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SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS VIA DEGREE THEORY FOR MULTIVALUED PERTURBATIONS OF $(S)_+$ MAPS

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Abstract. We consider a nonlinear Neumann problem driven by the p-Laplacian differential operator and with a nonsmooth potential function (hemivariational inequality). Using a degree-theoretic approach based on the degree map for certain multivalued perturbations of $(S)_+$-operators, we prove the existence of a nontrivial smooth solution.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary ∂Z. In this paper we consider the following nonlinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) \text{ a.e. on } Z, \\ \dfrac{\partial x}{\partial n_p} = 0 \text{ on } \partial Z, \ \ 1 < p < \infty. \end{cases} \tag{1.1}$$

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Here $j(z, x)$ is a potential function which is locally Lipschitz in $x \in \mathbb{R}$, not necessarily smooth, and by $\partial j(z, x)$ we denote the generalized subdifferential of $j(z, \cdot)$ (see Section 2). Also $\frac{\partial x}{\partial n_p}$ equals $\|Dx\|^p(Dx, n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial Z$. Using degree theory, we prove the existence of a nontrivial solution for problem (1.1).

Elliptic problems, driven by the p-Laplacian differential operator, have attracted a lot of interest in the past decade. However, most works deal with problems which have a smooth potential (i.e. $j(z, \cdot) \in C^1(\mathbb{R})$) and Dirichlet boundary conditions. The study of the corresponding Neumann problem has lagged behind. In this direction, we have the works of Anello-Cordaro [3], Arcoya-Orsina [4], Binding-Drabek-Huang [5], Faraci [9], Godoy-Gossez-Paczka [12], Huang [13] (problems with a smooth potential) and Filippakis-Gasinski-Papageorgiou [10], Marano-Motreanu [20], Papageorgiou-Smyrlis [21], Papalini [22], [23] (problems with a nonsmooth potential). In all the aforementioned works the approach is variational.

As we already mentioned, in this paper the approach is degree theoretic based on the degree map for multivalued perturbations of $(S)_+$-operators introduced and studied by Hu-Papageorgiou [14]. Our hypotheses do not involve the Ambrosetti-Rabinowitz condition, common in Neumann problems in order to check the PS-condition and, moreover, implying that the "slopes" $\left\{\frac{u}{|x|^{p-2}x} : u \in \partial j(z, x)\right\}$ cross the principal eigenvalue $\lambda_1 = 0$ of $(-\triangle_p, W^{1,p}(Z))$ as we move from 0 to $\pm\infty$. As a result our hypotheses are distinct from the ones employed in previous works on the subject. In addition, the degree-theoretic method of this paper is used for the first time on Neumann problems with the p-Laplacian.

2. Mathematical Background

Let X be a Banach space, X^* its topological dual, and by $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz if, for every $x \in X$, we can find a neighborhood U of x and a constant $k > 0$ (depending on U) such that

$$|\varphi(z) - \varphi(y)| \leq k\|z - y\| \text{ for all } z, y \in U.$$

If φ is a continuous convex function or if $\varphi \in C^1(X)$, then φ is locally Lipschitz.

For a locally Lipschitz function φ, we define the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x; h) = \limsup_{x' \to x,\ \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x;h)$ is sublinear and continuous and is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is the generalized subdifferential of φ. If $\varphi : X \to \mathbb{R}$ is continuous and convex, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, namely

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x+h) - \varphi(x) \text{ for all } h \in X\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

A multifunction (set-valued map) $G : X \to 2^{X^*}\backslash\{\emptyset\}$ is said to be upper semicontinuous (usc for short), if for every closed set $C \subseteq X^*$,

$$G^-(C) = \{x \in X : G(x) \cap C \neq \emptyset\}$$

is closed in X. The generalized subdifferential, $x \to \partial\varphi(x)$, is a usc multifunction from X with the norm topology into X^* with the w^*-topology. We say that a multifunction $G : X \to 2^{X^*}\backslash\{\emptyset\}$ belongs to class (P), if it is usc with $G(x)$ closed and convex for every $x \in X$ and such that

$$G(A) = \bigcup_{x \in A} G(x)$$

is relatively compact in X^* for every $A \subseteq X$ bounded.

Recall that, if $G : D \subseteq X \to 2^{X^*}\backslash\{\emptyset\}$ is an usc multifunction with closed and convex values, then for every $\varepsilon > 0$, we can find a continuous map $g_\varepsilon : D \to X^*$ such that

$$g_\varepsilon(x) \in G((x + B_\varepsilon) \cap D) + B^*_\varepsilon \text{ for all } x \in D \text{ and } g_\varepsilon(D) \subseteq \overline{\text{conv}}G(D)$$

with $B_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ and $B^*_\varepsilon = \{x^* \in X^* : \|x^*\| < \varepsilon\}$ (see Hu-Papageorgiou [15], Theorem 4.41, page 106). In particular, if G belongs to class (P), then the continuous approximate selector g_ε is a compact map.

Now we will define the degree map which we will use in the study of problem (1.1). Let X be a reflexive Banach space. By the Troyanski renorming theorem (see Gasinski-Papageorgiou [11], page 911), we can equivalently renorm X so that both X and X^* are locally uniformly convex with Frechet differentiable norms. So, in what follows, we assume that both X and X^* are locally uniformly convex. Then the duality map $\mathcal{F} : X \to X^*$, defined by

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

is a homeomorphism.

An operator $A : X \to X^*$, which is single valued and everywhere defined, is said to be of type $(S)_+$, if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, one has $x_n \to x$ in X.

Let U be a bounded open set in X and let $A : \overline{U} \to X^*$ be a demicontinuous operator of type $(S)_+$. Let $\{X_\alpha\}_{\alpha\in J}$ be the family of all finite-dimensional subspaces of X and let A_α be the Galerkin approximation of A with respect to X_α; that is,

$$\langle A_\alpha(x), y\rangle_{X_\alpha} = \langle A(x), y\rangle \text{ for all } x \in \overline{U} \cap X_\alpha \text{ and all } y \in X_\alpha.$$

Then, for $x^* \notin A(\partial U)$, $\deg_{(S)_+}(A, U, x^*)$ is defined by

$$\deg_{(S)_+}(A, U, x^*) = d_B(A_\alpha, U \cap X_\alpha, x^*)$$

for X_α large enough (in the sense of inclusion), with $d_B(\cdot, \cdot, \cdot)$ being the classical Brouwer degree.

If X is separable and A is bounded (maps bounded sets to bounded ones), then we can use only a countable subfamily $\{X_n\}_{n\geq 1}$ of $\{X_\alpha\}_{\alpha\in J}$ such that $\overline{\bigcup_{n\geq 1} X_n} = X$. For more on the degree map "deg"$_{(S)+}$, we refer the reader to Browder [6].

If $G : X \to 2^{X^*}\backslash\{\emptyset\}$ is a multifunction in the class (P), then for every $x^* \notin (A + G)(\partial U)$, $\deg(A + G, U, x^*)$ is defined by

$$\deg(A + G, U, x^*) = \deg_{(S)_+}(A + g_\varepsilon, U, x^*),$$

for $\varepsilon > 0$ small, where g_ε is the continuous ε-approximate selector of G as described earlier. For more on the degree map deg just defined, we refer the reader to Hu-Papageorgiou [14] (see also Hu-Papageorgiou [15], Section 4.4).

One of the fundamental properties of a degree map is the homotopy invariance property. For this reason, below we give the admissible homotopies for A and G.

Definition 2.1. [a] *A one-parameter family $\{A_t\}_{t\in[0,1]}$ of locally bounded maps from $\overline{U}$ into X^* is said to be a "homotopy of class $(S)_+$", if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \xrightarrow{w} x$ and for any $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ with $t_n \to t$ for which*

$$\limsup_{n\to\infty}\langle A_{t_n}(x_n), x_n - x\rangle \leq 0,$$

we have $x_n \to x$ in X and $A_{t_n}(x_n) \xrightarrow{w} A_t(x)$ in X^* as $n \to \infty$.

[b] *A one-parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t : \overline{U} \to 2^{X^*}\backslash\{\emptyset\}$ is said to be a "homotopy of class (P)", if $(t, x) \to G_t(x)$ is usc from $[0, 1]\times\overline{U}$*

into $2^{X^*}\backslash\{\emptyset\}$, *for every* $(t,x)\in[0,1]\times\overline{U}$ *the set* $G_t(x)$ *is closed, convex and*

$$\overline{\cup\{G_t(x) : t\in[0,1], x\in\overline{U}\}}$$

is compact in X^*.

Then the homotopy invariance for the degree map "deg" can be formulated as follows:

"If $\{A_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ such that for every $t\in[0,1]$ A_t is bounded, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $x^* : [0,1]\to X^*$ is a continuous map such that

$$x_t^* \notin (A_t+G_t)(\partial U) \text{ for all } t\in[0,1],$$

then $\deg(A_t+G_t,U,x_t^*)$ is independent of $t\in[0,1]$."

Both degree maps "$\deg_{(S)_+}$" and "deg" have all the usual properties (such as homotopy invariance, solution property, additivity of the domain, excision property).

Finally, let us recall some basic facts about the spectrum of the negative p-Laplacian with Neumann boundary condition. For details we refer to An Lê [18] and Gasinski-Papageorgiou [11]. Let $Z\subseteq\mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z, let $m\in L^\infty(Z)_+$, $m\neq 0$ and consider the following nonlinear, weighted (with weight m) eigenvalue problem:

$$\begin{cases} -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) = \widehat{\lambda}m(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z, \\ \dfrac{\partial x}{\partial n_p}=0 \text{ on } \partial Z, \ \widehat{\lambda}\in\mathbb{R}, \ 1<p<\infty. \end{cases} \tag{2.1}$$

Every $\widehat{\lambda}\in\mathbb{R}$, for which problem (2.1) has a nontrivial solution, is said to be an eigenvalue of $(-\triangle_p, W^{1,p}(Z), m)$ and the nontrivial solution is an eigenfunction corresponding to this eigenvalue. The linear subspace formed by the eigenfunctions corresponding to an eigenvalue $\widehat{\lambda}\in\mathbb{R}$ is the eigenspace corresponding to $\widehat{\lambda}$.

Problem (2.1) has a smallest (principal) eigenvalue denoted by $\widehat{\lambda}_1(m)$ which is equal to zero, it is isolated and it is simple (i.e. the corresponding eigenspace is one dimensional). There is a variational characterization of $\widehat{\lambda}_1(m)=0$, via the Rayleigh quotient, namely

$$0=\widehat{\lambda}_1(m)=\inf\Big[\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x\in W_0^{1,p}(Z),\ x\neq 0\Big]. \tag{2.2}$$

Evidently constant functions realize the infimum in (2.2).

In addition to $\widehat{\lambda}_1=\widehat{\lambda}_1(m)$, the Liusternik-Schnirelmann theory gives a whole strictly increasing sequence $\{\widehat{\lambda}_k=\widehat{\lambda}_k(m)\}_{k\geq 1}\subseteq\mathbb{R}_+$, for which the nonlinear eigenvalue problem (2.1) has a nontrivial solution. We have that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$ and the $\widehat{\lambda}_k$'s are called the "variational eigenvalues" of $(-\triangle_p, W^{1,p}(Z), m)$.

If $p = 2$ (linear eigenvalue problem), then the variational eigenvalues are all the eigenvalues of $(-\triangle, H^1(Z), m)$. If $p \neq 2$ (nonlinear eigenvalue problem), then we do not know if this is the case. However, since $\widehat{\lambda}_1 = 0$ is isolated and the set $\sigma(p,m)$ of eigenvalues of $(-\triangle_p, W^{1,p}(Z), m)$ is closed, if we set

$$\widehat{\lambda}_2^* = \inf\{\widehat{\lambda} : \widehat{\lambda} \in \sigma(p,m),\ \widehat{\lambda} > \widehat{\lambda}_1\},$$

then $\widehat{\lambda}_2^* \in \sigma(p,m)$ and in fact $\widehat{\lambda}_2^* = \widehat{\lambda}_2$. So the second eigenvalue of $(-\triangle_p, W^{1,p}(Z), m)$ and the second variational eigenvalue coincide.

Moreover, if we set

$$\begin{aligned} &\varphi_m(x) = \int_Z m|x|^p dz,\ \psi_m(x) = \int_Z m|x|^p dz + \|Dx\|_p^p \ \text{ for all } \ x \in W^{1,p}(Z), \\ &S(\psi_m) = \{x \in W^{1,p}(Z) : \psi_m(x) = 1\} \qquad (2.3) \\ &\text{and } \mathcal{A}_k = \{C \subseteq S(\psi_m) : C \text{ is compact, symmetric and } \gamma(C) \geq k\}, \end{aligned}$$

with γ being the Krasnoselskii genus (see Gasinski-Papageorgiou [11], page 679), then

$$\frac{1}{\widehat{\lambda}_k(m) + 1} = \sup_{C \in \mathcal{A}_k} \inf_{x \in C} \varphi_m(x) \ \text{ for all } \ k \in \{2, 3, \cdots\}. \qquad (2.4)$$

The maxmin-expressions in (2.4) give the variational expressions for the eigenvalues $\widehat{\lambda}_k(m)$, $k \geq 2$, and are nonlinear generalizations of the well-known Courant maxmin-characterizations of the eigenvalues of the linear problem. Finally, if $m \equiv 1$, then we write $\lambda_k = \widehat{\lambda}_k$ for all $k \geq 1$.

3. Nontrivial Solutions

Our hypotheses on the nonsmooth potential function $j(z,x)$ are the following:

<u>$H(j)$</u>: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $u \in \partial j(z,x)$, we have

$$|u| \leq \alpha(z) + c|x|^{p-1} \ \text{ with } \ \alpha \in L^\infty(Z)_+,\ c > 0;$$

(iv) there exist functions $\theta, \widehat{\theta} \in L^\infty(Z)_+$ such that $0 \le \theta(z) \le \widehat{\theta}(z) < \lambda_2$ a.e. on Z, the first inequality is strict on a set of positive measure and

$$\theta(z) \le \liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \widehat{\theta}(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exist functions $\eta, \widehat{\eta} \in L^\infty(Z)$ such that $\widehat{\eta}(z) \le \eta(z) \le 0$ a.e. on Z, the second inequality is strict on a set of positive measure and

$$\widehat{\eta}(z) \le \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \le \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \le \eta(z)$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$.

Remark 3.1. Hypotheses (iv) and (v) are nonuniform nonresonance conditions at $\pm\infty$ and at 0 respectively with respect to the principal eigenvalue $\lambda_1 = 0$ of $(-\triangle_p, W^{1,p}(Z))$. When $p = 2$, hypotheses $H(j)$ incorporate, in our framework of analysis, the so-called "asymptotically linear problems" (at both zero and infinity), which, since the appearance of the pioneering work of Amann-Zehnder [1], have attracted a lot of interest, primarily for the Dirichlet problem. A simple nonsmooth locally Lipschitz potential function which satisfies hypotheses $H(j)$, but does not fit in the setting of the papers mentioned in the Introduction, is the following. For simplicity we drop the z-dependence:

$$j(x) = \min\{\frac{1}{p}|x|^p, \frac{1}{\theta}|x|^\theta\} - \frac{\mu}{p}|x|^p$$

with $1 < p < \theta$ and $\max\{1-\lambda_2, 0\} < \mu < 1$.

Let $N : L^p(Z) \to 2^{L^{p'}}(Z)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the multivalued Nemitsky operator corresponding to the generalized subdifferential $\partial j(z,x)$, namely

$$N(x) = \{u \in L^{p'}(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z\}.$$

Proposition 3.2. *If hypotheses $H(j)$ hold, then N has nonempty, weakly compact and convex values and it is usc from $L^p(Z)$ with the norm topology into $L^{p'}(Z)$ with the weak topology (denoted by $L^{p'}(Z)_w$).*

Proof. Clearly N has weakly compact and convex values (see hypothesis $H(j)$ (iii)). What is not immediately clear is that the values are nonempty. To show this, let $x \in L^p(Z)$ and consider a sequence $\{s_n\}_{n\ge 1}$ of simple functions such that $s_n(z) \to x(z)$ almost everywhere on Z as $n \to \infty$ and

$$|s_n(z)| \le |x(z)| \text{ a.e. on } Z \text{ for all } n \ge 1.$$

Note that for every $x \in \mathbb{R}$, $z \to \partial j(z,x)$ is a graph-measurable multifunction. Hence, for every $n \geq 1$, $z \to \partial j(z, s_n(z))$ is graph measurable too. Invoking the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [15], page 158), we can find $f_n : Z \to \mathbb{R}$ a Borel measurable function such that

$$f_n(z) \in \partial j(z, s_n(z)) \text{ a.e. on } Z, \text{ for al } n \geq 1.$$

Hypothesis $H(j)(iii)$ implies that $\{f_n\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded. So we may assume that $f_n \xrightarrow{w} f$ in $L^{p'}(Z)$. Since $\partial j(z,\cdot)$ is a usc multifunction with closed and convex values, from Proposition 3.9, page 694, of Hu-Papageorgiou [15], we have

$$f(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \quad \Rightarrow f \in N(x), \text{ i.e., } N(x) \neq \emptyset.$$

Because of hypothesis $H(j)(iii)$, the multifunction N is locally weakly compact. Also recall that the weak topology on bounded subsets of $L^{p'}(Z)$ is metrizable. To show the upper semicontinuity of N, it suffices to show that $\mathrm{Gr}N = \{(x,f) \in L^p(Z) : f \in N(x)\}$ is sequentially closed in $L^p(Z) \times L^{p'}(Z)_w$ (see Hu-Papageorgiou [15], page 43). To this end let $\{(x_n, f_n)\}_{n\geq 1} \subseteq \mathrm{Gr}N$ and assume that $x_n \to x$ in $L^p(Z)$, and $f_n \xrightarrow{w} f$ in $L^{p'}(Z)$. Now, by passing to a suitable subsequence if necessary, we may assume that $x_n(z) \to x(z)$ a.e. on Z. Since $f_n(z) \in \partial j(z, x_n(z))$ a.e. on Z for all $n \geq 1$ as before exploiting the upper semicontinuity of $\partial j(z,\cdot)$, we obtain $f(z) \in \partial j(z, x(z))$ a.e. on Z, hence $(x,f) \in \mathrm{Gr}N$. □

Since $W^{1,p}(Z)$ is embedded compactly and densely in $L^p(Z)$, $L^{p'}(Z)$ is embedded compactly and densely in $W^{1,p}(Z)^*$. So from Proposition 3.2 above we infer:

Corollary 3.3. *If hypotheses $H(j)$ hold, then $N : W^{1,p}(Z) \to 2^{W^{1,p}(Z)^*} \setminus \{\emptyset\}$ is a multifunction of class (P).*

Next let $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ be the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W^{1,p}(Z).$$

Hereafter by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W^{1,p}(Z), W^{1,p}(Z)^*)$.

Proposition 3.4. *A is a demicontinuous, bounded, $(S)_+$-operator.*

Proof. Clearly A is demicontinuous, bounded, and monotone, hence it is maximal monotone. Consider $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ and assume that

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \leq 0. \tag{3.1}$$

Since A is maximal monotone, it is generalized pseudomonotone (see Gasinski-Papageorgiou [11], page 330) and so from (3.1) we infer that

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle, \quad \Rightarrow \|Dx_n\|_p \to \|Dx\|_p.$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z,\mathbb{R}^N)$ and, $L^p(Z,\mathbb{R}^N)$ being uniformly convex, has the Kadec-Klee property. Hence it follows that $Dx_n \to Dx$ in $L^p(Z,\mathbb{R}^N)$. Moreover, from the compact embedding of $W^{1,p}(Z)$ into $L^p(Z)$, we have $x_n \to x$ in $L^p(Z)$. Therefore we conclude that $x_n \to x$ in $W^{1,p}(Z)$, which proves that A is an $(S)_+$-operator. $\square$

From Corollary 3.3 and Proposition 3.4, we see that we can speak about $\deg(A-N, B_r(0), 0)$ for every $r > 0$ (see Section 2).

The next proposition extends a monotonicity result of Anane-Tsouli [2]. The result of Anane-Tsouli [2] was about the second eigenvalue of the "Dirichlet" p-Laplacian, while here we prove the result for the "Neumann" p-Laplacian.

Proposition 3.5. *If $m, m_1 \in L^\infty(Z)$, $m \neq 0$ and $m(z) < m_1(z)$ almost everywhere on Z, then $\widehat{\lambda}_2(m_1) < \widehat{\lambda}_2(m)$.*

Proof. Let $u_2 \in C^1(\overline{Z})$ be an eigenfunction corresponding to $\widehat{\lambda}_2(m_1)$. We know that u_2 necessarily changes sign and it has two nodal domains (see An Lê [18]). We set

$$Z_+ = \{u_2 > 0\} \text{ and } Z_- = \{u_2 < 0\}.$$

Then we define

$$v_\pm(z) = \begin{cases} \frac{u_2(z)}{(\int_{Z_\pm} m_1|u_2|^p dz)^{1/p}} & \text{if } z \in Z_\pm \\ 0 & \text{otherwise.} \end{cases}$$

We have $v_\pm \in W^{1,p}(Z) \cap C(\overline{Z})$. Clearly the elements $v_\pm$ are linearly independent and so if Y=span$\{v_+, v_-\}$, then $\dim Y = 2$ and if $v \in Y$, then $v = \beta_1 v_+ + \beta_2 v_-$ with $\beta_1, \beta_2 \in \mathbb{R}$. Evidently

$$v \to \varphi_{m_1}(v)^{1/p} = \Big(\int_Z m_1|v|^p dz\Big)^{1/p} = (|\beta_1|^p + |\beta_2|^p)^{1/p}$$

is an equivalent norm on Y and so the set

$$S_2 = \{v \in Y : \varphi_{m_1}(v) = \frac{1}{\widehat{\lambda}_2(m_1)+1}\}$$

is homeomorphic by an odd homeomorphism to the unit sphere of $\mathbb{R}^2$, which we know has Krasnoselskii genus 2 (see Gasinski-Papageorgiou [11], page

680). Therefore,

$$\gamma(S_2) = 2 \ \ (\gamma \text{ being the Krasnoselskii genus}).$$

From the definition of $\widehat{\lambda}_2(m_1)$ (see (2.4)), we have

$$\begin{aligned}
&\psi_{m_1}(v_\pm) = \left(\widehat{\lambda}_2(m_1)+1\right)\varphi_{m_1}(v_\pm),\\
&\Rightarrow \psi_{m_1}(v) = \left(\widehat{\lambda}_2(m_1)+1\right)(|\beta_1|^p\varphi_{m_1}(v_+) + |\beta_2|^p\varphi_{m_1}(v_-)\\
&\qquad = \left(\widehat{\lambda}_2(m_1)+1\right)(|\beta_1|^p + |\beta_2|^p)\\
&\qquad = \left(\widehat{\lambda}_2(m_1)+1\right)\varphi_{m_1}(v) = 1 \ \text{ for all } \ v = \beta_1 v_+ + \beta_2 v_- \in S_2,\\
&\quad \Rightarrow S_2 \subseteq S(\psi_{m_1}) \ \ (\text{see } (2.3)).
\end{aligned}$$

Using once more (2.4), we have

$$\inf_{v\in S_2} \varphi_{m_1}(v) = \frac{1}{\widehat{\lambda}_2(m_1)+1}. \tag{3.2}$$

But note that, for $v \in Y$, we have

$$\begin{aligned}
&\varphi_m(v) < \varphi_{m_1}(v),\\
&\Rightarrow \frac{1}{\widehat{\lambda}_2(m)+1} < \frac{1}{\widehat{\lambda}_2(m_1)+1} \ \ (\text{see (3.2) and recall that } S_2 \text{ is compact}),\\
&\Rightarrow \widehat{\lambda}_2(m_1) < \widehat{\lambda}_2(m).
\end{aligned}$$

□

Proposition 3.6. *If* $m \in L^\infty(Z)_+ = \{m \in L^\infty(Z) : m(z) \geq 0 \ \ a.e. \ on \ \ Z\}$, $m \neq 0$, *then there exists* $\xi > 0$ *such that* $\gamma(x) = \|Dx\|_p^p + \int_Z m|x|^p dz \geq \xi\|x\|^p$ *for all* $x \in W^{1,p}(Z)$.

Proof. Evidently $\gamma \geq 0$ and the function γ is positively p-homogeneous. We argue indirectly. Suppose the proposition is not true. Then we can find $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ with $\|x_n\| = 1$ such that $\gamma(x_n) \to 0$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W^{1,p}(Z) \ \text{ and } \ x_n \to x \ \text{ in } \ L^p(Z).$$

Then we have

$$\|Dx\|_p^p \leq \liminf_{n\to\infty} \|Dx_n\|_p^p \ \text{ and } \ \int_Z m|x_n|^p dz \to \int_Z m|x|^p dz \ \text{ as } \ n \to \infty.$$

So in the limit as $n \to \infty$, we have

$$\gamma(x) = \|Dx\|_p^p + \int_Z m|x|^p dz \leq \lim_{n\to\infty} \gamma(x_n) = 0, \tag{3.3}$$

$$\Rightarrow \|Dx\|_p^p \leq -\int_Z m|x|^p dz \leq 0, \quad \Rightarrow x \equiv \sigma \in \mathbb{R}.$$

If $\sigma \neq 0$, then from (3.3) we have $\|Dx\|_p^p \leq -|\sigma|^p \int_Z m dz < 0$, a contradiction. So $\sigma = 0$ and we have $x_n \to 0$ in $L^p(Z)$ and $\|Dx_n\|_p \to 0$ as $n \to \infty$. Therefore $x_n \to 0$ in $W^{1,p}(Z)$, a contradiction to the fact that $\|x_n\| = 1$ for all $n \geq 1$. This proves the proposition. □

As we already mentioned we can speak about $\deg(A-N, B_r(0), 0)$ for any $r > 0$. In the next proposition, we compute the degree for $r > 0$ small.

Proposition 3.7. *If hypotheses $H(j)$ hold, then there exists $\rho_0 > 0$ such that for all $0 < \rho \leq \rho_0$ we have deg$(A - N, B_\rho(0), 0) = 1$.*

Proof. Let $m_0 \in L^\infty(Z)_+$, $m_0 \neq 0$ such that $\eta(z) \leq -m_0(z)$ almost everywhere on Z. Also let $K : W^{1,p}(Z) \to W^{1,p}(Z)^*$ be the completely continuous operator defined by $K(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot) \in L^{p'}(Z) \subseteq W^{1,p}(Z)^*$ for all $x \in W^{1,p}(Z)$. We consider the admissible homotopy $h_1 : [0,1] \times W^{1,p}(Z) \to 2^{W^{1,p}(Z)^*} \backslash \{\emptyset\}$ defined by

$$h_1(\beta, x) = A(x) + (1-\beta)m_0 K(x) - \beta N(x)$$

for all $(\beta, x) \in [0,1] \times W^{1,p}(Z)$ (see Section 2).

Claim: There exists $\rho_0 > 0$ such that $0 \notin h_1(\beta, x)$ for all $\beta \in [0,1]$ and all $x \in \partial B_\rho$ with $0 < \rho \leq \rho_0$.

Suppose that the claim is not true. Then we can find $\{\beta_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that

$$\beta_n \to \beta \text{ in } [0,1], \; \|x_n\| \to 0 \text{ and } 0 \in h_1(\beta_n, x_n) \text{ for all } n \geq 1.$$

We have

$$A(x_n) + (1-\beta)m_0 K(x_n) = \beta_n u_n \text{ with } u_n \in N(x_n) \text{ for all } n \geq 1. \tag{3.4}$$

We set $y_n = \dfrac{x_n}{\|x_n\|}$, $n \geq 1$. Then we may assume (at least for a subsequence), that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(Z) \text{ and } y_n \to y \text{ in } L^p(Z) \text{ as } n \to \infty.$$

Dividing (3.4) by $\|x_n\|^{p-1}$, we obtain

$$A(y_n) + (1-\beta_n)m_0 K(y_n) = \beta_n \frac{u_n}{\|x_n\|^{p-1}} \text{ for all } n \geq 1. \tag{3.5}$$

By virtue of hypothesis $H(j)(v)$, we can find $\delta > 0$ such that

$$\widehat{\eta}(z) - 1 \leq \frac{u}{|x|^{p-2}x} \leq \eta(z) + 1 \tag{3.6}$$

for almost all $z \in Z$, all $|x| < \delta$ and all $u \in \partial j(z,x)$. Moreover, from hypothesis $H(j)(iii)$, we have

$$|u| \leq \alpha(z) + c|x|^{p-1} \leq \Big(\frac{\alpha(z)}{\delta^{p-1}} + c\Big)|x|^{p-1} \quad \text{for a.a. } z \in Z, \text{ all } |x| \geq \delta \text{ and all } u \in \partial j(z,x). \tag{3.7}$$

Combining (3.6) and (3.7), since $\eta, \widehat{\eta} \in L^\infty(Z)$, we can say that

$$|u| \leq c_1|x|^{p-1} \text{ for some } c_1 > 0, \text{ a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x), \tag{3.8}$$

$$\Rightarrow \frac{|u_n(z)|}{\|x_n\|^{p-1}} \leq c_1|y_n(z)|^{p-1} \text{ a.e. on } Z, \text{ for all } n \geq 1,$$

$$\Rightarrow \Big\{\frac{u_n}{\|x_n\|^{p-1}}\Big\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

Therefore, we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(Z).$$

Given $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$C^+_{\varepsilon,n} = \{z \in Z : x_n(z) > 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)^{p-1}} \leq \eta(z) + \varepsilon\}$$

$$\text{and } C^-_{\varepsilon,n} = \{z \in Z : x_n(z) < 0,\ \widehat{\eta}(z) - \varepsilon \leq \frac{u_n(z)}{|x_n(z)|^{p-2}x_n(z)} \leq \eta(z) + \varepsilon\}.$$

Since $x_n \to 0$ in $W^{1,p}(Z)$, we may assume that $x_n(z) \to 0$ almost everywhere on Z. Hence $x_n(z) \to 0^+$ almost everywhere on $\{y > 0\}$ and $x_n(z) \to 0^-$ almost everywhere on $\{y < 0\}$. Because of hypothesis $H(j)(v)$, we have

$$\chi_{C^+_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y > 0\} \text{ and } \chi_{C^-_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

Note that

$$\|(1 - \chi_{C^+_{\varepsilon,n}})\frac{u_n}{\|x_n\|^{p-1}}\|_{L^{p'}(\{y>0\})} \to 0$$

$$\text{and } \|(1 - \chi_{C^-_{\varepsilon,n}})\frac{u_n}{\|x_n\|^{p-1}}\|_{L^{p'}(\{y<0\})} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \chi_{C^+_{\varepsilon,n}}\tfrac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(\{y > 0\}) \text{ and } \chi_{C^-_{\varepsilon,n}}\tfrac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h_0 \text{ in } L^{p'}(\{y < 0\}) \text{ as } n \to \infty. \tag{3.9}$$

From the definitions of the sets $C^+_{\varepsilon,n}$ and $C^-_{\varepsilon,n}$, we have

$$\chi_{C^+_{\varepsilon,n}}(z)(\widehat{\eta}(z) - \varepsilon)y_n(z)^{p-1} \leq \chi_{C^+_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|^{p-1}} = \chi_{C^+_{\varepsilon,n}}(z)\frac{u_n(z)}{x_n(z)^{p-1}}y_n(z)^{p-1}$$

$$\leq \chi_{C^+_{\varepsilon,n}}(z)(\eta(z)+\varepsilon)y_n(z)^{p-1} \tag{3.10}$$

and

$$\chi_{C^-_{\varepsilon,n}}(z)(\eta(z)+\varepsilon)|y_n(z)|^{p-2}y_n(z) \leq \chi_{C^-_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|^{p-1}} \tag{3.11}$$

$$= \chi_{C^-_{\varepsilon,n}}(z)\frac{u_n(z)}{|x_n(z)|^{p-2}x_n(z)}|y_n(z)|^{p-2}y_n(z)$$

$$\leq \chi_{C^-_{\varepsilon,n}}(z)(\widehat{\eta}(z)-\varepsilon)|y_n(z)|^{p-2}y_n(z) \text{ a.e. on } Z.$$

Passing to the limit as $n \to \infty$ in (3.10) and (3.11) and using Proposition 3.9, page 694, of Hu-Papageorgiou [15], together with (3.9), we obtain

$$(\widehat{\eta}(z)-\varepsilon)y(z)^{p-1} \leq h_0(z) \leq (\eta(z)+\varepsilon)y(z)^{p-1} \text{ a.e. on } \{y>0\},$$

$$(\eta(z)+\varepsilon)|y(z)|^{p-2}y(z) \leq h_0(z) \leq (\widehat{\eta}(z)-\varepsilon)|y(z)|^{p-2}y(z) \text{ a.e. on } \{y<0\}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and so

$$\widehat{\eta}(z)y(z)^{p-1} \leq h_0(z) \leq \eta(z)y(z)^{p-1} \text{ a.e. on } \{y>0\}, \tag{3.12}$$

$$\eta(z)|y(z)|^{p-2}y(z) \leq h_0(z) \leq \widehat{\eta}(z)|y(z)|^{p-2}y(z) \text{ a.e. on } \{y<0\}. \tag{3.13}$$

Moreover, from (3.8) it is clear that

$$h_0(z) = 0 \text{ a.e. on } \{y=0\}. \tag{3.14}$$

From (3.12), (3.13) and (3.14), we deduce that

$$h_0(z) = g_0(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z, \tag{3.15}$$

with $g_0 \in L^\infty(Z)$, $\widehat{\eta}(z) \leq g_0(z) \leq \eta(z)$ a.e. on Z.

We return to (3.8) and act with the test function $y_n - y \in W^{1,p}(Z)$. We obtain

$$\langle A(y_n), y_n - y\rangle + (1-\beta_n)\int_Z m_0|y_n|^{p-2}y_n(y_n-y)dz$$

$$= \beta_n \int_Z \frac{u_n}{\|x_n\|^{p-1}}(y_n-y)dz \text{ for all } n \geq 1. \tag{3.16}$$

Clearly, we have

$$\int_Z m_0|y_n|^{p-2}y_n(y_n-y)dz,\ \int_Z \frac{u_n}{\|x_n\|^{p-1}}(y_n-y)dz \to 0 \text{ as } n\to\infty.$$

Now, if we pass to the limit as $n \to \infty$, in (3.16), we obtain

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0. \tag{3.17}$$

But by Proposition 3.4 A is an $(S)_+$-operator. Thus from (3.17) we infer that

$$y_n \to y \text{ in } W^{1,p}(Z);\ \text{ i.e., } \|y\| = 1. \tag{3.18}$$

Therefore, we can pass to the limit as $n \to \infty$ in (3.5) and obtain

$$\begin{aligned} &A(y) + (1-\beta)m_0 K(y) = \beta g_0 K(y) \ \text{ (see (3.15))},\\ \Rightarrow &A(y) + m_0 K(y) = \beta(g_0 + m_0)K(y). \end{aligned}$$

Acting with the test function $y \in W^{1,p}(Z)$, we obtain

$$\begin{aligned} &\|Dy\|_p^p + \int_Z m_0 |y|^p dz = \beta \int_Z (g_0 + m_0)|y|^p dz,\\ \Rightarrow &\xi \|y\|^p \le \beta \int_Z (g_0 + m_0)|y|^p dz \le 0\\ &\text{(see Proposition 3.6 and recall the choice of } m_0 \in L^\infty(Z)_+)\\ \Rightarrow &y = 0, \end{aligned}$$

a contradiction to the fact that $\|y\| = 1$ (see (3.18)).

This proves the Claim. Then we can use the homotopy invariance of the degree map and obtain

$$\deg(A - N, B_\rho, 0) = \deg_{(S)_+}(A + m_0 K, B_\rho, 0) \ \text{ for all } \ 0 < \rho \le \rho_0. \tag{3.19}$$

Next we compute $\deg_{(S)_+}(A + m_0K, B_\rho, 0)$. For this purpose, we consider the homotopy $h_2 : [0,1] \times W^{1,p}(Z) \to W^{1,p}(Z)^*$ defined by

$$h_2(\beta, x) = \beta A(x) + \beta m_0 K(x) + (1-\beta)\mathcal{F}(x)$$

for all $(\beta, x) \in [0,1] \times W^{1,p}(Z)$, where $\mathcal{F} : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is the duality map of the Sobolev space $W^{1,p}(Z)$ (see Section 2). Since the Sobolev spaces $W^{1,p}(Z), W^{1,p}(Z)^*$ are uniformly convex, it follows that $\mathcal{F}$ is a homeomorphism and also is maximal monotone, coercive and of type $(S)_+$. Therefore we see that $h_2(\beta, x)$ is a homotopy of type $(S)_+$ (see Browder [6], Proposition 12). We claim that $0 \notin h_2(\beta, x)$ for all $\beta \in [0,1]$, all $x \in \partial B_\rho$ and all $\rho > 0$. Indeed, if this is not the case, then for some $\beta \in [0,1]$ and some $x \ne 0$, we have

$$\begin{aligned} &\beta A(x) + \beta m_0 K(x) + (1-\beta)\mathcal{F}(x) = 0,\\ \Rightarrow &\beta \|Dx\|_p^p + \beta \int_Z m_0 |x|^p dz + (1-\beta)\|x\|^2 = 0,\\ \Rightarrow &\beta\xi \|x\|^p + (1-\beta)\|x\|^2 = 0 \ \text{ (see Proposition 3.6)},\\ \Rightarrow &x = 0, \ \text{ a contradiction.} \end{aligned}$$

As a result, the homotopy invariance property implies that

$$\deg_{(S)_+}(A+m_0K,B_\rho,0)=\deg_{(S)_+}(\mathcal{F},B_\rho,0)=1 \text{ for all } \rho>0. \tag{3.20}$$

Combining (3.19) and (3.20), we conclude that

$$\deg(A-N,B_\rho,0)=1 \text{ for all } 0<\rho\le\rho_0.$$ □

Next we compute the degree $\deg(A-N,B_r,0)$ for $r>0$ big.

Proposition 3.8. *If hypotheses $H(j)$ hold, then there exists $R_0>0$ such that for all $R\ge R_0$ we have deg$(A-N,B_R,0)=-1$.*

Proof. As in the proof of Proposition 3.7, $K: W^{1,p}(Z)\to W^{1,p}(Z)^*$ is defined by

$$(Kx)(\cdot)=|x(\cdot)|^{p-2}x(\cdot)\in L^{p'}(Z)\subseteq W^{1,p}(Z)^* \ \ (\tfrac{1}{p}+\tfrac{1}{p'}=1).$$

We consider the admissible homotopy $h_3:[0,1]\times W^{1,p}(Z)\to W^{1,p}(Z)^*$ defined by

$$h_3(\beta,x)=A(x)-(1-\beta)mK(x)-\beta N(x)$$

for all $(\beta,x)\in[0,1]\times W^{1,p}(Z)$, where $m\in L^\infty(Z)_+$, $\theta(z)\le m(z)\le\widehat{\theta}(z)$ almost everywhere on Z. The admissibility of h_3 follows from the complete continuity of K and Corollary 3.3.

Claim: There exists $R_0>0$ such that $0\notin h_3(\beta,x)$ for all $\beta\in[0,1]$, all $x\in\partial B_R$ and all $R\ge R_0$.

Again we argue indirectly. Suppose we can find $\{\beta_n\}_{n\ge1}\subseteq[0,1]$ and $\{x_n\}_{n\ge1}\subseteq W^{1,p}(Z)$ such that

$$\beta_n\to\beta\in[0,1],\ \ \|x_n\|\to\infty \text{ and } 0\in h_3(\beta_n,x_n) \text{ for all } n\ge1.$$

We have

$$A(x_n)=(1-\beta_n)mK(x_n)+\beta_n u_n \text{ with } u_n\in N(x_n),\ \ n\ge1. \tag{3.21}$$

We set $y_n=\dfrac{x_n}{\|x_n\|}$, $n\ge1$. Now we may assume that

$$y_n\xrightarrow{w}y \text{ in } W^{1,p}(Z) \text{ and } y_n\to y \text{ in } L^p(Z) \text{ as } n\to\infty.$$

From (3.21), we have

$$A(y_n)=(1-\beta_n)mK(y_n)+\beta_n\frac{u_n}{\|x\|^{p-1}}. \tag{3.22}$$

Hypothesis $H(j)(iii)$ implies that

$$\frac{|u_n(z)|}{\|x_n\|^{p-1}}\le\frac{\alpha(z)}{\|x_n\|^{p-1}}+c|y_n(z)|^{p-1} \text{ a.e. on } Z \text{ for all } n\ge1, \tag{3.23}$$

$$\Rightarrow \{\frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

Thus we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^{p'}(Z) \text{ as } n \to \infty.$$

For $\varepsilon > 0$ and $n \geq 1$, we introduce the sets

$$D^+_{\varepsilon,n} = \{z \in Z : x_n(z) > 0,\ \theta(z) - \varepsilon \leq \frac{u_n(z)}{x_n(z)^{p-1}} \leq \widehat{\theta}(z) + \varepsilon\}$$

$$\text{and } D^-_{\varepsilon,n} = \{z \in Z : x_n(z) < 0,\ \theta(z) - \varepsilon \leq \frac{u_n(z)}{|x_n(z)|^{p-2}x_n(z)} \leq \widehat{\theta}(z) + \varepsilon\}.$$

Note that $x_n(z) \to +\infty$ almost everywhere on $\{y > 0\}$ and $x_n(z) \to -\infty$ almost everywhere on $\{y < 0\}$. So by virtue of hypothesis $H(j)(iv)$, we have

$$\chi_{D^+_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y > 0\} \text{ and } \chi_{D^-_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } \{y < 0\}.$$

It follows that

$$\|(1 - \chi_{C^+_{\varepsilon,n}})\frac{u_n}{\|x_n\|^{p-1}}\|_{L^{p'}(\{y>0\})} \to 0,$$

$$\|(1 - \chi_{C^-_{\varepsilon,n}})\frac{u_n}{\|x_n\|^{p-1}}\|_{L^{p'}(\{y<0\})} \to 0.$$

From the definitions of the sets $D^+_{\varepsilon,n}$ and $D^-_{\varepsilon,n}$, we have

$$\chi_{D^+_{\varepsilon,n}}(z)(\theta(z) - \varepsilon)y_n(z)^{p-1} \leq \chi_{D^+_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|^{p-1}}$$
$$= \chi_{D^+_{\varepsilon,n}}(z)\frac{u_n(z)}{x_n(z)^{p-1}}y_n(z)^{p-1} \leq \chi_{D^+_{\varepsilon,n}}(z)(\widehat{\theta}(z) + \varepsilon)y_n(z)^{p-1} \text{ a.e. on } Z$$

and

$$\chi_{D^-_{\varepsilon,n}}(z)(\theta(z) - \varepsilon)|y_n(z)|^{p-2}y_n(z) \geq \chi_{D^-_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|^{p-1}}$$
$$= \chi_{D^-_{\varepsilon,n}}(z)\frac{u_n(z)}{|x_n(z)|^{p-2}x_n(z)}|y_n(z)|^{p-2}y_n(z)$$
$$\geq \chi_{D^-_{\varepsilon,n}}(z)(\widehat{\theta}(z) + \varepsilon)|y_n(z)|^{p-2}y_n(z) \quad \text{a.e. on } Z.$$

If we let $n \to \infty$ and use Proposition 3.9, page 694, of Hu-Papageorgiou [15], we obtain

$$(\theta(z) - \varepsilon)y(z)^{p-1} \leq h(z) \leq (\widehat{\theta}(z) + \varepsilon)y(z)^{p-1} \text{ a.e. on } \{y > 0\},$$
$$(\widehat{\theta}(z) + \varepsilon)|y(z)|^{p-2}y(z) \leq h(z) \leq (\theta(z) - \varepsilon)|y(z)|^{p-2}y(z) \text{ a.e. on } \{y < 0\}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and we obtain

$$\theta(z)y(z)^{p-1} \le h(z) \le \widehat{\theta}(z)y(z)^{p-1} \text{ a.e. on } \{y > 0\}, \tag{3.24}$$

$$\widehat{\theta}(z)|y(z)|^{p-2}y(z) \le h(z) \le \theta(z)|y(z)|^{p-2}y(z) \text{ a.e. on } \{y < 0\}. \tag{3.25}$$

Moreover, from (3.23), we see that

$$h(z) = 0 \text{ a.e. on } \{y = 0\}. \tag{3.26}$$

From (3.24),(3.25) and (3.26), it follows that

$$h(z) = g(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z$$

with $g \in L^\infty(Z)_+$, $\theta(z) \le g(z) \le \widehat{\theta}(z)$ almost everywhere on Z.

We return to (3.22) and act with the test function $y_n - y \in W^{1,p}(Z)$. Now we obtain

$$\langle A(y_n), y_n - y\rangle = (1-\beta_n)\int_Z m|y_n|^{p-2}y_n(y_n-y)dz + \beta_n \int_Z \frac{u_n}{\|x_n\|^{p-1}}(y_n-y)dz.$$

We have

$$\int_Z m|y_n|^{p-2}y_n(y_n-y)dz,\ \int_Z \frac{u_n}{\|x_n\|^{p-1}}(y_n-y)dz \to 0 \text{ as } n \to \infty.$$

It follows that

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0, \quad \Rightarrow y_n \to y \text{ in } W_0^{1,p}(Z) \text{ (see Proposition 3.4).}$$

Hence $\|y\| = 1$. We pass to the limit as $n \to \infty$ in (3.22) and obtain

$$A(y) = (1-\beta)mK(y) + \beta gK(y).$$

Let $\widehat{m} = (1-\beta)m + \beta g$. Then $\widehat{m} \in L^\infty(Z)_+$, $\theta(z) \le \widehat{m}(z) \le \widehat{\theta}(z)$ almost everywhere on Z and $A(y) = \widehat{m}K(y)$. For every $\psi \in C_c^1(Z)$, we have

$$\int_Z \|Dy\|^{p-2}(Dy, D\psi)_{\mathbb{R}^N}dz = \int_Z \widehat{m}|y|^{p-2}y\psi dz. \tag{3.27}$$

From the representation theorem for the elements of the dual space $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$ (see for example Gasinski-Papageorgiou [11], page 212), we have

$$\operatorname{div}(\|Dy\|^{p-2}Dy) \in W^{-1,p'}(Z).$$

If by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$, from (3.27) we have

$$\langle \text{-div}(\|Dy\|^{p-2}Dy), \psi\rangle_0 = \langle \widehat{m}|y|^{p-2}y, \psi\rangle_0 \text{ for all } \psi \in C_c^1(Z). \tag{3.28}$$

Recall that $C_c^1(Z)$ is dense in $W_0^{1,p}(Z)$. Thus from (3.28), it follows that

$$-\operatorname{div}(\|Dy(z)\|^{p-2}Dy(z)) = \widehat{m}(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z. \tag{3.29}$$

Using the nonlinear Green's identity of Kenmochi [16] and Casas-Fernandez [7] (see also Gasinski-Papageorgiou [11], page 211), for every $\xi \in W^{1,p}(Z)$ we have

$$\int_Z \|Dy\|^{p-2}(Dy, D\xi)_{\mathbb{R}^N}dz + \int_Z \xi \mathrm{div}(\|Dy\|^{p-2}Dy)dz = \langle \frac{\partial y}{\partial n_p}, \gamma_0(\xi)\rangle_{\partial Z}. \tag{3.30}$$

Here by γ_0 we denote the trace map and by $\langle\cdot,\cdot\rangle_{\partial Z}$ the duality brackets for the pair $(W^{\frac{1}{p'},p}(\partial Z), W^{-\frac{1}{p'},p'}(\partial Z))$. Since the range of the trace map is $W^{\frac{1}{p'},p}(\partial Z)$, from (3.27),(3.29) and (3.30), we infer that

$$\frac{\partial y}{\partial n_p} = 0 \ \text{ in } \ W^{-\frac{1}{p'},p'}(\partial Z). \tag{3.31}$$

From Theorem 7.1, page 286 of Ladyzhenskaya-Uraltseva [17] we have that $y \in L^\infty(Z)$. Then we can apply Theorem 2 of Lieberman [19] and obtain $y \in C^{1,\beta}(\overline{Z})$ for some $0 < \beta < 1$. So in (3.31) $\frac{\partial y}{\partial n_p} = 0$ is interpreted pointwise and finally we have

$$\begin{cases} -\mathrm{div}(\|Dy(z)\|^{p-2}Dy(z)) = \widehat{m}(z)|y(z)|^{p-2}y(z) \text{ a.e. on } Z, \\ \dfrac{\partial y}{\partial n_p}(z) = 0 \ \text{ for all } \ z \in \partial Z, \ \ 1 < p < \infty. \end{cases} \tag{3.32}$$

Since $\theta(z) \le \widehat{m}(z) \le \widehat{\theta}(z)$ almost everywhere on Z, from Proposition 3.5 we have

$$1 = \widehat{\lambda}_2(\lambda_2) < \widehat{\lambda}_2(\widehat{m}).$$

Also we know that $\widehat{\lambda}_1(\widehat{m}) = 0$. Hence $\lambda = 1$ is not an eigenvalue of $(-\triangle_p, W^{1,p}(Z), \widehat{m})$. Therefore, from (3.32), we deduce that $y = 0$, which contradicts the fact that $\|y\| = 1$. This proves the claim.

We use the homotopy invariance of the degree map and have

$$\deg(A - N, B_R, 0) = \deg_{(S)_+}(A - mK, B_R, 0) \ \text{ for all } \ R \ge R_0. \tag{3.33}$$

For a possibly smaller $m \in L^\infty(Z)_+$, we can always take a number $\mu > 0$ so that $m(z) \le \mu$ almost everywhere on Z and the interval $(0, \mu)$ contains no eigenvalues of $(-\triangle_p, W^{1,p}(Z))$ (recall $\lambda_1 = 0$ is isolated). So from our choice of $m \in L^\infty(Z)_+$ (see also hypothesis $H(j)(iv)$) and Drabek [8], we have

$$\deg_{(S)_+}(A - mK, B_R, 0) = -1 \ \text{ for all } \ R > 0. \tag{3.34}$$

From (3.33) and (3.34) it follows that $\deg(A - N, B_R, 0) = -1$ for all $R \ge R_0$. □

Theorem 3.9. *If hypotheses $H(j)$ hold, then problem* (1.1) *has at least one nontrivial solution $x \in C^1(\overline{Z})$.*

Proof. Let $\rho_0, R_0 > 0$ be as in Propositions 3.7 and 3.8 respectively. We choose $\rho \in (0, \rho_0]$ and $R \in [R_0, \infty)$ such that $\rho < R$. From the additivity of the domain property of the degree map, we have

$$\begin{aligned} &\deg(A - N, B_R, 0) = \deg(A - N, B_\rho, 0) + \deg(A - N, B_R \backslash \overline{B}_\rho, 0), \\ \Rightarrow &\deg(A - N, B_R \backslash \overline{B}_\rho, 0) = -2 \ \text{(see Propositions 3.7 and 3.8)}. \end{aligned}$$

Then from the solution property of the degree map, we obtain $x \in B_R \backslash \overline{B}_\rho$, hence $x \neq 0$, such that

$$A(x) = u \ \text{ with } \ u \in N(x).$$

As in the proof of Proposition 3.8, via the nonlinear Green's identity, we obtain

$$\left\{ \begin{array}{l} -\mathrm{div}(\|Dx(z)\|^{p-2} Dx(z)) = u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ \dfrac{\partial x}{\partial n_p} = 0 \ \text{ on } \ \partial Z. \end{array} \right. \tag{3.35}$$

From (3.35) and the nonlinear regularity theory (see Lieberman [19]), we conclude that $x \in C^1(\overline{Z}),\ x \neq 0$. □

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