

$$\Psi_2(y,\tau) = \frac{1}{2}\left(1 + 2K^{2\nu-2}e^{-\eta\tau}\right)\Psi_{SE}\left(K^{-\nu}e^{3\eta\tau/2}y\right).$$

Lemma 2.5. *It holds that*

$$\Psi_1(y,\tau) < \Psi_2(y,\tau) < 2 \quad \text{for } y \in (0, Ke^{-\eta\tau}] \text{ and } \tau \geq \tau_0,$$

if $K \gg 1$ *and* $\tau_0 \gg 1$.

The proof of this lemma is found in Section 5.3.

3. Proof of Theorem 1

In this section, we shall prove Theorem 1. Recall that $N \geq 11$, $\nu = \{-(N-2)+\sqrt{(N-10)(N-2)}\}/4 \in [-3/2,-1)$ and that $\eta = \lambda_2/(-2\nu+1)$.

Let

$$\sigma = \frac{1}{2} - \frac{1}{6(2\lambda_2+1)}.$$

Theorem 1 follows from the following proposition. First, we shall prove the following proposition.

Proposition 3.1. *For any* $0 < \varepsilon \ll 1$, $K \gg 1$. *There exists a solution* Ψ *to* (1.7) *and* (1.8) *satisfying the following.*
(*i*) *It holds that*

$$\Psi_1(y,\tau) < \Psi(y,\tau) < \Psi_2(y,\tau) \quad \text{for } 0 < y < Ke^{-\eta\tau} \quad \text{and} \quad \tau \geq \tau_0.$$

(*ii*) *It holds that*

$$\left|\Psi(y,\tau) - 2 + e^{-\lambda_2\tau}\varphi_2(y)\right| \leq \varepsilon e^{-\lambda_2\tau}\left(y^{2\nu} + y^{2\lambda_2}\right) \quad \text{for } Ke^{-\eta\tau} \leq y \leq e^{\sigma\tau} \text{ and } \tau \geq \tau_0.$$

(*iii*) *It holds that*

$$|\Psi(y,\tau) - 2| < \varepsilon \quad \text{for } y > e^{\sigma\tau} \text{ and } \tau \geq \tau_0.$$

For $\tau_1 \geq \tau_0$ and $\theta \in (0,1]$, we define $\mathcal{A}(\tau_0,\tau_1;\theta)$ as the set of functions $h \in C([\tau_0,\tau_1]; L^\infty((0,\infty)))$ satisfying

$$\left|h(y,\tau) - \Psi_\infty(y) + e^{-\lambda_2\tau}\varphi_2(y)\right| < \theta\varepsilon e^{-\lambda_2\tau}\left(y^{2\nu} + y^{2\lambda_2}\right)$$
$$\text{for } y \in [Ke^{-\eta\tau}, e^{\sigma\tau}] \quad \text{and} \quad \tau \in [\tau_0,\tau_1].$$

Lemma 3.1. *Let* $K \gg \tilde{K} \gg 1$ *and* $\tau_0 \gg 1$. *For* $\tau \geq \tau_0$, *it holds that*

$$\Psi_1(ke^{-\eta\tau},\tau) < 2 - e^{-\lambda_2\tau}\varphi_2(ke^{-\eta\tau}) + \sum_{j=0}^{1} \tilde{d}_j e^{-\lambda_2\tau}\varphi_j(ke^{-\eta\tau}) < \Psi_2(ke^{-\eta\tau},\tau)$$

for $k \in [\tilde{K}, K]$ *and* $|\tilde{d}_j| \leq \varphi_{00}/(4\varphi_{0j})$ $(j = 0,1)$.

Proof. If $\tau_0 \gg 1$, it holds that

$$\Psi_1(ke^{-\eta\tau}, \tau) < 2 - \frac{c_{ST}}{2k^2} e^{-\eta\tau}, \tag{3.1}$$

$$\Psi_2(ke^{-\eta\tau}, \tau) > 2 - \frac{4(N-2)}{K^{-2\nu}k^2} e^{-\eta\tau} > 2 - \frac{4(N-2)}{k^{-2\nu+2}} e^{-\eta\tau} \tag{3.2}$$

and that for $j = 0, 1, 2$

$$e^{-\lambda_2\tau}\varphi_j(ke^{-\eta\tau}) = \varphi_{0j}k^{2\nu}e^{-\eta\tau} + O(e^{-3\eta\tau}) > 0.$$

From those, we get this lemma. □

From Lemma 3.1, we obtain the existence of a constant $\theta_0 \in (0,1)$ satisfying

$$\begin{aligned} &\theta_0\Psi_1(\tilde{K}e^{-\eta\tau_0}, \tau_0) + (1-\theta_0)\Psi_2(\tilde{K}e^{-\eta\tau_0}, \tau_0) \\ &= \ 2 - e^{-\lambda_2\tau_0}\varphi_2(\tilde{K}e^{-\eta\tau_0}) + \sum_{j=0}^{1} d_j e^{-\lambda_2\tau_0}\varphi_j(\tilde{K}e^{-\eta\tau_0}). \end{aligned} \tag{3.3}$$

For $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$, let us put $\tilde{\sigma} = \frac{1}{2}\left(\frac{1}{2} + \sigma\right)$ and

$$\phi_0(y) = \phi(y, \tau_0) = \sum_{j=0}^{1} d_j\varphi_j(y) - e^{-\lambda_2\tau_0}\tilde{\varphi}_2(y), \tag{3.4}$$

where d_j $(j = 0, 1)$ and $\tilde{\varphi}_2$ satisfy the following:

($\phi_0$1) $\displaystyle\sum_{j=0}^{1} |d_j| < \varepsilon\theta_1 e^{-\lambda_2\tau_0}$, where

$$\theta_1 = \left\{ \max\left(\sup_{0<y<\infty} \frac{|\varphi_0(y)|}{y^{2\nu} + y^{2\lambda_2}}, \sup_{0<y<\infty} \frac{|\varphi_1(y)|}{y^{2\nu} + y^{2\lambda_2}}, 1 \right) + 1 \right\}^{-1}.$$

($\phi_0$2) For $y \in (0, \tilde{K}e^{-\eta\tau_0}]$

$$\tilde{\varphi}_2(y) = e^{\lambda_2\tau_0}\Big\{\Psi_\infty(y) - [\theta_0\Psi_1(y,\tau_0) + (1-\theta_0)\Psi_2(y,\tau_0)] + \sum_{j=0}^{1} d_j\varphi_j(y)\Big\},$$

where θ_0 is the constant in (3.3).

($\phi_0$3) For $y \in [\tilde{K}e^{-\eta\tau_0}, e^{\tilde{\sigma}\tau_0}]$, $\tilde{\varphi}_2(y) = \varphi_2(y)$.

($\phi_0$4) For $y \geq e^{\tilde{\sigma}\tau_0}$,

$$\tilde{\varphi}_2(y) = e^{\lambda_2\tau_0}\Big(\sum_{j=0}^{1} d_j\varphi_j(y) - \phi_0(e^{\tilde{\sigma}\tau_0}) \Big)$$

$$= \quad e^{\lambda_2\tau_0}\Big[\sum_{j=0}^{1} d_j\varphi_j(y) - \Big(\sum_{j=0}^{1} d_j\varphi_j(e^{\tilde\sigma\tau_0}) - e^{-\lambda_2\tau_0}\tilde\varphi_2(e^{\tilde\sigma\tau_0})\Big)\Big].$$

Let ϕ_0 be the function defined by (3.4), ($\phi_0$1), ($\phi_0$2), ($\phi_0$3) and ($\phi_0$4), $\Psi(y,\tau) = \Psi(y,\tau;d)$ be the solution to (1.9) and (1.10) and let $\phi(y,\tau) = \phi(y,\tau;d) = \Psi(y,\tau) - \Psi_\infty(y)$.

The solution ϕ in $C([\tau_0,\infty);L^\infty(0,\infty))$ to (1.11) and (1.12) is unique, and the radial function u_0 corresponding to ϕ_0 is in $L^\infty(\mathbf{R}^N;1)$.

Then, the radial solution (u,v) to (1.1) in $C([0,T);L^\infty(\mathbf{R}^N;1)\times L^\infty(\mathbf{R}^N;-1/2))$ is equivalent to the solution ϕ to (1.11) and (1.12) in $C([\tau_0,\infty);L^\infty(0,\infty))$ with the initial function ϕ_0 defined by (3.4), ($\phi_0$1), ($\phi_0$2), ($\phi_0$3) and ($\phi_0$4). Moreover, it follows from the positivity of u that $\phi > -2$ in $(0,\infty)\times[\tau_0,\infty)$.

We define $\mathcal{U}(\tau_0,\tau_1)$ as the set of $d = (d_0,d_1) \in \mathbf{R}^2$ satisfying $\Psi \in \mathcal{A}(\tau_0,\tau_1,1)$, where $\Psi(\cdot,\tau_0) = \Psi_\infty + \phi_0$.

We define $P(d;\tau_0,\tau_1) = (p_0,p_1)$ as $p_j = \langle\phi(\cdot,\tau_1),\varphi_j\rangle$ $(j = 0,1)$. The regularity and continuity of the solution ϕ to (1.11) and (1.12) in $L^\infty((0,\infty))$ with respect to d in ($\phi_0$1) follows from the regularity and continuity of the radial solution (u,v) to (1.1) with respect to the radial initial function u_0 in $L^\infty(\mathbf{R}^N;1)$. Thus, P is continuous with respect to d, τ_0 and τ_1.

We define

$$\widehat{\mathcal{U}(\tau_0,\tau_1)} = \Big\{d\in\mathbf{R}^2 : \sum_{j=0}^{1}|d_j| \le \varepsilon\theta_1 e^{-\lambda_2\tau_0},\ \Psi\in\overline{\mathcal{A}(\tau_0,\tau_1;1)}\Big\}.$$

In general, it holds that $\widehat{\mathcal{U}(\tau_0,\tau_1)} \supset \overline{\mathcal{U}(\tau_0,\tau_1)}$.

Proposition 3.2. *Let $K \gg \tilde K \gg 1$ and $\tau_0 \gg 1$, and let Ψ be a solution to* (1.9) *and* (1.10). *If $\Psi\in\overline{\mathcal{A}(\tau_0,\tau_1;1)}$, it holds that*

$$\Psi_1(y,\tau) < \Psi(y,\tau) < \Psi_2(y,\tau) \quad \textit{for } y\in(0,Ke^{-\eta\tau}] \ \textit{ and } \tau\in[\tau_0,\tau_1] \tag{3.5}$$

and that

$$|\Psi(y,\tau)-2| < \varepsilon \quad \textit{for } y \ge e^{\sigma\tau} \ \textit{ and } \ \tau\ge\tau_0. \tag{3.6}$$

Proof. It follows from $\Psi\in\overline{\mathcal{A}(\tau_0,\tau_1;1)}$ and (1.10) that $\Psi_1(0,\cdot) = \Psi(0,\cdot) = \Psi_2(0,\cdot)$ in $[\tau_0,\tau_1]$ and that

$$\Psi_1(Ke^{-\eta\tau},\tau) < \Psi(Ke^{-\eta\tau},\tau) < \Psi_2(Ke^{-\eta\tau},\tau) \quad \text{for } \tau\in[\tau_0,\tau_1],$$

if $K\gg\tilde K\gg 1$ and $\tau_0\gg 1$. From a comparison theorem, we get (3.5), if Ψ_1 and Ψ_2 satisfy that

$$\mathcal{M}(\Psi_1)\le 0 \ \text{ and } \ \mathcal{M}(\Psi_2)\ge 0 \quad \text{for } y\in(0,Ke^{-\eta\tau}) \text{ and } \tau\in[\tau_0,\tau_1]. \tag{3.7}$$

Let us put $\zeta = e^{\eta\tau}y$, $s = e^{2\eta\tau}/(2\eta)$, $\tilde{\Psi}(\zeta,s) = \Psi(y,\tau)$ and $\tilde{\Psi}_i(\zeta,s) = \Psi_i(y,\tau)$ $(i=1,2)$. We observe that

$$\begin{aligned}\tilde{\mathcal{M}}(\tilde{\Psi}) &= e^{-2\eta\tau}\mathcal{M}(\Psi) = \tilde{\Psi}_s - \tilde{\Psi}_{\zeta\zeta} - \frac{(N-3)}{\zeta}\tilde{\Psi}_\zeta + \frac{2(N-2)}{\zeta^2}\tilde{\Psi} \\ &\quad + \frac{1}{2\eta s}\left(\tfrac{1}{2}+\eta\right)\zeta\tilde{\Psi}_\zeta - \frac{1}{\zeta^2}\tilde{\Psi}\left\{(N-2)\tilde{\Psi} + \zeta\tilde{\Psi}_\zeta\right\} = 0.\end{aligned}$$

Putting $\kappa_1(s) = K^{-(\nu+1)/\nu}(2\eta s)^{-1/(4\nu)}$, we have that

$$\begin{aligned}\tilde{\mathcal{M}}(\tilde{\Psi}_1) &= -\frac{\tilde{c}_{ST}}{2K^2 s\sqrt{2\eta s}}\Psi_{ST}(\kappa_1(s)\zeta) + \kappa_1'(s)\zeta\Big(1 + \frac{\tilde{c}_{ST}}{K^2\sqrt{2\eta s}}\Big)\Psi_{ST}'(\kappa_1(s)\zeta) \\ &\quad + \frac{1}{2\eta s}\big(\tfrac{1}{2}+\eta\big)\kappa_1(s)\zeta\Big(1+\frac{\tilde{c}_{ST}}{K^2\sqrt{2\eta s}}\Big)\Psi_{ST}'(\kappa_1(s)\zeta) \qquad (3.8) \\ &\quad + \left\{\Big(1+\frac{\tilde{c}_{ST}}{K^2\sqrt{2\eta s}}\Big) - \Big(1+\frac{\tilde{c}_{ST}}{K^2\sqrt{2\eta s}}\Big)^2\right\}\frac{1}{\zeta^2} \\ &\qquad \times \Psi_{ST}(\kappa_1(s)\zeta)\left\{(N-2)\Psi_{ST}(\kappa_1(s)\zeta) + \kappa_1\zeta\Psi_{ST}'(\kappa_1(s)\zeta)\right\} \\ &\quad < -(N-2)\frac{\tilde{c}_{ST}}{K^2\zeta^2\sqrt{2\eta s}}\Psi_{ST}(\kappa_1(s)\zeta)^2 + \frac{C\kappa_1(s)\zeta}{s}\Psi_{ST}'(\kappa_1(s)\zeta)\end{aligned}$$

with some $C > 0$.

Here and henceforth, C represents a positive constant which is independent of $\mathcal{C}_1 = \{\varepsilon, \tau_0, \tau, \tau_1, K, \tilde{K}, \delta, \{\varphi_j\}_{j\geq 1}\}$. Then, each C may be different from the other C's.

It holds that

$$\frac{1}{K^2} \leq \inf_{0<y<K}\frac{\Psi_{ST}(y)}{y^2} \quad \text{and that} \quad \sup_{0<y<K}\frac{\Psi_{ST}'(y)}{y} = \sup_{0<y<\infty}\frac{\Psi_{ST}'(y)}{y} < \infty,$$

if $K \gg 1$. Combining this with (3.8) implies that

$$\tilde{\mathcal{M}}(\tilde{\Psi}_1) \leq 0 \quad \text{in } (0,K]\times[s_0,\infty), \qquad (3.9)$$

where $s_0 = e^{2\eta\tau_0}/(2\eta)$, if $\tau_0 \gg 1$.

Putting $\kappa_2(s) = K^{-\nu}e^{\eta\tau/2} = K^{-\nu}(2\eta s)^{1/4}$, we observe that

$$\begin{aligned}\tilde{\mathcal{M}}(\tilde{\Psi}_2) &= -\frac{K^{2\nu-2}}{2s\sqrt{2\eta s}}\Psi_{SE}(\kappa_2(s)\zeta) + \Big(\tfrac{1}{2} + \frac{K^{2\nu-2}}{\sqrt{2\eta s}}\Big)\kappa_2'(s)\zeta\Psi_{SE}'(\kappa_2(s)\zeta) \\ &\quad -\Big(\tfrac{1}{2}+\frac{K^{2\nu-2}}{\sqrt{2\eta s}}\Big)\tfrac{1}{2}\kappa_2^3(s)\zeta\Psi_{SE}'(\kappa_2(s)\zeta) \\ &\quad +\left\{\Big(\tfrac{1}{2}+\frac{K^{2\nu-2}}{\sqrt{2\eta s}}\Big) - \Big(\tfrac{1}{2}+\frac{K^{2\nu-2}}{\sqrt{2\eta s}}\Big)^2\right\}\frac{\Psi_{SE}(\kappa_2(s)\zeta)}{\zeta^2}\end{aligned}$$

$$\times \left\{(N-2)\Psi_{SE}(\kappa_2(s)\zeta) + \kappa_2(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta)\right\}$$
$$+\frac{1}{2\eta s}\left(\tfrac{1}{2}+\eta\right)\left(\tfrac{1}{2}+\frac{K^{2\nu-2}}{\sqrt{2\eta s}}\right)\kappa_2(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta). \tag{3.10}$$

Since it holds that

$$y\Psi'_{SE}(y) = \frac{4(N-2)\Psi_{SE}(y)}{2(N-2)+y^2} \quad \text{for } y>0,$$

for $\zeta \in (0,K]$ we observe that

$$\begin{aligned}
\frac{1}{4}\kappa_2^3(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta) &= \frac{(N-2)}{4\zeta^2}\Psi_{SE}(\kappa_2(s)\zeta)^2,\\
\frac{K^{2\nu-2}}{2\sqrt{2\eta s}}\kappa_2^3(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta) &< \frac{\Psi_{SE}(\kappa_2(s)\zeta)}{6\zeta^2}\kappa_2(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta),\\
\frac{K^{4\nu-4}}{2\eta s}\frac{(N-2)}{\zeta^2}\Psi_{SE}(\kappa_2(s)\zeta)^2 &< \frac{\Psi_{SE}(\kappa_2(s)\zeta)}{24\zeta^2}\kappa_2(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta),
\end{aligned}$$

and that

$$\frac{K^{2\nu-2}}{2s\sqrt{2\eta s}}\Psi_{SE}(\kappa_2(s)\zeta) < \frac{1}{2}\kappa'_2(s)\zeta\Psi'_{SE}(\kappa_2(s)\zeta),$$

if $K \gg 1$ and $\tau_0 \gg 1$. Combining those with (3.10) implies that

$$\tilde{\mathcal{M}}(\tilde{\Psi}_2) \geq 0 \quad \text{in } (0,K]\times[s_0,\infty),$$

if $K \gg 1$ and $\tau_0 \gg 1$. Combining this with (3.9) implies (3.7).

Since it holds that

$$\mathcal{N}\Big(\frac{2+\varepsilon}{4}\Psi_{SE}-2\Big) = \Big\{\frac{2+\varepsilon}{4} - \Big(\frac{2+\varepsilon}{4}\Big)^2\Big\}F(y,\Psi_{SE}) > 0,$$

we observe that

$$\int_{e^{\sigma\tau}}^{\infty}\Big[\mathcal{N}(\phi) - \mathcal{N}\Big(\frac{2+\varepsilon}{4}\Psi_{SE}-2\Big)\Big]\Big(\phi - \frac{2+\varepsilon}{4}\Psi_{SE}+2\Big)_+ y^{N-1}e^{-y^2/4}dy \leq 0.$$

Combining this with $\phi > -2$, $\phi \in C([\tau_0,\tau_1];L^\infty((0,\infty)))$ and

$$\phi(e^{\sigma\tau},\tau) < \frac{2+\varepsilon}{4}\Psi_{SE}(e^{\sigma\tau}) - 2 \quad \text{for } \tau \geq \tau_0 \gg 1$$

implies that

$$\begin{aligned}
&\frac{d}{d\tau}\Big\|\Big(\phi(\cdot,\tau) - \frac{2+\varepsilon}{4}\Psi_{SE}+2\Big)_+\Big\|_\tau^2 + \Big\|\frac{d}{dy}\Big(\phi(\cdot,\tau) - \frac{2+\varepsilon}{4}\Psi_{SE}+2\Big)_+\Big\|_\tau^2\\
&\leq C\left(1+\|\phi(\cdot,\tau)\|_\infty^2\right)\Big\|\Big(\phi(\cdot,\tau) - \frac{2+\varepsilon}{4}\Psi_{SE}+2\Big)_+\Big\|_\tau^2,
\end{aligned}$$

where

$$\|f\|_\tau = \left(\int_{e^{\sigma\tau}}^\infty |f(y)|^2 y^{N-1} e^{-y^2/4} dy \right)^{1/2}$$

and

$$\|\phi(\cdot,\tau)\|_\infty = \text{ess.sup}_{0<y<\infty} |\phi(y,\tau)|.$$

From this and $\phi \in C([\tau_0, \tau_1]; L^\infty((0,\infty)))$, we get that

$$\left\| \left(\phi(\cdot,\tau) - \frac{2+\varepsilon}{4} \Psi_{SE} + 2 \right)_+ \right\|_\tau^2 = 0.$$

Putting

$$\phi_{sub}(\tau) = -\frac{\varepsilon}{2} \exp\left(\frac{(N-2)}{\sigma} \left[e^{-2\sigma\tau_0} - e^{-2\sigma\tau} \right] \right),$$

we observe that $\mathcal{N}(\phi_{sub}) < 0$ for $y \in [e^{\sigma\tau}, \infty)$ and $\tau \geq \tau_0$ and that $\phi_0(y) > -\varepsilon/2$ for $y \in [e^{\sigma\tau}, \infty)$ and $\tau \geq \tau_0$, if $\tau_0 \gg 1$.

By using those and an argument similar to the above, we get that

$$\left\| (\phi(\cdot,\tau) - \phi_{sub}(\tau))_- \right\|_\tau^2 = 0.$$

(3.6) follows from those. Thus, we get this proposition. □

We will show the following proposition in the next section.

Proposition 3.3. *Suppose $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$. For $\tau_1 > \tau_0$ and $d \in \widehat{\mathcal{U}(\tau_0,\tau_1)}$, if $P(d; \tau_0, \tau_1) = 0$, then it holds that $\Psi \in \mathcal{A}(\tau_0, \tau_1, \theta)$ for some $\theta \in (0,1)$.*

Lemma 3.2. *Suppose that $d \in \widehat{\mathcal{U}(\tau_0,\tau_1)}$ satisfies $P(d; \tau_0, \tau_1) = 0$. Then, for any $\delta > 0$ it holds that*

$$\sum_{j=0}^{1} |d_j| \leq \delta e^{-\eta\tau_0/3} e^{-\lambda_2\tau_0},$$

if $\tau_0 \gg 1$.

Proof. Putting $\phi(y,\tau) = \Psi(y,\tau) - \Psi_\infty(y)$, it follows from $\Psi \in \overline{\mathcal{A}(\tau_0, \tau_1; 1)}$ that

$$\left| \phi(y,\tau) + e^{-\lambda_2\tau} \varphi_2(y) \right| \leq \varepsilon e^{-\lambda_2\tau} (y^{2\nu} + y^{2\lambda_2}) \quad \text{for } y \in \left[K e^{-\eta\tau}, e^{\sigma\tau} \right] \text{ and } \tau \in [\tau_0, \tau_1].$$

It follows from $P(d; \tau_0, \tau_1) = 0$ and the definition of the solution ϕ that

$$\langle \phi(\cdot,\tau_0), \varphi_j \rangle = -\int_{\tau_0}^{\tau_1} e^{\lambda_j(s-\tau_0)} \langle F(\cdot, \phi(\cdot,s)), \varphi_j \rangle ds.$$

We observe that for $j=0,1$

$$\langle \phi(\cdot,\tau_0),\varphi_j\rangle = d_j - e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_j\rangle. \tag{3.11}$$

From $(\phi_0 2)$, $(\phi_0 3)$, $(\phi_0 4)$, $|\phi(\cdot,\tau_0)|\le\varepsilon$ in $[e^{\sigma\tau_0},\infty)$ and $N+4\nu-5\ge 0$, we observe that for $j\neq 2$ and $0<\delta\ll 1$

$$\begin{aligned}
&|\langle\tilde\varphi_2,\varphi_j\rangle|^2 = |\langle\varphi_2-\tilde\varphi_2,\varphi_j\rangle|^2 \le \|\varphi_2-\tilde\varphi_2\|^2\\
&\le \int_0^{\tilde K e^{-\eta\tau}} e^{2\lambda_2\tau_0}\left(2+Ce^{-\lambda_2\tau_0}y^{2\nu}\right)^2 y^{N-1}e^{-y^2/4}dy + \int_{e^{\sigma\tau}}^\infty y^{4\lambda_2}y^{N-1}e^{-y^2/4}dy\\
&\le Ce^{2\lambda_2\tau_0}\tilde K^N e^{-N\eta\tau} + C\tilde K^{N+4\nu}e^{-(N+4\nu)\eta\tau_0} + C\exp\left(-\tfrac18 e^{2\sigma\tau}\right)\\
&\le C\tilde K^N e^{-3\eta\tau_0} < \frac{\delta^2}{16}e^{-2\eta\tau_0},
\end{aligned} \tag{3.12}$$

if $\tau_0\gg 1$.

For $\varphi\in H^1_w$, it holds that

$$\begin{aligned}
\int_0^\infty F(y,\phi(y,\tau))\varphi(y)y^{N-1}e^{-y^2/4}dy &= \int_0^\infty \frac{N-2}{2y^2}\phi^2\varphi y^{N-1}e^{-y^2/4}dy\\
&\quad - \int_0^\infty \frac{1}{2y}\varphi_y\phi^2 y^{N-1}e^{-y^2/4}dy + \int_0^\infty \frac14\phi^2\varphi y^{N-1}e^{-y^2/4}dy\\
&= \int_0^{Ke^{-\eta\tau}} H(y,\phi,\varphi)dy + \int_{Ke^{-\eta\tau}}^{e^{\sigma\tau}} H(y,\phi,\varphi)dy + \int_{e^{\sigma\tau}}^\infty H(y,\phi,\varphi)dy\\
&= I + II + III,
\end{aligned} \tag{3.13}$$

where

$$H(y,\phi,\varphi) = \left\{\left(\frac14+\frac{N-2}{2y^2}\right)\varphi - \frac{1}{2y}\varphi_y\right\}\phi^2 y^{N-1}e^{-y^2/4}.$$

Noticing $N+4\nu-5\ge 0$ and using Lemma 2.2, we observe that

$$\begin{aligned}
|I| &\le C\|\varphi\|_1\Big[\int_0^{Ke^{-\eta\tau}}\left(1+\frac{1}{y^2}\right)|\phi|^4 y^{N-1}e^{-y^2/4}dy\Big]^{1/2}\\
&\le CK^{(N-2)/2}e^{-\eta\tau/2}e^{-\lambda_2\tau}\|\varphi\|_1.
\end{aligned} \tag{3.14}$$

Combining

$$|II| \le C\Big[\int_{Ke^{-\eta\tau}}^{e^{\sigma\tau}}\left(1+\frac{1}{y^2}\right)|\phi|^4 y^{N-1}e^{-y^2/4}dy\Big]^{1/2}\|\varphi\|_1$$

with

$$\int_{Ke^{-\eta\tau}}^{e^{\sigma\tau}}\left(1+\frac{1}{y^2}\right)|\phi|^4 y^{N-1}e^{-y^2/4}dy$$

$$\leq \quad C\int_{Ke^{-\eta\tau}}^{e^{\sigma\tau}} e^{-4\lambda_2\tau}\Big(1+\frac{1}{y^2}\Big)(y^{8\nu}+y^{8\lambda_2})y^{N-1}e^{-y^2/4}dy$$
$$\leq Ce^{-4\lambda_2\tau}\int_{Ke^{-\eta\tau}}^{e^{\sigma\tau}}\Big(K^{4\nu}e^{-4\nu\eta\tau}y^{N+4\nu-3}+y^{N+8\lambda_2-1}\Big)\,e^{-y^2/4}dy$$

and $N+4\nu-5\geq 0$ implies that

$$|II|\leq CK^{2\nu}e^{-(\lambda_2+\eta)\tau}\|\varphi\|_1. \tag{3.15}$$

It follows from Lemma 2.2 that

$$|III| \quad \leq \quad C\Big(\int_{e^{\sigma\tau}}^{\infty}\Big(|\varphi(y)|^2+|\varphi_y(y)|^2\Big)y^{N-1}e^{-y^2/4}dy\Big)^{1/2}$$
$$\times\Big(\int_{e^{\sigma\tau}}^{\infty}|\phi|^4y^{N-1}e^{-y^2/4}dy\Big)^{1/2}\leq C\|\varphi\|_1\exp\big(-\tfrac{1}{8}e^{2\sigma\tau}\big).$$

Combining this with (3.13), (3.14) and (3.15) implies that

$$\|F(\cdot,\phi(\cdot,\tau))\|_{-1}\leq\frac{\delta}{20}e^{-\eta\tau/3}e^{-\lambda_2\tau}\qquad\text{for }\tau\geq\tau_0, \tag{3.16}$$

if $\tau_0\gg 1$. From this, the fact that $\lambda_0<\lambda_1<0$, and (2.6), we observe that for $j=0,1$

$$\int_{\tau_0}^{\tau_1}e^{\lambda_j(s-\tau_0)}\,|\langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle|\,ds\leq\int_{\tau_0}^{\tau_1}\sqrt{19}e^{\lambda_j(s-\tau_0)}\|F(\cdot,\phi(\cdot,s))\|_{-1}ds$$
$$\leq\frac{\sqrt{19}\delta}{20[\lambda_2+(\eta/3)-\lambda_j]}e^{-\lambda_j\tau_0}e^{-[\lambda_2+(\eta/3)-\lambda_j]\tau_0}\leq\frac{\delta}{4}e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}, \tag{3.17}$$

if $\tau_0\gg 1$. From this, (3.11), and (3.12), we get this lemma. □

Proposition 3.4. *Suppose* $\mathcal{U}(\tau_0,\tau_1)\neq\emptyset$ *for* τ_0 *and* τ_1 *with* $\tau_0<\tau_1$*. Then, it holds that* $\deg(P(\cdot;\tau_0,\tau_1),0,\mathcal{U}(\tau_0,\tau_1))=1$*, if* $K\gg\tilde{K}\gg 1$ *and* $\tau_0\gg 1$*.*

Proof. From (3.11), we have

$$p_j(d;\tau_0,\tau_0)=d_j-e^{-\lambda_2\tau_0}\langle\varphi_j,\tilde{\varphi}_2\rangle\qquad\text{for }j=0,1.$$

Noticing that $\Psi(\cdot,\tau_0)\in\mathcal{A}(\tau_0,\tau_0;1)$ for any $d\in\widehat{\mathcal{U}(\tau_0,\tau_1)}$, for $d\in\partial\mathcal{U}(\tau_0,\tau_0)$ we have $\sum_{j=1}^{1}|d_j|=\varepsilon\theta_1e^{-\lambda_2\tau_0}$ or $|d_j|\geq(\varepsilon\theta_1/2)e^{-\lambda_2\tau_0}$ for some $j\in\{0,1\}$. Combining this with (3.12) implies that

$$I(d)+\theta(P(d;\tau_0,\tau_0)-I(d))$$
$$=\quad(d_0-\theta e^{-\lambda_2\tau_0}\langle\varphi_0,\tilde{\varphi}_2\rangle,d_1-\theta e^{-\lambda_2\tau_0}\langle\varphi_1,\tilde{\varphi}_2\rangle)\neq 0$$

for $\theta\in[0,1]$, $d\in\partial\mathcal{U}(\tau_0,\tau_0)$ and $\tau_0\gg 1$. Then, we have

$$\deg(P(\cdot;\tau_0,\tau_0),0,\mathcal{U}(\tau_0,\tau_0))=\deg(I,0,\mathcal{U}(\tau_0,\tau_0))=1.$$

We assume that $P(d;\tau_0,\tau)=0$ for some $d\in\widehat{\mathcal{U}(\tau_0,\tau)}$ and $\tau\in[\tau_0,\tau_1]$.

From Proposition 3.3 and Lemma 3.2, we observe that $\Psi\in\mathcal{A}(\tau_0,\tau,\theta)$ for some $\theta\in(0,1)$ and that $|d_j|\le(\varepsilon\theta_1/3)e^{-\lambda_2\tau_0}$ for $j=0,1$, if $\tau_0\gg1$. It follows from those that $d\in\mathcal{U}(\tau_0,\tau)$. Thus, we obtain $P(\partial\mathcal{U}(\tau_0,\tau);\tau_0,\tau)\neq0$ for $\tau\in[\tau_0,\tau_1]$. By the continuity of the solution with respect to $d\in\mathbf{R}^2$, we have that

$$\deg(P(\cdot;\tau_0,\tau_1),0,\mathcal{U}(\tau_0,\tau_1))=\deg(P(\cdot;\tau_0,\tau_0),0,\mathcal{U}(\tau_0,\tau_0))=1,$$

if $K\gg\tilde{K}\gg1$ and $\tau_0\gg1$. Thus, we get this lemma. □

Proposition 3.5. *Suppose $K\gg1$ and $\tau_0\gg1$. For any $\tau_1>\tau_0$, it holds that $\mathcal{U}(\tau_0,\tau_1)\neq\emptyset$.*

Proof. Putting $d=0$, we observe that $\Psi(\cdot,\cdot;0)\in\mathcal{A}(\tau_0,\tau_0;\theta)$ for any $0<\theta<1$. Combining this with the continuity of solutions implies that

$$\Psi(\cdot,\cdot;0)\in\mathcal{A}(\tau_0,\tau_0+h,1/2)\quad\text{for some } h>0.$$

This means that $0\in\mathcal{U}(\tau_0,\tau)$ for $\tau\in[\tau_0,\tau_0+h]$. Putting

$$\tau^*=\sup\{\tau\ge\tau_0:\mathcal{U}(\tau_0,\tau_1)\neq\emptyset\},$$

it holds that $\tau^*>\tau_0$. We suppose that $\tau^*<\infty$. Taking a sequence $\{\tau_n\}$ with $\tau_n\nearrow\tau^*$ as $n\to\infty$, for each n there is a $d_n\in\mathcal{U}(\tau_0,\tau_n)$ such that $P(d_n,\tau_0,\tau_n)=0$ by Proposition 3.4. Since $\{d_n\}$ is bounded, we can assume that $d^*=\lim_{n\to\infty}d_n$ exists, without loss of generality. Then, we observe that $d^*\in\overline{\mathcal{U}(\tau_0,\tau^*)}\subset\widehat{\mathcal{U}(\tau_0,\tau^*)}$ and that $P(d^*,\tau_0,\tau^*)=0$. Combining this with Proposition 3.3 and Lemma 3.2 implies that $\Psi\in\mathcal{A}(\tau_0,\tau^*;\theta)$ for some $\theta\in(0,1)$ and that $d^*\in\mathcal{U}(\tau_0,\tau^*)$. By the continuity of solutions, for some $h>0$ it holds that $\Psi\in\mathcal{A}(\tau_0,\tau^*+h;1)$. This means $d^*\in\mathcal{U}(\tau_0,\tau^*+h)$. It contradicts the definition of τ^*. Then, we obtain that $\tau^*=\infty$. Thus, we get this proposition. □

Proof of Proposition 3.1. If we show $\overline{\mathcal{A}(\tau_0,\infty;1)}\neq\emptyset$, we have this proposition by Proposition 3.2.

Putting $\tau_n=\tau_0+n$, we have $\mathcal{U}(\tau_0,\tau_n)\neq\emptyset$ and $\deg(P(\cdot;\tau_0,\tau_n),0,\mathcal{U}(\tau_0,\tau_n))=1$, by Propositions 3.4 and 3.5. Then, for each n there exists a $d_n\in\mathcal{U}(\tau_0,\tau_n)$ satisfying $P(d_n;\tau_0;\tau_n)=0$. We can assume the existence of the limit $d=\lim_{n\to\infty}d_n$, without loss of generality. Then, for some $\theta_n\in(0,1)$ it holds that $\Psi(\cdot,\cdot;d_n)\in\mathcal{A}(\tau_0,\tau_n;\theta_n)$. For any $N\ge1$, we observe that

$$\Psi(\cdot,\cdot;d_n)\in\mathcal{A}(\tau_0,\tau_n;\theta_n)\subset\mathcal{A}(\tau_0,\tau_N;1)\quad\text{for } n\ge N.$$

Then, we have $\Psi(\cdot,\cdot;d)\in\overline{\mathcal{A}(\tau_0,\tau_N;1)}$ for any $N\ge1$. This means that $\Psi(\cdot,\cdot;d)\in\overline{\mathcal{A}(\tau_0,\infty;1)}$. Thus, we have this proposition. □

Next we shall show Theorem 1.

Proof of Theorem 1. Recall $\tau = -\log(T-t)$, $r = y\sqrt{T-t}$ and

$$\begin{aligned}\Psi(y,\tau) &= \frac{1}{(T-t)^{(N-2)/2}y^{N-2}}\int_0^{y\sqrt{T-t}} u(\tilde{r},t)\tilde{r}^{N-1}d\tilde{r}\\ &= \frac{1}{y^{N-2}}\int_0^y z(\xi,\tau)\xi^{N-1}d\xi,\end{aligned}$$

where $z(y,\tau) = (T-t)u(r,t)$.

Take $A > 0$, $R > 0$, $Ke^{-\eta A^2}e^{-\eta\tau} < R < A$ and $\tau \gg 1$. Put $P(\zeta,\theta) = \phi(R\zeta, R^2\theta+\tau)$.

It follows from $P_\theta = R^2\phi_\tau$, $P_\zeta = R\phi_y$ and $P_{\zeta\zeta} = R^2\phi_{yy}$ that

$$P_\theta = P_{\zeta\zeta} + \left(\frac{N-1}{\zeta} - \frac{R^2}{2}\zeta\right)P_\zeta + \frac{2(N-2)}{\zeta^2}P + \frac{1}{\zeta^2}P\{(N-2)P + \zeta P_\zeta\}.$$

From (3.5), (3.1) and (3.2), we observe that

$$|\phi(y,\tau)| \le \tilde{C}e^{-\eta\tau} \le \tilde{C}e^{-\lambda_2\tau}y^{2\nu}$$

for $y \in [(K/2)e^{-\eta A^2}e^{-\eta\tau}, Ke^{-\eta\tau}]$ and $\tau \ge \tau_0$.

Here and henceforth, $\tilde{C}$ represents a positive constant depending only on $\tilde{\mathcal{C}} = \{K, \tilde{K}, A\}$. Then, each $\tilde{C}$ may be different from the other $\tilde{C}$'s. Combining this with [13, Theorem 10.1 in Chapter IV] and $\Psi \in \overline{\mathcal{A}(\tau_0,\infty;1)}$ implies that

$$|P_\zeta(1,1)| \le \tilde{C}\max_{1/2<\zeta<2, 1/2<\theta<2}|P(\zeta,\theta)| \le \tilde{C}e^{-\lambda_2\tau}(R^{2\nu} + R^{2\lambda_2})$$

and that

$$\left|z(R,\tau+R^2) - \frac{(N-2)}{R^2}\Psi(R,\tau+R^2)\right| \le \tilde{C}\frac{e^{-\eta\tau}}{R^2}.$$

Replacing $\tau + R^2$ and R by τ and y, respectively, we get that

$$\left|z(y,\tau) - \frac{(N-2)}{y^2}\Psi(y,\tau)\right| \le \tilde{C}\frac{e^{-\eta\tau}}{y^2} \quad \text{for } Ke^{-\eta\tau} < y < A \text{ and } \tau \ge \tau_0$$

or

$$\left|u(r,t) - \frac{2(N-2)}{r^2}\right| \le \frac{\tilde{C}}{r^2}(T-t)^\eta$$

for $K(T-t)^{(2\eta+1)/2} < r < A(T-t)^{1/2}$ and $0 < t < T$ with a positive constant $\tilde{C}$. Then, we get (ii) of Theorem 1.

Let us put $\tilde{P}(\zeta,\theta) = \Psi(R\zeta, R^2\theta+\tau)$. Noticing $Ke^{-\eta\tau} \ll 1$ for $\tau \gg 1$ and using (1.9),

$$0 \le \Psi(y,\tau) \le 2\Psi_2(y,\tau) \quad \text{for } 0 \le y \le 2Ke^{-\eta\tau} \text{ and } \tau \ge \tau_0,$$

and an argument similar to the above, we observe that

$$|\tilde{P}_\zeta(1,1)| \leq C \max_{1/2<\zeta<2, 1/2<\theta<2} |\tilde{P}(\zeta,\theta)| \leq C\Psi_{SE}(2K^{-\nu}e^{3\eta R^2}e^{3\eta\tau/2}R)$$

$$\text{for } 0 < R \leq Ke^{-\eta\tau} \text{ and } \tau \gg 1,$$

$$\left|z(R,\tau+R^2)\right| \leq \frac{C}{R^2}\Psi_{SE}(2K^{-\nu}e^{3\eta R^2}e^{3\eta\tau/2}R)$$

for $0 < R \leq Ke^{-\eta\tau}$ and $\tau \gg 1$, and that

$u(r,t) \leq \tilde{C}(T-t)^{-(1+3\eta)}$ for $0 \leq r \leq K(T-t)^{(1+2\eta)/2}$ and $0 < T-t \ll 1$.

For $y \in (0, Ke^{-\eta\tau}]$, we have $\Psi(y,\tau) > \Psi_1(y,\tau)$ or

$$\frac{1}{y^2}\Psi(y,\tau) > \frac{(\kappa_3(\tau)e^{\eta\tau})^2}{(\kappa_3(\tau)e^{\eta\tau})^2 y^2}\Psi_{ST}(\kappa_3(\tau)e^{\eta\tau}y),$$

where

$$\kappa_3(\tau) = K^{-(\nu+1)/\nu}e^{-\eta\tau/(2\nu)}. \tag{3.18}$$

Taking $y \to 0$ and changing variables, we get that

$$u(0,t) \geq \tilde{C}(T-t)^{-1+(\lambda_2/\nu)}.$$

By using an argument similar to the above, it follows from $\Psi(\cdot,\tau) < \Psi_2(\cdot,\tau)$ in $(0, Ke^{-\eta\tau}]$ that

$$u(0,t) \leq \frac{2}{(N-2)K^{2\nu}}(T-t)^{-(1+3\eta)}.$$

Then, we get (i) of Theorem 1.

We shall show (iii) of Theorem 1. It follows from $\Psi \in \overline{\mathcal{A}(\tau_0,\infty;1)}$ and (3.6) that

$$|M(r,t)-2| \leq \varepsilon \quad \text{for } r \geq \frac{A}{2}(T-t)^{1/2}.$$

Let us take $A \gg 1$, $R \geq A/2$ and $t \in (0,T)$. Putting

$$V(\zeta,\theta) = M(A^{-1}\sqrt{T-t}(\zeta+R), A^{-2}(T-t)\theta + t) - 2$$

and using (2.3), we observe that

$$V_\theta = V_{\zeta\zeta} + \frac{N-1}{\zeta+R}V_\zeta + \frac{2(N-2)}{(\zeta+R)^2}V + \frac{1}{(\zeta+R)^2}V\{(N-2)V + (\zeta+R)V_\zeta\}.$$

It follows from [13, Theorem 10.1 in Chapter IV] that

$$|V_\zeta(1,1)| \leq \tilde{C} \max_{1/2<\zeta<2, 1/2<\theta<2} |V(\zeta,\theta)| \leq \tilde{C}\varepsilon,$$

where $\tilde{C}$ is independent of $R \in [A/2, \infty)$. This means that

$$\Big|u(A^{-1}\sqrt{T-t}(1+R), A^{-2}(T-t)+t) - \frac{2(N-2)}{A^{-2}(T-t)(1+R)^2}\Big| \\ \leq \frac{\tilde{C}\varepsilon}{A^{-2}(T-t)(1+R)^2}.$$

Taking the larger $A > 0$, if necessary, and taking $\tilde{t} = A^{-2}(T-t)+t$ and $\tilde{r} = A^{-1}\sqrt{T-t}(1+R)$, we observe that $T - \tilde{t} = (1-A^{-2})(T-t)$ and that

$$\frac{1}{\sqrt{A^2-1}}\Big(1+\frac{A}{2}\Big)\sqrt{T-\tilde{t}} \leq \sqrt{T-\tilde{t}}.$$

Then, we obtain that

$$\Big|u(\tilde{r},\tilde{t}) - \frac{2(N-2)}{\tilde{r}^2}\Big| \leq \frac{\tilde{C}\varepsilon}{\tilde{r}^2}$$

for $\tilde{r} \in [\sqrt{T-\tilde{t}}, \infty)$ and $0 < T - \tilde{t} \ll 1$. Replacing $\tilde{r}$ and $\tilde{t}$ by r and t, respectively, we get (iii) of Theorem 1 in $[\sqrt{T-t}, \infty)$. Combining this with (ii) of Theorem 1 implies (iii) of Theorem 1. Thus, we finish the proof of this theorem. □

4. Proof of Proposition 3.3

In this section, we shall prove Proposition 3.3.

Recall $\alpha = 2\nu + (N-2)/2$.

A straightforward calculation yields the following proposition, and the proposition is shown in [8, 14] in the case where $2\alpha+1$ is a non-negative integer. Thus, we omit the proof.

Proposition 4.1. *Let v be the solution of the Cauchy problem*

$$\begin{cases} v_\tau = v_{yy} + \Big(\dfrac{2\alpha+1}{y} - \dfrac{y}{2}\Big)v_y & \text{in } (0,\infty)\times(0,\infty), \\ v(\cdot,0) = v_0 & \text{in } (0,\infty) \end{cases}$$

with a function $v_0 \in L^\infty((0,\infty))$. Then, v is expressed by

$$\begin{aligned} v(y,\tau) &= \frac{\exp\Big(\frac{\alpha}{2}\tau - \frac{e^{-\tau}y^2}{4(1-e^{-\tau})}\Big)y^{-\alpha}}{2(1-e^{-\tau})} \\ &\quad \times \int_0^\infty I_\alpha(\sqrt{Q}y)\xi^{\alpha+1}\exp\Big(-\frac{\xi^2}{4(1-e^{-\tau})}\Big)v_0(\xi)d\xi, \end{aligned}$$

where I_α is the modified Bessel function of order α and

$$Q = Q(\xi,\tau) = \frac{e^{-\tau}\xi^2}{4(1-e^{-\tau})^2}.$$

We get the following estimates. The proof is found in Section 5.4.

Lemma 4.1. *For $N \geq 11$, it holds that*

$$\left|I'_\alpha(x) - I_\alpha(x)\right| \leq \frac{C}{x} I_\alpha(x) \quad \text{for } x > 0 \tag{4.1}$$

and that

$$\frac{x^\alpha e^x}{C(x+1)^{(2\alpha+1)/2}} \leq I_\alpha(x) \leq \frac{Cx^\alpha e^x}{(x+1)^{(2\alpha+1)/2}} \quad \text{for } x > 0. \tag{4.2}$$

4.1. **Estimates in the case where $\tau_0 \leq \tau \leq \min(\tau_0+1, \tau_1)$.** We get that

$$\begin{aligned} \phi(\cdot,\tau_0) &= \sum_{j=0}^{1} \left(d_j - e^{-\lambda_2\tau_0}\langle\tilde\varphi_2, \varphi_j\rangle\right)\varphi_j - e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2 \\ &\quad - \sum_{j=3}^{\infty} e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_j\rangle\varphi_j. \end{aligned}$$

Putting

$$\begin{aligned} S_1(\cdot,\tau) &= -e^{-\lambda_2\tau}\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2, \\ S_2(\cdot,\tau) &= -\sum_{j\neq 2} e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_j\rangle e^{-\lambda_j(\tau-\tau_0)}\varphi_j + \sum_{j=0}^{1} d_j e^{-\lambda_j(\tau-\tau_0)}\varphi_j \end{aligned}$$

and

$$S_3(\cdot,\tau) = \int_{\tau_0}^{\tau} e^{-(\tau-s)\tilde A} F(\cdot,\phi(\cdot,s))ds,$$

it holds that

$$\begin{aligned} \phi(\cdot,\tau) &= e^{-(\tau-\tau_0)\tilde A}\phi(\cdot,\tau_0) + \int_{\tau_0}^{\tau} e^{-(\tau-s)\tilde A}F(\cdot,\phi(\cdot,s))ds \\ &= S_1 + S_2 + S_3. \end{aligned} \tag{4.3}$$

In particular, it holds that

$$\begin{aligned} S_2(\cdot,\tau_0) &= -e^{-\lambda_2\tau_0}\tilde\varphi_2 + e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2 + \sum_{j=0}^{1} d_j\varphi_j \\ &= \phi(\cdot,\tau_0) + e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2. \end{aligned}$$

4.1.1. *Estimates of S_2 in the case where $\tau_0 \le \tau \le \min(\tau_0+1, \tau_1)$.* Put

$$\sum(\tau_0, \tau_1^*) = \{(y,\tau) : Ke^{-\eta\tau} \le y \le e^{\sigma\tau},\ \tau_0 \le \tau \le \tau_1^*\},$$

where $\tau_1^* = \min(\tau_1, \tau_0+1)$.

Lemma 4.2. *For any $\delta \in (0,1]$, it holds that*

$$|S_2(y,\tau)| \le \delta e^{-\lambda_2\tau}\left(y^{2\nu} + y^{2\lambda_2}\right) \qquad \text{in } \sum(\tau_0, \tau_1^*),$$

if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$.

Proof. It holds that

$$S_2(\cdot,\tau) = e^{-(\tau-\tau_0)\tilde{A}} S_2(\cdot, \tau_0).$$

Putting $\mathcal{S}(y,\tau) = y^\beta S_2(y,\tau)$, we observe that

$$\begin{aligned} \mathcal{S}_\tau &= \mathcal{S}_{yy} + \Big\{(-2\beta + N - 1)\frac{1}{y} - \frac{y}{2}\Big\}\mathcal{S}_y \\ &\qquad + \{\beta(\beta+1) - \beta(N-1) + 2(N-2)\}\frac{1}{y^2}\mathcal{S} + \frac{\beta}{2}\mathcal{S}. \end{aligned}$$

Putting $\beta = \{(N-2) - \sqrt{(N-10)(N-2)}\}/2 = -2\nu$, $\mathcal{S}$ satisfies

$$\left(e^{\nu(\tau-\tau_0)}\mathcal{S}\right)_\tau = \left(e^{\nu(\tau-\tau_0)}\mathcal{S}\right)_{yy} + \Big\{\frac{2\alpha+1}{y} - \frac{y}{2}\Big\}\left(e^{\nu(\tau-\tau_0)}\mathcal{S}\right)_y.$$

Combining this with Proposition 4.1 implies that

$$S_2(y,\tau) = \int_0^\infty K(y,\xi,\tau-\tau_0) S_2(\xi,\tau_0)\,d\xi,$$

where

$$\begin{aligned} K(y,\xi,\theta) &= \frac{\exp\left(\frac{N-2}{4}\theta\right)}{2(1-e^{-(\tau-\tau_0)})}\frac{\xi^{N/2}}{y^{(N-2)/2}} \\ &\qquad \times \exp\Big(-\frac{e^{-\theta}y^2 + \xi^2}{4(1-e^{-\theta})}\Big) I_{(N+4\nu-2)/2}\Big(\frac{e^{-\theta/2}\xi y}{2(1-e^{-\theta})}\Big) \end{aligned}$$

and

$$\begin{aligned} S_2(y,\tau) &= \int_0^\infty K(y,\xi,\tau-\tau_0) S_2(\xi,\tau_0)\,d\xi \\ &= \int_0^{\tilde{K}e^{-\eta\tau_0}} K(y,\xi,\tau-\tau_0) S_2(\xi,\tau_0)\,d\xi \\ &\qquad + \int_{\tilde{K}e^{-\eta\tau_0}}^\infty K(y,\xi,\tau-\tau_0) S_2(\xi,\tau_0)\,d\xi = S_{2,1}(y,\tau) + S_{2,2}(y,\tau). \end{aligned}$$

Since it follows from Lemma 2.4 that

$$\Psi_{ST}(r) \geq 2 - 2c_{ST}r^{2\nu} \quad \text{for } r \geq r_0,$$

with some $r_0 > 0$, it holds that

$$\begin{aligned} |\Psi_1(y,\tau) - 2| &\leq 2\left(r_0 K^{(1+\nu)/\nu} e^{\lambda_2\tau/(2\nu)}\right)^{-2\nu} y^{2\nu} \\ &= \frac{2}{r_0^{2\nu}K^{2+2\nu}} e^{-\lambda_2\tau} y^{2\nu} \quad \text{for } y \in (0, r_0 K^{(1+\nu)/\nu} e^{\lambda_2\tau/(2\nu)}] \end{aligned}$$

and that

$$\begin{aligned} |\Psi_1(y,\tau) - 2| &\leq 2c_{ST}\left(K^{-(1+\nu)/\nu} e^{-\lambda_2\eta\tau/(2\nu)} y\right)^{2\nu} = \frac{2c_{ST}}{K^{2+2\nu}} e^{-\lambda_2\tau} y^{2\nu} \\ &\qquad \text{for } y \in [r_0 K^{(1+\nu)/\nu} e^{\lambda_2\tau/(2\nu)}, Ke^{-\eta\tau}]. \end{aligned}$$

Then, we observe that

$$|\Psi_1(y,\tau) - 2| \leq \frac{C}{K^{2+2\nu}} e^{-\lambda_2\tau} y^{2\nu} \quad \text{for } y \in (0, Ke^{-\eta\tau}] \text{ and } \tau \geq \tau_0. \tag{4.4}$$

For $y \in (0, Ke^{-\eta\tau_0}]$ and $\tau \in [\tau_0, \tau_1]$, it follows from this, Lemmas 2.4 and 2.5 and Proposition 3.2 that

$$\phi(y,\tau) \leq \Psi_2(y,\tau) - 2 \leq \Psi_2(Ke^{-\eta\tau}, \tau) - 2 < 0$$

and that

$$\phi(y,\tau) \geq \Psi_1(y,\tau) - 2 \geq -\frac{C}{K^{2+2\nu}} e^{-\lambda_2\tau} y^{2\nu}.$$

From those, we have

$$|\phi(y,\tau)| \leq \frac{C}{K^{2+2\nu}} e^{-\lambda_2\tau} y^{2\nu} \quad \text{for } y \in (0, Ke^{-\eta\tau}] \text{ and } \tau \in [\tau_0, \tau_1]. \tag{4.5}$$

From (3.12) and (4.5), we have

$$\begin{aligned} |S_2(y,\tau_0)| &= \left|\phi(y,\tau_0) + e^{-\lambda_2\tau_0}\langle\tilde{\varphi}_2 - \varphi_2, \varphi_2\rangle\varphi_2 + e^{-\lambda_2\tau_0}\varphi_2\right| \\ &\leq \frac{C}{K^{2(\nu+1)}} e^{-\lambda_2\tau_0} y^{2\nu} \quad \text{for } y \in (0, Ke^{-\eta\tau_0}], \end{aligned} \tag{4.6}$$

if $K \gg 1$ and $\tau_0 \gg 1$. By (4.2) in Lemma 4.1 and (4.6), we observe that

$$\begin{aligned} |S_{2,1}(y,\tau)| &\leq C\int_0^{\tilde{K}e^{-\eta\tau_0}} K(y,\xi,\tau-\tau_0)\frac{e^{-\lambda_2\tau_0}}{K^{2(\nu+1)}}\xi^{2\nu}d\xi \\ &\leq \frac{Ce^{-\lambda_2\tau_0}}{K^{2(\nu+1)}} \frac{\exp\left(\frac{N-2}{4}(\tau-\tau_0)\right)}{(1-e^{-(\tau-\tau_0)})} \exp\left(-\frac{e^{-(\tau-\tau_0)}y^2}{4(1-e^{-(\tau-\tau_0)})}\right) y^{2\nu} \end{aligned}$$

$$\times \int_0^{\tilde{K}e^{-\eta\tau_0}} \exp\Big(\frac{e^{-(\tau-\tau_0)/2}y\xi}{2(1-e^{-(\tau-\tau_0)})}\Big)\exp\Big(-\frac{\xi^2}{4(1-e^{-(\tau-\tau_0)})}\Big)$$
$$\times\Big(\frac{e^{-(\tau-\tau_0)/2}\xi}{2(1-e^{-(\tau-\tau_0)})}\Big)^{(N+4\nu-2)/2}\Big\{1+\frac{e^{-(\tau-\tau_0)/2}y\xi}{2(1-e^{-(\tau-\tau_0)})}\Big\}^{-(N+4\nu-1)/2}$$
$$\times\xi^{(N+4\nu)/2}d\xi = VI.$$

Putting $\zeta(\xi)=\xi/\sqrt{1-e^{-(\tau-\tau_0)}}$ and using $y \geq Ke^{-\eta\tau}$, we observe that

$$VI \leq \frac{Ce^{-\lambda_2\tau_0}}{K^{2(\nu+1)}}\exp\Big(\frac{\tilde{K}e^{-(\tau-\tau_0)/2}y^2}{2K(1-e^{-(\tau-\tau_0)})}\Big)\exp\Big(-\frac{e^{-(\tau-\tau_0)}y^2}{4(1-e^{-(\tau-\tau_0)})}\Big)$$
$$\times y^{2\nu}\int_0^\infty e^{-\zeta^2/4}\Big\{1+\frac{K\zeta^2}{2e\tilde{K}}\Big\}^{-(N+4\nu-1)/2}\zeta^{N+4\nu-1}d\zeta.$$

Putting $\tilde{\zeta}=\sqrt{K}\zeta/\sqrt{2e\tilde{K}}$, we observe that

$$\int_0^\infty e^{-\zeta^2/4}\Big\{1+\frac{K\zeta^2}{2e\tilde{K}}\Big\}^{-(N+4\nu-1)/2}\zeta^{N+4\nu-1}d\zeta$$
$$\leq \int_{\sqrt{2e\tilde{K}/K}}^\infty e^{-\zeta^2/4}\Big(\frac{2e\tilde{K}}{K}\Big)^{(N+4\nu-1)/2}d\zeta$$
$$+\int_0^1 \frac{\tilde{\zeta}^{N+4\nu-1}}{(1+\tilde{\zeta}^2)^{(N+4\nu-1)/2}}\Big(\frac{2e\tilde{K}}{K}\Big)^{(N+4\nu)/2}d\tilde{\zeta} \leq C\Big(\frac{\tilde{K}}{K}\Big)^{(N+4\nu-1)/2},$$

if $K \gg \tilde{K} \gg 1$. It follows from this that

$$|S_{2,1}(y,\tau)| \leq C\frac{\tilde{K}^{(N+4\nu-1)/2}}{K^{(N+8\nu+3)/2}}e^{-\lambda_2\tau_0}y^{2\nu} \quad \text{in } \textstyle\sum(\tau_0,\tau_1^*),$$

if $K \gg \tilde{K} \gg 1$. From this and $N+8\nu+1 \geq 0$, for any $\delta \in (0,1]$ we observe that

$$|S_{2,1}(y,\tau)| \leq \delta e^{-\lambda_2\tau_0}y^{2\nu} \quad \text{in } \textstyle\sum(\tau_0,\tau_1^*),$$

if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$.

Next, we shall estimate $S_{2,2}$. Since S_2 satisfies

$$S_2(y,\tau_0)=e^{-\lambda_2\tau_0}\langle\tilde{\varphi}_2-\varphi_2,\varphi_2\rangle\varphi_2+\sum_{j=0}^1 d_j\varphi_j \quad \text{for } y \in [\tilde{K}e^{-\eta\tau_0}, e^{\tilde{\sigma}\tau_0}],$$

we have

$$|S_2(y,\tau_0)| < \delta e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}\left(y^{2\nu}+y^{2\lambda_2}\right) \quad \text{for } \tilde{K}e^{-\eta\tau_0} \leq y \leq e^{\tilde{\sigma}\tau_0},$$

by Lemma 3.2 and (3.12). Note that $2e^{\sigma\tau_0} < e^{\tilde{\sigma}\tau_0}$ and that $2e^{-(\tau-\tau_0)/2}y \geq \tilde{K}e^{-\eta\tau_0}$ in $\sum(\tau_0, \tau_1^*)$, if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$. It follows from this that

$$\begin{aligned}|S_{2,2}(y,\tau)| &\leq C\delta e^{-\eta\tau_0/3}\int_0^{2e^{-(\tau-\tau_0)/2}y} K_1(y,\xi,\tau,\tau_0)d\xi \\ &\quad + C\delta e^{-\eta\tau_0/3}\int_{2e^{-(\tau-\tau_0)/2}y}^{e^{\tilde{\sigma}\tau_0}} K_1(y,\xi,\tau,\tau_0)d\xi + C\varepsilon\int_{e^{\tilde{\sigma}\tau_0}}^{\infty} K(y,\xi,\tau-\tau_0)d\xi \\ &= S_{2,2}^1 + S_{2,2}^2 + S_{2,2}^3 \quad \text{ in } \textstyle\sum(\tau_0,\tau_1^*),\end{aligned}$$

where $K_1(y,\xi,\tau,\tau_0) = e^{-\lambda_2\tau_0}K(y,\xi,\tau-\tau_0)(\xi^{2\nu}+\xi^{2\lambda_2})$. Since it follows from (4.2) in Lemma 4.1 that

$$\begin{aligned}K_1(y,\xi,\tau,\tau_0) &\leq Ce^{-\nu(\tau-\tau_0)}e^{-\lambda_2\tau_0}y^{2\nu}\exp\Big(-\frac{(\xi-e^{-(\tau-\tau_0)/2}y)^2}{4(1-e^{-(\tau-\tau_0)})}\Big) \\ &\quad \times\Big(1+\frac{e^{-(\tau-\tau_0)/2}\xi y}{2(1-e^{-(\tau-\tau_0)})}\Big)^{-(N+4\nu-1)/2}\frac{\xi^{N+4\nu-1}(1+\xi^{2\lambda_2-2\nu})}{(1-e^{-(\tau-\tau_0)})^{(N+4\nu)/2}}, \qquad (4.7)\end{aligned}$$

we have

$$\begin{aligned}&S_{2,2}^1(y,\tau) \leq C\delta e^{-\nu(\tau-\tau_0)}e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}y^{2\nu}\int_0^{2e^{-(\tau-\tau_0)/2}y}\exp\Big(-\frac{(\xi-e^{-(\tau-\tau_0)/2}y)^2}{4(1-e^{-(\tau-\tau_0)})}\Big) \\ &\times\Big\{1+\frac{\xi^2}{4(1-e^{-(\tau-\tau_0)})}\Big\}^{-(N+4\nu-1)/2}\frac{\xi^{N+4\nu-1}\left(1+\xi^{2(\lambda_2-\nu)}\right)}{\left(1-e^{-(\tau-\tau_0)}\right)^{(N+4\nu)/2}}d\xi \\ &\leq C\delta e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}\left(y^{2\nu}+y^{2\lambda_2}\right)\int_0^\infty\exp\Big(-\frac{1}{4}\Big(t-\frac{e^{-(\tau-\tau_0)/2}y}{\sqrt{(1-e^{-(\tau-\tau_0)})}}\Big)^2\Big)dt \\ &\leq C\delta e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}\left(y^{2\nu}+y^{2\lambda_2}\right).\end{aligned}$$

Noticing that

$$\xi - e^{-(\tau-\tau_0)/2}y \geq \frac{1}{2}\xi \quad \text{ for } 2e^{-(\tau-\tau_0)/2}y < \xi < e^{\tilde{\sigma}\tau_0}$$

and using (4.7), we have

$$\begin{aligned}S_{2,2}^2(y,\tau) &\leq C\delta e^{-\nu(\tau-\tau_0)}e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}y^{2\nu}\int_{2e^{-(\tau-\tau_0)/2}y}^{e^{\tilde{\sigma}\tau_0}}\exp\Big(-\frac{\xi^2}{16(1-e^{-(\tau-\tau_0)})}\Big) \\ &\quad\times\frac{\xi^{N+4\nu-1}\left(1+\xi^{2\lambda_2-2\nu}\right)}{\left(1-e^{-(\tau-\tau_0)}\right)^{(N+4\nu)/2}}d\xi \leq C\delta e^{-\eta\tau_0/3}e^{-\lambda_2\tau_0}y^{2\nu}.\end{aligned}$$

Noticing $\xi \geq e^{\tilde{\sigma}\tau_0} \geq 2y$ for $\xi \geq e^{\tilde{\sigma}\tau_0}$ and using (4.7), we have

$$S_{2,2}^3 \leq C\varepsilon e^{-\nu(\tau-\tau_0)} e^{-\lambda_2\tau_0} y^{2\nu} \int_{e^{\tilde{\sigma}\tau_0}}^{\infty} \exp\Big(- \frac{\xi^2}{16(1-e^{-(\tau-\tau_0)})}\Big) \cdot \frac{\xi^{N+4\nu-1}\left(1+\xi^{2\lambda_2-2\nu}\right)}{\left(1-e^{-(\tau-\tau_0)}\right)^{(N+4\nu)/2}} d\xi \leq C\varepsilon \exp\Big(- \frac{e^{\tilde{\sigma}\tau_0}}{32}\Big) e^{-\lambda_2\tau_0} y^{2\nu},$$

if $\tau_0 \gg 1$. Taking the larger $\tau_0 \gg 1$, if necessary, we have this lemma. □

4.1.2. *Estimates of* S_3 *in the case where* $\tau_0 \leq \tau \leq \min(\tau_0+1, \tau_1)$. Let us take a, $b \in [\tau_0, \tau_1]$ with $0 < b-a \leq 2$. For $(y,\tau) \in \sum(a,b)$, let us put

$$S_3(y,\tau;a) = \int_a^\tau e^{-(\tau-s)\tilde{A}} F(\cdot, \phi(\cdot, s)) ds.$$

Henceforth, C represents a positive constant which is independent of $\mathcal{C}_2 = \mathcal{C}_1 \cup \{a,b\}$. Then, each C may be different from the other C's.

We observe that

$$S_3(y,\tau;a) = \int_a^\tau \int_0^\infty K(y,\xi,\tau-s) \frac{N-2}{\xi^2} \phi(\xi,s)^2 d\xi ds + \int_a^\tau \int_0^\infty K(y,\xi,\tau-s) \frac{1}{\xi} \phi_\xi(\xi,s)\phi(\xi,s) d\xi ds = S_3^A(y,\tau;a) + S_3^B(y,\tau;a),$$

$$|S_3^A(y,\tau;a)| \leq C \int_a^\tau \int_0^\infty \frac{e^{-\nu(\tau-\tau_0)} y^{2\nu}}{(1-e^{-(\tau-s)})^{(N+4\nu)/2}} \exp\Big(- \frac{(e^{-(\tau-s)/2}y-\xi)^2}{4(1-e^{-(\tau-s)})}\Big) \times \Big\{1 + \frac{e^{-(\tau-s)/2}y\xi}{2(1-e^{-(\tau-s)})}\Big\}^{-(N+4\nu-1)/2} |\phi(\xi,s)|^2 \xi^{N+2\nu-3} d\xi ds$$

and that

$$S_3^B(y,\tau;a) = -\frac{1}{2} \int_a^\tau \frac{\exp\Big(\frac{N-2}{4}(\tau-s) - \frac{y^2}{4(1-e^{-(\tau-s)})}\Big)}{2(1-e^{-(\tau-s)})y^{(N-2)/2}} \times \int_0^\infty VII \cdot \exp\Big(- \frac{\xi^2}{4(1-e^{-(\tau-s)})}\Big) \phi(\xi,s)^2 \xi^{(N-2)/2} d\xi ds,$$

where

$$VII = \frac{N-2}{2\xi} I_{(N+4\nu-2)/2}(\sqrt{Q}y) + \frac{e^{-(\tau-s)/2}y}{2(1-e^{-(\tau-s)})} I'_{(N+4\nu-2)/2}(\sqrt{Q}y) - \frac{\xi}{2(1-e^{-(\tau-s)})} I_{(N+4\nu-2)/2}(\sqrt{Q}y).$$

From (4.1) in Lemma 4.1, we observe that

$$\begin{aligned}
&|S_3(y,\tau;a)| \le |S_3^A(y,\tau;a)| + |S_3^B(y,\tau;a)| \\
&\quad \le \quad C\int_a^\tau \int_0^\infty K_2(y,\xi,\tau-s)\phi^2(\xi,s)d\xi ds \\
&\quad \le \quad C\int_a^\tau \int_0^{\tilde{K}e^{-\eta s}} K_2(y,\xi,\tau-s)\phi^2(\xi,s)d\xi ds \\
&\qquad\quad +C\int_a^\tau \int_{\tilde{K}e^{-\eta s}}^\infty K_2(y,\xi,\tau-s)\phi^2(\xi,s)d\xi ds \\
&\quad = \quad S_{3,1}(y,\tau;a) + S_{3,2}(y,\tau;a),
\end{aligned}$$

where

$$\begin{aligned}
K_2(y,\xi,\theta) = \frac{e^{-\nu\theta}y^{2\nu}}{(1-e^{-\theta})^{(N+4\nu)/2}}\Big\{1+\frac{|e^{-\theta/2}y-\xi|\xi}{2(1-e^{-\theta})}\Big\}\exp\Big(-\frac{(e^{-\theta/2}y-\xi)^2}{4(1-e^{-\theta})}\Big) \\
\times\Big(1+\frac{e^{-\theta/2}y\xi}{2(1-e^{-\theta})}\Big)^{-(N+4\nu-1)/2}\xi^{N+2\nu-3}. \qquad (4.8)
\end{aligned}$$

For a, $b \in [\tau_0, \tau_1]$ with $a < b$, let us put

$$\textstyle\sum_O(a,b) = \{(y,\tau) : Ke^{-\eta\tau} \le y,\ a \le \tau \le b\}.$$

Lemma 4.3. *Let a and b satisfy $\tau_0 \le a < b \le \tau_1$ and $b - a \le 2$. For any $\delta \in (0,1]$, it holds that*

$$|S_{3,1}(y,\tau;a)| \le \delta e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu} \qquad \text{in } \textstyle\sum_O(a,b),$$

if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$.

Proof. Let us put $\theta = -1/(4\nu)$,

$$S_{3,1}^1(y,\tau;a) = \int_a^\tau \int_0^{\tilde{K}e^{-(1+\theta)\eta s}} K_2(y,\xi,\tau-s)\phi(\xi,s)^2 d\xi ds$$

and

$$S_{3,1}^2(y,\tau;a) = \int_a^\tau \int_{\tilde{K}e^{-(1+\theta)\eta s}}^{\tilde{K}e^{-\eta s}} K_2(y,\xi,\tau-s)\phi(\xi,s)^2 d\xi ds.$$

For $a \le s \le \tau \le b$ and $\tilde{K}e^{-(1+\theta)\eta s} \le \xi \le \tilde{K}e^{-\eta s} \ll Ke^{-\eta\tau} \le y$, it follows from (4.5) that

$$|\phi(\xi,s)| \quad \le \quad \frac{C\tilde{K}^{2\nu}}{K^{2+2\nu}}e^{-\eta s/2}.$$

Then, we have

$$S^2_{3,1}(y,\tau;a) \le C\int_a^\tau \int_{\tilde K e^{-(1+\theta)\eta s}}^{\tilde K e^{-\eta s}} \frac{e^{-\nu(\tau-s)}y^{2\nu}}{(1-e^{-(\tau-s)})} \exp\Big(-\frac{e^{-(\tau-s)}y^2}{16(1-e^{-(\tau-s)})}\Big)$$
$$\times \exp\Big(-\frac{K^2\xi^2}{16e^4\tilde K^2(1-e^{-(\tau-s)})}\Big)\Big\{1+\frac{K\xi^2}{2e^2\tilde K(1-e^{-(\tau-s)})}\Big\}^{-(N+4\nu-1)/2}$$
$$\times \frac{\xi^{N+2\nu-3}}{(1-e^{-(\tau-s)})^{(N+4\nu-2)/2}} \frac{C\tilde K^{2\nu}}{K^{4+4\nu}} e^{-\eta s/2}e^{-\lambda_2 s}\xi^{2\nu}d\xi ds.$$

Using this and putting $\zeta = \xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = u(s) = y^2/(1-e^{-(\tau-s)})$, we have that

$$S^2_{3,1}(y,\tau;a) \le C\frac{\tilde K^{2\nu}}{K^{4+4\nu}} e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu}$$
$$\times \int_{y^2/(1-e^{-(\tau-a)})}^\infty \frac{e^{(\tau-s)}}{u} e^{-u/(16e^2)} \int_0^{\tilde K e^2\sqrt{u}/K} \zeta^{N+4\nu-3} d\zeta du$$
$$\le \quad C\frac{\tilde K^{N+6\nu-2}}{K^{N+8\nu+2}} e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu}. \tag{4.9}$$

Since it holds that

$$y \ge Ke^{-\eta\tau} \ge \frac{K}{\tilde K e^{2\eta}} e^{\theta\eta s}\xi \quad \text{ for } \xi\in(0,\tilde K e^{-(1+\theta)\eta s}] \ \text{ and } \ a\le s\le\tau\le b,$$

we observe that

$$S^1_{3,1}(y,\tau;a) \le C\int_a^\tau \frac{e^{-\nu(\tau-s)}y^{2\nu}}{(1-e^{-(\tau-s)})^{(N+4\nu)/2}} \int_0^{\tilde K e^{-(1+\theta)\eta s}} \Big\{1+\frac{e^{-(\tau-s)/2}y\xi+\xi^2}{2(1-e^{-(\tau-s)})}\Big\}$$
$$\times \exp\Big(-\frac{y^2+[K/(e^{2\eta}\tilde K)]^2 e^{2\theta\eta s}\xi^2}{32e^2(1-e^{-(\tau-s)})}\Big)\Big\{1+\frac{Ke^{\theta\eta s}\xi^2}{4\tilde K e^{2\eta}(1-e^{-(\tau-s)})}\Big\}^{-(N+2\nu-1)/2}$$
$$\times \xi^{N+2\nu-3}\frac{C}{K^{2+2\nu}} e^{-\lambda_2 s}\xi^{2\nu}d\xi ds,$$

by (4.5). Putting $\tilde\zeta = e^{\theta\eta s}\xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = u(s) = y^2/(1-e^{-(\tau-s)})$, and noticing $\tilde K e^{-\eta s}/\sqrt{1-e^{-(\tau-s)}} \le (e^{2\eta}\tilde K/K)\sqrt{u}$ and $N+4\nu-5\ge 0$, we observe that

$$S^1_{3,1}(y,\tau;a) \le C\tilde K^{-2\nu}e^{-(N+4\nu-2)\theta\eta}e^{-\lambda_2\tau}y^{2\nu} \int_{y^2/(1-e^{-(\tau-a)})}^\infty \frac{e^{\tau-s}}{u}e^{-u/(32e^2)}$$
$$\times \int_0^{e^{2\eta}\tilde K\sqrt{u}/K} (1+\tilde\zeta\sqrt{u}+\tilde\zeta^2)\tilde\zeta^{N+4\nu-3} \exp\Big(-\frac{[K/(e^{2\eta}\tilde K)]^2}{32e^2}\tilde\zeta^2\Big) d\tilde\zeta du$$

$$\leq C\frac{\tilde{K}^{N+2\nu-2}}{K^{N+4\nu-2}}e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu}. \tag{4.10}$$

From (4.9), (4.10) and the fact that $N+4\nu-5\geq 0$, for any $\delta\in(0,1]$ we have

$$|S_{3,1}(y,\tau;a)|\leq \delta e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b),$$

if $K\gg\tilde{K}\gg 1$ and $\tau_0\gg 1$. Thus, we finish the proof of this lemma. □

From $\Psi\in\overline{\mathcal{A}(\tau_0,\tau_1;1)}$, we get that

$$\begin{aligned}|\phi(y,\tau)| &\leq e^{-\lambda_2\tau}|\varphi_2(y)|+\varepsilon e^{-\lambda_2\tau}\big(y^{2\nu}+y^{2\lambda_2}\big)\\ &\leq Ce^{-\lambda_2\tau}\big(y^{2\nu}+y^{2\lambda_2}\big) \quad \text{in } \textstyle\sum(\tau_0,\tau_1).\end{aligned} \tag{4.11}$$

Let us put

$$\chi(y,\tau)=\begin{cases} K^{-4-4\nu} & \text{if } y\in(0,Ke^{-\eta\tau}],\\ 1 & \text{if } y\in(Ke^{-\eta\tau},\infty).\end{cases}$$

It follows from (4.11) and (4.5) that

$$|\phi(y,\tau)|^2\leq Ce^{-2\lambda_2\tau}\big(\chi(y,\tau)y^{4\nu}+y^{4\lambda_2}\big) \quad \text{for } 0<y\leq e^{\sigma\tau} \text{ and } \tau_0\leq\tau\leq\tau_1.$$

From this and (3.6), we get that

$$\begin{aligned}|S_{3,2}(y,\tau;a)| &\leq C\int_a^\tau\int_{\tilde{K}e^{-\eta s}}^{e^{\sigma s}}K_2(y,\xi,\tau-s)e^{-2\lambda_2 s}\chi(y,s)\xi^{4\nu}d\xi ds\\ &+C\int_a^\tau\int_{\tilde{K}e^{-\eta s}}^{e^{\sigma s}}K_2(y,\xi,\tau-s)e^{-2\lambda_2 s}\xi^{4\lambda_2}d\xi ds\\ &+C\int_a^\tau\int_{e^{\sigma s}}^{\infty}K_2(y,\xi,\tau-s)\varepsilon^2 d\xi ds\\ &=S^1_{3,2}(\cdot,\cdot;a)+S^2_{3,2}(\cdot,\cdot;a)+S^3_{3,2}(\cdot,\cdot;a).\end{aligned}$$

The following lemma holds.

Lemma 4.4. *Let a and b satisfy $\tau_0\leq a<b\leq\tau_1$ and $b-a\leq 2$. For any $\delta\in(0,1]$, it holds that*

$$S^1_{3,2}(y,\tau;a)\leq\delta e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b),$$

if $K\gg\tilde{K}\gg 1$ and $\tau_0\gg 1$.

Proof. For $(y,\tau)\in\sum_O(a,b)$ and $a\leq s\leq\tau$, it holds that $2e^{2\eta}y\geq 2e^{2\eta}Ke^{-\eta\tau}\geq 2Ke^{-\eta s}$. Let us put

$$S^{1,1}_{3,2}(y,\tau;a)=\frac{1}{K^{4(\nu+1)}}\int_a^\tau\int_{\tilde{K}e^{-\eta s}}^{2e^{2\eta}\min(y,e^{\sigma s})}K_2(y,\xi,\tau-s)\xi^{4\nu}d\xi ds$$

and

$$S^{1,2}_{3,2}(y,\tau;a)=\int_a^\tau\int_{2e^{2\eta}\min(y,e^{\sigma s})}^{2e^{2\eta}e^{\sigma s}}K_2(y,\xi,\tau-s)\xi^{4\nu}d\xi ds.$$

Putting $\zeta=\xi/\sqrt{1-e^{-(\tau-s)}}$, $u=y^2/(1-e^{-(\tau-s)})$,

$$D_1=\Big\{\zeta\in\Big(\frac{\tilde{K}e^{-\eta s}}{\sqrt{1-e^{-(\tau-s)}}},2e^{2\eta}\sqrt{u}\Big):|e^{-(\tau-s)/2}\sqrt{u}-\zeta|\ge\frac{1}{2}e^{-(\tau-s)/2}\sqrt{u}\Big\}$$

and

$$D_2=\Big\{\zeta\in\Big(\frac{\tilde{K}e^{-\eta s}}{\sqrt{1-e^{-(\tau-s)}}},2e^{2\eta}\sqrt{u}\Big):|e^{-(\tau-s)/2}\sqrt{u}-\zeta|\le\frac{1}{2}e^{-(\tau-s)/2}\sqrt{u}\Big\},$$

we get that

$$\begin{aligned}
S^{1,1}_{3,2}(y,\tau;a)&\le C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}}{K^{4+4\nu}}\int_a^\tau\int_{\tilde{K}e^{-\eta s}/\sqrt{1-e^{-(\tau-s)}}}^{2e^{2\eta}y/\sqrt{1-e^{-(\tau-s)}}}\frac{e^{-\nu(\tau-s)}y^{2\nu}}{(1-e^{-(\tau-s)})}\\
&\quad\times\Big\{1+\frac{1}{2}\Big|\frac{e^{-(\tau-s)/2}y}{\sqrt{1-e^{-(\tau-s)}}}-\zeta\Big|\zeta\Big\}\exp\Big(-\frac{1}{4}\Big(\frac{e^{-(\tau-s)/2}y}{\sqrt{1-e^{-(\tau-s)}}}-\zeta\Big)^2\Big)\\
&\quad\times\Big\{1+\frac{e^{-(\tau-s)/2}y\zeta}{2\sqrt{1-e^{-(\tau-s)}}}\Big\}^{-(N+4\nu-1)/2}\zeta^{N+4\nu-3}d\zeta ds\\
&= C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}\int_{y^2/(1-e^{-(\tau-a)})}^\infty\int_{D_1}K_3(u,\zeta,\tau-s)d\zeta du\\
&\quad+C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}\int_{y^2/(1-e^{-(\tau-a)})}^\infty\int_{D_2}K_3(u,\zeta,\tau-s)d\zeta du\\
&= S^{1,1}_{3,2}(D_1;a)+S^{1,1}_{3,2}(D_2;a),
\end{aligned}\tag{4.12}$$

where

$$\begin{aligned}
K_3(u,\zeta,\theta)&=\frac{e^{(1-\nu)\theta}}{u}\Big\{1+\frac{1}{2}\Big|e^{-\theta/2}\sqrt{u}-\zeta\Big|\zeta\Big\}\\
&\quad\times\exp\Big(-\frac{1}{4}\Big(e^{-\theta/2}\sqrt{u}-\zeta\Big)^2\Big)\Big\{1+\frac{1}{2}e^{-\theta/2}\sqrt{u}\zeta\Big\}^{-(N+4\nu-1)/2}\zeta^{N+4\nu-3}.
\end{aligned}$$

Noticing $N+4\nu-5\ge 0$, and putting $t=t(\zeta)=\zeta-e^{-(\tau-s)/2}\sqrt{u}$, we observe that

$$S^{1,1}_{3,2}(D_1;a)\le C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}\int_0^\infty u^{(N+4\nu-5)/2}e^{-u/(32e^2)}$$

$$\times \int_{-e^{-(\tau-s)/2}\sqrt{u}/2}^{\infty}(1+|t|\sqrt{u})e^{-t^2/8}dtdu \le \frac{C\tilde{K}^{2\nu}e^{-2\nu\eta\tau}}{K^{4+4\nu}}y^{2\nu}. \tag{4.13}$$

Since we have that

$$\frac{1}{2}e^{-(\tau-s)/2}\sqrt{u} \le \zeta \le \frac{3}{2}e^{-(\tau-s)/2}\sqrt{u} \quad \text{for } \zeta \in D_2,$$

it holds that

$$\begin{aligned} S_{3,2}^{1,1}(D_2;a) \le{}& C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}\int_0^1\int_{D_2}\frac{1}{u}\Big(\frac{\zeta^2}{1+\zeta^2}\Big)^{(N+4\nu-3)/2}d\zeta du \\ &+C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}\int_1^{\infty}\frac{1}{u^2}\int_{|t|\le e^{-(\tau-s)/2}\sqrt{u}/2}\left\{1+\sqrt{u}t\right\}e^{-t^2/4}dtdu \\ \le{}& C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}. \end{aligned} \tag{4.14}$$

Combining (4.12) with (4.13) and (4.14) implies that

$$|S_{3,2}^{1,1}(y,\tau;a)| \le C\frac{\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}}{K^{4+4\nu}}. \tag{4.15}$$

Since it holds that $S_{3,2}^{1,2}(y,\tau;a)=0$ in $[e^{\sigma\tau},\infty)$, the estimate of $S_{3,2}^{1,2}(y,\tau;a)$ in $\sum_O(a,b)$ follows from the one in $\sum(a,b)$.

Since it holds that $N+4\nu-5\ge 0$ and that

$$\frac{1}{u}=\frac{1-e^{-(\tau-s)}}{y^2}\le K^{-2}e^{2\eta\tau} \quad \text{for } y \ge Ke^{-\eta\tau}, \tag{4.16}$$

we observe that

$$\begin{aligned} |S_{3,2}^{1,2}(y,\tau;a)| \le{}& C\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}\int_{y^2/(1-e^{-(\tau-a)})}^{\infty}\int_{2e^{2\eta}\sqrt{u}}^{\infty}\frac{(1+\zeta)}{u} \\ &\times e^{-\zeta^2/16}\left\{1+\frac{1}{2}e^{-(\tau-s)/2}\zeta\sqrt{u}\right\}^{-(N+4\nu-1)/2}\zeta^{N+4\nu-3}d\zeta du \\ \le{}& C\tilde{K}^{2\nu}e^{-2\nu\eta\tau}y^{2\nu}\frac{e^{\eta\tau/2}}{\sqrt{K}}\int_{y^2/(1-e^{-(\tau-a)})}^{\infty}\frac{du}{u^{3/4}(1+u)^{(N+4\nu-1)/2}} \\ \le{}& C\tilde{K}^{2\nu}K^{-1/2}e^{\lambda_2\tau}e^{-\eta\tau/2}y^{2\nu}. \end{aligned} \tag{4.17}$$

For any $\delta\in(0,1]$, it follows from (4.17) and (4.15) that

$$|S_{3,2}^{1}(y,\tau;a)| \le \delta e^{-\eta\tau/2}e^{-\lambda_2\tau}y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b),$$

if $K\gg\tilde{K}\gg 1$ and $\tau_0\gg 1$. Thus, we have this lemma. □

Lemma 4.5. *Let a and b satisfy $\tau_0 \le a < b \le \tau_1$ and $b - a \le 2$. For any $\delta \in (0,1]$, it holds that*

$$|S^2_{3,2}(y,\tau;a)| \le \delta e^{-3\lambda_2\tau/2}(y^{2\nu} + y^{2\lambda_2}) \quad \text{in } \textstyle\sum_O(a,b),$$

if $\tau_0 \gg 1$.

Proof. We observe that

$$\begin{aligned}
S^2_{3,2}(y,\tau;a) &\le C\int_a^\tau \int_{\tilde{K}e^{-\eta s}}^{2e^{2\sigma}\min(y,e^{\sigma s})} K_2(y,\xi,\tau-s)e^{-2\lambda_2 s}\xi^{4\lambda_2}d\xi ds \\
&+ C\int_a^\tau \int_{2e^{2\sigma}\min(y,e^{\sigma\tau})}^{2e^{2\sigma}e^{\sigma s}} K_2(y,\xi,\tau-s)e^{-2\lambda_2 s}\xi^{4\lambda_2}d\xi ds \\
&= S^{2,1}_{3,2}(y,\tau;a) + S^{2,2}_{3,2}(y,\tau;a). \qquad (4.18)
\end{aligned}$$

Since it holds that $S^{2,2}_{3,2}(y,\tau;a) = 0$ for $y \ge e^{\sigma\tau}$, the estimate of $S^{2,2}_{3,2}(\cdot,\cdot;a)$ in $\sum_O(a,b)$ follows from the one in $\sum(a,b)$.

Putting $\zeta = \xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = y^2/(1-e^{-(\tau-s)})$, it follows from (4.16) that

$$\begin{aligned}
S^{2,2}_{3,2}(y,\tau;a) &\le Ce^{-2\lambda_2\tau}y^{2\nu}\int_{y^2/(1-e^{-(\tau-a)})}^\infty \Big(1-e^{-(\tau-s)}\Big)^{2\lambda_2-\nu} \\
&\times \frac{1}{u}\int_{2e^{2\eta}\sqrt{u}}^\infty (1+\zeta^2)\exp\Big(-\frac{u+\zeta^2}{32}\Big)\frac{\zeta^{N+2\nu+4\lambda_2-3}}{\big(1+e^{-(\tau-s)/2}\zeta\sqrt{u}\big)^{(N+4\nu-1)/2}}d\zeta du.
\end{aligned}$$

Noticing $N + 4\nu - 5 \ge 0$, $\lambda_2 = (-2\nu+1)\eta$ and (4.16), for $\delta \in (0,1]$ we observe that

$$|S^{2,2}_{3,2}(y,\tau;a)| \le \frac{C}{K}e^{2\nu\eta\tau}e^{-\lambda_2\tau}y^{2\nu} \quad \text{in } \textstyle\sum(a,b).$$

Then, we have

$$|S^{2,2}_{3,2}(y,\tau;a)| \le \frac{C}{K}e^{2\nu\eta\tau}e^{-\lambda_2\tau}y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b). \qquad (4.19)$$

Putting $\zeta = \xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = y^2/(1-e^{-(\tau-s)})$, we observe that

$$\begin{aligned}
|S^{2,1}_{3,2}(y,\tau;a)| &\le Ce^{-2\lambda_2\tau}y^{2\nu}\int_{y^2/(1-e^{-(\tau-a)})}^\infty \int_{\tilde{K}e^{-\eta s}/\sqrt{1-e^{-(\tau-s)}}}^{2e^{2\sigma}\min(\sqrt{u},e^{\sigma s}/\sqrt{1-e^{-(\tau-s)}})} \frac{1}{u} \\
&\times \Big(1-e^{-(\tau-s)}\Big)^{\lambda_2/2}\Big\{1+|e^{-(\tau-s)/2}\sqrt{u}-\zeta|\zeta\Big\}
\end{aligned}$$

$$\times \exp\left(-\frac{1}{4}\left[e^{-(\tau-s)/2}\sqrt{u}-\zeta\right]^2\right)\frac{\zeta^{N+4\nu-3+\lambda_2}e^{\sigma\lambda_2 s}y^{2\lambda_2-2\nu}}{\left(1+e^{-(\tau-s)/2}\sqrt{u}\zeta\right)^{(N+4\nu-1)/2}}d\zeta du$$
$$\leq \quad Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}\int_{y^2/(1-e^{-(\tau-a)})}^{\infty}\int_{D_1}K_3(u,\zeta,\tau-s)d\zeta du$$
$$+Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}\int_{y^2/(1-e^{-(\tau-a)})}^{\infty}\int_{D_2}K_3(u,\zeta,\tau-s)d\zeta du$$
$$= \quad S_{3,2}^{2,1}(D_1;a)+S_{3,2}^{2,1}(D_2;a), \tag{4.20}$$

where

$$K_3(u,\zeta,\theta)\leq u^{-1}\Big(1+|e^{-\theta/2}\sqrt{u}-\zeta|\zeta\Big)$$
$$\times\exp\Big(-\frac{1}{4}\left[e^{-\theta/2}\sqrt{u}-\zeta\right]^2\Big)\frac{\zeta^{N+4\nu-3+\lambda_2}}{\left(1+e^{-\theta/2}\sqrt{u}\zeta\right)^{(N+4\nu-1)/2}}.$$

Noticing $N+4\nu-5\geq 0$, $\zeta\leq 2e^{2\eta}\sqrt{u}$ and $\lambda_2\in(0,1)$, we have

$$S_{3,2}^{2,1}(D_2;a)\leq Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}\int_{y^2/(1-e^{-(\tau-a)})}^{\infty}\frac{u^{(N+4\nu-5+\lambda_2)/2}(1+\sqrt{u})}{(1+u)^{(N+4\nu-1)/2}}du$$
$$\leq Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}. \tag{4.21}$$

Noticing $0\leq\tau-s\leq 2$, we observe that

$$S_{3,2}^{2,1}(D_1;a) \quad\leq\quad Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}$$
$$\times\int_0^{\infty}\int_0^{2e^{2\eta}\sqrt{u}}\frac{\zeta^{N+4\nu-3+\lambda_2}(1+\sqrt{u})}{u[1+(e^{-(\tau-s)/2}\sqrt{u}\zeta)/2]^{(N+4\nu-1)/2}}e^{-u/(32e^2)}d\zeta du$$
$$\leq\quad Ce^{\sigma\lambda_2\tau}e^{-2\lambda_2\tau}y^{2\lambda_2}.$$

From this, (4.18), (4.19), (4.20), (4.21) and the fact that $0<\sigma<1/2$, for $\delta\in(0,1]$ we obtain that

$$|S_{3,2}^2(y,\tau;a)|\leq\delta e^{-3\lambda_2\tau/2}(y^{2\nu}+y^{2\lambda_2})\quad\text{ in }\textstyle\sum_O(a,b),$$

if $\tau_0\gg 1$. Thus, we finish the proof of this lemma. □

Lemma 4.6. *Let a and b satisfy $\tau_0\leq a<b\leq\tau_1$ and $b-a\leq 2$. For $\delta\in(0,1]$, it holds that*

$$|S_{3,2}^3(y,\tau;a)|\leq\delta e^{-\tau/4}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2})\quad\text{ in }\textstyle\sum_O(a,b),$$

if $\tau_0\gg 1$.

Proof. Let us put

$$
\begin{aligned}
&|S^3_{3,2}(y,\tau;a)| \le C\varepsilon^2 \int_a^\tau \int_{e^{\sigma s}}^{\max(4y,e^{\sigma s})} K_2(y,\xi,\tau-s)d\xi ds\\
&+ C\varepsilon^2 \int_a^\tau \int_{\max(4y,e^{\sigma s})}^\infty K_2(y,\xi,\tau-s)d\xi ds\\
&= S^{3,1}_{3,2}(y,\tau;a) + S^{3,2}_{3,2}(y,\tau;a).
\end{aligned}
$$

Since it holds that for $y \le e^{\sigma\tau}/2$ and $e^{\sigma s} \le \xi$

$$
\begin{aligned}
K_2(y,\xi,\tau-s) &\le \frac{Ce^{-\nu(\tau-s)}y^{2\nu}}{(1-e^{-(\tau-s)})^{(N+4\nu)/2}}\\
&\quad \times \exp\Big(-\frac{e^{2\sigma s}+\xi^2}{32(1-e^{-(\tau-s)})}\Big)\Big\{1+\frac{\xi^2}{2(1-e^{-(\tau-s)})}\Big\}\xi^{N+2\nu-3}\\
&\le \quad Cy^{2\nu}e^{-(N+4\nu)\sigma s}\xi^{N+2\nu-3}\exp\Big(-\frac{e^{2\sigma s}}{64}\Big)e^{-\xi^2/32},
\end{aligned}
$$

we observe that

$$
\begin{aligned}
&|S^{3,2}_{3,2}(y,\tau;a)|\\
&\le C\varepsilon^2 \exp\Big(-\frac{e^{2\sigma\tau}}{64e^{4\sigma}}\Big)y^{2\nu}\int_a^\tau e^{-(N+4\nu)\sigma s}\int_{4y}^\infty \xi^{N+2\nu-3}e^{-\xi^2/32}d\xi ds\\
&\le C\exp\Big(-\frac{e^{2\sigma\tau}}{64e^{4\sigma}}\Big)y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b)\cap\big\{(y,\tau)\,\big|\, y\le \frac{e^{\sigma\tau}}{2}\big\}. \qquad (4.22)
\end{aligned}
$$

It holds that for $e^{\sigma\tau}/2 \le y \le \xi/4$

$$
\begin{aligned}
K_2(y,\xi,\tau-s) &\le \frac{Ce^{-\nu(\tau-s)}y^{2\nu}}{(1-e^{-(\tau-s)})^{(N+4\nu)/2}}\exp\Big(-\frac{y^2+\xi^2}{16(1-e^{-(\tau-s)})}\Big)\\
&\quad \times\Big\{1+\frac{y^2}{1-e^{-(\tau-s)}}\Big\}^{-(N+4\nu-1)/2}\Big\{1+\frac{\xi^2}{1-e^{-(\tau-s)}}\Big\}\xi^{N+2\nu-3}.
\end{aligned}
$$

Using this and putting $\zeta = \xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = y^2/(1-e^{-(\tau-s)})$, we then observe that

$$
\begin{aligned}
&|S^{3,2}_{3,2}(y,\tau;a)| \le C\varepsilon^2 e^{-(2\nu+3)\sigma\tau}\exp\Big(-\frac{e^{\sigma\tau}}{32}\Big)y^{2\nu}\\
&\quad \times\int_{e^{2\sigma\tau}/[2(1-e^{-(\tau-a)})]}^\infty \int_{4\sqrt{u}}^\infty \frac{1}{u}\zeta^{N+2\nu-3}(1+\zeta^2)\exp\Big(-\frac{u+\zeta^2}{32}\Big)d\zeta du\\
&\le \quad C\exp\Big(-\frac{e^{\sigma\tau}}{32}\Big)y^{2\nu} \quad \text{in } \big\{(y,\tau)\in \textstyle\sum_O(a,b) : y\ge \frac{e^{\sigma\tau}}{2}\big\}.
\end{aligned}
$$

Combining this with (4.22) implies that for $\delta \in (0,1]$

$$|S_{3,2}^{3,2}(y,\tau;a)| \le \frac{\delta}{2} e^{-\tau/4} e^{-\lambda_2 \tau} y^{2\nu} \quad \text{in } \textstyle\sum_O(a,b), \tag{4.23}$$

if $\tau_0 \gg 1$.

Putting $\zeta = \xi/\sqrt{1-e^{-(\tau-s)}}$ and $u = y^2/(1-e^{-(\tau-s)})$, for $e^{\sigma(\tau-2)}/4 \le y$ we observe that

$$\begin{aligned} S_{3,2}^{3,1}(y,\tau;a) &\le C \int_{y^2/(1-e^{-(\tau-a)})}^{\infty} \int_{e^{\sigma s}/\sqrt{1-e^{-(\tau-s)}}}^{4e^{2\sigma}\sqrt{u}} u^{\nu-1} \\ &\quad \times \exp\Big(-\frac{1}{8}(e^{-(\tau-s)/2}\sqrt{u} - \zeta)^2 \Big) \frac{\zeta^{N+2\nu-3}}{(1+\zeta^2)^{(N+4\nu-2)/2}} d\zeta du \\ &\le \ C \int_{e^{2\sigma\tau}/(16e^{4\sigma})}^{\infty} \frac{u^{(N+4\nu-5)/2}}{(1+u)^{(N+4\nu-2)/2}} du \le Ce^{-(2\lambda_2+1)\sigma\tau} y^{2\lambda_2}. \end{aligned}$$

Since it holds that $S_{3,2}^{3,1}(\cdot,\cdot;a) = 0$ in $\sum_O(a,b) \cap \{y \le e^{\sigma(\tau-2)}/4\}$ and $(2\lambda_2 + 1)\sigma = (3\lambda_2+1)/3$, for $\delta \in (0,1]$ we have

$$|S_{3,2}^{3,1}(y,\tau;a)| \le \frac{\delta}{2} e^{-\tau/4} e^{-\lambda_2\tau} y^{2\lambda_2} \quad \text{in } \textstyle\sum_O(a,b),$$

if $\tau_0 \gg 1$. Combining this with (4.23) implies that for $\delta \in (0,1]$

$$S_{3,2}^{3}(y,\tau) \le \delta e^{-\tau/4} e^{-\lambda_2\tau}(y^{2\nu} + y^{2\lambda_2}) \quad \text{in } \textstyle\sum_O(a,b),$$

if $\tau_0 \gg 1$. Thus, we get this lemma. □

Let us put $\mu = \min(\eta/3, 1/4, \lambda_2/2)$.

From Lemmas 4.3, 4.4, 4.5 and 4.6, we get the following lemma.

Lemma 4.7. *Let a and b satisfy $\tau_0 \le a < b \le \tau_1$ and $b - a \le 2$. For $\delta \in (0,1]$, it holds that*

$$|S_3(y,\tau;a)| \le \delta e^{-\mu\tau} e^{-\lambda_2\tau}(y^{2\nu} + y^{2\lambda_2}) \quad \textit{in } \textstyle\sum_O(a,b),$$

if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$.

4.1.3. *End of estimates where $\tau_0 \le \tau \le \min(\tau_0+1, \tau_1)$.* By using (3.12), for $\delta \in (0,1]$ we have

$$\Big| S_1(y,\tau) + e^{-\lambda_2\tau}\varphi_2(y) \Big| \le \delta e^{-\lambda_2\tau}(y^{2\nu} + y^{2\lambda_2}) \quad \text{in } \textstyle\sum(\tau_0, \tau_1^*),$$

if $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$. By this, Lemmas 4.2 and 4.7 with $a = \tau_0$, and $b = \tau_1^*$, we finish the proof of Proposition 3.3, in the case where $\tau_0 \le \tau \le \min(\tau_1, \tau_0+1)$, $K \gg \tilde{K} \gg 1$ and $\tau_0 \gg 1$.

4.1.4. *Estimates in the case where* $\tau_0 + 1 \le \tau \le \tau_1$. Henceforth, K, $\tilde{K}$, τ_0 and δ are positive constants determined in Section 4.1

Throughout this subsection, we assume that $P(d; \tau_0, \tau_1) = 0$ for a $d \in \mathcal{U}(\tau_0, \tau_1)$ and $\tau_0 + 1 \le \tau \le \tau_1$.

Since it follows from (4.3) that

$$\phi(\cdot, \tau_1) = e^{-(\tau_1-\tau_0)\tilde{A}}\phi(\cdot, \tau_0) + \int_{\tau_0}^{\tau_1} e^{-(\tau_1-s)\tilde{A}} F(\cdot, \phi(\cdot, s))ds,$$

then we observe that for $j = 0, 1$

$$\langle \phi(\cdot, \tau_1), \varphi_j \rangle = \langle e^{-(\tau_1-\tau_0)\tilde{A}}\phi(\cdot, \tau_0), \varphi_j \rangle + \int_{\tau_0}^{\tau_1} e^{-(\tau_1-s)\tilde{A}} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle ds$$

or

$$e^{-\lambda_j(\tau-\tau_0)}\langle \phi(\cdot, \tau_0), \varphi_j \rangle = -\int_{\tau_0}^{\tau_1} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle ds. \tag{4.24}$$

For $j = 2, 3, 4, \cdots$, it holds that

$$\begin{aligned}
&\langle \phi(\cdot, \tau), \varphi_j \rangle = e^{-\lambda_j(\tau-\tau_0)}\langle \phi(\cdot, \tau_0), \varphi_j \rangle + \int_{\tau_0}^{\tau} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle ds \\
&= -e^{-\lambda_2\tau_0} e^{-\lambda_j(\tau-\tau_0)}\langle \tilde{\varphi}_2, \varphi_j \rangle + \int_{\tau_0}^{\tau} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle ds. \qquad (4.25)
\end{aligned}$$

Let us put $\ell = [\tau - \tau_0]$ and $\tilde{\tau}_k = k + \tau_0$ $(k = 0, 1, 2, \cdots, \ell + 1)$, where $[x]$ is the maximum integer which is equal or less than x.

By (4.3), (4.24) and (4.25), we have

$$\begin{aligned}
&\phi(\cdot, \tau) = -e^{-\lambda_2\tau}\langle \tilde{\varphi}_2, \varphi_2 \rangle \varphi_2 - \sum_{j=0}^{1} \int_{\tau}^{\tau_1} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle \varphi_j ds \\
&- \sum_{j=3}^{\infty} e^{-\lambda_2\tau_0} e^{-\lambda_j(\tau-\tau_0)}\langle \tilde{\varphi}_2, \varphi_j \rangle \varphi_j + \sum_{j=2}^{\infty} \int_{\tau_0}^{\tau} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle \varphi_j ds \\
&= -M_0(y, \tau) - M_1(y, \tau) - M_2(y, \tau) + M_4(y, \tau) \qquad (4.26)
\end{aligned}$$

for $\tau \in [\tilde{\tau}_1, \tilde{\tau}_2) \cap [\tau_0, \tau_1]$ and that

$$\begin{aligned}
&\phi(\cdot, \tau) = -e^{-\lambda_2\tau}\langle \tilde{\varphi}_2, \varphi_2 \rangle \varphi_2 - \sum_{j=0}^{1} \int_{\tau}^{\tau_1} e^{-\lambda_j(\tau-s)} \langle F(\cdot, \phi(\cdot, s)), \varphi_j \rangle \varphi_j ds \\
&- \sum_{j=3}^{\infty} e^{-\lambda_2\tau_0} e^{-\lambda_j(\tau-\tau_0)}\langle \tilde{\varphi}_2, \varphi_j \rangle \varphi_j
\end{aligned}$$

$$+\sum_{i=0}^{k-1} e^{-(\tau-\tilde{\tau}_{i+1})\tilde{A}} \sum_{j=2}^{\infty} \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} e^{-\lambda_j(\tilde{\tau}_{i+1}-s)} \langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle \varphi_j ds$$

$$+\sum_{j=2}^{\infty} \int_{\tilde{\tau}_k}^{\tau} e^{-\lambda_j(\tau-s)} \langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle \varphi_j ds \tag{4.27}$$

$$= -M_0(y,\tau) - M_1(y,\tau) - M_2(y,\tau) + \sum_{i=0}^{k-1} e^{-(\tau-\tilde{\tau}_{i+1})\tilde{A}} M_{3,i}(y) + M_4(y,\tau)$$

for $\tau \in [\tilde{\tau}_{k+1}, \tilde{\tau}_{k+2}) \cap [\tau_0, \tau_1]$ and $k = 1, 2, 3, \ldots, \ell - 1$.

By using an argument similar to the way of showing (3.17), we observe that

$$|M_1(y,\tau)| \le \frac{\delta}{2} e^{-\eta\tau/3} e^{-\lambda_2\tau} (y^{2\nu} + y^{2\lambda_2}) \quad \text{in } (0,\infty) \times [\tau_0, \tau_1]. \tag{4.28}$$

Let us put

$$\textstyle\sum_I(a,b) = \left\{(y,\tau) \mid 0 < y \le e^{(\tau-a)/2},\ a \le \tau \le b\right\}.$$

Lemma 4.8. *For $j = 0, 1, 2, \cdots$, it holds that*

$$|\varphi_j(y)| \le C(j+1)^{\alpha/2} y^{2\nu} \left(1 + \frac{y^2}{4}\right)^j \quad \textit{in } (0,\infty).$$

Lemma 4.8 will be proved in Section 5.5.

Henceforth, C represents a positive constant which is independent of $\mathcal{C}_3 = \mathcal{C}_1 \cup \{\{i\}_{0\le i\le k-1}, \{k\}_{1\le k\le \ell-1}, \ell\}$. Then, C may be different from the other C's.

Lemma 4.9. *It holds that*

$$|M_2(y,\tau)| \le C\delta e^{-\eta\tau_0} e^{-\lambda_2\tau} (y^{2\nu} + y^{2\lambda_2}) \quad \textit{in } \textstyle\sum_I(\tau_0+1, \tau_1).$$

Proof. In this proof, we suppose that $j \ge 3$. It follows from $\tau \ge \tau_0 + 1$ that

$$|M_2(y,\tau)| \le e^{-\lambda_2\tau} \sum_{j=3}^{\infty} |\langle \tilde{\varphi}_2, \varphi_j\rangle| |\varphi_j(y)| e^{-(\lambda_j-\lambda_2)(\tau-\tau_0)}. \tag{4.29}$$

Noticing the estimate of (3.12) is independent of $j \ge 3$ and using (4.29), we have

$$|M_2(y,\tau)| \le \frac{\delta}{4} e^{-\eta\tau_0} e^{-\lambda_2\tau} \sum_{j=3}^{\infty} |\varphi_j(y)| e^{-(j-2)(\tau-\tau_0)}$$

for $y > 0$ and $\tau \geq \tau_0 + 1$. It follows from Lemma 4.8 that

$$|M_2(y,\tau)| \leq C\delta e^{-\eta\tau_0} e^{-\lambda_2\tau} y^{2\nu}\Big(1+\frac{y^2}{4}\Big)^2 \sum_{j=3}^{\infty} j^{\alpha/2}\Big(1+\frac{y^2}{4}\Big)^{j-2} e^{-(j-2)(\tau-\tau_0)}$$

$$\leq C\delta e^{-\eta\tau_0} e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \sum_{j=3}^{\infty} j^{\alpha/2}\Big[\frac{1}{e}+\frac{1}{4}\Big(e^{-(\tau-\tau_0)/2}y\Big)^2\Big]^{j-2}.$$

Combining this with

$$\frac{1}{e}+\frac{1}{4}\Big(e^{-(\tau-\tau_0)/2}y\Big)^2 \leq \frac{3}{4} \quad \text{in } \textstyle\sum_I(\tau_0+1,\tau_1)$$

implies that

$$|M_2(y,\tau)| \leq C\delta e^{-\eta\tau_0} e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \quad \text{in } \textstyle\sum_I(\tau_0+1,\tau_1).$$

Thus, we have this lemma. □

Lemma 4.10. *Suppose that $\tau \in [\tilde\tau_{k+1}, \tilde\tau_{k+2}]$ for $k = 1, 2, 3, \cdots, \ell - 1$. For $i = 0, 1, 2, \cdots, k-1$, it holds that*

$$\Big|e^{-(\tau-\tilde\tau_{i+1})\tilde A} M_{3,i}(y)\Big| \leq C\delta e^{-\mu\tilde\tau_{i+1}} e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \quad \textit{in } (0, e^{(\tau-\tilde\tau_{i+1})/2}].$$

Proof. It holds that

$$\Big|e^{-(\tau-\tilde\tau_{i+1})\tilde A} M_{3,i}(y)\Big| = \Big|\sum_{j=2}^{\infty}\int_{\tilde\tau_i}^{\tilde\tau_{i+1}} e^{-\lambda_j(\tau-s)}\langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle\varphi_j ds\Big|$$

$$\leq \int_{\tilde\tau_i}^{\tilde\tau_{i+1}} e^{-\lambda_2(\tau-s)}\Big(\sum_{j=2}^{\infty}|\varphi_j(y)|^2(1+\lambda_j^3)e^{-2(\lambda_j-\lambda_2)(\tau-s)}\Big)^{1/2}$$

$$\times\Big(\sum_{j=2}^{\infty}\frac{1}{1+\lambda_j^3}|\langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle|^2\Big)^{1/2} ds. \tag{4.30}$$

It follows from Lemma 4.8 , $\tau - \tilde\tau_{i+1} \geq \tau - \tilde\tau_k \geq 1$, $\tilde\tau_{i+1} \geq s$, (2.5) and the fact that

$$\frac{\Gamma(j+p+1)}{j!} \leq Cj^p \quad \text{for } p \in \mathbf{R} \text{ and } j \geq \max(p,1) \tag{4.31}$$

that

$$\sum_{j=2}^{\infty}|\varphi_j(y)|^2(1+\lambda_j^3)e^{-2(\lambda_j-\lambda_2)(\tau-s)}$$

$$\le Cy^{4\nu}\big(1+\frac{y^2}{4}\big)^4\sum_{j=2}^{\infty}j^{\alpha+3}\Big[\frac{1}{\sqrt{e}}+\frac{1}{4}(e^{-(\tau-\tilde{\tau}_{i+1})/2}y)^2\Big]^{2(j-2)}\le C\big(y^{2\nu}+y^{2\lambda_2}\big)^2$$

and that

$$\sum_{j=2}^{\infty}\frac{|\langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle|^2}{1+\lambda_j^3}\le C\|F(\cdot,\phi(\cdot,s))\|_{-1}^2.$$

Combining this with (4.30) implies that

$$\big|e^{-(\tau-\tilde{\tau}_{i+1})\tilde{A}}M_{3,i}(y)\big|\le C\Big[\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}}e^{-\lambda_2(\tau-s)}\|F(\cdot,\phi(\cdot,s))\|_{-1}ds\Big](y^{2\nu}+y^{2\lambda_2})$$

in $(0,e^{(\tau-\tilde{\tau}_{i+1})/2}]$. This lemma follows from this and (3.16). □

Lemma 4.11. *Suppose* $\tau\in[\tilde{\tau}_{k+1},\tilde{\tau}_{k+2})\cap[\tau_0,\tau_1]$ *and* $k=0,1,2,\cdots,\ell-1$. *It holds that*

$$|M_4(y,\tau)|\le C\delta e^{-\mu\tilde{\tau}_{k+1}}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2})\quad \textit{in}\ \ [Ke^{-\eta\tau},e^{\sigma\tau}].$$

Proof. We have

$$\begin{aligned}M_4(y,\tau)&=S_3(y,\tau;\tilde{\tau}_k)-\sum_{j=0}^{1}\int_{\tilde{\tau}_k}^{\tau}e^{-\lambda_j(\tau-s)}\langle F(\cdot,\phi(\cdot,s)),\varphi_j\rangle ds\varphi_j(y)\\&=S_3(y,\tau;\tilde{\tau}_k)-M_{4,1}(y,\tau).\end{aligned}\tag{4.32}$$

Notice that $\tau-\tilde{\tau}_k\le\tilde{\tau}_{k+2}-\tilde{\tau}_k=2$. From Lemma 4.7 with $a=\tilde{\tau}_k$ and $b=\tau$, we have

$$|S_3(y,\tau;\tilde{\tau}_k)|\le\delta e^{-\mu\tau}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2})\quad\text{in }[Ke^{-\eta\tau},e^{\sigma\tau}].\tag{4.33}$$

By using the way of showing (3.17), we observe that

$$\begin{aligned}|M_{4,1}(y,\tau)|&\le\sum_{j=0}^{1}\sqrt{19}\int_{\tilde{\tau}_k}^{\tau}e^{-\lambda_j(\tau-s)}\|F(\cdot,\phi(\cdot,s))\|_{-1}ds|\varphi_j(y)|\\&\le C\delta e^{-\mu\tilde{\tau}_{k+1}}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2})\quad\text{in }(0,\infty).\end{aligned}\tag{4.34}$$

This lemma follows from this, (4.32) and (4.33). □

Lemma 4.12. *For* $i=0,1,2,\cdots,k-1$, *it holds that*

$$|M_{3,i}(y)|\le\frac{C}{K^{2+2\nu}}e^{-\lambda_2\tilde{\tau}_{i+1}}y^{2\nu}\quad\textit{for }y\in(0,Ke^{-\eta\tilde{\tau}_{i+1}}),\tag{4.35}$$

and that

$$|M_{3,i}(y)|\le C\delta e^{-\mu\tilde{\tau}_{i+1}}e^{-\lambda_2\tilde{\tau}_{i+1}}(y^{2\nu}+y^{2\lambda_2})\ \ \textit{for }y\in[Ke^{-\eta\tilde{\tau}_{i+1}},\infty).\tag{4.36}$$

Proof. By using the way of showing that

$$\begin{aligned}\sum_{j=0}^{1}\int_{\tilde\tau_i}^{\tilde\tau_{i+1}} e^{-\lambda_j(\tilde\tau_{i+1}-s)}\,|\langle F(\cdot,\phi(\cdot,\tau)),\varphi_j\rangle|\,ds|\varphi_j(y)| \\ \le\ C\delta e^{-\mu\tilde\tau_{i+1}}e^{-\lambda_2\tilde\tau_{i+1}}(y^{2\nu}+y^{2\lambda_2}) \quad \text{in } (0,\infty).\end{aligned}$$

Combining this and

$$M_{3,i}(y)=S_3(y,\tilde\tau_{i+1};\tilde\tau_i)-\sum_{j=0}^{1}\int_{\tilde\tau_i}^{\tilde\tau_{i+1}} e^{-\lambda_j(\tilde\tau_{i+1}-s)}\langle F(\cdot,\phi(\cdot,\tau)),\varphi_j\rangle ds\varphi_j(y) \tag{4.37}$$

with Lemma 4.7 implies (4.36). We observe that

$$\begin{aligned}\phi(y,\tilde\tau_{i+1}) \ &= \ -M_0(y,\tilde\tau_{i+1})-M_1(y,\tilde\tau_{i+1})-M_2(y,\tilde\tau_{i+1}) \\ &\quad +\sum_{m=0}^{i-1}e^{-(\tilde\tau_{i+1}-\tilde\tau_{m+1})\tilde A}M_{3,m}(y)+M_{3,i}(y).\end{aligned} \tag{4.38}$$

Combining (4.38) with $\delta\in(0,1]$, (3.12), (4.5), (4.28) and Lemmas 4.9 and 4.10 implies that

$$\begin{aligned}|M_{3,i}(y)| \ &\le\ \Big(\frac{C}{K^{2+2\nu}}+C\delta e^{-\eta\tau_0}+\frac{\delta}{2}e^{-\eta\tilde\tau_{i+1}/3}+C\delta e^{-\eta\tau_0/2} \\ &\qquad +\sum_{m=0}^{i-1}C\delta e^{-\mu\tilde\tau_{m+1}}\Big)e^{-\lambda_2\tilde\tau_{i+1}}(y^{2\nu}+y^{2\lambda_2}) \\ &\le\ \frac{C}{K^{2+2\nu}}e^{-\lambda_2\tilde\tau_{i+1}}(y^{2\nu}+y^{2\lambda_2}).\end{aligned}$$

Thus, we finish the proof of this lemma. □

Lemma 4.13. *It holds that*

$$|M_2(y,\tau_0)|\le\frac{C}{K^{2+2\nu}}e^{-\lambda_2\tau_0}y^{2\nu} \quad \textit{in } (0,Ke^{-\eta\tau_0}),$$

$$|M_2(y,\tau_0)|\le C\delta e^{-\mu\tau_0}e^{-\lambda_2\tau_0}(y^{2\nu}+y^{2\lambda_2}) \quad \textit{in } [Ke^{-\eta\tau_0},e^{\sigma\tau_0}]$$

and that

$$|M_2(y,\tau_0)|\le Ce^{-\lambda_2\tau_0}y^{2\lambda_2} \quad \textit{in } (e^{\sigma\tau_0},\infty).$$

Proof. Since it holds that

$$M_2(y,\tau)=e^{-\lambda_2\tau_0}\Big(\tilde\varphi_2(y)-\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2(y)-\sum_{j=0}^{1}\langle\tilde\varphi_2,\varphi_j\rangle\varphi_j(y)\Big)$$

$$= -\phi(y,\tau_0) - e^{-\lambda_2\tau_0}\langle\tilde\varphi_2,\varphi_2\rangle\varphi_2(y) + \sum_{j=0}^{1}\left(d_j - \langle\tilde\varphi_2,\varphi_j\rangle\right)\varphi_j(y),$$

we observe that

$$|M_2(y,\tau_0)| \le \frac{C}{K^{2+2\nu}}e^{-\lambda_2\tau_0}y^{2\nu} \quad \text{in } (0, Ke^{-\eta\tau_0}]$$

and that

$$|M_2(y,\tau_0)| \le Ce^{-\lambda_2\tau_0}(y^{2\nu}+y^{2\lambda_2}) \quad \text{in } [e^{\sigma\tau_0},\infty).$$

By using

$$\tilde\varphi_2(y) - \langle\tilde\varphi_2,\varphi_2\rangle\varphi_2(y) - \sum_{j=0}^{1}\langle\tilde\varphi_2,\varphi_j\rangle\varphi_j(y) = \sum_{j=0}^{2}\langle\varphi_2-\tilde\varphi_2,\varphi_j\rangle\varphi_j(y)$$

in $[Ke^{-\eta\tau_0}, e^{\sigma\tau_0}]$ and (3.12), we have

$$|M_2(y,\tau_0)| \le C\delta e^{-\eta\tau_0}e^{-\lambda_2\tau_0}(y^{2\nu}+y^{2\lambda_2}) \quad \text{in } [Ke^{-\eta\tau_0}, e^{\sigma\tau_0}].$$

Thus, we finish the proof of this lemma. □

Lemma 4.14. *Let $k = 1,2,3,\cdots,\ell-1$ and $\tau\in[\tilde\tau_{k+1},\tau_1]$. Then, for $i = 0,1,2,\cdots,k-1$ it holds that*

$$|e^{-(\tau-\tilde\tau_{i+1})\tilde A}M_{3,i}(y)| \le C(K^{N+2\nu-2}e^{-\lambda_2\tau_0})\delta e^{-\mu\tilde\tau_{i+1}}e^{-\lambda_2\tau}y^{2\lambda_2} \quad \textit{for } y\in[e^{(\tau-\tilde\tau_{i+1})/2}, e^{\sigma\tau}].$$

Proof. It follows from the definition of $K(y,\xi,\theta)$ that

$$|e^{-(\tau-\tilde\tau_{i+1})\tilde A}M_{3,i}(y)| \le \int_0^{Ke^{-\eta\tilde\tau_{i+1}}} K(y,\xi,\tau-\tilde\tau_{i+1})|M_{3,i}(\xi)|d\xi$$

$$+\int_{Ke^{-\eta\tilde\tau_{i+1}}}^{e^{\sigma\tilde\tau_{i+1}}} K(y,\xi,\tau-\tilde\tau_{i+1})|M_{3,i}(\xi)|d\xi + \int_{e^{\sigma\tilde\tau_{i+1}}}^{\infty} K(y,\xi,\tau-\tilde\tau_{i+1})|M_{3,i}(\xi)|d\xi$$

$$= M_{3,i}^1(y,\tau) + M_{3,i}^2(y,\tau) + M_{3,i}^3(y,\tau).$$

From (4.36) and $\tau-\tilde\tau_{i+1}\ge 1$, we have

$$|M_{3,i}^2(y,\tau)| \le C\delta e^{-(\lambda_2+\mu)\tilde\tau_{i+1}}\int_{Ke^{-\eta\tilde\tau_{i+1}}}^{e^{\sigma\tilde\tau_{i+1}}} K(y,\xi,\tau-\tilde\tau_{i+1})(\xi^{2\nu}+\xi^{2\lambda_2})d\xi. \tag{4.39}$$

For $x>0$, putting

$$K_4(\xi,x) = \frac{\xi^{N+4\nu-1}}{(1+\xi x)^{(N+4\nu-1)/2}}(1+\xi^{2\lambda_2-2\nu})\exp\Big(-\frac{(x-\xi)^2}{4(1-e^{-1})}\Big)$$

and using $\tau - \tilde{\tau}_{i+1} \geq 1$, we observe that

$$K(y,\xi,\tau-\tilde{\tau}_{i+1}) \leq Ce^{-\nu(\tau-\tilde{\tau}_{i+1})}y^{2\nu}K_4(e^{-(\tau-\tilde{\tau}_{i+1})/2}y,\xi). \tag{4.40}$$

Putting

$$\begin{aligned} f(x) &= \int_0^\infty K_4(\xi,x)d\xi = \int_{|\xi-x|\leq x/2} K_4(\xi,x)d\xi + \int_{|\xi-x|\geq x/2} K_4(\xi,x)d\xi \\ &= f_1(x) + f_2(x), \end{aligned}$$

we have

$$f_1(x) \leq C(1+x^{2\lambda_2-2\nu})\int_{x/2}^{3x/2} \exp\Big(-\frac{(x-\xi)^2}{4(1-e^{-1})}\Big)d\xi \leq C(1+x^{2\lambda_2-2\nu})$$

and that

$$\begin{aligned} f_2(x) \leq C\int_0^\infty &\left[(\xi-x)^{N+4\nu-1}+x^{N+4\nu-1}\right]\left[1+(\xi-x)^{2\lambda_2-2\nu}+x^{2\lambda_2-2\nu}\right] \\ &\times \exp\Big(-\frac{(x-\xi)^2}{8(1-e^{-1})}\Big)\exp\Big(-\frac{x^2}{32(1-e^{-1})}\Big)d\xi \leq C. \end{aligned}$$

Noticing $e^{-(\tau-\tilde{\tau}_{i+1})/2}y \geq 1$ and using those, (4.39) and (4.40), for $\delta \in (0,1]$ we observe that

$$\begin{aligned} |M_{3,i}^2(y,\tau)| &\leq C\delta e^{-\nu(\tau-\tilde{\tau}_{i+1})}e^{-(\lambda_2+\mu)\tilde{\tau}_{i+1}}y^{2\nu}\left(e^{-(\tau-\tilde{\tau}_{i+1})/2}y\right)^{2\lambda_2-2\nu} \\ &\leq C\delta e^{-\mu\tilde{\tau}_{i+1}}e^{-\lambda_2\tau}y^{2\lambda_2} \qquad \text{for } y \geq e^{(\tau-\tilde{\tau}_{i+1})/2}. \end{aligned} \tag{4.41}$$

It follows from (4.35) and (4.40) that

$$\begin{aligned} |M_{3,i}^1(y,\tau)| &\leq Cy^{2\nu}\int_0^{Ke^{-\eta\tilde{\tau}_{i+1}}} K^{-2-2\nu}e^{-\nu(\tau-\tilde{\tau}_{i+1})}e^{-\lambda_2\tilde{\tau}_{i+1}}\xi^{N+4\nu-1}d\xi \\ &\leq CK^{N+2\nu-2}e^{-\nu(\tau-\tilde{\tau}_{i+1})}e^{-(N+4\nu)\eta\tilde{\tau}_{i+1}}e^{-\lambda_2\tilde{\tau}_{i+1}}y^{2\nu}. \end{aligned}$$

Combining this with

$$y^{-2\lambda_2+2\nu} \leq e^{(-\lambda_2+\nu)(\tau-\tilde{\tau}_{i+1})} \qquad \text{for } y \geq e^{(\tau-\tilde{\tau}_{i+1})/2}$$

implies that

$$|M_{3,i}^1(y,\tau)| \leq CK^{N+2\nu-2}e^{-(N+4\nu)\eta\tilde{\tau}_{i+1}}e^{-\lambda_2\tau}y^{2\lambda_2}.$$

Using $N+6\nu-2 \geq 0$, for $\delta > 0$ we have

$$|M_{3,i}^1(y,\tau)| \leq CK^{N+2\nu-2}\delta e^{-(\lambda_2+\eta)\tilde{\tau}_{i+1}}e^{-\lambda_2\tau}y^{2\lambda_2} \text{ for } y \in [e^{(\tau-\tilde{\tau}_{i+1})/2}, e^{\sigma\tau}]. \tag{4.42}$$

Since we treat the term $M^3_{3,i}$ in the region $\xi - e^{-(\tau-\tilde{\tau}_{i+1})/2}y \geq (1 - e^{-(1-2\sigma)/2})\xi$, then we have

$$\begin{aligned} |M^3_{3,i}(y,\tau)| &\leq C\delta e^{-\nu(\tau-\tilde{\tau}_{i+1})}y^{2\nu}\exp\Big(-\frac{1}{8}(1-e^{-(1-2\sigma)/2})^2e^{2\sigma\tilde{\tau}_{i+1}}\Big) \\ &\leq C\delta e^{-\lambda_2(\tau-\tilde{\tau}_{i+1})}\exp\Big(-\frac{1}{8}(1-e^{-(1-2\sigma)/2})^2e^{2\sigma\tilde{\tau}_{i+1}}\Big)y^{2\lambda_2}. \end{aligned}$$

Then, for $\delta > 0$ it holds that

$$|M^3_{3,i}(y,\tau)| \leq C\delta e^{-(\lambda_2+\eta)\tilde{\tau}_{i+1}}e^{-\lambda_2\tau}y^{2\lambda_2} \quad \text{for } y \in [e^{(\tau-\tilde{\tau}_{i+1})/2}, e^{\sigma\tau}].$$

By this, (4.42) and (4.41), we have this lemma. □

By using Lemma 4.13 and an argument similar to the proof of Lemma 4.14, we can proof the following lemma. Thus, we omit the proof.

Lemma 4.15. *It holds that*

$$|M_2(y,\tau)| \leq C(K^{N+2\nu-2}e^{-\lambda_2\tau_0})\delta e^{-\mu\tau_0}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \quad \textit{for } y \in [e^{(\tau-\tau_0)/2}, e^{\sigma\tau}] \quad \textit{and } \tau \in [\tau_0+1, \tau_1].$$

In the case where $\tau_0+1 \leq \tau \leq \tau_1$, τ is in $[\tilde{\tau}_{k+1}, \tilde{\tau}_{k+2})$ for a $k \in \{0,1,2,\ldots, \ell-1\}$, it follows from (3.12), (4.26), (4.27), (4.28) and Lemmas 4.9, 4.15, 4.10, 4.14 and 4.11 that

$$\begin{aligned} &|\phi(y,\tau) + e^{-\lambda_2\tau}\varphi_2(y)| \leq C\delta\Big[\tfrac{1}{4}e^{-\eta\tau_0} + \tfrac{1}{2}e^{-\eta\tau/3} + (1+K^{N+2\nu-2}e^{-\lambda_2\tau_0})e^{-2\eta\tau_0} \\ &+ (1+K^{N+2\nu-2}e^{-\lambda_2\tau_0})\sum_{i=0}^{k-1}e^{-\mu\tilde{\tau}_{i+1}} + e^{-\mu\tilde{\tau}_{k+1}}\Big]e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \\ &\leq C(1+K^{N+2\nu-2}e^{-\lambda_2\tau_0})\delta e^{-\mu\tau_0}e^{-\lambda_2\tau}(y^{2\nu}+y^{2\lambda_2}) \end{aligned} \tag{4.43}$$

for $y \in [Ke^{-\eta\tau}, e^{\sigma\tau}]$.

Replacing τ_0 by $\max(\tau_0, [\log(2C)]/\mu, (\log K^{N+2\nu-2}]/\lambda_2)$, Proposition 3.3 is shown in the case where $\tau_0+1 \leq \tau \leq \tau_1$, where C is the constant in (4.43).

4.2. **End of proof.** Proposition 3.3 follows from Sections 4.1 and 4.2.

5. Some technical results

In this section, we shall give the proofs of some lemmas that have been stated in the previous sections.

5.1. **Proof of Lemma 2.2.** For $g \in H^1_w \cap L^\infty((0,\infty))$ and $\beta > 0$, we observe that

$$\begin{aligned} &\int_0^\infty \frac{1}{y^2}|g(y)|^2 y^{N-1} e^{-y^2/4} dy \\ &= \frac{1}{N-2}\int_0^\infty \left(-2g(y)g_y(y) + \frac{y}{2}|g(y)|^2\right) y^{N-2} e^{-y^2/4} dy \\ &\leq \frac{\beta}{N-2}\int_0^\infty \frac{1}{y^2}|g(y)|^2 y^{N-1} e^{-y^2/4} dy + \frac{\|g_y\|^2}{\beta(N-2)} + \frac{\|g\|^2}{2(N-2)}. \end{aligned}$$

Then, it holds that

$$\left\{1 - \frac{\beta}{N-2}\right\} \int_0^\infty \frac{1}{y^2}|g(y)|^2 y^{N-1} e^{-y^2/4} dy \leq \frac{\|g_y\|^2}{\beta(N-2)} + \frac{\|g\|^2}{2(N-2)}.$$

Putting $\beta = (N-2)/2$, we have this lemma for $g \in H^1_w \cap L^\infty((0,\infty))$.

For $g \in H^1_w$ and $n \geq 1$, putting

$$g_n(x) = \begin{cases} n & \text{if } g(x) \geq n, \\ g(x) & \text{if } -n < g(x) < n, \\ -n & \text{if } -n \leq g(x), \end{cases}$$

it holds that $\|g_n\| \leq \|g\|$ and $\|g_n\|_1 \leq \|g\|_1$. From those and the monotone convergence theorem, we get this lemma. □

5.2. **Proof of Lemma 2.4.** Putting $r = e^s$ and $V(s) = \Psi_{ST}(r) - 2$, we have

$$\frac{d^2V}{ds^2} + (N-2)\frac{dV}{ds} + 2(N-2)V + V\left\{(N-2)V + \frac{dV}{ds}\right\} = 0. \tag{5.1}$$

Since the eigenvalues of the linear part of (5.1) are

$$\mu_\pm = \frac{1}{2}\left\{-(N-2) \pm \sqrt{(N-2)(N-10)}\right\},$$

we have

$$|V(s)|, \quad \left|\frac{d}{ds}V(s)\right| \leq Ce^{\mu_+ s}.$$

Putting $W(s) = V(s)\{(N-2)V(s) + V'(s)\}$, we have

$$\begin{aligned} V(s) &= D_1 e^{\mu_+ s} + D_2 e^{\mu_- s} - \frac{e^{\mu_+ s}}{\mu_+ - \mu_-}\int_0^s e^{-\mu_+ \xi} W(\xi) d\xi \\ &\quad + \frac{e^{\mu_- s}}{\mu_+ - \mu_-}\int_0^s e^{-\mu_- \xi} W(\xi) d\xi, \end{aligned}$$

$$\int_0^\infty e^{-\mu_+\xi} W(\xi) d\xi < \infty \tag{5.2}$$

and

$$\int_0^s e^{-\mu_-\xi} W(\xi) d\xi \le C \left(e^{(2\mu_+ - \mu_-)s} + 1 \right). \tag{5.3}$$

Then, we observe that

$$\begin{aligned} V(s) = &\Big(D_1 - \frac{1}{\mu_+ - \mu_-} \int_0^\infty e^{-\mu_+\xi} W(\xi) d\xi \Big) e^{\mu_+ s} + D_2 e^{\mu_- s} \\ &+ \frac{e^{\mu_+ s}}{\mu_+ - \mu_-} \int_s^\infty e^{-\mu_+\xi} W(\xi) d\xi + \frac{e^{\mu_- s}}{\mu_+ - \mu_-} \int_0^s e^{-\mu_-\xi} W(\xi) d\xi. \end{aligned} \tag{5.4}$$

We assume that

$$D_1 - \frac{1}{\mu_+ - \mu_-} \int_0^\infty e^{-\mu_+\xi} W(\xi) d\xi = 0. \tag{5.5}$$

We observe that $V'(s) \le \mu_- V(s)$ for $s \in \mathbf{R}$. In fact, if it holds that $V'(s) = \mu_- V(s)$ for some $s \in \mathbf{R}$, we observe that for such an $s \in \mathbf{R}$

$$\frac{d}{ds} V'(s) < \mu_-^2 V(s),$$

by using (5.1). Combining this with $V'(s) < \mu_- V(s)$ for $s \ll -1$ implies that

$$V'(s) \le \mu_- V(s) \qquad \text{for } s \in \mathbf{R}. \tag{5.6}$$

Combining this with $V < 0$ in $\mathbf{R}$ implies that

$$(N-2)V + \frac{dV}{ds} \le \{(N-2) + \mu_-\} V < 0 \qquad \text{in } \mathbf{R}$$

or $W > 0$ in $\mathbf{R}$. It follows from this, (5.4) and (5.5) that

$$0 > V(s) > D_2 e^{\mu_- s} \qquad \text{for } 0 \le s < \infty.$$

This implies that

$$\begin{aligned} &\Big| \int_0^s e^{-\mu_\pm\xi} \frac{dV}{d\xi}(\xi) V(\xi) d\xi \Big| \\ &= \Big| \frac{1}{2} V(s)^2 e^{-\mu_\pm s} - \frac{1}{2} V(0)^2 + \frac{\mu_\pm}{2} \int_0^s e^{-\mu_\pm\xi} V(\xi)^2 d\xi \Big| < \infty. \end{aligned}$$

Then, it holds that for $0 \le s < \infty$

$$\begin{aligned} V(s) \;\; = \;\; &\Big(D_2 + \frac{1}{\mu_+ - \mu_-} \int_0^\infty e^{-\mu_-\xi} W(\xi) d\xi \Big) e^{\mu_- s} \\ &+ \frac{e^{\mu_+ s}}{\mu_+ - \mu_-} \int_s^\infty e^{-\mu_+\xi} W(\xi) d\xi - \frac{e^{\mu_- s}}{\mu_+ - \mu_-} \int_s^\infty e^{-\mu_-\xi} W(\xi) d\xi \end{aligned}$$

and that

$$\frac{dV}{ds}(s) = \mu_- V(s) + e^{\mu_+ s}\int_s^\infty e^{-\mu_- \xi} W(\xi) d\xi \quad \text{for } 0 \le s < \infty.$$

From this and $W > 0$ in $\mathbf{R}$, we have $V' > \mu_- V$ in $[0, \infty)$. It contradicts (5.6). Thus, it holds that

$$D_1 - \frac{e^{\mu_+ s}}{\mu_+ - \mu_-}\int_0^\infty e^{-\mu_+ \xi} W(\xi) d\xi \neq 0.$$

Combining this, (5.2), (5.3), (5.4) and $\mu_- < \mu_+ < 0$ with $V < 0$, we have

$$V(s) = -c_{ST}(1 + o(1))e^{\mu_+ s} \quad \text{as } s \to \infty$$

and

$$V'(s) = -\mu_+ c_{ST}(1 + o(1))e^{\mu_+ s} \quad \text{as } s \to \infty,$$

with a positive constant c_{ST}.

From those, $\mu_+ = 2\nu$, $\Psi_{ST}(r) = 2 + V(s)$ and $\Psi'_{ST}(r) = e^{-s}V'(s)$, we get this lemma. □

5.3. **Proof of Lemma 2.5.** Ψ_2 satisfies

$$\Psi_2(y, \tau) \le \Psi_2(Ke^{-\eta\tau}, \tau) < 2 \quad \text{for } y \in (0, Ke^{-\eta\tau}] \text{ and } \tau \ge \tau_0.$$

Putting $\zeta = \kappa_3(\tau)e^{\eta\tau}y$, it holds that

$$\Psi_{SE}(K^{-\nu}e^{3\eta\tau/2}y) = \Psi_{SE}\Big(\frac{e^{\eta\tau/2}\zeta}{K^\nu \kappa_3(\tau)}\Big), \quad \Psi_{ST}(\kappa_3(\tau)e^{\eta\tau}y) = \Psi_{ST}(\zeta),$$

where $\kappa_3(\tau)$ is the function in (3.18).

Since Ψ_{SE} satisfies $\lim_{y\to 0} y^{-2}\Psi_{SE}(y) = 2/(N-2)$ and

$$\Psi''_{ST}(0)/2 = \lim_{y\to 0} y^{-2}\Psi_{ST}(y) > 0,$$

there exists $y_0 \in (0, 1)$ satisfying

$$\frac{\Psi_{SE}(y)}{y^2} \ge \frac{1}{N-2} \quad \text{and} \quad \frac{\Psi_{ST}(y)}{y^2} < \Psi''_{ST}(0) \quad \text{for } 0 < y \le y_0.$$

Taking the smaller $y_0 > 0$, if necessary, it follows from $\nu \in [-3/2, -1)$ and $e^{\eta\tau/2}/\kappa_3(\tau) = K^{(\nu+1)/\nu}e^{\eta\tau(\nu+1)/(2\nu)}$ that

$$\frac{1}{3}\Psi_{SE}\Big(\frac{e^{\eta\tau/2}\zeta}{K^\nu \kappa_3(\tau)}\Big) > \Psi''_{ST}(0)\zeta^2 > \Psi_{ST}(\zeta)$$

for $\zeta \in (0, y_0/\sqrt{4N\Psi''_{ST}(0)})$, if $\tau_0 \gg 1$. Let $y_1 = y_0/\sqrt{4N\Psi''_{SE}(0)}$. Taking the smaller $y_1 \in (0,1)$, if necessary, it follows from Lemma 2.4 that

$$\Psi_2\Big(\frac{\zeta}{e^{\eta\tau}\kappa_3(\tau)}, \tau\Big) \geq \Psi_2\Big(\frac{y_1}{e^{\eta\tau}\kappa_3(\tau)}, \tau\Big) \geq 2\Big\{1 - \frac{2(N-2)K^{2\nu}\kappa_3(\tau)^2}{e^{\eta\tau}y_1^2}\Big\}$$
$$> \Big(1 + \frac{\tilde{c}_{ST}}{K^2 e^{\eta\tau}}\Big)\Psi_{ST}\Big(\frac{1}{y_1}\Big) > \Psi_1(\zeta, \tau) \quad \text{for } y_1 < \zeta < y_1^{-1},$$

if $\tau_0 \gg 1$. Combining those with $e^{\eta\tau}\kappa_3(\tau) = K^{-(1+\nu)/\nu}e^{-\lambda_2\tau/(2\nu)}$ implies that

$$\Psi_1(y,\tau) < \Psi_2(y,\tau) \quad \text{for } y \in (0, K^{(1+\nu)/\nu}e^{\lambda_2\tau/(2\nu)}/y_1) \text{ and } \tau \geq \tau_0.$$

Putting $\xi = e^{\eta\tau}y$, for $K^{(1+\nu)/\nu}e^{\eta\tau/(2\nu)}/y_1 \leq \xi \leq K$ we observe that

$$\begin{aligned}
&\Psi_2(y,\tau) - \Psi_1(y,\tau)\\
&= \frac{1}{2}\Big(1 + \frac{2}{K^{-2\nu+2}}e^{-\eta\tau}\Big)\Psi_{SE}\Big(K^{-\nu}e^{\eta\tau/2}\xi\Big) - \Big(1 + \frac{\tilde{c}_{ST}}{K^2e^{\eta\tau}}\Big)\Psi_{ST}(\kappa_3(\tau)\xi)\\
&> 2\Big(1 + \frac{2}{K^{-2\nu+2}}e^{-\eta\tau}\Big)\Big(1 - 2(N-2)K^{2\nu}e^{-\eta\tau}\xi^{-2}\Big)\\
&\qquad -2\Big(1 + \frac{\tilde{c}_{ST}}{K^2}e^{-\eta\tau}\Big)\Big(1 - \frac{c_{ST}}{4}K^{-2\nu-2}e^{-\eta\tau}\xi^{2\nu}\Big)\\
&> -8(N-2)K^{2\nu}e^{-\eta\tau}\xi^{-2} - \frac{2\tilde{c}_{ST}}{K^2}e^{-\eta\tau} + \frac{c_{ST}}{2K^{2\nu+2}}e^{-\eta\tau}\xi^{2\nu} > 0,
\end{aligned}$$

if $K \gg 1$ and $\tau_0 \gg 1$. Thus, we get this lemma. □

5.4. **Proof of Lemma 4.1.** First, in the case where $N \geq 12$ we shall prove this lemma.

It follows from [21, page 204] that

$$I_\alpha(x) = \frac{e^x}{\Gamma(\alpha+\frac{1}{2})\Gamma(1/2)\sqrt{2x}}\sum_{m=0}^{\infty}\frac{\big(\frac{1}{2}-\alpha\big)_m}{m!(2x)^m}\int_0^{2x}t^{(2\alpha+2m-1)/2}e^{-t}dt, \quad (5.7)$$

where

$$(x)_m = x(x+1)\cdot\dots\cdot(x+m-1) \quad \text{and} \quad (x)_0 = 1.$$

By using (5.7), we have

$$\begin{aligned}
I'_\alpha(x) &= I_\alpha(x) - \frac{1}{2x}I_\alpha(x) + \frac{e^x}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(1/2)\sqrt{2x}}\\
&\quad\times\Big\{\sum_{m=0}^{\infty}(-1)\frac{\big(\frac{1}{2}-\alpha\big)_m}{m!(2x)^m}\left(\frac{m}{x}\right)\int_0^{2x}t^{(2\alpha+2m-1)/2}e^{-t}dt
\end{aligned}$$

$$+\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}-\alpha\right)_m}{m!(2x)^m}(2x)^{(2\alpha+2m-1)/2}e^{-2x}\Big\}. \tag{5.8}$$

For $N \geq 12$, we have $\alpha \geq \sqrt{5}$, 2α is not an integer, and

$$\begin{aligned}\frac{1}{(m-1)!}\left|\left(\tfrac{1}{2}-\alpha\right)_m\right| &\leq \frac{1}{(m-1)!}\left|\left(\tfrac{1}{2}-\alpha\right)_{[\alpha]+1}\times\left(\tfrac{3}{2}+[\alpha]-\alpha\right)_{m-[\alpha]-1}\right| \\ &\leq C\frac{\Gamma\left(m-\alpha+\frac{1}{2}\right)}{(m-1)!} \end{aligned} \tag{5.9}$$

for $m \gg 1$, where $[x]$ is the maximum integer which is equal or less than x. Combining those with (4.31) implies that

$$\left|\sum_{m=1}^{\infty}(-1)\frac{\left(\frac{1}{2}-\alpha\right)_m}{m!(2x)^m}\left(\frac{m}{x}\right)\int_0^{2x}t^{(2\alpha+2m-1)/2}e^{-t}dt\right| \leq \frac{C}{x}\int_0^{2x}t^{(2\alpha-1)/2}e^{-t}dt \tag{5.10}$$

and that

$$\left|\sum_{m=1}^{\infty}\frac{\left(\frac{1}{2}-\alpha\right)_m}{m!(2x)^m}(2x)^{(2\alpha+2m-1)/2}e^{-2x}\right| \leq Cx^{(2\alpha-1)/2}e^{-2x}. \tag{5.11}$$

Since it follows from [21, page 77 and page 203] that

$$I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty \ \text{ and that } \ I_\alpha(x) \sim \frac{(x/2)^\alpha}{\Gamma(\alpha+1)} \quad \text{as } x \to 0,$$

we observe (4.2) in Lemma 4.1.

Combining (4.2) with (5.8), (5.10) and (5.11) implies that

$$\left|I'_\alpha(x) - I_\alpha(x)\right| \leq \frac{C}{x}I_\alpha(x) \quad \text{for } x > 0$$

in the case where $N \geq 12$.

Next, we shall prove this lemma in the case where $N = 11$.

In the case where $N = 11$, it holds that $\alpha = 3/2$. Since it follows from [21, page 80] that

$$I_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\cosh x - \frac{1}{x}\sinh x\Big),$$

we have

$$\begin{aligned} I'_{3/2}(x) - I_{3/2}(x) &= -\frac{1}{2x}I_{3/2}(x) + \frac{1}{x}\sqrt{\frac{2}{\pi x}}\Big(\frac{1}{x}\sinh x - xe^{-x} - e^{-x}\Big) \\ &= -\frac{1}{2x}I_{3/2}(x) + \frac{1}{x}VI(x). \end{aligned}$$

Since we observe that $I_{3/2} > 0$ in $(0, \infty)$,

$$\lim_{x\to 0} \frac{VI(x)}{I_{3/2}(x)} = 2 \quad \text{and that} \quad \lim_{x\to\infty} \frac{VI(x)}{I_{3/2}(x)} = 0,$$

we obtain

$$\left| I'_{3/2}(x) - I_{3/2}(x) \right| \leq \frac{C}{x} I_{3/2}(x).$$

Thus, we have this lemma. □

5.5. **Proof of Lemma 4.8.** It holds that

$$\begin{aligned} |L_j^\alpha(x)| &\leq \Big| \sum_{m=0}^{j} \frac{\Gamma(j+\alpha+1)x^m}{(j-m)!\Gamma(m+\alpha+1)m!} \Big| \\ &\leq \frac{\Gamma(j+\alpha+1)}{j!} \sum_{m=0}^{j} {}_jC_m x^m \leq \frac{\Gamma(j+\alpha+1)}{j!} (1+x)^j \qquad \text{for } x \geq 0. \end{aligned}$$

This lemma follows from this, (2.4) and (4.31). □

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