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CRITICAL FUNCTIONS AND ELLIPTIC PDE ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. We study in this work the existence of minimizing solutions to the critical-power type equation $\triangle_{\mathbf{g}} u + h.u = f.u^{\frac{n+2}{n-2}}$ on a compact Riemannian manifold in the limit case normally not solved by variational methods. For this purpose, we use a concept of "critical function" that was originally introduced by E. Hebey and M. Vaugon for the study of the second best constant in the Sobolev embeddings. Along the way, we prove an important estimate concerning concentration phenomenas when f is a non-constant function.

1. Introduction

In the beginning was the Yamabe problem:

Yamabe problem: *Given a compact Riemannian manifold* $(M, \mathbf{g})$ *of dimension* $n \geq 3$, *does there exist a metric* $\mathbf{g}'$ *conformal to* $\mathbf{g}$ *having constant scalar curvature*?

If we write $\mathbf{g}' = u^{\frac{4}{n-2}}.\mathbf{g}$ where $u > 0$ is a smooth function on M, the scalar curvatures are linked by the partial differential equation

$$\triangle_{\mathbf{g}} u + \frac{n-2}{4(n-1)} S_{\mathbf{g}}.u = \frac{n-2}{4(n-1)} S_{\mathbf{g}'}.u^{\frac{n+2}{n-2}},$$

where $S_{\mathbf{g}}$ is the scalar curvature of $\mathbf{g}$ and where $\triangle_{\mathbf{g}} = -\nabla^i \nabla_i$ is the Riemannian Laplacian of $\mathbf{g}$.

To solve the Yamabe problem, one therefore has to prove the existence of a solution $u > 0$ to this partial differential equation when $S_{\mathbf{g}'}$ is a constant. More generally, the prescribed curvature problems, which consist in deciding, given a smooth function f on M, if f is the scalar curvature of a metric conformal to $\mathbf{g}$, come down to proving the existence of a positive smooth solution u to the above equation when $S_{\mathbf{g}'}$ is replaced by f.

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These problems launched the study of elliptic PDEs on compact Riemannian manifolds of the form

$$(E_{h,f,\mathbf{g}}) : \triangle_{\mathbf{g}} u + h.u = f.u^{\frac{n+2}{n-2}}.$$

In all this paper M will be a compact Riemannian manifold of dimension $n \geqslant 3$, we will use the letter $\mathbf{g}$ or $\mathbf{g}'$ to denote a Riemannian metric on M, and h and f will always be smooth functions on M. We will always suppose the functions to be smooth, however, in the definitions and in most of the theorems, continuity is in general sufficient. Beside, we will keep this notation: the letter $\mathbf{g}$ for the metrics, letter h for the function on the left of equation $E_{h,f,\mathbf{g}}$, (defining the operator $\triangle_{\mathbf{g}} + h$), and letter f for the function on the right of the equation; the unknown function will be designated by u.

One of the possible methods to study these equations is the use of variational methods, which have the advantage of giving minimizing solutions, or solution of minimal energy. If one multiplies equation $(E_{h,f,\mathbf{g}})$ by u and integrate over M, one gets

$$\int_M |\nabla u|^2_{\mathbf{g}} \, dv_{\mathbf{g}} + \int_M h.u^2 dv_{\mathbf{g}} = \int_M f \, |u|^{\frac{2n}{n-2}} \, dv_{\mathbf{g}}.$$

The variational methods therefore lead us to consider the functional

$$I_{h,\mathbf{g}}(w) = \int_M |\nabla w|^2_{\mathbf{g}} \, dv_{\mathbf{g}} + \int_M h.w^2 dv_{\mathbf{g}}$$

defined for $w \in H_1^2(M)$, the Sobolev space of L^2 functions whose gradient is also in L^2, and the minimum of this functional $\lambda_{h,f,\mathbf{g}} = \inf_{w \in \mathcal{H}_f} I_{h,\mathbf{g}}(w)$ on the set

$$\mathcal{H}_f = \{w \in H_1^2(M) : \int_M f \, |w|^{\frac{2n}{n-2}} \, dv_{\mathbf{g}} = 1\}.$$

The Euler equation associated with the minimization problem of this functional by a function u such that $I_{h,\mathbf{g}}(u) = \inf_{w \in \mathcal{H}_f} I_{h,\mathbf{g}}(w)$ is indeed exactly

$$(E_{h,f,\mathbf{g}}) : \triangle_{\mathbf{g}} u + hu = \lambda_{h,f,\mathbf{g}}.f.u^{\frac{n+2}{n-2}},$$

where $\lambda_{h,f,\mathbf{g}}$ appears as a normalizing constant due to the condition

$$\int_M f \, |u|^{\frac{2n}{n-2}} \, dv_{\mathbf{g}} = 1.$$

It is sometimes useful to consider the functional

$$J_{h,f,\mathbf{g}}(w) = \frac{\int_M |\nabla w|^2_{\mathbf{g}} \, dv_{\mathbf{g}} + \int_M h.w^2 dv_{\mathbf{g}}}{\left(\int_M f \, |w|^{\frac{2n}{n-2}} \, dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}$$

and the subset of $H_1^2(M)$ where it is defined

$$\mathcal{H}_f^+ = \{w \in H_1^2(M) : \int_M f\,|w|^{\frac{2n}{n-2}}\,dv_{\mathbf{g}} > 0\}.$$

One then considers the minimization problem by a function u such that $J_{h,f,\mathbf{g}}(u) = \inf_{w\in\mathcal{H}_f^+} J_{h,f,\mathbf{g}}(w)$, the Euler equation being identical but without the normalizing constant. This functional sometimes presents the advantage of being homogeneous in the sense that $J_{h,f,\mathbf{g}}(c.w) = J_{h,f,\mathbf{g}}(w)$ for any constant c. One therefore sees that

$$\inf_{w\in\mathcal{H}_f} I_{h,\mathbf{g}}(w) = \inf_{w\in\mathcal{H}_f^+} J_{h,f,\mathbf{g}}(w) = \lambda_{h,f,\mathbf{g}}.$$

This functional J also has the particular quality, when $h = \frac{n-2}{4(n-1)} S_{\mathbf{g}}$, of being invariant by conformal changes of metrics; it is therefore especially useful when studying problems of prescribed scalar curvatures. We shall mostly use $I_{h,\mathbf{g}}$ and $\mathcal{H}_f$, but for some problems $J_{h,f,\mathbf{g}}$ will prove to be more convenient when we shall want to avoid the constraint $\int_M f\,|u|^{\frac{2n}{n-2}}\,dv_{\mathbf{g}} = 1$.

We will say that a function $u \in H_1^2(M)$ is a solution of minimal energy, or a minimizing solution, if either $I_{h,\mathbf{g}}(u) = \lambda_{h,f,\mathbf{g}}$ with $\int_M f u^{\frac{2n}{n-2}} = 1$, or $J_{h,f,\mathbf{g}}(u) = \lambda_{h,f,\mathbf{g}}$. Then, up to multiplying it by a constant, u is strictly positive and smooth, and it is a solution of

$$(E_{h,f,\mathbf{g}}) : \triangle_{\mathbf{g}} u + hu = \lambda_{h,f,\mathbf{g}}.f.u^{\frac{n+2}{n-2}}$$

with or without the normalizing constant which can always be suppressed just by multiplying again u by a constant. Please, note that we will use this notation $(E_{h,f,\mathbf{g}})$ and $\lambda_{h,f,\mathbf{g}}$ throughout all this article.

Th. Aubin discovered a very important relation between equation $(E_{h,f,\mathbf{g}})$ and the notion of best constant in the Sobolev imbedding theorems. Remember that the inclusion of $H_1^2(M)$ in $L^p(M)$ is compact for $p < \frac{2n}{n-2}$ and only continuous for $p = \frac{2n}{n-2}$, which is called the critical exponent for the Sobolev imbeddings and will be denoted $2^* = \frac{2n}{n-2}$. The continuous imbedding $H_1^2(M) \subset L^{2^*}(M)$ is expressed by the existence of two positive constants A and B such that

$$\forall u \in H_1^2(M) : \Big(\int_M |u|^{\frac{2n}{n-2}} dv_{\mathbf{g}}\Big)^{\frac{n-2}{n}} \le A \int_M |\nabla u|^2_{\mathbf{g}}\,dv_{\mathbf{g}} + B \int_M u^2 dv_{\mathbf{g}}. \quad (1)$$

The best first constant is the minimum A that one can put in (1) such that there exists B with (1) still true. It was proved by E. Hebey and M. Vaugon

[17] that this minimum is attained, and its value is known to be the same as for the sharp Euclidean Sobolev inequality,

$$A_{\min} = K(n,2)^2 = \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}},$$

where ω_n is the volume of the unit sphere of dimension n. One then takes $B_0(\mathbf{g})$ to be the minimum B such that (1) remains true with $A_{\min}$; it is proved that $B_0(\mathbf{g}) < +\infty$ [17]. The inequality, for all $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{\frac{2n}{n-2}}\,dv_{\mathbf{g}}\Big)^{\frac{n-2}{n}} \leq K(n,2)^2 \int_M |\nabla u|^2_{\mathbf{g}}\,dv_{\mathbf{g}} + B_0(\mathbf{g}) \int_M u^2 dv_{\mathbf{g}} \quad (2)$$

is then sharp with respect to both the first and second constants, in the sense that none of them can be lowered. If the value of the best constant $A_{\min} = K(n,2)^2$ is known and independent of the manifold $(M,\mathbf{g})$, on the other hand, $B_0(\mathbf{g})$, as the notation indicates, depends on the geometry and its study is difficult; it is for this purpose that "critical functions" were introduced by E. Hebey and M. Vaugon [18]. When there shall be no risk of confusion, these constants will be denoted by K and B_0.

As a remark, note that because of the compactness of the inclusion $H_1^2(M) \subset L^p(M)$ for $p < 2^*$, standard variational methods and elliptic theory give rapidly the existence of minimizing solutions of the equation $\triangle_{\mathbf{g}} u + hu = f.u^{p-1}$ when $\triangle_{\mathbf{g}} + h$ is a coercive operator. The case $p = 2^*$ is therefore already a limit case. (Very little is known for $p > 2^*$ without additional hypothesis, like, e.g., invariance by symmetry, see [14].)

The best constants in the Sobolev embedding appeared in the study of equations $(E_{h,f,\mathbf{g}})$ when Th. Aubin proved the following theorem:

Theorem (Aubin)**.** *For any Riemannian manifold $(M,\mathbf{g})$ of dimension $n \geqslant 3$, any function h such that $\triangle_{\mathbf{g}} + h$ is a coercive operator, and any function f such that $\sup_M f > 0$, one always has*

$$\lambda_{h,f,\mathbf{g}} \leq \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}.$$

Furthermore, if this inequality is strict, then there exists a minimizing solution for $(E_{h,f,\mathbf{g}})$.

This theorem is the starting point of all this work. It proves the existence of minimizing solutions to equation $(E_{h,f,\mathbf{g}})$ under the hypothesis

$$\lambda_{h,f,\mathbf{g}} < \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}.$$

Our work is essentially concerned with the problem of the existence of minimizing solutions to these equations ($E_{h,f,\mathbf{g}}$) in the "critical case" where

$$\lambda_{h,f,\mathbf{g}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}},$$

a problem which is normally not solved by variational methods. It is for the study of this problem that we are now going to define the "critical functions."

Let us first review the data:

Data: Throughout this article, $(M, \mathbf{g})$ will be a compact Riemannian manifold of dimension $n \geqslant 3$. We let $f : M \to \mathbb{R}$ be a fixed smooth function such that $\sup_M f > 0$. Let also $h : M \to \mathbb{R}$ be a smooth function with the additional hypothesis that the operator $\triangle_{\mathbf{g}} + h$ is coercive if f is not positive on all of M. (Remember that continuity of h and f is sufficient in the definitions and in most of the theorems. Also, if $f \leqslant 0$ on M, classical variational methods already give a lot of results for the existence of solutions; therefore $\sup f > 0$ is the most interesting case.)

Definition 1. *With these data, and with the above notation, we say that*

- *h is weakly critical for f and $\mathbf{g}$ if $\lambda_{h,f,\mathbf{g}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$*
- *h is subcritical for f and $\mathbf{g}$ if $\lambda_{h,f,\mathbf{g}} < \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$*
- *h is **critical** for f and $\mathbf{g}$ if h is weakly critical and if for any function $k \leq h$, $k \neq h$ such that $\triangle_{\mathbf{g}} + k$ is coercive, k is subcritical.*

Using the theorem of Th. Aubin, we can give an equivalent definition of critical functions. Indeed, using this theorem, it is easy to see that if h is weakly critical and ($E_{h,f,\mathbf{g}}$) has a minimizing solution u, then h is a critical function; just note that for $k \leq h$, $k \neq h$, $I_{k,\mathbf{g}}(u) < I_{h,\mathbf{g}}(u)$. Therefore, we can give the following equivalent definition:

Definition 2. *A function h is critical for f and $\mathbf{g}$ if*

- *for any continuous function $k \leq h$, $k \neq h$ such that $\triangle_{\mathbf{g}} + k$ is coercive (which is the case as soon as k is close enough to h in C^0), ($E_{k,f,\mathbf{g}}$) has a minimizing solution,*
- *for any continuous function $k' \geq h$, $k' \neq h$, ($E_{k',f,\mathbf{g}}$) has **no** minimizing solution.*

Remark. If h is weakly critical for a positive function f, necessarily, $\triangle_{\mathbf{g}} + h$ is coercive; just use the Sobolev inequality.

Critical functions are thus introduced as "separating" functions giving rise to an equation having minimizing solutions, and functions giving rise to an equation that cannot have any such solution. We therefore have transformed the problem of the existence of minimizing solutions when

$$\lambda_{h,f,\mathbf{g}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$$

to the problem of existence of minimizing solutions to $(E_{h,f,\mathbf{g}})$ when h is a critical function.

Before passing to the theorems proved in this work, we have to give two very important properties of critical functions.

First, they transform in conformal changes of metric exactly like scalar curvature; indeed, let $u \in C^\infty(M)$, $u > 0$ and $\mathbf{g}' = u^{\frac{4}{n-2}}\mathbf{g}$ a metric conformal to $\mathbf{g}$. Let also h be a smooth function. We set

$$h' = \frac{\triangle_{\mathbf{g}} u + h.u}{u^{\frac{n+2}{n-2}}}.$$

Then, some computations show that h is critical for f and $\mathbf{g}$ if and only if h' is critical for f and $\mathbf{g}'$.

Second, we come back to the evaluation of $\lambda_{h,f,\mathbf{g}}$. Th. Aubin introduced, in the functional $J_{h,f,\mathbf{g}}$, the following test functions

$$\psi_k(Q) = \begin{cases} (\frac{1}{k} + r^2)^{-\frac{n-2}{2}} - (\frac{1}{k} + \delta^2)^{-\frac{n-2}{2}} & \text{if } r < \delta \\ 0 & \text{if } r \geq \delta, \end{cases}$$

where $\delta < injM$ (the injectivity radius of M), $P \in M$ is a fixed point, $k \in \mathbb{N}^*$, and where $r = d_{\mathbf{g}}(P,Q)$. When $dimM = n \geqslant 4$, we get, if P is a point where f is maximum on M,

$$J_{h,f,\mathbf{g}}(\psi_k) = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}\Big\{1+ \\ + \frac{1}{n(n-4)}\Big(\frac{4(n-1)}{n-2}h(P) - S_{\mathbf{g}}(P) + \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}\Big)\frac{1}{k}\Big\} + o(\frac{1}{k}).$$

We therefore get the following important proposition:

Proposition 1. *If $dimM \geq 4$ and if h is weakly critical for f and $\mathbf{g}$ (thus in particular if it is critical), as $\lambda_{h,f,\mathbf{g}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$, necessarily, if P is a point of maximum of f*

$$\frac{4(n-1)}{n-2}h(P) \geq S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

Remark. If f is constant on M, this means that $\frac{4(n-1)}{n-2}h \geq S_{\mathbf{g}}$ on all of M. Note also that in dimension 4, the term $\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$ disappears.

2. Statement of the results

In all that follows, we will make the following hypothesis:

Hypothesis (H) We now suppose that $dimM = n \geqslant 4$. We suppose that all our functions h are such that $\triangle_{\mathbf{g}} + h$ is coercive. Also, f will always be a smooth function such that $\sup_M f > 0$. We will denote $\max f = \{x \in M : f(x) = \sup_M f\}$.

Our first theorem concerns the existence of minimizing solutions to $(E_{h,f,\mathbf{g}})$ when h is critical.

Theorem. *If h is a critical function for f and $\mathbf{g}$, ($h, f, \mathbf{g}$ satisfying **H**), and if for all points P where f is a maximum on M, we have*

$$\frac{4(n-1)}{n-2}h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)},$$

then there exist a minimizing solution for $(E_{h,f,\mathbf{g}})$.

This theorem is an immediate consequence of the following result, more general but more technical in its statement. (Just take $h_t = h - t$ to get the theorem above.)

Theorem 1. *Let h be a weakly critical function for f and $\mathbf{g}$, (assuming hypothesis **H**). If, for all point P where f is maximum, we have*

$$\frac{4(n-1)}{n-2}h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)},$$

and if there exists a family of functions (h_t), $h_t \lneqq h$, h_t being sub-critical for all t in a neighbourhood of a real $t_0 \in \mathbb{R}$, and such that $h_t \underset{t \to t_0}{\to} h$ in $C^{0,\alpha}$, then there exists a minimizing solution for $(E_{h,f,\mathbf{g}})$, and therefore, h is critical for f and $\mathbf{g}$.

E. Hebey and M. Vaugon, in the context of their study of $B_0(\mathbf{g})$, proved this theorem in the case where f is constant, and as they did, we base our computations on the article of Djadli and Druet [10]. The presence of a non-constant function f on the right of equation $(E_{h,f,\mathbf{g}})$ introduces new difficulties in the proof, and requires the use of very powerful estimates concerning concentration phenomena, called C^0-*theory*, due to Druet and Robert, available in [13]; the use of C^0-*theory* was kindly suggested to us by E. Hebey. Also, an alternate proof, not using C^0-*theory*, thus in some sense more elementary, but requiring the additional hypothesis that the Hessian of f be non-degenerate at its points of maximum on M, will, as a "byproduct," prove another very important estimate concerning these concentration phenomena, not available without heavy hypotheses in the case when f is a constant function; this estimate concerns the speed of convergence to a concentration point (see subsection 4.2), is of independent interest, and was obtained in the author's PhD thesis [8] to prove Theorem 1.

The next natural question is of course to ask if there exist critical functions. The answer, which is positive, will appear to be a consequence of Theorem 1. We will say that a set $E \subset M$ is *thin* if $M - E$ contains a dense open subset.

Theorem 2. *Being given the manifold $(M, \mathbf{g})$ and a non-constant function f, there exist infinitely many functions h critical for f and $\mathbf{g}$, which satisfy, in each maximum point P of f,*

$$\frac{4(n-1)}{n-2} h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}. \qquad (*)$$

By Theorem 1, these critical functions are such that $(E_{h,f,\mathbf{g}})$ have minimizing solutions. Also, if the set of maximum points of f is thin and if $\int_M f > 0$, there exist critical functions h that are strictly positive and satisfy the strict inequality $(*)$.

These first theorems lead us to modify slightly our vision of critical functions. Note that, in equation $(E_{h,f,\mathbf{g}})$, there are three data that one can modify; the functions h and f, of course, but also the metric $\mathbf{g}$ in a conformal class, as, by the conformal Laplacian transformation formula, the equation is changed in a similar one if we change $\mathbf{g}$ in $\mathbf{g}' = u^{\frac{4}{n-2}}.\mathbf{g}$. This lead us to the following definition

$(h, f, \mathbf{g})$ *is a critical triple if h is a critical function for f and $\mathbf{g}$.*

We shall say that the triple $(h, f, \mathbf{g})$ has minimizing solutions if $(E_{h,f,\mathbf{g}})$ has; we can also speak of weakly critical or sub-critical triples.We then asked ourselves the following question:

Being given two of the three data of a triple, can one find the third to obtain a critical triple?

For example, the problem of the existence of critical functions can be formulated in the following manner: if we are given the function f and the metric $\mathbf{g}$, can we complete the triple $(., f, \mathbf{g})$ by a function h to obtain a critical triple $(h, f, \mathbf{g})$?

We address the two other questions, first fixing h and f and seeking a conformal metric $\mathbf{g}'$, and then fixing the function h and the metric $\mathbf{g}$ and seeking a function f. We obtain answers expressed by the following two theorems:

Theorem 3. *On the manifold* $(M, \mathbf{g})$*, let there be given a function* h *and a function* f*, satisfying (**H**). We suppose that the set of maximum points of* f *is thin. Then, there exists a metric* $\mathbf{g}'$ *conformal to* $\mathbf{g}$ *such that* $(h, f, \mathbf{g}')$ *is a critical triple. Moreover, we can find* $\mathbf{g}'$ *such that* $(h, f, \mathbf{g}')$ *has minimizing solutions.*

This theorem was proved by E. Humbert and M. Vaugon in the case $f = cst = 1$ and M not conformally diffeomorphic to the sphere, [19]. Their method works in the case of a non-constant function f and an arbitrary manifold once it is proved that we can suppose the existence of positive critical functions satisfying the strict inequality (*) in Theorem 2, a result we included in this theorem (note that, as $\sup f > 0$, we can always find a metric $\mathbf{g}'$ conformal to $\mathbf{g}$ such that $\int_M f dv_{\mathbf{g}'} > 0$). In fact, when M is not conformally diffeomorphic to the sphere and $S_{\mathbf{g}}$ is constant, it can be proved that $B_0(\mathbf{g})K(n,2)^{-2}$ is a critical (constant) function for 1 and $\mathbf{g}$, and it is obviously positive. We will discuss weaker hypotheses for this theorem, as well as the problem of existence of positive critical functions in section 6.

The last question brings us to the following answer when the dimension of M is greater than 5, a requirement which is linked to the fact that $\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$ disappears in dimension 4 in the inequality of Proposition 1.

Theorem 4. *Let there be given a manifold* $(M, \mathbf{g})$ *of dimension* $n \geq 5$*, and a function* h *such that* $\triangle_{\mathbf{g}} + h$ *is coercive. Then, there exists a non-constant function* f *such that* $(h, f, \mathbf{g})$ *is critical with minimizing solutions if, and only if,* $(h, 1, \mathbf{g})$ *is a sub-critical triple* (*where* 1 *is the constant function* 1).

Note that if $(h, 1, \mathbf{g})$ is weakly critical, then either this triple has minimizing solutions, in which case it is a critical triple, or there is no non-constant function f such that $(h, f, \mathbf{g})$ is critical with minimizing solutions (see the proof and what follows). The proof of this theorem is quite difficult, and makes use of the method developed for the proof of Theorem 1. Also, this proof brought us to make some more remarks about critical functions. First, it is easily seen, by using the functional J, that if $(h, f, \mathbf{g})$ is a critical triple, then, for any constant $c > 0$, $(h, c.f, \mathbf{g})$ is also a critical triple. It would therefore be more appropriate to speak of a triple $(h, [f], \mathbf{g})$ where $[f] = \{c.f : c > 0\}$ could be called the "class" of f. Note for example that we can always suppose that $\sup f = 1$; also, to compare two triples $(h, f, \mathbf{g})$ and $(h, f', \mathbf{g})$, one has to suppose that $\sup f = \sup f'$. Note also that on $[f]$, the quotient $\frac{\triangle_{\mathbf{g}} f}{f}$ is constant. Second, in the proof of Theorem 4, we had to approximate the function f by a family (f_t), unlike Theorem 1 where we used a family (h_t) approaching h. This suggested another possible definition of critical functions, dual to the first one in the sense that we exchange the role of h and f.

Definition 3. *Let $(M, \mathbf{g})$ be of dimension $n \geq 3$ and h be such that $\triangle_{\mathbf{g}} + h$ is coercive. We shall say that a smooth function f such that $\sup_M f > 0$ is critical for h and $\mathbf{g}$ if*

(a) $\lambda_{h,f,\mathbf{g}} = \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}$

(b) *for any smooth function f' such that $\sup f = \sup f'$ and $f' \gneqq f$, $\lambda_{h,f',\mathbf{g}} < \frac{1}{K(n,2)^2 (\sup_M f')^{\frac{n-2}{n}}}$.*

Remark. If $\sup f = \sup f'$ and $f' \lneqq f$, then $\lambda_{h,f',\mathbf{g}} = \frac{1}{K(n,2)^2 (\sup_M f')^{\frac{n-2}{n}}}$ as $J_{h,f',\mathbf{g}}(w) \geqslant J_{h,f,\mathbf{g}}(w)$ for any function w.

It is then natural to ask if the two definitions are equivalent ($\mathbf{g}$ being fixed)

Is f critical for h if, and only if, h is critical for f ?

This question seems quite difficult. A positive answer would justify the concept of critical triple. Remember that, because of Proposition 1, we have in both cases, when P is a point where f is maximum on M,

$$\frac{4(n-1)}{n-2} h(P) \geqslant S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

We obtain the following theorem:

Theorem 5. *Let $(M, \mathbf{g})$ be a compact manifold of dimension $n \geq 5$, and let h be a function such that $\triangle_{\mathbf{g}} + h$ is coercive. Let f be a smooth function such that $\sup_M f > 0$. We suppose that for any point P where f is maximum on M,*

$$\frac{4(n-1)}{n-2} h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

Then, f is critical for h if, and only if, h is critical for f.

Remark. If 1 is critical for h, then every non-constant function f, such that $\sup f = 1$, is weakly critical for h with *no* minimizing solutions. Indeed, here again if a function f is weakly critical for h with a minimizing solution, then f is critical.

There is an interesting consequence of Theorems 4 and 5. We said in the introduction that an important application of equations $(E_{h,f,\mathbf{g}})$ was the study of prescribed scalar curvature: being given a smooth function f on the manifold $(M, \mathbf{g})$, is f the scalar curvature of a metric conformal to $\mathbf{g}$? The theorem of Th. Aubin shows that if f is sub-critical for $S_{\mathbf{g}}$, then f is a scalar curvature. Theorem 4 applied to $h = \frac{n-2}{4(n-1)} S_{\mathbf{g}}$ shows that

On a compact manifold $(M, \mathbf{g})$ *not conformaly diffeomorphic to the sphere, there exist scalar curvatures of metrics conformal to* $\mathbf{g}$ *that are only weakly critical* (*more precisely critical*).

Another application, remarked by E. Hebey, is to the study of *Sobolev inequalities in the presence of a twist.* See the article [9] for more details on the construction of twisted metrics.

The previous theorems all deal with manifolds of dimension at least 4, or even 5. We will give results concerning the dimension 3 in the last section. They are very interesting, but they are rapid generalizations of results obtained by O. Druet in the case $f =$ *constant* [11], the introduction of a non-constant f introducing this time no real difficulties. We prefer therefore to state them at the end, with no proof, sending the reader to the article of O. Druet or to our PhD thesis [8], available online, for more details.

3. The three main tools

We want to present here the three main tools used in the proof of our various theorems. These tools were developed by several persons since M. Vaugon and P.L. Lions, essentially E. Hebey, O. Druet F. Robert, M. Struwe, E. Humbert and Z. Faget, among others.

3.1. **The concentration point.**

To prove the existence of a solution $u > 0$ to our equation

$$(E_{h,f,\mathbf{g}}) : \triangle_{\mathbf{g}} u + h.u = \lambda.f.u^{\frac{n+2}{n-2}},$$

the idea will often be to associate a family of equations having minimizing solutions $u_t > 0$

$$E_t : \triangle_{\mathbf{g}} u_t + h_t.u_t = \lambda_t.f.u_t^{\frac{n+2}{n-2}}$$

with $h_t \to h$ in $C^{0,\alpha}(M)$ and $\lambda_t \to \lambda$ a converging sequence of real numbers, in such a way that for some $u \in H_1^2$ $u_t \to u$ strongly in L^p, $p < 2^*$, and $u_t \rightharpoonup u$ weakly in H_1^2 with a constraint

$$\int_M f.u_t^{2^*} dv_{\mathbf{g}} = 1.$$

To simplify, we will suppose that all convergences are for $t \to t_0 = 1$. The difficulty will be to prove that u is not the trivial zero solution, as then, by the maximum principle, we have $u > 0$. We will proceed by contradiction, and suppose $u \equiv 0$. The idea is then that, because of the condition $\int_M f.u_t^{2^*} = 1$, all the "mass" of the functions u_t, which converge to 0 in L^p, $p < 2^*$, concentrates around a point of the manifold. We thus define:

Definition 4. *$x_0 \in M$ is a point of concentration of the sequence (u_t) if for any $\delta > 0$:*

$$\limsup_{t \to t_0} \int_{B(x_0,\delta)} u_t^{2^*} dv_{\mathbf{g}} > 0.$$

It is easy to see that because M is compact and we require $\int_M f.u_t^{2^*} dv_{\mathbf{g}} = 1$, there exists at least one point of concentration. We will show that there exists only one point of concentration, that it is a point where f is maximum, and that there exists a sequence of points x_t converging to a point $x_0 \in M$ such that $u_t(x_t) = \max_M u_t \to +\infty$, and $u_t \to 0$ in $C^0_{loc}(M - \{x_0\})$. In fact the idea is that one can work "as if" the functions u_t have compact support in a small neigbourhood of x_0 when t is close to t_0.

3.2. **Blow-up analysis.** Thanks to the concentration point, one brings back the study of the family u_t converging to 0, to what happens around x_0. The idea of *blow-up analysis* is to do a "change of scale" around x_0; we will call a *blow-up* with center x_t and coefficient k_t the following sequence of charts and changes of metrics. We consider, for δ small enough,

$$\begin{array}{ccccc} B(x_t,\delta) & \overset{\exp_{x_t}^{-1}}{\to} & B(0,\delta) \subset \mathbb{R}^n & \overset{\psi_{k_t}}{\to} & B(0,k_t\delta) \subset \mathbb{R}^n \\ & & x & \mapsto & k_t x \\ \mathbf{g} & \to & \mathbf{g}_t = \exp_{x_t}^* \mathbf{g} & \to & \widetilde{\mathbf{g}}_t = k_t^2 (\psi_{k_t}^{-1})^* \mathbf{g}_t, \end{array}$$

where $\exp_{x_t}^{-1}$ is the chart deduced from the exponential map in x_t. We set

$$\overline{u}_t = u_t \circ \exp_{x_t};\ \overline{f_t} = f \circ \exp_{x_t};\ \overline{h_t} = h_t \circ \exp_{x_t}.$$

We have

$$\begin{aligned} \triangle_{\mathbf{g}_t} \overline{u}_t + \overline{h}_t.\overline{u}_t &= \lambda_t \overline{f}_t.\overline{u}_t^{\frac{n+2}{n-2}} \\ \int_{B(0,r)} \overline{u}_t^{\alpha} dv_{\mathbf{g}_t} &= \int_{B(x_t,r)} u_t^{\alpha} dv_{\mathbf{g}} \text{ for all } \alpha \geq 1. \end{aligned}$$

We then set $m_t = \max_M u_t$; $\widetilde{u}_t = m_t^{-1} \overline{u}_t \circ \psi_{k_t}^{-1}$; $\widetilde{h}_t = \overline{h}_t \circ \psi_{k_t}^{-1}$; $\widetilde{f}_t = \overline{f}_t \circ \psi_{k_t}^{-1}$; $\widetilde{\mathbf{g}}_t = k_t^2 (\exp_{x_t} \circ \psi_{k_t}^{-1})^* \mathbf{g}$, so in particular $\widetilde{u}_t(x) = m_t^{-1} \overline{u}_t(\frac{x}{k_t})$ and $\widetilde{\mathbf{g}}_t(x) = \exp_{x_t}^* \mathbf{g}(\frac{x}{k_t})$. Then

$$(\widetilde{E}_t) \quad : \quad \triangle_{\widetilde{\mathbf{g}}_t} \widetilde{u}_t + \frac{1}{k_t^2} \widetilde{h}_t.\widetilde{u}_t = \frac{m_t^{\frac{4}{n-2}}}{k_t^2} \lambda_t \widetilde{f}_t.\widetilde{u}_t^{\frac{n+2}{n-2}} \tag{3}$$

$$and \quad : \quad \int_{B(0,k_t r)} \widetilde{u}_t^{\alpha} dv_{\widetilde{\mathbf{g}}_t} = \frac{k_t^n}{m_t^{\alpha}} \int_{B(x_t,r)} u_t^{\alpha} dv_{\mathbf{g}}.$$

We will mostly use the following parameters: we consider a sequence of points (x_t) such that $m_t = \max_M u_t = u_t(x_t) := \mu_t^{-\frac{n-2}{2}}$ and $k_t = \mu_t^{-1}$. μ_t will appear to be a fundamental parameter in the study of concentration phenomena. Denoting by (x^i) the coordinates in $\mathbb{R}^n$, one has

$$(\widetilde{E}_t) \quad : \quad \triangle_{\widetilde{\mathbf{g}}_t} \widetilde{u}_t + \mu_t^2.\widetilde{h}_t.\widetilde{u}_t = \lambda_t \widetilde{f}_t.\widetilde{u}_t^{\frac{n+2}{n-2}} \tag{4}$$

$$and \quad : \quad \int_{B(0,\mu_t^{-1} r)} x^{i_1}...x^{i_p}.\widetilde{u}_t^{\alpha} dv_{\widetilde{\mathbf{g}}_t} = \mu_t^{-p-n+\alpha\frac{n-2}{2}} \int_{B(0,r)} x^{i_1}...x^{i_p} \overline{u}_t^{\alpha} dv_{\mathbf{g}_t}.$$

A very important result is that when $\mu_t \to 0$ and therefore $k_t \to +\infty$, the components of $\widetilde{\mathbf{g}}_t$ converge in C^2_{loc} to those of the Euclidean metric, and $(\widetilde{E}_t)$ "converges" to the equation

$$\triangle_e \widetilde{u} = \lambda f(x_0).\widetilde{u}^{\frac{n+2}{n-2}}$$

in the sense that $\widetilde{u}_t \to \widetilde{u}$ in $C^2_{loc}(\mathbb{R}^n)$. It is known, then, that

$$\widetilde{u} = (1 + \frac{\lambda f(x_0)}{n(n-2)} |x|^2)^{-\frac{n-2}{2}}.$$

3.3. **The iteration process.** The idea of the Möser iteration process is to multiply the equations (E_t) by successive powers u_t^k of the functions u_t and to integrate over M to obtain bounds on increasing L^p-norms of the u_t. To localize the study around the concentration point x_0, which is a maximum point for f, we shall in fact multiply the equations by $\eta^2 u_t^k$ where η is a cut-off function equal to 1 (respectively 0) on a ball $B(x_0, r)$ where $f \geq 0$, and equal to 0 (respectively 1) on $M \backslash B(x_0, 2r)$, and where $k \geq 1$, then integrate by parts. We will therefore be able to study blow-up around x_0 using this method. We get, after some integrations by parts, and using equation (E_t)

$$\begin{aligned} &\frac{4k}{(k+1)^2} \int_M \left| \nabla(\eta u_t^{\frac{k+1}{2}}) \right|^2 \\ &= \lambda_t \int_M f \eta^2 u_t^{\frac{n+2}{n-2}} u_t^k + \int_M (\frac{2}{k+1} |\nabla \eta|^2 + \frac{2(k-1)}{(k+1)^2} \eta \triangle \eta - \eta^2 h_t) u_t^{k+1}, \end{aligned} \tag{5}$$

where the integrals are taken with the measure $dv_{\mathbf{g}}$. Then using the Hölder inequality, if $f \geq 0$ on $Supp\, \eta$ we obtain

$$\lambda_t \int_M f \eta^2 u_t^{\frac{n+2}{n-2}} u_t^k \leq \lambda_t (\sup_{Supp\, \eta} f)^{\frac{n-2}{n}} \Big(\int_{Supp\, \eta} f u_t^{\frac{2n}{n-2}} \Big)^{\frac{2}{n}} \Big(\int_M (\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{n}}.$$

Then using the Sobolev inequality

$$\Big(\int_M (\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le K(n,2)^2 \int_M \Big|\nabla(\eta u_t^{\frac{k+1}{2}})\Big|^2 + B\int_M \eta u_t^{k+1}$$

with $B > 0$. Therefore,

$$\begin{aligned}&\frac{4k}{(k+1)^2}(\int_M (\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\\ &\le \lambda_t K(n,2)^2(\sup_{Supp\,\eta} f)^{\frac{n-2}{n}}.(\int_{Supp\,\eta} f u_t^{\frac{2n}{n-2}})^{\frac{2}{n}}.(\int_M (\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\\ &+\int_M (\frac{4k}{(k+1)^2}B\eta + \frac{2}{k+1}|\nabla\eta|^2 + \frac{2(k-1)}{(k+1)^2}\eta\triangle\eta - \eta^2 h_t)u_t^{k+1}.\end{aligned}$$

Then

$$Q(t,k,\eta).(\int_M (\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \le (\frac{4k}{(k+1)^2}B + C_0 + C_\eta)\int_{Supp\,\eta} u_t^{k+1}, \quad (6)$$

where

$$Q(t,k,\eta) = \frac{4k}{(k+1)^2} - \lambda_t K(n,2)^2(\sup_{Supp\,\eta} f)^{\frac{n-2}{n}}.(\int_{Supp\,\eta} f.u_t^{2^*})^{\frac{2}{n}},$$

where we recall that $2^* = \frac{2n}{n-2}$ and where $C_0\,et\,C_\eta$ are constants independent of k and t and such that for all $k \ge 1, t$

$$\Big\|\frac{2}{k+1}|\nabla\eta|^2 + \frac{2(k-1)}{(k+1)^2}\eta\triangle\eta\Big\|_{L^\infty(M)} \le C_\eta \quad\text{and}\quad \Big\|h_t\Big\|_{L^\infty(M)} \le C_0.$$

If the sign of f changes on $Supp\,\eta$, we go back to Hölder's inequality

$$\lambda_t\int_M f\eta^2 u_t^{\frac{n+2}{n-2}} u_t^k \le \lambda_t(\sup_{Supp\,\eta}|f|).(\int_{Supp\,\eta} u_t^{\frac{2n}{n-2}})^{\frac{2}{n}}.(\int_M(\eta u_t^{\frac{k+1}{2}})^{\frac{2n}{n-2}})^{\frac{n-2}{n}}$$

to obtain (6) with

$$Q(t,k,\eta) = \frac{4k}{(k+1)^2} - \lambda_t K(n,2)^2(\sup_{Supp\,\eta}|f|).(\int_{Supp\,\eta} u_t^{2^*})^{\frac{2}{n}}. \quad (7)$$

One can also replace $\sup_{Supp\,\eta}|f|$ by $\sup_M f$.

The goal is to show that (ηu_t) is bounded in $L^{\frac{k+1}{2}2^*}$ and therefore that we can extract a sub-sequence converging strongly in L^{2^*}.

Remark. Those three tools also work for more general equations that we can associate to $(E_{h,f,\mathbf{g}}) : \triangle_{\mathbf{g}} u + h.u = \mu_h.f.u^{\frac{n+2}{n-2}}$, like, e.g., $E_t : \triangle_{\mathbf{g}} u_t + h_t.u_t = \lambda_t.f_t.u_t^{q_t-1}$ where $q_t \to 2^*$ and $f_t \to f$ in some L^p, still with $h_t \to h$ in $C^{0,\alpha}(M)$ and $\lambda_t \to \lambda$.

4. Proof of Theorem 1

4.1. **Setup.** Let h be a weakly critical function for f and $\mathbf{g}$ such that for any $P \in M$ where f is maximum on M we have

$$h(P) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(P) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$$

and such that there exists a family $(h_t), h_t \lneqq h, h_t$ sub-critical for every t, and satisfying $h_t \underset{t \to t_0}{\rightarrow} h$ in $C^{0,\alpha}$. To simplify, we suppose that $t_0 = 1$ and that $t \to 1$. Then for every t

$$\lambda_t := \lambda_{h_t, f, \mathbf{g}} < \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}$$

and there exists a family u_t of minimizing solutions of the equations

$$E_t : \triangle_{\mathbf{g}} u_t + h_t . u_t = \lambda_t . f . u_t^{\frac{n+2}{n-2}} \text{ with } \int_M f u_t^{2^*} dv_{\mathbf{g}} = 1.$$

We then see, as $\triangle_{\mathbf{g}} + h$ is coercive, that the sequence (u_t) is bounded in H_1^2 (just multiply E_t by u_t and integrate on M). Thus, there exists a function $u \in H_1^2$, $u \geq 0$ such that, after extracting a subsequence, $u_t \overset{H_1^2}{\rightharpoonup} u$, $u_t \overset{L^2}{\rightarrow} u$, $u_t \overset{p.p.}{\rightarrow} u$, and we can suppose

$$\lambda_t \overset{\leq}{\rightarrow} \lambda \leqslant \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}.$$

In particular, $u_t \overset{L^p}{\rightarrow} u$, for all $p < 2^* = \frac{2n}{n-2}$ as the inclusion of H_1^2 in L^p is compact for all $p < 2^*$. Therefore, u is a weak solution of

$$\triangle_{\mathbf{g}} u + h.u = \lambda . f . u^{\frac{n+2}{n-2}}$$

and by standard elliptic theory, u is C^∞. The maximum principle then tells us that either $u > 0$ or $u \equiv 0$.

If $u > 0$ then, using elliptic theory and an iteration process, and the fact that h is weakly critical, one can prove that

$$\lambda = \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}$$

and then that u is a minimizing positive solution of

$$\triangle_{\mathbf{g}} u + h.u = \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}} . f . u^{\frac{n+2}{n-2}} \quad \text{with} \quad \int_M f u^{2^*} dv_{\mathbf{g}} = 1$$

and the theorem is proved.

If $u \equiv 0$, we will show that there is a concentration phenomena. All the study that follows will aim at finding a contradiction. From now, we suppose that we are in this case $u \equiv 0$.

4.2. **Concentration phenomena.** In this section we study the behavior of a family of $C^{2,\alpha}$ solutions (u_t) of

$$\triangle_{\mathbf{g}} u_t + h_t u_t = \lambda_t f u_t^{\frac{n+2}{n-2}} \text{ with } \int_M f u_t^{\frac{2n}{n-2}} dv_{\mathbf{g}} = 1,$$

where f is a smooth function such that $\sup_M f > 0$. We also suppose that $h_t \to h$ in $C^{0,\alpha}$ where h is such that $\triangle_{\mathbf{g}} + h$ is coercive. The sequence (u_t) is bounded in H_1^2, therefore, up to a subsequence, $u_t \rightharpoonup u$ weakly in H_1^2, and we suppose that $u \equiv 0$; that is, $u_t \to 0$ in any L^p for $p < 2^*$. We also make the following "minimal energy" hypothesis

$$\lambda_t \leq \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}$$

and we can suppose that $\lambda_t \to \lambda$. All these hypotheses are satisfied by the u_t of the preceding section. The results of this section are valid for $dimM = 3$, except L^2-concentration, valid for $\dim M \geq 4$. In all this text, c, C are constants independent of t and δ.

Proposition 2. *There exists, after extraction of a subsequence, exactly one concentration point x_0, and it is a point where f is maximum on M. Moreover*

$$\forall \delta > 0, \ \overline{\lim_{t \to 1}} \int_{B(x_0,\delta)} f u_t^{2^*} dv_{\mathbf{g}} = 1.$$

Proof. We apply the iteration process. First, as M is compact, there exists at least one point of concentration. Otherwise, we could cover M by a finite number of balls $B(x_i,\delta)$ such that $\lim_{t\to 1} \int_{B(x_i,\delta)} u_t^{2^*} = 0$, and we would have $\lim_{t \to 1} \int_M u_t^{2^*} = 0$, which would contradict

$$1 = \int_M f u_t^{2^*} dv_{\mathbf{g}} \leq \sup |f| \int_M u_t^{2^*} dv_{\mathbf{g}}.$$

The principle of the iteration process is the following: if we find, for a point x, a cut-off function η equal to 1 around x such that $Q(t,k,\eta) \geq Q > 0$, we get, using formula (6) or (7), that $(\eta u_t^{\frac{k+1}{2}})$ is bounded in L^{2^*}, and therefore we can extract a subsequence such that (ηu_t) converges strongly to 0 in L^{2^*}; thus x cannot be a concentration point.

Let us prove now that we can do this for a point x such that $f(x) \leq 0$. If $f(x) < 0$, we choose δ small enough such that $f < 0$ on $B(x,\delta)$ and we choose η with support in $B(x,\delta)$. As (u_t) is bounded in H_1^2 and thus in L^{2^*}, we get, using formula (5), that for any k such that $1 \le k \le 2^* - 1$

$$\frac{4k}{(k+1)^2}\int_M \left|\nabla(\eta u_t^{\frac{k+1}{2}})\right|^2 \le \int_M \left(\frac{2}{k+1}\left|\nabla\eta\right|^2 + \frac{2(k-1)}{(k+1)^2}\eta\triangle\eta - \eta^2 h_t\right)u_t^{k+1} \le C_1,$$

where C_1 is independent of t. Therefore for any k such that $1 \le k \le 2^* - 1$ there exists C_2 independent of t such that

$$\int_M \left|\nabla(\eta u_t^{\frac{k+1}{2}})\right|^2 \le C_2.$$

Therefore, $(\eta u_t^{\frac{k+1}{2}})$ is bounded in H_1^2 and, using the Sobolev inequality, $(\eta u_t^{\frac{k+1}{2}})$ is bounded in L^{2^*} for any k such that $1 \le k \le 2^* - 1$.

If $f(x) = 0$, by continuity of f and choosing δ small enough, we get in (7) that for any k such that $1 \le k \le 2^* - 1$, $Q(t,k,\eta) \ge Q > 0$. Therefore, as we said, here again $(\eta u_t^{\frac{k+1}{2}})$ is bounded in L^{2^*}, and therefore we can extract a subsequence such that (ηu_t) converges strongly to 0 in L^{2^*}. Thus, when $f(x) \le 0$, x cannot be a concentration point.

Now, let x be a concentration point, then $f(x) > 0$ as we just saw. For $\delta > 0$ such that $f \ge 0$ on $B(x,\delta)$, set

$$\limsup_{t\to 1}\int_{B(x,\delta)} f u_t^{2^*} = a_\delta.$$

Then as $\int_M f u_t^{2^*} = 1$ and as $\overline{lim}\int_B f u_t^{2^*} = 0$ if $f \leqslant 0$ on B from what we saw above, necessarily, $a_\delta \le 1$. Suppose that there exists $\delta > 0$ such that $a_\delta < 1$. Because

$$\lambda_t \underset{\leqslant}{\to} \lambda \leqslant \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$$

we get

$$\overline{\lim_{t\to 1}}\lambda_t K(n,2)^2(\sup_M f)^{\frac{n-2}{n}} a_\delta < 1.$$

Beside, $\frac{4k}{(k+1)^2} \underset{\substack{k\to 1\\ >}}{\to} 1$. Therefore, for k close to 1 such that

$$\overline{\lim_{t\to 1}}\lambda_t K(n,2)^2(\sup_M f)^{\frac{n-2}{n}} a_\delta < \frac{4k}{(k+1)^2}$$

we get, taking η with support in $B(x,\delta)$, that in formula (6), $Q(t,k,\eta) \ge Q > 0$ for all t, where Q is independent of t. So, as before, x cannot be a concentration point, and we have a contradiction. Thus $a_\delta = 1$, for all

$\delta > 0$. Therefore, x is the only concentration point, that we will now denote x_0. The same reasoning shows that, necessarily,

$$\lambda = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}.$$

In the same way, if $f(x_0) \neq \sup_M f$, there exists $\delta > 0$ such that $\sup_{B(x_0,\delta)} f < \sup_M f$. But $\lambda_t \leq \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$, so

$$\overline{\lim_{t\to 1}} \lambda_t K(n,2)^2 (\sup_{B(x_0,\delta)} f)^{\frac{n-2}{n}} (\int_{B(x_0,\delta)} f u_t^{2^*})^{\frac{2^*-2}{2^*}} < 1.$$

Then for k close enough to 1, taking η with support in $B(x_0,\delta)$, we get, in (6), $Q(t,k,n) \geq Q > 0$ for all t, and once again we have a contradiction. Therefore, $f(x_0) = \sup_M f > 0$.

Note that this is the main particular quality introduced by the function f on the right of equation $(E_{h,f,\mathbf{g}})$. It gives a precise location for the concentration point.

The next propositions concerning the concentration phenomenon are now quite standard, even though they are mostly published in the case $f =$ *constant* and often with few details. We shall therefore give possible proofs, referring to the books [12] and [13] for more information, the presence of a function f introducing only slight modifications that we will indicate when necessary.

Proposition 3. *$u_t \to 0$ in $C^0_{loc}(M - \{x_0\})$.*

Proof. It is a typical application of the iteration process in standard elliptic theory. For the first step, let $q > 0$ be fixed. We prove that, for any $\delta > 0$, there exists $C = C(\delta, q)$ independent of t such that for t close enough to 1

$$\|u_t\|_{L^q(M\setminus B(x_0,\delta))} \leq C \|u_t\|_{L^2(M)}. \tag{8}$$

To apply the iteration process, we build a sequence $\eta_1, ..., \eta_m$ of m cut-off functions such that $\eta_j = 0$ on $B(x_0,\delta/2)$ and $\eta_j = 1$ on $M\setminus B(x_0,\delta)$ and such that $M\setminus B(x_0,\delta) \subset ... \subset \{\eta_{j+1} = 1\} \subset Supp\, \eta_{j+1} \subset \{\eta_j = 1\} \subset ... \subset M\setminus B(x_0,\delta/2)$ and where m is chosen such that $2(\frac{2^*}{2})^m > q$. We set $q_1 = 2$ and $q_j = (\frac{2^*}{2})q_{j-1}$. The iteration process (6), (7), gives that

$$Q(t, q_j - 1, \eta_j).(\int_M (\eta_j u_t^{\frac{q_j}{2}})^{2^*})^{\frac{n-2}{n}} \leq (\frac{4(q_j-1)}{{q_j}^2} B + C_0 + C_{\eta_j}) \int_{Supp\, \eta_j} u_t^{q_j}.$$

But for $j \le m$ we have $\frac{4(q_j-1)}{{q_j}^2} \ge c > 0$ and, from Proposition 2, $\int_{Supp\,\eta_j} u_t^{2^*} \to 0$, therefore, in (7),

$$Q(t, q_j - 1, \eta_j) \ge c > 0, \forall j.$$

Thus there exists a neighborhood V_j of 1 and a constant $C_j > 0$ such that for $t \in V_j$

$$\left(\int_M (\eta_j u_t^{\frac{q_j}{2}})^{2^*}\right)^{\frac{n-2}{n}} \le C_j \int_{Supp\,\eta_j} u_t^{q_j}.$$

Then by construction of the η_j we have

$$\left(\int_{\{\eta_j=1\}} u_t^{q_j \frac{2^*}{2}}\right)^{\frac{n-2}{n}} \le C_j \int_{\{\eta_{j-1}=1\}} u_t^{q_j}$$

and thus

$$\|u_t\|_{L^q(M\backslash B(x_0,\delta))} \le C(\prod_{j=1}^m C_j)\, \|u_t\|_{L^2(M)} \quad \forall t \in V_1 \cap ... \cap V_m.$$

Second step: By the Gilbarg-Trudinger theorem (8.25) [16], we have, if u is a solution of an equation $E : \triangle_{\mathbf{g}} u + h.u = F$, where $\triangle_{\mathbf{g}} + h$ is coercive, and if $\omega \subset\subset \omega'$ are two open sets, for $r > 1$, $q > n/2$

$$\sup_{\omega} u \le c\, \|u\|_{L^r(\omega')} + c'\, \|F\|_{L^q(\omega')}\,. \tag{9}$$

This theorem is also an application of the iteration process. We apply it to $E_t : \triangle_{\mathbf{g}} u_t + h_t.u_t = \lambda_t.f.u_t^{\frac{n+2}{n-2}}$ and to $\omega \subset\subset \omega' \subset M\backslash\{x_0\}$.

Then, with the first step applied to $q\frac{n+2}{n-2}$, and choosing $\omega = M\backslash B(x_0,\delta)$, $\omega' = M\backslash B(x_0,\delta/2)$, $r = 2$, $q > n/2$ we obtain

$$\sup_{M\backslash B(x_0,\delta)} u_t \le c\, \|u_t\|_{L^2(\omega')} + c'\lambda_t\, \|u_t\|^{\frac{n+2}{n-2}}_{L^{q\frac{n+2}{n-2}}(\omega')} \le c\, \|u_t\|_{L^2(M)} + c''\, \|u_t\|^{\frac{n+2}{n-2}}_{L^2(M)}\,.$$

But $\|u_t\|_{L^2(M)} \to 0$, thus we have the result.

We recall now the notation of subsection (3.2) and we consider a sequence of points (x_t) such that $m_t = \max_M u_t = u_t(x_t) := \mu_t^{-\frac{n-2}{2}}$. From Proposition 3, $x_t \to x_0$ and $\mu_t \to 0$. Remember that $\overline{u}_t, \overline{f}_t, \overline{h}_t, \mathbf{g}_t$ are the functions and the metric "viewed" in the chart $\exp_{x_t}^{-1}$, and $\widetilde{u}_t, \widetilde{h}_t, \widetilde{f}_t, \widetilde{\mathbf{g}}_t$ are the functions and the metric after blow-up. From now on, all the blow-ups will be made on balls $B(x_t,\delta)$ where $f \ge 0$, which is possible as $f(x_0) > 0$.

Proposition 4. *For all $R > 0$, $\lim_{t\to 1} \int_{B(x_t,R\mu_t)} f u_t^{2^*} dv_{\mathbf{g}} = 1 - \varepsilon_R$ where $\varepsilon_R \underset{R\to+\infty}{\to} 0$.*

Proof. This is a direct application of blow-up analysis in x_t with $k_t = \mu_t^{-1}$: $\widetilde{u}_t \to \widetilde{u} = (1 + \frac{\lambda f(x_0)}{n(n-2)} |x|^2)^{-\frac{n-2}{2}} = (1 + \frac{f(x_0)^{\frac{2}{n}}}{K(n,2)^2 n(n-2)} |x|^2)^{-\frac{n-2}{2}}$ in $C^2_{loc}(\mathbb{R}^n)$. Then

$$\int_{B(x_t,R\mu_t)} f u_t^{2^*} dv_{\mathbf{g}} = \int_{B(0,R)} \widetilde{f}_t.\widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} \underset{t\to 1}{\to} f(x_0)(\int_{B(0,R)} \widetilde{u}^{2^*} dx) \underset{R\to\infty}{\to} 1.$$

Proposition 5 (Weak estimates, first part)**.** *There exists $C > 0$ such that for all $x \in M : d_{\mathbf{g}}(x, x_t)^{\frac{n-2}{2}} u_t(x) \le C$.*

Proof. Define $w_t(x) = d_{\mathbf{g}}(x, x_t)^{\frac{n-2}{2}} u_t(x)$. We want to prove that there exists $C > 0$ such that $\sup_M w_t \le C$. By contradiction, we suppose that (for a subsequence) $\sup_M w_t \to +\infty$. Let y_t be a point where w_t is maximum. M being compact, $d_{\mathbf{g}}(x, x_t)$ is bounded, therefore $u_t(y_t) \to \infty$, and thus from Proposition 3, $y_t \to x_0$. Besides, the definition of μ_t gives

$$d_{\mathbf{g}}(y_t, x_t)\mu_t^{-1} \to +\infty.$$

We do a *blow-up* with center y_t and coefficient $k_t = u_t(y_t)^{\frac{2}{n-2}}$. If $x \in B(0, 2)$, then

$$\begin{aligned} d_{\mathbf{g}}(x_t, \exp_{y_t}(u_t(y_t)^{-\frac{2}{n-2}} x) &\ge d_{\mathbf{g}}(y_t, x_t) - 2u_t(y_t)^{-\frac{2}{n-2}} \\ &\ge u_t(y_t)^{-\frac{2}{n-2}} (w_t(y_t)^{\frac{2}{n-2}} - 2) \sim d_{\mathbf{g}}(y_t, x_t) \end{aligned}$$

as $w_t(y_t) \to \infty$ and $u_t(y_t) \to \infty$. Therefore, for t close to 1,

$$d_{\mathbf{g}}(x_t, \exp_{y_t}(u_t(y_t)^{-\frac{2}{n-2}} x) \ge \frac{1}{2} d_{\mathbf{g}}(y_t, x_t).$$

As a consequence, for any $R > 0$ and t close to 1, $B(y_t, 2u_t(y_t)^{-\frac{2}{n-2}}) \cap B(x_t, R\mu_t) = \emptyset$. Thus, by Proposition 4,

$$\begin{aligned} &\int_{B(0,2)} \widetilde{f}_t.\widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} = \int_{B(y_t, 2u_t(y_t)^{-\frac{2}{n-2}})} f u_t^{2^*} dv_{\mathbf{g}} \le \int_{M \setminus B(x_t, R\mu_t)} f u_t^{2^*} dv_{\mathbf{g}} \\ &\le \int_M f u_t^{2^*} dv_{\mathbf{g}} - \int_{B(x_t,R\mu_t)} f u_t^{2^*} dv_{\mathbf{g}} \underset{t\to 1, R\to\infty}{\longrightarrow} 0. \end{aligned}$$

But the iteration process then gives that for $1 \le k \le 2^* - 1$

$$\int_{B(0,1)} \widetilde{u}_t^{\frac{k+1}{2} 2^*} dv_{\widetilde{\mathbf{g}}_t} \to 0$$

and by iteration we obtain that for all $p \ge 1$, $\int_{B(0,1)} \widetilde{u}_t^p dv_{\widetilde{\mathbf{g}}_t} \to 0$. We deduce that $\|\widetilde{u}_t\|_{L^\infty(B(0,1))} \to 0$ whereas $\widetilde{u}_t(0) = 1$. Thus we have a contradiction.

Proposition 6 (Weak estimates, second part)**.** *For all $\varepsilon > 0$, there exists $R > 0$ such that for all $x \in M, t$,*

$$d_{\mathbf{g}}(x, x_t) \geq R\mu_t \Rightarrow d_{\mathbf{g}}(x, x_t)^{\frac{n-2}{2}} u_t(x) \leq \varepsilon.$$

Proof. We use the same method, supposing the existence of an $\varepsilon_0 > 0$ and $y_t \in M$ such that $\lim_{t\to 1} d_{\mathbf{g}}(y_t, x_t)\mu_t^{-1} = +\infty$ and

$$w_t(y_t) = d_{\mathbf{g}}(y_t, x_t)^{\frac{n-2}{2}} u_t(y_t) \geq \varepsilon_0.$$

We do a blow-up with center y_t and coefficient $k_t = u_t(y_t)^{\frac{2}{n-2}}$ and with $m_t = u_t(y_t)$. Then, as in Proposition 5, for any $R > 0$ and t close to 1, $B(y_t, \frac{1}{2}\varepsilon_0^{\frac{2}{n-2}} u_t(y_t)^{-\frac{2}{n-2}}) \cap B(x_t, R\mu_t) = \emptyset$. Therefore, as previously

$$\int_{B(0,\frac{1}{2}\varepsilon_0^{\frac{2}{n-2}})} \widetilde{f}_t.\widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} \to 0$$

and we obtain in the same way a contradiction.

Proposition 7 (L^2-concentration)**.** *If $\dim M \geq 4$, for all $\delta > 0$*

$$\lim_{t\to 1} \frac{\int_{B(x_0,\delta)} u_t^2 dv_{\mathbf{g}}}{\int_M u_t^2 dv_{\mathbf{g}}} = 1.$$

Proof. We first use the two first steps of the proof of Proposition 3 to show that there exists $c > 0$ such that

$$\sup_{M\setminus B(x_0,\delta)} u_t \leq c \left\| u_t \right\|_{L^2(M)}.$$

Indeed, going over what we did there,

$$\begin{aligned}\sup_{M\setminus B(x_0,\delta)} u_t &\leq c \left\| u_t \right\|_{L^2(\omega')} + c'\lambda_t \left\| u_t^{\frac{n+2}{n-2}} \right\|_{L^q(\omega')} \\ &\leq c \left\| u_t \right\|_{L^2(M)} + c'\lambda_t^q \sup_{\omega'}(u_t^{\frac{n+2}{n-2}-1}) \left\| u_t \right\|_{L^q(\omega')} \leq c'' \left\| u_t \right\|_{L^2(M)}\end{aligned}$$

as we know now that $\sup_{\omega'}(u_t^{\frac{n+2}{n-2}-1}) \to 0$ and that, on the other hand, the first step of the proof of Proposition 3 gives $\left\| u_t \right\|_{L^q(\omega')} \leq C \left\| u_t \right\|_{L^2(M)}$.

Third step: Using the fact that

$$\left\| u_t \right\|^2_{L^2(M\setminus B(x_0,\delta))} \leq \sup_{M\setminus B(x_0,\delta)} u_t \int_{M\setminus B(x_0,\delta)} u_t \leq c \left\| u_t \right\|_{L^2(M)} \left\| u_t \right\|_{L^1(M)}, \quad (10)$$

we now want to prove that

$$\left\| u_t \right\|_{L^1(M)} \leq c \left\| u_t \right\|^{2^*-1}_{L^{2^*-1}(M)}. \quad (11)$$

If $h > 0$, we get the result by integrating equation E_t. Otherwise, for any $q \in (2, 2^*)$, there exists $\varphi > 0$, a solution of $\triangle_{\mathbf{g}}\varphi + h\varphi = \lambda_{h,f,\mathbf{g}}.f.\varphi^{q-1}$. We set

$$\mathbf{g}' = \varphi^{\frac{4}{n-2}}\mathbf{g} \text{ and } \overline{h_t} = \frac{\triangle_{\mathbf{g}}\varphi + h_t\varphi}{\varphi^{\frac{n+2}{n-2}}}.$$

Then for t close to 1

$$\overline{h_t} = \varphi^{q-2^*} - (h - h_t)\varphi^{2-2^*} \geq \varepsilon_0 > 0.$$

Besides, by conformal invariance, and using E_t, we have

$$\triangle_{\mathbf{g}'}\overline{u_t} + \overline{h_t}.\overline{u_t} = \lambda_t f.\overline{u_t}^{\frac{n+2}{n-2}},$$

where $\overline{u_t} = \varphi^{-1}.u_t$. Integrating, we obtain

$$\varepsilon_0 \int_M \overline{u_t} dv_{\mathbf{g}'} \leq \lambda_t \sup f \int_M \overline{u_t}^{\frac{n+2}{n-2}} dv_{\mathbf{g}'}$$

and thus there exists $C > 0$ such that for t close to 1

$$\|u_t\|_{L^1(M)} \leq C \|u_t\|_{L^{2^*-1}(M)}^{2^*-1},$$

where the norms are now relative to $dv_{\mathbf{g}}$.

Fourth step: We conclude using Hölder's inequality. If $n = \dim M \geq 6$, then

$$\|u_t\|_{L^{2^*-1}(M)}^{2^*-1} \leq \|u_t\|_{L^2(M)}^{\frac{n+2}{n-2}} Vol_{\mathbf{g}}(M)^{\frac{n-6}{2(n-2)}}.$$

With (10) and (11), we obtain

$$\lim_{t\to 1} \frac{\|u_t\|^2_{L^2(M\setminus B(x_0,\delta))}}{\|u_t\|^2_{L^2(M)}} = 0,$$

which proves the result. If $n = 5$, Hölder's inequality gives

$$\|u_t\|_{L^{2^*-1}(M)}^{2^*-1} \leq \|u_t\|_{L^2(M)}^{\frac{3}{2}} \|u_t\|_{L^{2^*}(M)}^{\frac{5}{6}}$$

and we also conclude using (10) and (11). If now $n = 4$, we have to use Proposition 6 and the associated blow-up. See [13] or [8].

Proposition 8 (Strong estimates)**.** *For any ν, $0 < \nu < n - 2$, there exists a constant $C(\nu) > 0$ such that*

$$\forall x \in M : d_{\mathbf{g}}(x, x_t)^{n-2-\nu} \mu_t^{-\frac{n-2}{2}+\nu} u_t(x) \leq C(\nu).$$

Proof. The proof requires the use of the Green's function and the weak estimates. The idea is due to O. Druet and E. Hebey [12]. We recall first the property of the Green's function. If $\triangle_{\mathbf{g}}+h$ is a coercive operator, there exists a unique function (at least C^2 with our hypothesis) $G_h : M \times M\backslash\{(x,x) : x \in M\} \to \mathbb{R}$ symmetric and positive, such that in the sense of distributions, we have for all $x \in M$

$$\triangle_{\mathbf{g},y} G_h(x,y) + h(y)G_h(x,y) = \delta_x. \tag{12}$$

Furthermore, there exists $c > 0$, $\rho > 0$ such that for all (x,y) with $0 < d_{\mathbf{g}}(x,y) < \rho$,

$$\frac{c}{d_{\mathbf{g}}(x,y)^{n-2}} \leq G_h(x,y) \leq \frac{c^{-1}}{d_{\mathbf{g}}(x,y)^{n-2}} \tag{13}$$

$$\frac{|\nabla_y G_h(x,y)|}{G_h(x,y)} \geq \frac{c}{d_{\mathbf{g}}(x,y)} \tag{14}$$

c and ρ vary continuously with h

$$G_h(x,y) d_{\mathbf{g}}(x,y)^{n-2} \to \frac{1}{(n-2)\omega_{n-1}} \text{ when } d_{\mathbf{g}}(x,y) \to 0. \tag{15}$$

To prove these strong estimates, it is sufficient, considering (13), to prove that $\mu_t^{\frac{n-2}{2}-(n-2)(1-\nu)} u_t(x) \leq c' G_h^{1-\nu}(x,x_t)$ (just change ν by $(n-2)\nu$). First, notice that, using for example the weak estimates, the strong estimates are true in any ball $B(x_t, R\mu_t)$ where R is fixed. We therefore have to prove the estimates in the manifold with boundary $M\backslash B(x_t, R\mu_t)$ whose boundary is $b(M\backslash B(x_t, R\mu_t)) = bB(x_t, R\mu_t)$. For ν small, there exists $\varepsilon_0 > 0$ such that the operator

$$\triangle_{\mathbf{g}} + \frac{h - 2\varepsilon_0}{1-\nu}$$

is still coercive; let $\widetilde{G}$ be its Green's function. To prove our estimate, we apply the maximum principle to $L_t\varphi = \triangle_{\mathbf{g}}\varphi + h_t\varphi - \lambda_t f u_t^{2^*-2}\varphi$ and to $x \longmapsto \widetilde{G}^{1-\nu}(x,x_t) - c\mu_t^{\frac{n-2}{2}-(n-2)(1-\nu)} u_t(x)$. As $L_t u_t = 0$ with $u_t > 0$, L_t satisfies the maximum principle (see [6]). Using (12), the fact that $\delta_{x_t}(x) = 0$ on $M\backslash B(x_t, R\mu_t)$ and the fact that for t close to 1, $h_t - h \geq -\varepsilon_0$ (as $h_t \to h$ in C^0), some computations give that for all $x \in M\backslash B(x_t, R\mu_t)$

$$\frac{L_t \widetilde{G}^{1-\nu}}{\widetilde{G}^{1-\nu}}(x,x_t) \geq \varepsilon_0 - \lambda_t f(x) u_t(x)^{2^*-2} + \nu(1-\nu)\left|\frac{\nabla \widetilde{G}}{\widetilde{G}}\right|^2 (x,x_t). \tag{16}$$

We now separate $M\backslash B(x_t, R\mu_t)$ into two parts using a ball $B(x_t, \rho)$ where $\rho > 0$ is as in (13) and (14). For t close to 1, $\rho > R\mu_t$. $R > 0$ will be fixed later.

1) As $u_t \to 0$ in $C^0_{loc}(M\backslash\{x_0\})$, (16) gives for t close to 1

$$\forall x \in M\backslash B(x_t,\rho) : L_t\widetilde{G}^{1-\nu}(x,x_t) \geq 0.$$

2) Using the weak estimates (second part), in $B(x_t,\rho)\backslash B(x_t,R\mu_t)$

$$d_{\mathbf{g}}(x,x_t)^2 u_t(x)^{2^*-2} \leq \varepsilon_R,$$

where $\varepsilon_R \underset{R\to\infty}{\to} 0$. Then, with (14) and (16), for R big enough

$$\frac{L_t\widetilde{G}^{1-\nu}}{\widetilde{G}^{1-\nu}}(x,x_t) \geq \varepsilon_0 - \lambda_t f(x) u_t(x)^{2^*-2} + \nu(1-\nu)\frac{c}{d_{\mathbf{g}}(x,x_t)^2}$$

$$\geq \varepsilon_0 - \lambda_t(\sup_{B(x_t,\rho)} f).\frac{\varepsilon_R}{d_{\mathbf{g}}(x,x_t)^2} + \nu(1-\nu)\frac{c}{d_{\mathbf{g}}(x,x_t)^2} \geq \varepsilon_0 + \frac{c'}{d_{\mathbf{g}}(x,x_t)^2} \geq 0.$$

We have proved that in $M\backslash B(x_t,R\mu_t)$ and for any constant $C_t > 0$ which can depend on t

$$L_t(C_t.\widetilde{G}^{1-\nu}(x,x_t)) = C_t.L_t\widetilde{G}^{1-\nu}(x,x_t) \geq 0 = L_t u_t.$$

At last, on the boundary $b(M\backslash B(x_t,R\mu_t))$, using (13), we obtain

$$\widetilde{G}^{1-\nu}(x,x_t) \geq \frac{c}{d_{\mathbf{g}}(x,x_t)^{(n-2)(1-\nu)}} = \frac{c}{(R\mu_t)^{(n-2)(1-\nu)}}.$$

So, if we let $C_t = c^{-1}R^{(n-2)(1-\nu)}\mu_t^{(n-2)(1-\nu)-\frac{n-2}{2}}$, we have for $x \in bB(x_t,R\mu_t)$ $= b(M\backslash B(x_t,R\mu_t))$

$$C_t.\widetilde{G}^{1-\nu}(x,x_t) \geq \mu_t^{-\frac{n-2}{2}} = \sup u_t \geq u_t(x).$$

Therefore, by the maximum principle

$$C_t.\widetilde{G}^{1-\nu}(x,x_t) \geq u_t(x) \text{ in } M\backslash B(x_t,R\mu_t)$$

which can be rewritten

$$\widetilde{G}^{1-\nu}(x,x_t) \geq C_t^{-1}u_t(x) = c\ \mu_t^{\frac{n-2}{2}-(n-2)(1-\nu)}u_t(x)$$

and therefore, using (13),

$$d_{\mathbf{g}}(x,x_t)^{(n-2)(1-\nu)}\mu_t^{\frac{n-2}{2}-(n-2)(1-\nu)}u_t(x) \leq c$$

which gives the strong estimates by replacing ν with $(n-2)\nu$.

Proposition 9 (Corollary: Strong L^p-concentration)**.** *For all $R > 0$, $\delta > 0$ and $p > \frac{n}{n-2}$*

$$\lim_{t\to 1}\frac{\int_{B(x_t,R\mu_t)} u_t^p dv_{\mathbf{g}}}{\int_{B(x_t,\delta)} u_t^p dv_{\mathbf{g}}} = 1 - \varepsilon_R \quad \textit{where} \quad \varepsilon_R \underset{R\to+\infty}{\to} 0.$$

Proof. Just apply the strong estimates to a blow-up in x_t. By blow-up formulae

$$\int_M u_t^p dv_{\mathbf{g}} \geq \int_{B(x_t,\mu_t)} u_t^p dv_{\mathbf{g}} = \mu_t^{n-\frac{n-2}{2}p} \int_{B(0,1)} \widetilde{u}_t^p dv_{\widetilde{g}_t} \geq C\mu_t^{n-\frac{n-2}{2}p}.$$

On the other hand, by the strong estimates

$$\begin{aligned}\int_{M\setminus B(x_t,R\mu_t)} u_t^p dv_{\mathbf{g}} &\leq C\mu_t^{p\frac{n-2}{2}} \int_{M\setminus B(x_t,R\mu_t)} d_{\mathbf{g}}(y_t,x)^{(2-n)p} dv_{\mathbf{g}} \\ &\leq C\mu_t^{n-p\frac{n-2}{2}} R^{n+(2-n)p}\end{aligned}$$

as soon as $p > \frac{n}{n-2}$. Dividing, we obtain the corollary.

At this point, to carry on the proof of Theorem 1, we need a powerful extension of the strong estimates, called C^0- *theory*, which is in fact a complete control of the sequence $d_{\mathbf{g}}(x,x_t)^{n-2}\mu_t^{-\frac{n-2}{2}} u_t(x)$; it is expressed by the next theorem of Druet and Robert, and proved in arbitrary energy in [13].

Another approach, also accessible at this point and originally used in the author's PhD thesis, is to prove another very important estimate concerning the "speed" of convergence of (x_t) to x_0, but it requires the additional hypothesis that the Hessian of f be non-degenerate at the points of maximum of f; it will be our Theorem 6, whose proof is independent of the theorem of Druet-Robert, only requiring the results up to Proposition 9, and appears as a byproduct of an alternative proof of Theorem 1. It is however of independent interest, as it is a very important estimate concerning concentration phenomena which has been studied by various authors.

We now state the theorem of Druet and Robert and refer for its proof to [13], the function f introducing no difficulties. It says first that one can take $\nu = 0$ in the strong estimates, but also that one has somehow the reverse estimate.

Theorem (Druet, Robert)**.** *For any $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that, up to a subsequence, for any t and any $x \in B(x_0, \delta_\varepsilon)$*

$$(1-\varepsilon)B_t(x) \leq u_t(x) \leq (1+\varepsilon)B_t(x),$$

where

$$B_t(x) = \mu_t^{-\frac{n-2}{2}} \Big(1 + \frac{\lambda f(x_0)}{n(n-2)} \frac{d_{\mathbf{g}}(x_t,x)^2}{\mu_t^2}\Big)^{-\frac{n-2}{2}}$$

is the "standard bubble."

Note that, in the proof of Theorem 1, we will need the minoration $(1-\varepsilon)B_t(x) \leq u_t(x)$, which is a stronger result than $u_t(x) \leq (1+\varepsilon)B_t(x)$ which must first be proved to get the minoration.

Finally, we come to our main result concerning the concentration phenomenon, which is the "missing link" between the sequence (x_t) and x_0.

Theorem 6 (Second fundamental estimate)**.** *Suppose that $dimM \geq 5$ and that the Hessian of the function f is non-degenerate at each of its points of maximum. Then, there exists a constant C such that for all t*

$$\frac{d_{\mathbf{g}}(x_t, x_0)}{\mu_t} \leq C.$$

Moreover, if for each point P of maximum of f we have

$$h(P) = \frac{n-2}{4(n-1)} S_{\mathbf{g}}(P) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)},$$

then more precisely $\frac{d_{\mathbf{g}}(x_t, x_0)}{\mu_t} \to 0$.

To understand the significance of this theorem, note that the weak and strong estimates, the strong L^p-concentration and the estimates in the theorem of Druet-Robert, are "centered" in x_t. Theorem 6 allows one to "translate" these estimates in x_0 in the sense that one can now replace x_t by x_0. This estimate, called by Zoé Faget "second fundamental estimate" (the "first one" being the strong estimate), joined with the estimates of $C^0 - \textit{theory}$ presented in the theorem of Druet and Robert above, gives a complete description of the behavior of a sequence of solutions of the equations $\triangle_{\mathbf{g}} u_t + h_t u_t = \lambda_t f u_t^{\frac{n+2}{n-2}}$ in the spirit of the study of Palais-Smale sequences associated to these equations. It has been studied, for example, by Druet and Robert in the case $f = \textit{constant} = 1$ in [12] where they require strong hypotheses on the shape of the functions h_t and on the geometry of the manifold near the concentration point, or by Hebey in the Euclidean setting. Intuitively, it seems that our hypothesis on f "fixes" the position of the concentration point, and so we get a control on the distance between x_t and x_0. Also, our method seems to be applicable to other settings, see e.g. [15] and [9].

4.3. **Proof of Theorem 1.** Remember that $\overline{u}_t, \overline{f}_t, \overline{h}_t, \mathbf{g}_t$ are the functions and the metric "viewed" in the chart $\exp_{x_t}^{-1}$, and $\widetilde{u}_t, \widetilde{h}_t, \widetilde{f}_t, \widetilde{\mathbf{g}}_t$ are the functions and the metric after blow-up with center x_t and coefficient $k_t = \mu_t^{-1}$. From now, all the blow-ups will be made on balls $B(x_t, \delta)$ where $f \geq 0$, which is possible as $f(x_0) > 0$.

Let also η be a cut-off function on $\mathbb{R}^n$ equal to 1 on the Euclidean ball $B(0, \delta/2)$, and equal to 0 on $\mathbb{R}^n \backslash B(0, \delta)$, $0 \leq \eta \leq 1$ with $|\nabla \eta| \leq C.\delta^{-1}$ where

δ is chosen small enough to have $f \geq 0$ on the balls $B(x_t, \delta)$. The Sobolev inequality gives on the one hand

$$(\int_{B(0,\delta)} (\eta \overline{u}_t)^{2^*} dx)^{\frac{2}{2^*}} \leq K(n,2)^2 \int_{B(0,\delta)} |\nabla(\eta \overline{u}_t)|_e^2 \, dx, \tag{17}$$

where $|.|_e$ is the Euclidean metric of the associated measure dx.

On the other hand, integration by parts gives, noting that $|\nabla \eta| = \Delta \eta = 0$ on $B(0, \delta/2)$,

$$\int_{B(0,\delta)} |\nabla(\eta \overline{u}_t)|_e^2 \, dx \leq \int_{B(0,\delta)} \eta^2 \overline{u}_t \triangle_e \overline{u}_t dx + C.\delta^{-2} \int_{B(0,\delta)\backslash B(0,\delta/2)} \overline{u}_t^2 dx.$$

Denoting by $\mathbf{g}_t^{ij}$ the components of $\mathbf{g}_t$ and by $\Gamma(\mathbf{g}_t)_{ij}^k$ the associated Christoffel symbols, we write

$$\triangle_e \overline{u}_t = \triangle_{\mathbf{g}_t} \overline{u}_t + (\mathbf{g}_t^{ij} - \delta^{ij}) \partial_{ij} \overline{u}_t - \mathbf{g}_t^{ij} \Gamma(\mathbf{g}_t)_{ij}^k \partial_k \overline{u}_t.$$

We get from this inequality, using this expression of the Laplacian, equation $E_t : \triangle_{\mathbf{g}} u_t + h_t.u_t = \lambda_t.f.u_t^{\frac{n+2}{n-2}}$ "viewed" in the chart $\exp_{x_t}^{-1}$, and using the fact that $|\nabla \eta| = \Delta \eta = 0$ on $B(0, \delta/2)$ and with some integration by parts

$$\begin{aligned}
&\int_{B(0,\delta)} |\nabla(\eta \overline{u}_t)|_e^2 \, dx \leq \lambda_t \int_{B(0,\delta)} \eta^2 \overline{f}_t \overline{u}_t^{2^*} dx - \int_{B(0,\delta)} \eta^2 \overline{h}_t \overline{u}_t^2 dx \\
&+ C.\delta^{-2} \int_{B(0,\delta)\backslash B(0,\delta/2)} \overline{u}_t^2 dx - \int_{B(0,\delta)} \eta^2 (\mathbf{g}_t^{ij} - \delta^{ij}) \partial_i \overline{u}_t \partial_j \overline{u}_t dx \\
&+ \frac{1}{2} \int_{B(0,\delta)} (\partial_k (\mathbf{g}_t^{ij} \Gamma(\mathbf{g}_t)_{ij}^k + \partial_{ij} \mathbf{g}_t^{ij})(\eta \overline{u}_t^2) dx.
\end{aligned}$$

Using the Sobolev inequality (17) and the fact that $\lambda_t \leq \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}}$, we obtain at last

$$\int_{B(0,\delta)} \overline{h}_t (\eta \overline{u}_t)^2 dx \leq A_t + B_t + C_t + C.\delta^{-2} \int_{B(0,\delta)\backslash B(0,\delta/2)} \overline{u}_t^2 dx, \tag{18}$$

where

$$B_t = \frac{1}{2} \int_{B(0,\delta)} (\partial_k (\mathbf{g}_t^{ij} \Gamma(\mathbf{g}_t)_{ij}^k + \partial_{ij} \mathbf{g}_t^{ij})(\eta \overline{u}_t^2) dx$$

$$C_t = \Big| \int_{B(0,\delta)} \eta^2 (\mathbf{g}_t^{ij} - \delta^{ij}) \partial_i \overline{u}_t \partial_j \overline{u}_t dx \Big|$$

$$A_t = \frac{1}{K(n,2)^2 (\sup_M f)^{\frac{n-2}{n}}} \int_{B(0,\delta)} \overline{f}_t \eta^2 \overline{u}_t^{2^*} dx - \frac{1}{K(n,2)^2} (\int_{B(0,\delta)} (\eta \overline{u}_t)^{2^*} dx)^{\frac{2}{2^*}}.$$

These computations were developed in the article of Djadli and Druet [10]. Our goal is to use L^2-concentration (Proposition 7) to obtain a contradiction; we shall divide (18) by $\int_{B(0,\delta)} \overline{u}_t^2 dx$ and take the limit when $t \to t_0 = 1$.

L^2-concentration first gives

$$\frac{C.\delta^{-2} \int_{B(0,\delta)\setminus B(0,\delta/2)} \overline{u}_t^2 dx}{\int_{B(0,\delta)} \overline{u}_t^2 dx} \underset{t\to 1}{\to} 0.$$

Z. Djadli and O. Druet [10] showed (see also [10] for full details)

$$\overline{\lim_{t\to 1}} \frac{C_t}{\int_{B(0,\delta)} \overline{u}_t^2 dx} \le \varepsilon_\delta, \quad \text{where } \varepsilon_\delta \to 0 \text{ when } \delta \to 0.$$

Furthermore, as $x_t \to x_0$ we have $\lim_{t\to 1}(\partial_k(\mathbf{g}_t^{ij}\Gamma(\mathbf{g}_t)_{ij}^k + \partial_{ij}\mathbf{g}_t^{ij})(0) = \frac{1}{3}S_{\mathbf{g}}(x_0)$, therefore, using L^2-concentration,

$$\overline{\lim_{t\to 1}} \frac{B_t}{\int_{B(0,\delta)} \overline{u}_t^2 dx} = \frac{1}{6}S_{\mathbf{g}}(x_0) + \varepsilon_\delta.$$

It is the expression A_t which will give

$$\frac{n-2}{4(n-1)}S_{\mathbf{g}}(x_0) - \frac{1}{6}S_{\mathbf{g}}(x_0) \quad \text{and} \quad \frac{(n-2)(n-4)}{8(n-1)}\frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}.$$

By Hölder's inequality

$$\int_{B(0,\delta)} \overline{f}_t \eta^2 \overline{u}_t^{2^*} dx \le \Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dx\Big)^{\frac{2}{n}} \Big(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}}.$$

In addition

$$dx \le (1 + \frac{1}{6}Ric(x_t)_{ij}x^i x^j + C\,|x|^3)dv_{\mathbf{g}_t}.$$

Using this development and $(1+x)^\alpha \le 1 + \alpha x$ for $0 < \alpha \le 1$,

$$\begin{aligned}&\Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dx\Big)^{\frac{2}{n}} \\ &\le \Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t}\Big)^{\frac{2}{n}} + \frac{1}{(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t})^{\frac{n-2}{n}}} \frac{2}{n}\{S_t\} + C\{S_t\}^2,\end{aligned}$$

where

$$\{S_t\} = \frac{1}{6}Ric(x_t)_{ij} \int_{B(0,\delta)} x^i x^j \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} + C\int_{B(0,\delta)} |x|^3\, \overline{u}_t^{2^*} dv_{\mathbf{g}_t}.$$

We deduce

$$A_t \le \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}(A_t^1 + A_t^2),$$

where

$$A_t^1 = \Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} \Big)^{\frac{2}{n}} \Big(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx \Big)^{\frac{n-2}{n}} - \Big(\sup f \int_{B(0,\delta)} (\eta \overline{u}_t)^{2^*} dx \Big)^{\frac{n-2}{n}}$$

and

$$A_t^2 = \frac{2(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}}}{n(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t})^{\frac{n-2}{n}}} \times \Big\{ \frac{1}{6} Ric(x_t)_{ij} \int_{B(0,\delta)} x^i x^j \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} + C \int_{B(0,\delta)} |x|^3 \, \overline{u}_t^{2^*} dv_{\mathbf{g}_t} \Big\} (1 + \varepsilon_\delta)$$

as $\{S_t\} \to 0$ when $\delta \to 0$ uniformly in t. A_t^2 will give, by developing the metric, $S_{\mathbf{g}}(x_0)$ while A_t^1 will give, by developing f, $-\frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$.

Note that for any $\alpha \in H_1^2(B(x_0, 2\delta))$

$$\lim_{t\to 1} \frac{\int_{B(x_t,\delta)} \alpha dx}{\int_{B(x_t,\delta)} \alpha dv_{\mathbf{g}_t}} = 1 + O(\delta^2) = 1 + \varepsilon_\delta.$$

We start by studying A_t^2 :

1) We have $\lim_{t\to 1} \frac{(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}}}{(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t})^{\frac{n-2}{n}}} = 1 + \varepsilon_\delta$.

2) Using the weak estimates (Proposition 5), $|x|^2 \, \overline{u}_t^{2^*} \leq c \overline{u}_t^2$, from which we get

$$\frac{\int_{B(0,\delta)} |x|^3 \, \overline{u}_t^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \leq C.\varepsilon_\delta.$$

3) Using the blow-up formulas we write, for all $R > 0$,

$$\int_{B(0,\delta)} x^i x^j \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} = \int_{B(0,R\mu_t)} x^i x^j \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} + \int_{B(0,\delta)\setminus B(0,R\mu_t)} x^i x^j \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t}$$

$$= \mu_t^2 \int_{B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} + \mu_t^2 \int_{B(0,\delta\mu_t^{-1})\setminus B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}$$

and

$$\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t} = \mu_t^2 \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}.$$

Using the weak estimates again, we get

$$\int_{B(0,\delta\mu_t^{-1})\setminus B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} \leq \varepsilon_R . \int_{B(0,\delta\mu_t^{-1})\setminus B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t},$$

thus,

$$\frac{\int_{B(0,\delta\mu_t^{-1})\setminus B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} \leq \varepsilon_R,$$

where $\varepsilon_R \to 0$ when $R \to +\infty$. Now, if $i \neq j$

$$\overline{\lim_{t\to 1}} \frac{|\int_{B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}|}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} \leq \overline{\lim_{t\to 1}} \frac{|\int_{B(0,R)} x^i x^j \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}|}{\int_{B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} = 0$$

because

$$\widetilde{u}_t \to \widetilde{u} = \left(1 + \frac{f(x_0)^{\frac{2}{n}}}{K(n,2)^2 n(n-2)} |x|^2\right)^{-\frac{n-2}{2}} \quad \text{in} \quad C^0(B(0,R))$$

and $\widetilde{u}$ is radial (see Subsection 3.2).

If $i = j$

$$\frac{\int_{B(0,R)} x^i x^i \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} = \frac{\int_{B(0,R)} (x^i)^2 \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} . \frac{\int_{B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}}.$$

But as soon as $n > 4$, using strong L^2-concentration (Proposition 9), we obtain

$$\lim_{R\to\infty} \overline{\lim_{t\to 1}} \frac{\int_{B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} = 1,$$

therefore,

$$\begin{aligned} \lim_{R\to\infty} \overline{\lim_{t\to 1}} \frac{\int_{B(0,R)} x^i x^i \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} &= f(x_0) \frac{\int_{\mathbb{R}^n} (x^i)^2 . \widetilde{u}^2 dx}{\int_{\mathbb{R}^n} \widetilde{u}^2 dx} \\ &= f(x_0)^{\frac{n-2}{n}} K(n,2)^2 \frac{n(n-4)}{4(n-1)} \end{aligned}$$

and thus

$$\overline{\lim_{t\to 1}} \frac{1}{f(x_0)^{\frac{n-2}{n}} K(n,2)^2} \frac{A_t^2}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} = \frac{n-4}{12(n-1)} S_{\mathbf{g}}(x_0) + \varepsilon_\delta,$$

which, with $\overline{\lim}_{t\to 1} \frac{B_t}{\int_{B(0,\delta)} \overline{u}_t^2 dx} = \frac{1}{6} S_{\mathbf{g}}(x_0) + \varepsilon_\delta$ gives

$$\overline{\lim}_{t\to 1}\Big(\frac{1}{f(x_0)^{\frac{n-2}{n}} K(n,2)^2} \frac{A_t^2}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} + \frac{B_t}{\int_{B(0,\delta)} \overline{u}_t^2 dx}\Big) = \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) + \varepsilon_\delta.$$

If $n = 4$ we write

$$\lim_{R\to\infty} \overline{\lim}_{t\to 1} \frac{\int_{B(0,R)} x^i x^i \widetilde{f}_t \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}}{\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} \le f(x_0)^{\frac{n-2}{n}} K(n,2)^2 \frac{n-4}{4(n-1)}$$

and we get the conclusion by distinguishing two cases, $S_{\mathbf{g}}(x_0) < 0$ or $S_{\mathbf{g}}(x_0) \ge 0$, the proof being finished as $\frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$ does not appear in dimension 4 (see the end of the proof).

Let us now consider A_t^1.

$$A_t^1 = \Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t}\Big)^{\frac{2}{n}} \Big(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}} - \Big(\sup f \int_{B(0,\delta)} (\eta \overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}}.$$

We write $f_t = f(x_0) + g_t$. Remembering that $f(x_0) = \sup f$, we have $g_t(x_0) = 0$ and $g_t \le 0$. Using $(1+x)^\alpha \le 1 + \alpha x$ for $0 < \alpha \le 1$

$$\Big(\int_{B(0,\delta)} \overline{f}_t (\eta \overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}} \le \Big(\int_{B(0,\delta)} f(x_0) (\eta \overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}} + \frac{n-2}{n} \frac{\int_{B(0,\delta)} \overline{g}_t (\eta \overline{u}_t)^{2^*} dx}{(\int_{B(0,\delta)} f(x_0)(\eta \overline{u}_t)^{2^*} dx)^{\frac{2}{n}}},$$

where $\overline{g}_t$ is g_t in the exponential chart in x_t. We now use the theorem of Druet and Robert to write, in $B(0,\delta)$, $\overline{u}_t \ge (1-\varepsilon_\delta)\overline{B}_t$, where $\overline{B}_t$ is B_t in the exponential chart in x_t. Because $g_t \le 0$, we have

$$\int_{B(0,\delta)} \overline{g}_t (\eta \overline{u}_t)^{2^*} dx \le (1-\varepsilon_\delta) \int_{B(0,\delta)} \overline{g}_t (\eta \overline{B}_t)^{2^*} dx.$$

Combining this with the expansion above and the fact that

$$\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t} \le 1,$$

we obtain

$$A_t^1 \le (1-\varepsilon_\delta) \frac{n-2}{n} \frac{\int_{B(0,\delta)} \overline{g}_t (\eta \overline{B}_t)^{2^*} dx}{(\int_{B(0,\delta)} f(x_0)(\eta \overline{u}_t)^{2^*} dx)^{\frac{2}{n}}}.$$

We now expand g_t noting that $\partial_i \overline{g}_t = \partial_i \overline{f}_t$ and $\partial^2_{ij}\overline{g}_t = \partial^2_{ij}\overline{f}_t$.

$$\overline{g}_t(x) \le g_t(x_t) + x^i \partial_i \overline{f}_t(x_t) + \frac{1}{2}\partial_{kl}\overline{f}_t(x_t).x^k x^l + c\,|x|^3 .$$

Thus,

$$\int_{B(0,\delta)} \overline{g}_t (\eta \overline{B}_t)^{2^*} dx \le g_t(x_t) \int_{B(0,\delta)} (\eta\overline{B}_t)^{2^*} dx + \partial_i \overline{f}_t(x_t) \int_{B(0,\delta)} x^i (\eta \overline{B}_t)^{2^*} dx$$
$$+ \frac{1}{2}\partial_{kl}\overline{f}_t(x_t) \int_{B(0,\delta)} x^k x^l (\eta\overline{B}_t)^{2^*} dx + C \int_{B(0,\delta)} |x|^3 \,(\eta\overline{B}_t)^{2^*} dx.$$

Now, first $g_t(x_t) \le 0$, and second, and this is the main point for which we need the theorem of Druet and Robert (see the reason at the beginning of the next section), as $\overline{B}_t$ is radial, we have

$$\partial_i \overline{f}_t(x_t) \int_{B(0,\delta)} x^i (\eta\overline{B}_t)^{2^*} dx = 0.$$

Therefore, introducing all this in the last inequality for A^1_t, we have

$$\overline{\lim_{t\to 1}} \frac{A^1_t}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \le$$

$$\tfrac{n-2}{n}(1-\varepsilon_\delta) \overline{\lim_{t\to 1}} \frac{\frac{1}{2}\partial_{kl}\overline{f}_t(x_t)\int_{B(0,\delta)} x^k x^l (\eta\overline{B}_t)^{2^*} dv_{\mathbf{g}_t} + C\int_{B(0,\delta)} |x|^3 \,(\eta\overline{B}_t)^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}}$$

where we have replaced dx by $dv_{\mathbf{g}_t}$ using the remark made at the beginning of the study of A^2_t.

Now, as for A^2_t, we write

$$\begin{aligned} \overline{\lim_{t\to 1}} \frac{\int_{B(0,\delta)} x^k x^l (\eta\overline{B}_t)^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} &= f(x_0)^{\frac{-2}{n}} K(n,2)^2 \frac{n-4}{4(n-1)} \text{ if } k = l \\ &= 0 \text{ if } k \ne l \end{aligned}$$

and therefore,

$$\frac{1}{K(n,2)^2 f(x_0)^{\frac{n-2}{n}}} \frac{n-2}{n} \overline{\lim_{t\to 1}} \frac{\frac{1}{2}\partial_{kl}\overline{f}_t(x_t)\int_{B(0,\delta)} x^k x^l (\eta\overline{B}_t)^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}}$$

$$= \frac{1}{f(x_0)} \frac{(n-2)(n-4)}{4(n-1)} \sum_l \frac{1}{2}\partial_{ll}\overline{f}_1(0) = -\frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$$

as $\triangle_{\mathbf{g}} f(x_0) = -\sum_l \partial_{ll}\overline{f}_1(0)$ in the exponential chart in x_0. Also

$$\frac{\int_{B(0,\delta)} |x|^3\, \overline{u}_t^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \le C.\varepsilon_\delta.$$

Thus, we have proved that dividing inequality (18) by $\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}$ and letting t go to 1, we get

$$h(x_0) + \varepsilon_\delta \le \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)} + \varepsilon_\delta.$$

Letting δ tend to 0

$$h(x_0) \le \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)},$$

which contradicts our hypothesis

$$h(x_0) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$$

when x_0 is a point of maximum of f. This proves that $u \not\equiv 0$, and therefore $u_t \to u > 0$, a minimizing solution for $(E_{h,f,\mathbf{g}})$, and thus the weakly critical function h is in fact critical.

4.4. **Alternate proof, proof of the fundamental estimate.** As we saw in the last part of the proof, the difficulty introduced by the presence of the function f is to control the first derivatives of f, $\partial_i f(x_t)$, as blow-up gives

$$\int_{B(0,\delta)} \partial_i f(x_t) x^i \overline{u}_t^{2^*} dv_{\mathbf{g}_t} = \mu_t \int_{B(0,\delta\mu_t^{-1})} \partial_i f(x_t) x^i \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t}$$

to be divided by

$$\mu_t^2 \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t},$$

and it would be necessary to control $\frac{\partial_i f(x_t)}{\mu_t}$, which seems to be difficult. But thanks to the theorem of Druet and Robert, we can replace $u(t)$ by $B(t)$ near x_t, and after blow-up

$$\mu_t \int_{B(0,\delta\mu_t^{-1})} \partial_i f(x_t) x^i \widetilde{B}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t} = 0$$

as $\widetilde{B}_t$ is radial. Of course, the proof is then short, but the proof of the theorem of Druet and Robert is quite involved, even though the strong estimates (Proposition 8) are the first step.

The other way to get over the problem of the first derivatives of f is to expand f in x_0 as then $\partial_i f(x_0) = 0$ because x_0 is a point of maximum of f. But then, one has to transpose the weak and strong estimates from x_t to x_0, which, as we said in the section about concentration phenomenon, requires that we prove the following estimate

$$\frac{d_g(x_t, x_0)}{\mu_t} \leq C.$$

As we said, this estimate is important and of independent interest, as it gives a complete description of the sequence (u_t). This is why we give this alternate proof of Theorem 1, even though it requires an additional hypothesis. This proof, which gives at the same time the proof of Theorem 1 and of the estimate, is, we think, interesting, and is available directly after Proposition 9; i.e., it does not require the theorem of Druet and Robert.

We now make the hypothesis that the Hessian of f is non-degenerate at its points of maximum. We also suppose now that $dimM \geq 5$, even though our proof gives Theorem 1 in dimension 4.

Let us note that $x_0(t) = \exp_{x_t}^{-1}(x_0) = (x_0^1(t), ..., x_0^n(t))$, which is possible as soon as t is close enough to 1 for a fixed radius δ. Then $x_0(t) \to 0$ when $t \to 1$. The point $x_0(t)$ is a locally strict maximum of $\overline{f}_t$. We will let δ go to 0 at the end of the reasoning, after having taken the limit when $t \to 1$.

The expansion of $\overline{f}_t$ in $x_0(t)$ gives

$$\begin{aligned}\overline{f}_t(x) &\leq f(x_0) + \frac{1}{2}\partial_{kl}\overline{f}_t(x_0(t))(x^k - x_0^k(t))(x^l - x_0^l(t)) + c\,|x - x_0(t)|^3 \\ &:= f(x_0) + T_t,\end{aligned}$$

(T_t analogous to a Taylor expansion) where $(\partial_{kl}\overline{f}_t(x_0))$ is a negative definite matrix (we shall write < 0). The letters c, C will always be constants independent of t and δ. Remember that

$$\begin{aligned}A_t^1 = &\Big(\int_{B(0,\delta)} \overline{f}_t \overline{u}_t^{2^*} dv_{\mathbf{g}_t}\Big)^{\frac{2}{n}} \Big(\int_{B(0,\delta)} \overline{f}_t (\eta\overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}} \\ &- \Big(\sup f \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}}.\end{aligned}$$

Introducing the expansion of $\overline{f}_t$ in $x_0(t)$, and using again the fact that $(1 + x)^\alpha \leq 1 + \alpha x$ for $0 < \alpha \leq 1$, we get

$$\Big(\int_{B(0,\delta)} \overline{f}_t (\eta\overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}} \leq \Big(\int_{B(0,\delta)} f(x_0)(\eta\overline{u}_t)^{2^*} dx\Big)^{\frac{n-2}{n}}$$

$$+\frac{\frac{n-2}{n}}{(\int_{B(0,\delta)} f(x_0)(\eta\overline{u}_t)^{2^*}dx)^{\frac{2}{n}}}\{F_t\},$$

where

$$\{F_t\} = \frac{1}{2}\partial_{kl}\overline{f}_t(x_0(t))\int_{B(0,\delta)}(x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*}dx$$
$$+ C\int_{B(0,\delta)}|x - x_0(t)|^3(\eta\overline{u}_t)^{2^*}dx;$$

remembering that $\sup_M f = f(x_0)$ and that $\int_{B(0,\delta)}\overline{f}_t\overline{u}_t^{2^*}dv_{\mathbf{g}_t} \leq 1$,

$$A_t^1 \leq \frac{n-2}{n}\frac{(\int_{B(0,\delta)}\overline{f}_t\overline{u}_t^{2^*}dv_{\mathbf{g}_t})^{\frac{2}{n}}}{(\int_{B(0,\delta)} f(x_0)(\eta\overline{u}_t)^{2^*}dx)^{\frac{2}{n}}}\{F_t\}. \tag{19}$$

Therefore, we obtain

$$\overline{\lim_{t\to 1}}\frac{A_t^1}{\int_{B(0,\delta)}\overline{u}_t^2 dv_{\mathbf{g}_t}} \leq \frac{n-2}{n}(1+\varepsilon_\delta)$$

$$\times\overline{\lim_{t\to 1}}\frac{\frac{1}{2}\partial_{kl}\overline{f}_t(x_0)\int_{B(0,\delta)}(x^k-x_0^k(t))(x^l-x_0^l(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}+C\int_{B(0,\delta)}|x-x_0(t)|^3(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}}{\int_{B(0,\delta)}\overline{u}_t^2 dv_{\mathbf{g}_t}},$$

where we write $\partial_{kl}\overline{f}_t(x_0)$ for $\partial_{kl}\overline{f}_t(x_0(t))$. Considering the expansion

$$\overline{f}_t(x) \leq f(x_0) + \frac{1}{2}\partial_{kl}\overline{f}_t(x_0(t)).(x^k - x_0^k(t))(x^l - x_0^l(t)) + c\,|x - x_0(t)|^3,$$

note that by the regularity of $\exp_{x_t}^{-1}\circ\exp_{x_0}$ with respect to all the variables, we can suppose that c is independent of t. Moreover,

$$c\,|x - x_0(t)|^3 \leq c'\,|x - x_0(t)|\sum_k(x^k - x_0^k(t))^2 \leq 2\delta c'\sum_k(x^k - x_0^k(t))^2,$$

where we recall that δ is the radius of the ball of integration. We can then write

$$\overline{f}_t(x) \leq f(x_0) + (\frac{1}{2}\partial_{kl}\overline{f}_t(x_0(t)) + \delta C_{kl})(x^k - x_0^k(t))(x^l - x_0^l(t))$$

where $C_{kl} = c\delta_{kl} = c$ if $k = l$ and $C_{kl} = 0$ if $k \neq l$ (δ_{kl} is the Krönecker symbol) is independent of t.

We introduce one more notation:

$$D_{kl}(t,\delta) = \frac{1}{2}\partial_{kl}\overline{f}_t(x_0(t)) + \delta C_{kl}.$$

Then

1) $\lim\limits_{\delta\to 0}\lim\limits_{t\to 1} D_{kl}(t,\delta) = \frac{1}{2}\partial_{kl}\overline{f}_1(x_0(1))$ where $\overline{f}_1 = f \circ \exp_{x_0}^{-1}$ and $x_0(1) = 0 = \exp_{x_0}^{-1}(x_0)$.

2) For any δ small enough and for all t close to 1, $D_{kl}(t,\delta)$ is still negative definite.

$D_{kl}(t,\delta)$ is the Hessian of f in $x_0(t)$ perturbed on its diagonal by the third-order terms. It is for the second point that we need the hypothesis that the Hessian of f is non-degenerate. Thus,

$$\begin{aligned}&\frac{1}{2}\partial_{kl}\overline{f}_t(x_0)\int_{B(0,\delta)}(x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}\\&+C\int_{B(0,\delta)}|x-x_0(t)|^3(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}\\&\le D_{kl}(t,\delta)\int_{B(0,\delta)}(x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}.\end{aligned}$$

Let

$$\{F_t'\} = D_{kl}(t,\delta)\int_{B(0,\delta)}(x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}.$$

We have

$$\begin{aligned}&\overline{\lim_{t\to 1}}\frac{A_t^1}{\int_{B(0,\delta)}v_t^2 dv_{\mathbf{g}_t}}\\&\le \frac{n-2}{n}\overline{\lim_{t\to 1}}\frac{D_{kl}(t,\delta)\int_{B(0,\delta)}(x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}}{\int_{B(0,\delta)}\overline{u}_t^2 dv_{\mathbf{g}_t}}(1+\varepsilon_\delta).\end{aligned}$$

In the expansion of $D_{kl}(t,\delta)(x^k - x_0^k(t))(x^l - x_0^l(t))$, we are interested in the first term, i.e $D_{kl}(t,\delta)x^k x^l$ (look back at how we obtained $S_g(x_0)$ in A_t^2), and we are going to show that the other terms can be neglected. The idea is to reorganize the expansion of $\{F_t'\}$ and to use the fact $D_{kl}(t,\delta)$ is a negative bilinear form

$$\begin{aligned}\{F_t'\} = {}& D_{kl}(t,\delta)\int_{B(0,\delta)}x^k x^l(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t} + D_{kl}(t,\delta)x_0^k(t)x_0^l(t)\int_{B(0,\delta)}(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}\\&- D_{kl}(t,\delta)\int_{B(0,\delta)}(x^k x_0^l(t) + x^l x_0^k(t))(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}.\end{aligned}$$

We rewrite the two last terms (suppressing some δ and t and all integrals being taken with respect to $dv_{\mathbf{g}_t}$)

$$D_{kl}.x_0^k x_0^l\int_{B(0,\delta)}(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t} - D_{kl}\int_{B(0,\delta)}(x^k x_0^l + x^l x_0^k)(\eta\overline{u}_t)^{2^*}dv_{\mathbf{g}_t}$$

$$= D_{kl}\Big[x_0^k x_0^l \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} - x_0^l \int_{B(0,\delta)} x^k (\eta\overline{u}_t)^{2^*} - x_0^k \int_{B(0,\delta)} x^l (\eta\overline{u}_t)^{2^*}\Big]$$

$$= D_{kl}\Big[x_0^k (\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}} . x_0^l (\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}}$$

$$-x_0^l (\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}} \frac{\int_{B(0,\delta)} x^k (\eta\overline{u}_t)^{2^*}}{(\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}}} - x_0^k (\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}} \frac{\int_{B(0,\delta)} x^l (\eta\overline{u}_t)^{2^*}}{(\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*})^{\frac{1}{2}}}\Big].$$

Thus, setting

$$\varepsilon^k(t) = \int_{B(0,\delta)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}, \qquad z_t = (\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^{\frac{1}{2}},$$

the expression above becomes

$$D_{kl}.x_0^k x_0^l \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t} - D_{kl} \int_{B(0,\delta)} (x^k x_0^l + x^l x_0^k)(\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}$$

$$= D_{kl}\Big[x_0^k(t).z_t.x_0^l(t).z_t - x_0^l(t).z_t.\frac{\varepsilon^k(t)}{z_t} - x_0^k(t).z_t.\frac{\varepsilon^l(t)}{z_t}\Big]$$

$$= D_{kl}\Big[(x_0^k(t).z_t - \frac{\varepsilon^k(t)}{z_t})(x_0^l(t).z_t - \frac{\varepsilon^l(t)}{z_t}) - \frac{\varepsilon^k(t)\varepsilon^l(t)}{z_t^2}\Big].$$

By this method of reorganization of the Hessian, we have obtained

$$\frac{1}{2}\partial_{kl}\overline{f}_t(x_0) \int_{B(0,\delta)} (x^k - x_0^k(t))(x^l - x_0^l(t))(\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}$$
$$+ C \int_{B(0,\delta)} |x - x_0(t)|^3 (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}$$
$$\leq D_{kl}(t,\delta) \int_{B(0,\delta)} x^k x^l (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}$$
$$+ D_{kl}(t,\delta)(x_0^k(t).z_t - \frac{\varepsilon^k(t)}{z_t})(x_0^l(t).z_t - \frac{\varepsilon^l(t)}{z_t}) - D_{kl}(t,\delta)\frac{\varepsilon^k(t)\varepsilon^l(t)}{z_t^2}$$
$$\leq D_{kl}(t,\delta) \int_{B(0,\delta)} x^k x^l (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t} - D_{kl}(t,\delta)\frac{\varepsilon^k(t)\varepsilon^l(t)}{z_t^2}$$

because, and this is the fundamental point,

$$D_{kl}(t,\delta)\omega^k\omega^l \leq 0 \;\; \forall \omega = (\omega^1, ..., \omega^n),$$

which allows us to suppress from the inequality

$$D_{kl}(t,\delta)(x_0^k(t).z_t - \frac{\varepsilon^k(t)}{z_t})(x_0^l(t).z_t - \frac{\varepsilon^l(t)}{z_t}).$$

It is this term that will give us the estimate $\frac{d_{\mathbf{g}}(x_t,x_0)}{\mu_t} \leq C$ (see below).

We have therefore obtained

$$\overline{\lim_{t\to 1}} \frac{A_t^1}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}}$$

$$\leq \frac{n-2}{n} \overline{\lim_{t\to 1}} \frac{D_{kl}(t,\delta)\int_{B(0,\delta)} x^k x^l (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t} - D_{kl}(t,\delta)\frac{\varepsilon^k(t)\varepsilon^l(t)}{z_t^2}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}}(1+\varepsilon_\delta).$$

Now, as for A_t^2, we write

$$\begin{aligned}\overline{\lim_{t\to 1}} \frac{\int_{B(0,\delta)} x^k x^l (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} &= f(x_0)^{\frac{-2}{n}} K(n,2)^2 \frac{n-4}{4(n-1)} \text{ if } k = l \\ &= 0 \text{ if } k \neq l\end{aligned}$$

and therefore,

$$\begin{aligned}&\frac{1}{K(n,2)^2 f(x_0)^{\frac{n-2}{n}}} \frac{n-2}{n} \overline{\lim_{t\to 1}} \frac{D_{kl}(t,\delta)\int_{B(0,\delta)} x^k x^l (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t}}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \\ &= \frac{1}{f(x_0)} \frac{(n-2)(n-4)}{4(n-1)} \sum_l (\frac{1}{2}\partial_{ll}\overline{f}_1(0) + c_{ll}\delta) \\ &= -\frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)} + \varepsilon_\delta\end{aligned}$$

as $\triangle_{\mathbf{g}} f(x_0) = -\sum_l \partial_{ll}\overline{f}_1(0)$ in the exponential chart in x_0.

At last, let us show that the residual term can be neglected.

$$\left|\varepsilon^k(t)\varepsilon^l(t)\right| \leq \frac{1}{2}(\varepsilon^k(t)^2 + \varepsilon^l(t)^2).$$

But

$$\begin{aligned}\varepsilon^k(t)^2 &= (\int_{B(0,\delta)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2 \\ &= (\int_{B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t} + \int_{B(0,\delta)\backslash B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2 \\ &\leq 2(\int_{B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2 + 2(\int_{B(0,\delta)\backslash B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2.\end{aligned}$$

The blow-up formulas give, for a fixed R,

$$\frac{(\int_{B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \leq \frac{(\mu_t \int_{B(0,R)} x^k \widetilde{u}_t^{2^*} dv_{\widetilde{\mathbf{g}}_t})^2}{\mu_t^2 \int_{B(0,R)} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}} \underset{t\to 1}{\to} \frac{(\int_{B(0,R)} x^k \widetilde{u}^{2^*} dx)^2}{\int_{B(0,R)} \widetilde{u}^2 dx} = 0$$

because $\widetilde{u}$ is radial.

At last, using the weak estimates, $d_{\mathbf{g}}(x,x_t)^{\frac{n-2}{2}} u_t(x) \leq \varepsilon$ if $d_{\mathbf{g}}(x,x_t) \geq R\mu_t$, and using Hölder's inequality

$$\begin{aligned}(\int_{B(0,\delta)\backslash B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2 &\leq \varepsilon_R^2 (\int_{B(0,\delta)\backslash B(0,R\mu_t)} \overline{u}_t^{2\frac{n-1}{n-2}} dv_{\mathbf{g}_t})^2 \\ &\leq \varepsilon_R^2 (\int_{B(0,\delta)\backslash B(0,R\mu_t)} \overline{u}_t^2 dv_{\mathbf{g}_t})(\int_{B(0,\delta)\backslash B(0,R\mu_t)} \overline{u}_t^{\frac{2n}{n-2}} dv_{\mathbf{g}_t});\end{aligned}$$

therefore,

$$\frac{(\int_{B(0,\delta)\backslash B(0,R\mu_t)} x^k (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t})^2}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \leq \varepsilon_R^2 (\int_{B(0,\delta)\backslash B(0,R\mu_t)} \overline{u}_t^{\frac{2n}{n-2}} dv_{\mathbf{g}_t}) \leq c\varepsilon_R^2,$$

where $\varepsilon_R \to 0$ when $R \to \infty$. Remarking that because x_0 is a concentration point

$$z_t^2 = \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dv_{\mathbf{g}_t} \geq \int_{B(x_0,\delta/4)} u_t^{2^*} dv_{\mathbf{g}} \geq c > 0$$

we have obtained

$$\frac{\left|\varepsilon^k(t)\varepsilon^l(t)\right|}{z_t^2 \int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \underset{t\to 1}{\to} 0.$$

We have therefore obtained once again that

$$h(x_0) + \varepsilon_\delta \leq \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)} + \varepsilon_\delta.$$

Letting δ tend to 0,

$$h(x_0) \leq \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)},$$

which contradicts our hypothesis

$$h(x_0) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$$

when x_0 is a point of maximum of f. This proves that $u_t \to u > 0$, a minimizing solution for $(E_{h,f,\mathbf{g}})$, and therefore the weakly critical function h is in fact critical.

We now prove the estimate

$$\frac{d_{\mathbf{g}}(x_t, x_0)}{\mu_t} \le C.$$

Going back to the computations above, we have obtained

$$\begin{aligned} h(x_0) \quad \le \quad & \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)} + \varepsilon_\delta \qquad (20) \\ & + \overline{\lim_{t\to 1}} \frac{n-2}{n} \frac{D_{kl}(t,\delta)(x_0^k(t).z_t - \frac{\varepsilon^k(t)}{z_t})(x_0^l(t).z_t - \frac{\varepsilon^l(t)}{z_t})}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}}, \end{aligned}$$

where $D_{kl}(t,\delta)$ is negative definite for t close to 1 and for all δ small enough, and where we recall that $x_0(t) = \exp_{x_t}^{-1}(x_0) = (x_0^1(t), ..., x_0^n(t))$. So, there exists a $\lambda > 0$ such that for all $\omega \in \mathbb{R}^n$

$$D_{kl}(t,\delta)\omega^k\omega^l \le -\lambda \sum_k |\omega^k|^2$$

and so

$$D_{kl}(t,\delta) \frac{(x_0^k(t).z_t - \frac{\varepsilon^k(t)}{z_t})(x_0^l(t).z_t - \frac{\varepsilon^l(t)}{z_t})}{(\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t})^{\frac{1}{2}} (\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t})^{\frac{1}{2}}} \le$$

$$-\lambda \sum_k \Big| \frac{x_0^k(t).z_t}{(\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t})^{\frac{1}{2}}} - \frac{\varepsilon^k(t)}{z_t(\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t})^{\frac{1}{2}}} \Big|^2.$$

Moreover, we already proved that

$$\frac{\varepsilon^k(t)^2}{z_t^2 \int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \underset{t\to 1}{\to} 0$$

as we also have $z_t = (\int_{B(0,\delta)} \overline{u}_t^{2^*} dv_{\mathbf{g}_t})^{\frac{1}{2}}$, and therefore as x_0 is a concentration point $0 < c \le \liminf z_t \le \limsup z_t \le c' < +\infty$. Therefore, necessarily, because of (20), for all k, there exists a constant $C > 0$ such that for $t \to 1$

$$\frac{x_0^k(t)}{(\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t})^{\frac{1}{2}}} \le C.$$

Now

$$\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t} = \mu_t^2 \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t}.$$

But the strong estimates give that

$$\overline{\lim_{t\to 1}}\int_{B(0,\delta\mu_t^{-1})}\widetilde{u}_t^2 dv_{\widetilde{\mathbf{g}}_t} < +\infty,$$

therefore,

$$\int_{B(0,\delta)}\overline{u}_t^2 dv_{\mathbf{g}_t} \sim C\mu_t^2$$

from where we have for all k $\frac{x_0^k(t)}{\mu_t} \le C'$ and so $\frac{d_{\mathbf{g}}(x_t,x_0)}{\mu_t} \le C$. If we have furthermore that at the points of maximum of f

$$h(P) = \frac{n-2}{4(n-1)}S_{\mathbf{g}}(P) - \frac{(n-2)(n-4)}{8(n-1)}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)},$$

then we have more precisely that $\frac{d_{\mathbf{g}}(x_t,x_0)}{\mu_t} \to 0$.

Remark. Note that when concentration occurs we have:

$$h(x_0) \le \frac{n-2}{4(n-1)}S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)}\frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}.$$

5. Critical triple 1: existence of critical functions

The idea behind proving the existence of critical functions (Theorem 2) is to find, being given the manifold $(M,\mathbf{g})$ and the function f, a subcritical function h_0 and a weakly critical function h_1 and then to join these two functions by a continuous path; Theorem 1 then shows that this path must "cross" the set of critical functions.

Note first that, by the sharp Sobolev inequality (2), $B_0(\mathbf{g})K(n,2)^{-2}$ is a weakly critical function for any manifold $(M,\mathbf{g})$ and any function f. Also, it is known that

$$B_0(\mathbf{g})K(n,2)^{-2} \ge \frac{n-2}{4(n-1)}\sup_M S_{\mathbf{g}}.$$

Therefore, for any $\alpha > 0$, and for any point P where f is maximum on M, we have

$$B_0(\mathbf{g})K(n,2)^{-2} + \alpha \ge \frac{n-2}{4(n-1)}S_{\mathbf{g}}(P) - \frac{(n-2)(n-4)}{8(n-1)}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

Now, we are going to modify the weakly critical function $B_0(\mathbf{g})K(n,2)^{-2}+\alpha$ by the test functions presented in the introduction. They can be seen under the following form: for any $x \in M$ and any $\delta > 0$ small enough, there exists a sequence of functions (ψ_k) with compact support in $B(x,\delta)$ such that for any function h

$$J_{h,1,\mathbf{g}}(\psi_k)=\frac{\int_M|\nabla\psi_k|^2\,dv_{\mathbf{g}}+\int_M h.\psi_k{}^2dv_{\mathbf{g}}}{\left(\int_M|\psi_k|^{\frac{2n}{n-2}}\,dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}\underset{k\to\infty}{\to}\frac{1}{K(n,2)^2}$$

and $\int_M\psi_k^{\frac{2n}{n-2}}dv_{\mathbf{g}}=1$, this last condition being obtained by multiplying the functions in the introduction by suitable constants. We will use the functional J here, as

$$\int_M f.\psi_k^{\frac{2n}{n-2}}dv_{\mathbf{g}}\neq 1.$$

Let then ψ_k be one of these functions, where k and $B(x,\delta)$ will be fixed later. We consider, for $t>0$ the sequence

$$h_t=B_0(\mathbf{g})K(n,2)^{-2}+\alpha-t.\psi_k^{\frac{4}{n-2}}.$$

First, we seek a condition for $\triangle_{\mathbf{g}}+h_t$ to be coercive. Noting $B_0K^{-2}=B_0(\mathbf{g})K(n,2)^{-2}$, and taking all integrals for the measure $dv_{\mathbf{g}}$, we have for $u\in H_1^2$

$$\begin{aligned}\int_M(|\nabla u|^2_{\mathbf{g}}+h_tu^2)\qquad&=\int_M(|\nabla u|^2_{\mathbf{g}}+B_0K^{-2}.u^2)-(t-\alpha)\int_M\psi_k^{\frac{4}{n-2}}.u^2\\&\geq K(n,2)^{-2}(\int_M u^{\frac{2n}{n-2}})^{\frac{n-2}{n}}-(t-\alpha)\int_M\psi_k^{\frac{4}{n-2}}.u^2\end{aligned}$$

by the Sobolev inequality. But using Hölder's inequality

$$\int_M\psi_k^{\frac{4}{n-2}}.u^2\leq(\int_M\psi_k^{\frac{2n}{n-2}})^{\frac{n-2}{n}}(\int_M u^{\frac{2n}{n-2}})^{\frac{2}{n}}=(\int_M u^{\frac{2n}{n-2}})^{\frac{2}{n}}$$

as $\int_M\psi_k^{\frac{2n}{n-2}}=1$. Thus, using Hölder's inequality again to get the existence of a constant $C>0$ such that

$$C\int_M u^2\leq\Big(\int_M u^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}$$

we have as soon as $K(n,2)^{-2}-(t-\alpha)>0$

$$\begin{aligned}\int_M(|\nabla u|^2_{\mathbf{g}}+h_tu^2)\qquad&\geq(K(n,2)^{-2}-(t-\alpha))(\int_M u^{\frac{2n}{n-2}})^{\frac{n-2}{n}}\\&\geq(K(n,2)^{-2}-(t-\alpha))C\int_M u^2.\end{aligned}$$

So $\triangle_{\mathbf{g}} + h_t$ is coercive as soon as $t - \alpha < K(n,2)^{-2}$; we then fix t_1 such that $\alpha < t_1 < K(n,2)^{-2} + \alpha$.

We now want to fix ψ_k so that h_{t_1} is subcritical for f. We pick first x close enough to a point x_0 of maximum of f and δ small enough such that $f > 0$ on $B(x,\delta)$, to obtain

$$J_{h_{t_1},f,\mathbf{g}}(\psi_k) = \frac{\int_M |\nabla\psi_k|^2 + \int_M B_0K^{-2}.{\psi_k}^2 - (t_1-\alpha)\int_M \psi_k^{\frac{2n}{n-2}}}{\left(\int_M f\,|\psi_k|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}$$

$$\leq \frac{J_{B_0K^{-2},1}(\psi_k)}{(\underset{B(x,\delta)}{Inf}\ f)^{\frac{n-2}{n}}} - \frac{t_1-\alpha}{(\underset{B(x,\delta)}{\sup}\ f)^{\frac{n-2}{n}}} \leq \frac{J_{B_0K^{-2},1}(\psi_k)}{(\underset{B(x,\delta)}{Inf}\ f)^{\frac{n-2}{n}}} - \frac{t_1-\alpha}{(\sup_M f)^{\frac{n-2}{n}}}.$$

For any $\varepsilon > 0$, by continuity of f, we can choose x close enough to a point of maximum x_0 and δ small enough such that $B(x,\delta)\cap\{x : f(x) = Maxf\} = \emptyset$ and

$$\frac{1}{(\inf_{B(x,\delta)} f)^{\frac{n-2}{n}}} \leq \frac{1}{(\sup_M f)^{\frac{n-2}{n}}} + \varepsilon,$$

x and δ being fixed, we can now choose k large enough to have

$$J_{B_0K^{-2},1}(\psi_k) \leq K(n,2)^{-2} + \varepsilon.$$

Therefore, choosing ε small enough, we see that because $\frac{t_1-\alpha}{(\sup_M f)^{\frac{n-2}{n}}} > 0$

$$J_{h_{t_1},f,\mathbf{g}}(\psi_k) < \frac{1}{K(n,2)^{-2}(\underset{M}{\sup}f)^{\frac{n-2}{n}}}$$

and therefore h_{t_1} is subcritical for f. We now set

$$t_0 = Inf\{t \leq t_1 : \lambda_{h_t} < \frac{1}{K(n,2)^2(\underset{M}{\sup}f)^{\frac{n-2}{n}}}\}\ .$$

Then $t_0 \geq 0$, and

$$\lambda_{h_{t_0}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}} \quad \text{and} \quad \lambda_{h_t} < \frac{1}{K(n,2)^2(\underset{M}{\sup}f)^{\frac{n-2}{n}}} \quad \text{if} \quad t > t_0.$$

Furthermore, for all t, $t_0 \leq t \leq t_1$,

$$\frac{4(n-1)}{n-2}h_{t_0}(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}}f(P)}{f(P)} \quad \text{for} \quad P \in \{x : f(x) = Maxf\}$$

because $B(x,\delta)\cap\{x : f(x) = Maxf\} = \emptyset$. At last, $h_t \underset{t\to t_0}{\to} h_{t_0}$ in $C^{0,\alpha}$, and $\triangle_{\mathbf{g}} + h_{t_0}$ is coercive. Therefore, by Theorem 1, h_{t_0} is critical and $(E_{h,f,\mathbf{g}})$ has minimizing solutions.

Now, we prove that if $\{x : f(x) = Maxf\}$ is thin and if $\int_M f > 0$, there exist positive critical functions. We start again with $h = B_0(\mathbf{g})K(n,2)^{-2}+\alpha$, with $\alpha > 0$. For all P where f is maximum on M

$$\frac{4(n-1)}{n-2}B_0(\mathbf{g})K(n,2)^{-2} + \alpha > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}}f(P)}{f(P)}$$

as

$$B_0(\mathbf{g})K(n,2)^{-2} \geq \frac{(n-2)}{4(n-1)}MaxS_{\mathbf{g}}.$$

As f is not constant, there exists η with support in $M\backslash\{x : f(x) = Maxf\}$ and such that $0 \leq \eta \leq 1$. Let

$$c = \left(\int_M f dv_{\mathbf{g}}\right)^{-\frac{n-2}{n}}.$$

That is where we need $\int_M f dv_{\mathbf{g}} > 0$. We have $\int fc^{2^*}dv_{\mathbf{g}} = 1$. For $t \in \mathbb{R}^+$ we set $h_t = B_0K^{-2} + \alpha - t\eta$. Then $h_t = B_0K^{-2} + \alpha$ on $\{x : f(x) = Maxf\}$, and

$$\begin{aligned}I_{h_t}(c) &= \int_M (B_0K^{-2} + \alpha)c^2 dv_{\mathbf{g}} - c^2 t\int_M \eta dv_{\mathbf{g}}\\ &= \Big(\int_M f dv_{\mathbf{g}}\Big)^{-\frac{2}{2^*}}\Big((B_0K^{-2} + \alpha)Vol_{\mathbf{g}}(M) - t\int_M \eta dv_{\mathbf{g}}\Big).\end{aligned}$$

So, if t is large enough,

$$I_{h_t}(c) < \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}.$$

We also want h_t to be positive on M. By the definition of h_t and because $\sup_M \eta = 1$, it is the case if

$$t < B_0(\mathbf{g})K(n,2)^{-2} + \alpha. \tag{21}$$

But we also want

$$I_{h_t}(c) < \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}$$

which requires

$$t > \frac{1}{\int_M \eta dv_{\mathbf{g}}}\Big((B_0K^{-2} + \alpha)Vol_{\mathbf{g}}(M) - \frac{\left(\int_M f dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}\Big). \tag{22}$$

We can find such a t if

$$\frac{1}{\int_M \eta dv_{\mathbf{g}}}\Big((B_0K^{-2}+\alpha)Vol_{\mathbf{g}}(M)-\frac{\left(\int_M f dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}\Big) < B_0K^{-2}+\alpha$$

which can be written

$$\int_M \eta dv_{\mathbf{g}} > Vol_{\mathbf{g}}(M) - \frac{K^{-2}\left(\int_M f dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}{(B_0K^{-2}+\alpha)(\sup_M f)^{\frac{n-2}{n}}}. \tag{23}$$

Remember that we want η to have support in $M\backslash\{x : f(x) = Maxf\}$ with $0 \le \eta \le 1$. But we made the hypothesis that $\{x : f(x) = Maxf\}$, the set of maximum points of f, is a thin set. We can therefore find such a function η with $\int_M \eta dv_{\mathbf{g}}$ as close as we want to $Vol_{\mathbf{g}}(M)$. As

$$\frac{\left(\int_M f dv_{\mathbf{g}}\right)^{\frac{n-2}{n}}}{B_0(\mathbf{g})(\sup_M f)^{\frac{n-2}{n}}} > 0$$

we can find η satisfying (23) and a real t, denoted t_1, satisfying (21) and (22).

On the set $\{x : f(x) = \max f\}$, $h_t = B_0K^{-2} + \alpha$, so for all $P \in \{x : f(x) = \max f\}$

$$\frac{4(n-1)}{n-2}h_t(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

We then set

$$t_0 = Inf\{t \le t_1 : \lambda_{h_t} < \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}}\} .$$

Necessarily, $t_0 < t_1$. We recall that (see Section 1)

$$\lambda_{h,f,\mathbf{g}} = \lambda_h = \inf_{w\in\mathcal{H}_f} I_h(w).$$

Therefore,

$$\lambda_{h_{t_0}} = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}} \quad\text{and}\quad \lambda_{h_t} < \frac{1}{K(n,2)^2(\sup\limits_M f)^{\frac{n-2}{n}}} \quad\text{if}\quad t > t_0.$$

Furthermore, for all t, $t_0 \le t \le t_1$, $h_t > 0$ on M and

$$\frac{4(n-1)}{n-2}h_{t_0}(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)} \quad\text{for}\quad P \in \{x : f(x) = Maxf\} .$$

At last $h_t \underset{t\to t_0}{\to} h_{t_0}$ in C^0, and as $h_{t_0} > 0$, $\triangle_{\mathbf{g}} + h_{t_0}$ is coercive. Therefore by Theorem 1, h_{t_0} is critical and $(E_{h,f,\mathbf{g}})$ has minimizing solutions.

Remark. The preceding proofs also show, by replacing B_0K^{-2} by h, that if $(h, f, \mathbf{g})$ is weakly critical, and if

$$\frac{4(n-1)}{n-2}h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)} \text{ for } P \in \{x : f(x) = Max f\}$$

then there exists $h' \leqslant h$ such that(h', f, g) is critical.

If we only have

$$\frac{4(n-1)}{n-2}h(P) \geq S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)} \text{ for } P \in \{x : f(x) = Max f\},$$

then for any $\varepsilon > 0$ there exists $h' \leq h + \varepsilon$ such that (h', f, g) is critical.

Weaker hypotheses are sufficient to prove the existence of positive critical functions; for example, it suffices that the *boundary* of the set $Maxf$ be a set of null measure; see [8] for full details.

6. Critical triple 2

We want to prove here Theorem 3. This theorem relies on the transformation formula for a critical function in a conformal change of metric (seen at the end of the introduction)

$(h', f, \mathbf{g}' = u^{\frac{4}{n-2}}\mathbf{g})$ is critical if and only if $(h = h'u^{\frac{4}{n-2}} - \frac{\triangle_{\mathbf{g}} u}{u}, f, \mathbf{g})$ *is critical.*

We set, for $u \in C^\infty_+(M) = \{u \in C^\infty(M) : u > 0\}$, $F_{h'}(u) = h'u^{\frac{4}{n-2}} - \frac{\triangle_{\mathbf{g}} u}{u}$. Then

$$(h', f, \mathbf{g}') \textit{ is critical if and only if } (F_{h'}(u), f, \mathbf{g}) \textit{ is critical.}$$

To prove the theorem, we therefore have to prove the existence of a function h such that

1) $\triangle_{\mathbf{g}} u + h.u = h'u^{\frac{n+2}{n-2}}$ has a solution $u > 0$, and

2) $(h, f, \mathbf{g})$ is critical.

Indeed, in this case $h = F_{h'}(u)$ and h' is critical for f and $\mathbf{g}' = u^{\frac{4}{n-2}}\mathbf{g}$.

E. Humbert and M. Vaugon proved this theorem in the case $f = cste$ and for a manifold not conformaly diffeomorphic to the sphere [19]. Their method relies on the fact that for such a manifold, after a first conformal change of metric, $B_0(\mathbf{g})K(n, 2)^{-2}$ is a critical function (we will denote these two constants K and B_0). In fact, a careful study of their proof shows that what is needed is in fact that B_0K^{-2} be positive. But we proved in the previous section the existence of positive critical functions under a geometric hypothesis concerning f. Recall that our proof will work on the sphere, but only for a non-constant function f.

The principle of the proof of E. Humbert and M. Vaugon is the following. We know that there exists a sequence (h_t) of sub-critical functions for f and $\mathbf{g}$ such that $h_t \overset{C^2}{\to} h$ where $(h, f, \mathbf{g})$ is critical and such that for any point P where f is maximum on M

$$\frac{4(n-1)}{n-2} h(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}.$$

For a sequence $q_t \to 2^*$, $q_t < 2^*$ we build a sequence $u_t > 0$ of solutions of

$$\triangle_{\mathbf{g}} u + h.u = h' u^{q_t - 1} \text{ with } \int h' u_t^{q_t} dv_{\mathbf{g}} \leq C \text{ independant of } t$$

such that $u_t \overset{H_1^2}{\rightharpoonup} u \geqslant 0$. Here again, if $u > 0$, then u is solution (up to a multiplicative constant) of $\triangle_{\mathbf{g}} u + h.u = h' u^{\frac{n+2}{n-2}}$ and we are done.

Now, if $u = 0$, one shows that the u_t concentrate and that using this phenomenon, one can find a t_0 close to 1 (if e.g. $t \to 1$) and a real s large, such that $F_{h'}(u_{t_0})$ is sub-critical and $F_{h'}(u_{t_0}^s)$ is weakly critical, with furthermore

$$\frac{4(n-1)}{n-2} F_{h'}(u_{t_0}^s)(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$$

at any point P where f is maximum. Then, considering the path $t \to F_{h'}(u_{t_0}^{ts})$ and using Theorem 1, we get the existence of a critical function on this path. It is to obtain the conditions on $F_{h'}(u_{t_0}^s)$ at the maximum points of f that we need the existence of positive critical functions.

We will now give the scheme of the proof, referring for complete details to the article of E. Humbert and M. Vaugon or to our PhD thesis available online, and we will only indicate the modifications due to our function f and the necessity of having positive critical functions.

First, we said that we will need positive critical functions. Their existence was proved under the hypothesis that $Maxf$ is thin and that $\int_M f dv_{\mathbf{g}} > 0$. But $\sup_M f > 0$, so, after making if necessary a first conformal change of metric, we can suppose that $\int_M f dv_{\mathbf{g}} > 0$, and we supposed in the hypothesis of Theorem 3 that $Maxf$ is thin, and therefore we can suppose that we have positive critical function for f and $\mathbf{g}$.

Then, we fix some (more) notation

$$J_{h,h',\mathbf{g},q}(w) = \frac{\int_M |\nabla w|^2 dv_{\mathbf{g}} + \int_M h.w^2 dv_{\mathbf{g}}}{\left(\int_M h' |w|^q dv_{\mathbf{g}}\right)^{\frac{2}{q}}}; \qquad \inf_{w \in \mathcal{H}^+_{h',q}} J_{h,h',\mathbf{g},q}(w) := \lambda_{h,h',\mathbf{g},q}$$

where

$$\mathcal{H}^+_{h',q} = \{w \in H_1^2(M) : w > 0 \text{ and } \int_M h'.w^q dv_{\mathbf{g}} > 0\}$$

and

$$\Omega_{h,h',\mathbf{g},q} = \{u \in \mathcal{H}^+_{h',q} : J_{h,h',\mathbf{g},q}(u) = \lambda_{h,h',\mathbf{g},q},\ \int_M h'.w^q dv_{\mathbf{g}} = (\lambda_{h,h',\mathbf{g},q})^{\frac{q}{q-2}}\}.$$

Let (h_t) be a sequence of sub-critical functions for f and $\mathbf{g}$ such that $h_t \overset{C^2}{\to} h$ where $(h, f, \mathbf{g})$ is critical, with $\triangle_{\mathbf{g}} + h_t$ coercive. We know that we can find such a sequence with $h_t > 0$ and $h > 0$, and also

$$\frac{4(n-1)}{n-2} h_t(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$$

for all $P \in Maxf$. But here, we can say more, and that is where the existence of positive critical functions is crucial. Indeed, for any constant $c > 0$, if $\mathbf{g}' = c\mathbf{g}$, then $S_{\mathbf{g}'} = c^{-1} S_{\mathbf{g}}$ and $\triangle_{\mathbf{g}'} = c^{-1} \triangle_{\mathbf{g}}$ and by the transformation formula for critical functions

h is (sub-, weakly) critical for f and $\mathbf{g}$ if and only if $c^{-1}h$ is (sub-, weakly) critical for f and $\mathbf{g}'$.

Therefore, up to multiplying $\mathbf{g}$ by a constant, we can, for any constant $C > 0$, suppose

$$\begin{aligned} h_t &> C \text{ on } M \\ \frac{4(n-1)}{n-2} h_t(P) - S_{\mathbf{g}}(P) + \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)} &> C \ \ \forall P \in Maxf \end{aligned}$$

and $(h, f, \mathbf{g})$ has minimizing solutions.

We can now follow the method exposed above; we only give the scheme of the proof.

First step. Thanks to the compactness of the inclusion $H_1^2 \subset L^q$, it is known that for all $q < 2^*$ and $u \in \Omega_{h,h',\mathbf{g},q}$, u is a solution of $\triangle_{\mathbf{g}} u + h.u = h'u^{q-1}$. Using this fact, in the first step, one proves the following:

There exist sequences $(q_i), (t_i)$, such that $2 < q_i < 2^*$, $q_i \to 2^*$, $t_i \to 1$, $h_{t_i} \to h$ and a sequence $(v_i) \in \Omega_{h_{t_i},h',\mathbf{g},q_i}$ such that $(F_{h'}(v_i), f, \mathbf{g})$ is sub-critical. We note $J_i = J_{h_{t_i},h',\mathbf{g},q_i}$ and $\lambda_i = \lambda_{h_{t_i},h',\mathbf{g},q_i}$. Then

$$J_i(v_i) = \lambda_i \text{ and } \int h' v_i^{q_i} dv_{\mathbf{g}} = \lambda_i^{\frac{q_i}{q_i-2}}$$

and v_i is a positive solution of

$$\triangle_{\mathbf{g}} v_i + h_{t_i}.v_i = h' v_i^{q_i - 1}.$$

The sequence (v_i) is bounded in H_1^2 and thus there exists $v \in H_1^2$ such that $v_i \overset{H_1^2}{\rightharpoonup} v$, $v_i \overset{L^2}{\to} v$ and $v_i \overset{L^{2^*-2}}{\longrightarrow} v$. Once again, we have two possibilities, $v \equiv 0$ or $v > 0$.

Second step. If $v > 0$, as we said above, the proof is over, up to a subsequence, $v_i \overset{C^2}{\to} v$ and so on the one hand $F_{h'}(v_i) \to F_{h'}(v)$, and on the other hand

$$F_{h'}(v_i) = h_{t_i} + h'(v_i^{\frac{4}{n-2}} - v_i^{q_i-2}) \to h;$$

that is, $F_{h'}(v) = h$ which is critical for f and $\mathbf{g}$ with minimizing solutions. Thus h' is critical for f and $\mathbf{g}' = v^{\frac{4}{n-2}}\mathbf{g}$, with minimizing solutions.

The rest of the proof is therefore concerned with the case $v \equiv 0$.

Third step. One proves that there is a concentration phenomenon.

a) One first shows that $0 < c \leqslant \overline{\lim}\lambda_i \leqslant K^{-2}(\sup_M h')^{-\frac{n-2}{n}}$.

b) Second, one shows that

$$0 < \lambda^{\frac{n}{2}}(\sup_M h')^{-1} \leqslant \overline{\lim} \int_M v_i^{q_i} dv_{\mathbf{g}} \leqslant K^{2^*}\lambda^{\frac{n2^*}{4}} \leqslant K^{-n}(\sup_M h')^{-\frac{n}{2}},$$

where $\lambda > 0$ is such that, after extraction, $\lambda_i \to \lambda$.

c) We say that $x \in M$ is a concentration point if

$$\forall r > 0 \quad \overline{\lim} \int_{B(x,r)} v_i^{q_i} dv_{\mathbf{g}} > 0.$$

Using a), b), and methods analogous to section 4.2, one gets the following:

First, as M is compact, there exists at least one concentration point $x \in M$. Then, using the iteration process, one shows that

$$\overline{\lim} \int_{B(x,r)} v_i^{q_i} dv_{\mathbf{g}} \geqslant K^{-n}(\sup_M h')^{-\frac{n}{2}}.$$

d) Therefore using the method of section 4.2, we get

1) $\overline{\lim} \int_{B(x,r)} v_i^{q_i} dv_{\mathbf{g}} = K^{-n}(\sup_M h')^{-\frac{n}{2}}$, for all $r > 0$;

2) x is the only concentration point, denoted x_0;

3) $\lambda = K^{-2}(\sup_M h')^{-\frac{n-2}{n}}$;

4) x_0 is a point of maximum of h';

5) $v_i \to 0$ in $C^2_{loc}(M - \{x_0\})$.

Fourth step. We know now that the sequence (v_i) concentrates at x_0 and that for any i $F'_{h'}(v_i)$ is sub-critical for f and $\mathbf{g}$. We would like to find a v_{i_0}, a function $v > 0$ and a continuous path from v_{i_0} to v such that $F_{h'}(v)$ is weakly critical for f and $\mathbf{g}$ and such that

$$\frac{4(n-1)}{n-2}F_{h'}(v)(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$$

for all $P \in Maxf$. Then, Theorem 1 will tell us that on the path u_t from v_{i_0} to v there exists a u_t such that $F_{h'}(u_t)$ is critical for f and $\mathbf{g}$.

That is where we are going to use the existence of positive critical functions.

Let $s \geqslant 1$ and let v be a positive function. Then

$$\triangle_{\mathbf{g}}(v^s) = sv^{s-1} \triangle_{\mathbf{g}} v - s(s-1)v^{s-2} |\nabla v|^2_{\mathbf{g}}.$$

Thus,

$$F_{h'}(v_i^s) = h' v_i^{s\frac{4}{n-2}} + sh_{t_i} - sh' v_i^{q_i-2} + s(s-1)\frac{|\nabla v|^2_{\mathbf{g}}}{v_i^2}$$

and therefore

$$F_{h'}(v_i^s) \geqslant sh_{t_i} + h'(v_i^{s\frac{4}{n-2}} - sv_i^{q_i-2}).$$

Now on $\{x \in M : h'(x) \leqslant 0\}$, $v_i \to 0$ uniformly because $x_0 \in Max\, h'$ and $h'(x_0) > 0$ as we have supposed that $\triangle_{\mathbf{g}} + h'$ is coercive. Furthermore if $s \geqslant 1$, then $s\frac{4}{n-2} \geqslant q_i - 2$. Thus, for i large enough $F_{h'}(v_i^s) \geqslant sh_{t_i}$ on $\{x \in M : h'(x) \leqslant 0\}$ and on $\{x \in M : h'(x) > 0\}$.

We consider the function of a real variable defined for $x \geqslant 0$ by

$$\beta_{i,s}(x) = x^{s\frac{4}{n-2}} - sx^{q_i-2} = x^{q_i-2}(x^{s\frac{4}{n-2}-q_i+2} - s).$$

An easy study of this function shows that, for $x \geqslant 0, \beta_{i,s}(x) \geqslant -s$. But $F_{h'}(v_i^s) \geqslant sh_{t_i} + h'\beta_{i,s}(v_i)$, therefore,

$$F_{h'}(v_i^s) \geqslant sh_{t_i} - sh' \text{ on } \{x \in M : h'(x) > 0\}.$$

We can therefore write

$$F_{h'}(v_i^s) \geqslant s(h_{t_i} - \sup_M h') \text{ on } \{x \in M : h'(x) > 0\}.$$

We now use our work from the beginning of the proof, which says that, for any $C > 0$, we can suppose that $h_t > C$ on M and

$$\frac{4(n-1)}{n-2}h_t(P) - S_{\mathbf{g}}(P) + \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)} > C, \quad \forall P \in Maxf.$$

Then, first, if we suppose that $h > \sup_M h'$ on M, we see that for i and s large enough

$$F_{h'}(v_i^s) \geqslant B_0(\mathbf{g})K(n,2)^{-2} \tag{24}$$

and therefore $F_{h'}(v_i^s)$ is weakly critical for f and $\mathbf{g}$. Beside, for all $t \in [1, s]$ we also have

$$F_{h'}(v_i^t) \geqslant t(h_{t_i} - \sup_M h') \geqslant h_{t_i} - \sup_M h' > 0$$

so $\triangle_{\mathbf{g}} + F_{h'}(v_i^t)$ is coercive.

Secondly, if we also suppose that

$$\frac{4(n-1)}{n-2} h_t(P) - S_{\mathbf{g}}(P) + \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)} > \frac{4(n-1)}{n-2} \sup_M h' \quad \forall P \in Maxf$$

we have for all $t \in [1, s]$

$$\frac{4(n-1)}{n-2} F_{h'}(v_i^t)(P) > S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)} \quad \forall P \in Maxf \tag{25}$$

as soon as i is large enough.

We therefore fix i and s large enough to have (24) and (25) and we consider

$$s_0 = \inf\{t > 1 : F_{h'}(v_i^t) \text{ is weakly critical}\}.$$

We then apply Theorem 1 to the path $t \in [1, s_0] \mapsto F_{h'}(v_i^t)$ to obtain that $F_{h'}(v_i^{s_0})$ is critical for f and $\mathbf{g}$, with minimizing solutions. Therefore h' is critical for f and $\mathbf{g}' = (v_i^{s_0})^{\frac{4}{n-2}} \mathbf{g}$ with minimizing solutions.

This ends the proof.

7. Critical triple 3

Let $(M, \mathbf{g})$ be a compact Riemannian manifold of dimension $n \geqslant 3$. Let h be a fixed C^∞ function such that $\triangle_{\mathbf{g}} + h$ is coercive. The problem we want to study is the following: *can we find a function f such that $(h, f, \mathbf{g})$ is a critical triple ?*

We first make a remark. If $h \geqslant B_0(\mathbf{g}) K(n,2)^{-2}$, then h is weakly critical for any function f, and there cannot exist a function f such that $(h, f, \mathbf{g})$ is subcritical. But more important is the next observation:

If there exists a non-constant function f such that $(h, f, \mathbf{g})$ is critical with a minimizing solution u, then $(h, 1, \mathbf{g})$ is sub-critical.

Indeed, as we saw in section 1, we can suppose that $\sup f = 1$. Then, as $u > 0$

$$J_{h,1}(u) < J_{h,f}(u) = \frac{1}{K(n,2)^2 (\sup f)^{\frac{2}{2^*}}} = \frac{1}{K(n,2)^2}$$

and therefore h is subcritical for 1.

We want to prove that, at least if $dimM \geq 5$, this necessary condition is sufficient; i.e we want to prove Theorem 4. We thus suppose now that $(h, 1, \mathbf{g})$ *is sub-critical.*

The proof will proceed in two steps:

First step. We prove that there exists a function $f \in C^\infty(M)$ such that $\sup_M f = 1$, with $\triangle_{\mathbf{g}} f$ being as large as we want in its maximum points, and such that $(h, f, \mathbf{g})$ is weakly critical.

Second step. Being given this function f, we prove that there exists on the path $t \to f_t = t.1 + (1-t)f$ a function for which h is critical.

First step. We proceed by contradiction. We suppose that for any $f \in C^\infty(M)$ such that $\sup_M f > 0$, $(h, f, \mathbf{g})$ is sub-critical. Then, for all such functions, there exists a positive solution u to the equation

$$\triangle_{\mathbf{g}} u + h.u = \lambda.f.u^{\frac{n+2}{n-2}},$$

where

$$\lambda = \inf_{w \in \mathcal{H}_f} I_{h,\mathbf{g}}(w) \ \ and \ \ \int_M f.u^{\frac{2n}{n-2}} dv_{\mathbf{g}} = 1.$$

The metric $\mathbf{g}$ being fixed, we will not write $dv_{\mathbf{g}}$ in the integrals.

The idea is to build a family of functions f_t whose Laplacians tend to infinity at the maximum points. One of these function will then give a weakly critical triple $(h, f_t, \mathbf{g})$. Furthermore, our proof holding for any subsequence of this family, this function will have a Laplacian as large as we want in its point of maximum.

In $\mathbb{R}^n$, we build for $t \to 0$ a family (P_t) of C^∞ functions, similar to a regularizing sequence, such that

$$0 \leqslant P_t \leqslant 1, \quad P_t(x) = P_t(|x|), \quad P_t(0) = 1, \quad \|\nabla P_t\| \sim \frac{c_1}{t} \ \text{ on } \ B(0,t)$$

$$|\triangle P_t(0)| \sim \frac{c_2}{t^2}, \quad Supp\, P_t = B(0,t).$$

Let now x_0 be a point of M such that $h(x_0) > 0$; this point exists because $\triangle_{\mathbf{g}} + h$ is coercive. We define

$$f_t = P_t \circ \exp_{x_0}^{-1}.$$

We are therefore supposing that, for all t, $(h, f_t, \mathbf{g})$ is sub-critical and we are looking for a contradiction. For all t we have a solution $u_t > 0$ of

$$(E_t): \ \triangle_{\mathbf{g}} u_t + h.u_t = \lambda_t.f_t.u_t^{\frac{n+2}{n-2}}$$

with $\int f_t u_t^{2^*} dv_{\mathbf{g}} = 1$ and $\lambda_t < K^{-2}(\sup_M f_t)^{-\frac{n-2}{n}} = K^{-2}$. Then, (u_t) is bounded in $H_1^2(M)$ when $t \to 0$. So (u_t) is bounded in L^{2^*} and $(u_t^{2^*-1})$

is bounded in $L^{\frac{2^*}{2^*-1}}$. After extraction of a subsequence, if $f_t \xrightarrow{L^2} f$ and $u_t \xrightarrow{L^2} u$, then $f_t u_t^{2^*-1} \rightharpoonup f u^{2^*-1}$. But here, $f_t \xrightarrow{L^p} 0$, therefore the equation (E_t) "converges" to $\triangle_{\mathbf{g}} u + h.u = 0$ in the sense that u is a solution of this equation. But $\triangle_{\mathbf{g}} + h$ is coercive, therefore $u = 0$; i.e., $u_t \to 0$ in L^p for $p < 2^*$.

The sequence (u_t) therefore concentrates in the sense we saw in subsection 4.2. But in subsection 4.2, the function f on the right-hand side of the equation was constant and it was on the left-hand side that we had a sequence (h_t). However the results we saw there remain true, only the blow-up necessary for the weak estimates requires a new treatment. We will go over these results, only detailing the new difficulties.

a) There exists, up to a subsequence of (u_t), *exactly one concentration point and it is the point* x_0 *where the* f_t *are maximum on* M. *Moreover,*

$$\forall \delta > 0,\ \overline{\lim_{t\to 1}} \int_{B(x_0,\delta)} f_t u_t^{2^*} = 1.$$

The method of subsection 4.2 works here. More precisely, as $Supp\, f_t = B(x_0, t)$, we have for all $\delta > 0$ and as soon as $t < \delta$

$$\int_{B(x_0,\delta)} f_t u_t^{2^*} = 1.$$

We can also suppose that

$$\lambda_t \to \lambda = K^{-2} (\sup_M f_t)^{-\frac{n-2}{n}} = K^{-2}.$$

b) $u_t \to 0$ *in* $C^0_{loc}(M - \{x_0\})$. The proof is the same as in subsection 4.2.

c) weak estimates. We consider a sequence of points (x_t) such that

$$m_t = \underset{M}{Max}\, u_t = u_t(x_t) := \mu_t^{-\frac{n-2}{2}}.$$

From the previous point, $x_t \to x_0$ and $\mu_t \to 0$. Remember that $\overline{u}_t, \overline{f}_t, \overline{h}_t, \mathbf{g}_t$ are the functions and the metric seen in the chart $\exp_{x_t}^{-1}$, and $\widetilde{u}_t$, $\widetilde{h}_t$, $\widetilde{f}_t, \widetilde{\mathbf{g}}_t$ are the functions after blow-up with center x_t and coefficient $k_t = \mu_t^{-1}$.

Reviewing the proof of the weak estimates in section 4.2, we see that it will work here if we obtain

$$\forall R > 0\ \ : \lim_{t\to 0} \int_{B(x_t, R\mu_t)} f_t u_t^{2^*} dv_{\mathbf{g}} = 1 - \varepsilon_R \ \textit{ where }\ \varepsilon_R \underset{R\to+\infty}{\to} 0.$$

This relation is itself proved using blow-up theory once it is proved that $\widetilde{u}_t \overset{C^2_{loc}(\mathbb{R}^n)}{\to} \widetilde{u}$ where $\widetilde{u}$ is a solution of

$$\triangle_e \widetilde{u} = K^{-2} \widetilde{u}^{\frac{n+2}{n-2}}.$$

This is where we have the main difficulty due to the presence of a family (f_t). Indeed, after blow-up, the equation

$$(E_t):\ \triangle_{\mathbf{g}} u_t + h.u_t = \lambda_t.f_t.u_t^{\frac{n+2}{n-2}}$$

becomes

$$(\widetilde{E}_t)\ :\ \triangle_{\widetilde{\mathbf{g}}_t} \widetilde{u}_t + \mu_t^2.\widetilde{h}_t.\widetilde{u}_t = \lambda_t \widetilde{f}_t.\widetilde{u}_t^{\frac{n+2}{n-2}}$$

and to obtain that this equation "converges" to

$$\triangle_e \widetilde{u} = K^{-2} \widetilde{u}^{\frac{n+2}{n-2}}$$

we need to show that $(\widetilde{f}_t)$ is simply convergent to 1 (which is obvious when we have a constant function f on the right-hand side of $(E_{h,f,\mathbf{g}})$). As the sequence $(\widetilde{f}_t)$ is uniformly bounded by 1 on $\mathbb{R}^n$ (considering we have extended $\widetilde{f}_t$ by 0 on $\mathbb{R}^n \backslash B(0, \delta\mu_t^{-1})$), we have, using e.g. Theorem 8.25 of Gilbard-Trudinger [16] and Ascoli's theorem, the existence of a function $\widetilde{u} \in C^0(\mathbb{R}^n)$ such that, after extraction, $\widetilde{u}_t \overset{C^0_{loc}(\mathbb{R}^n)}{\to} \widetilde{u}$, with $\widetilde{u}(0) = 1$.

We are going to prove that $\widetilde{f}_t \overset{a.e.}{\to} 1$ on $\mathbb{R}^n$ in two steps (we will prove a little bit more):

1) There exists $\widetilde{f} \in L^2_{loc}(\mathbb{R}^n)$ such that $\widetilde{f}_t \overset{a.e.}{\to} \widetilde{f}$ on $\mathbb{R}^n$.

2) $\widetilde{f} = 1$ almost everywhere on $\mathbb{R}^n$.

First step. We have $\widetilde{f}_t(x) = \overline{f}_t(\mu_t x)$ and $\left|\nabla \overline{f}_t\right| \leq \frac{c}{t}$. Therefore

$$|\nabla \widetilde{f}_t| \leq c.\frac{\mu_t}{t}.$$

We consider two cases:

a) If $(\frac{\mu_t}{t})$ is bounded, then for any compact set $K \subset\subset \mathbb{R}^n$, $(\widetilde{f}_t)$ is bounded in $H_1^{n+1}(K)$ (where $n = \dim M$). Thus, by compactness of the inclusion $H_1^{n+1}(K) \subset C^{0,\alpha}(K)$ for some $\alpha > 0$, up to a subsequence, there exists $\widetilde{f}_K \in C^{0,\alpha}(K)$ such that $\widetilde{f}_t \overset{C^{0,\alpha}(K)}{\to} \widetilde{f}_K$. By diagonal extraction, we construct $\widetilde{f} \in C^{0,\alpha}(\mathbb{R}^n)$ such that $\widetilde{f}_t \overset{C^{0,\alpha}(K')}{\to} \widetilde{f}$ for any compact set K' of $\mathbb{R}^n$, and moreover $\widetilde{f} \in H^{n+1}_{1,loc}(\mathbb{R}^n)$. So $\widetilde{f}_t \overset{a.e.}{\to} \widetilde{f}$ *on* $\mathbb{R}^n$.

b) If $\frac{\mu_t}{t} \to +\infty$, the support of $\widetilde{f}_t$ is

$$Supp \widetilde{f}_t = B(\frac{x_0(t)}{\mu_t}, \frac{t}{\mu_t}),$$

where $x_0(t) = \exp_{x_t}^{-1}(x_0)$.

If $(\frac{|x_0(t)|}{\mu_t})$ is bounded, there is after extraction a subsequence $\frac{x_0(t)}{\mu_t} \to P \in \mathbb{R}^n$; and therefore, $\widetilde{f}_t \overset{C^0_{loc}(\mathbb{R}^n - \{P\})}{\to} 0$. If $\frac{|x_0(t)|}{\mu_t} \to \infty$, then $\widetilde{f}_t \overset{C^0_{loc}(\mathbb{R}^n)}{\to} 0$ In both cases, $\widetilde{f}_t \overset{p.p}{\to} 0$ *on* $\mathbb{R}^n$.

In case a, $\widetilde{u}$ is a weak solution of

$$\triangle_e \widetilde{u} = K^{-2} \widetilde{f} \widetilde{u}^{\frac{n+2}{n-2}}$$

with $\widetilde{f} \geq 0$ as $\widetilde{f}_t \geq 0$, and $\widetilde{f} \in H^{n+1}_{1,loc}(\mathbb{R}^n) \subset C^{0,\alpha}(\mathbb{R}^n)$.

In case b, $\widetilde{u}$ is a weak solution of

$$\triangle_e \widetilde{u} = 0.$$

In both cases, elliptic theory and standard regularity theorems give the C^2 regularity of $\widetilde{u}$, and therefore $\triangle_e \widetilde{u} \geq 0$. The maximum principle then shows that either $\widetilde{u} \equiv 0$ or $\widetilde{u} > 0$. But $\widetilde{u}(0) = 1$ thus $\widetilde{u} > 0$.

Second step. We start using the iteration process: for some cut-off function η equal to 1 near x_0, we multiply (E_t) by $\eta^2 u_t$, integrate and use the Sobolev inequality to obtain, remembering that $\lambda_t < K^{-2}(\sup_M f_t)^{-\frac{n-2}{n}}$ and that $\sup f_t = 1$,

$$(\int_M (\eta u_t)^{2^*})^{\frac{2}{2^*}} \leq \lambda_t K^2 \int_M \eta^2 f_t u_t^{2^*} + c \int_{Supp\,\eta} u_t^2.$$

We take $\eta = 1$ on $B(x_0, \frac{3}{2}\delta)$ and $\eta = 0$ on $M \backslash B(x_0, 2\delta)$. Then for t close to 0 $Supp\, f_t \subset B(x_0, t) \subset B(x_t, \delta) \subset B(x_0, \frac{3}{2}\delta)$. So

$$(\int_{B(x_t,\delta)} {u_t}^{2^*})^{\frac{2}{2^*}} \leq \int_{B(x_t,\delta)} f_t u_t^{2^*} + c \int_M u_t^2$$

and after blow-up

$$(\int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^{2^*})^{\frac{2}{2^*}} \leq \int_{B(0,\delta\mu_t^{-1})} \widetilde{f}_t \widetilde{u}_t^{2^*} + c \int_M u_t^2 = 1 + c \int_M u_t^2.$$

But $\int_M u_t^2 \to 0$, therefore,

$$\overline{\lim_{t \to 0}} \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^{2^*} \leq 1.$$

Beside, we know that $\widetilde{f}_t \overset{a.e.}{\to} \widetilde{f}$ with $\widetilde{f} \leq 1$ and $\widetilde{u}_t(0) = 1$. Let us suppose that there exists a set $A \subset \mathbb{R}^n$ with $mes(A) > 0$ such that $\widetilde{f} < 1$ on A and write $\mathbb{R}^n = A \cup B$ with $\widetilde{f} = 1$ almost everywhere on B. Then, as $\widetilde{f}_t \geq 0$ and as $\widetilde{u}_t \overset{C^2}{\to} \widetilde{u} > 0$,

$$\begin{aligned} 1 = \int_{B(0,\delta\mu_t^{-1})} \widetilde{f}_t \widetilde{u}_t^{2^*} &\leqslant \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})\cap A} \widetilde{f}_t \widetilde{u}_t^{2^*} + \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})\cap B} \widetilde{f}_t \widetilde{u}_t^{2^*} \\ &< \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})\cap A} \widetilde{u}_t^{2^*} + \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})\cap B} \widetilde{u}_t^{2^*} = \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^{2^*} \end{aligned}$$

so

$$1 < \overline{\lim_{t\to 0}} \int_{B(0,\delta\mu_t^{-1})} \widetilde{u}_t^{2^*}$$

which is a contradiction, and therefore $\widetilde{f}_t \overset{a.e.}{\to} 1$ on $\mathbb{R}^n$.

Thus, as we said, $(\widetilde{E}_t) \ : \ \triangle_{\widetilde{\mathbf{g}}_t} \widetilde{u}_t + \mu_t^2 . \widetilde{h}_t . \widetilde{u}_t = \lambda_t \widetilde{f}_t . \widetilde{u}_t^{\frac{n+2}{n-2}}$ "converges" to $\triangle_e \widetilde{u} = K^{-2} \widetilde{u}^{\frac{n+2}{n-2}}$ in the sense that $\widetilde{u}_t \overset{C^2_{loc}(\mathbb{R}^n)}{\to} \widetilde{u}$, where $\widetilde{u}$ is a solution of $\triangle_e \widetilde{u} = K^{-2} \widetilde{u}^{\frac{n+2}{n-2}}$. As $\widetilde{u}(0) = 1$, $\widetilde{u}(x) = (1 + \frac{K^{-2}}{n(n-2)} |x|^2)^{-\frac{n-2}{2}}$. Now, we can proceed exactly as in subsection 4.2. We have

$$\forall R > 0 : \lim_{t\to 0} \int_{B(x_t, R\mu_t)} f_t u_t^{2^*} dv_{\mathbf{g}} = 1 - \varepsilon_R \ \ where \ \ \varepsilon_R \underset{R\to+\infty}{\to} 0,$$

then there exists $C > 0$ such that for all $x \in M \ : \ d_{\mathbf{g}}(x, x_t)^{\frac{n-2}{2}} u_t(x) \leq C$ and for all $\varepsilon > 0$, there exists $R > 0$ such that for all t, for all $x \in M$: $d_{\mathbf{g}}(x, x_t) \geq R\mu_t \Rightarrow d_{\mathbf{g}}(x, x_t)^{\frac{n-2}{2}} u_t(x) \leq \varepsilon$.

d) We have here again the L^2-concentration. If $\dim M \geq 4$,

$$\forall \delta > 0 \ : \ \lim_{t\to 0} \frac{\int_{B(x_0,\delta)} u_t^2 dv_{\mathbf{g}}}{\int_M u_t^2 dv_{\mathbf{g}}} = 1.$$

e) We also have the strong estimates. For $0 < \nu < \frac{n-2}{2}$, there exists $C(\nu) > 0$ such that for all $x \in M \ : \ d_{\mathbf{g}}(x, x_t)^{n-2-\nu} \mu_t^{-\frac{n-2}{2}+\nu} u_t(x) \leq C$, and therefore the strong L^p-concentration: for all $R > 0, \delta > 0$ and $p > \frac{n}{n-2}$ where $n = \dim M$

$$\lim_{t\to 0} \frac{\int_{B(x_t, R\mu_t)} u_t^p dv_{\mathbf{g}}}{\int_{B(x_t,\delta)} u_t^p dv_{\mathbf{g}}} = 1 - \varepsilon_R \quad \text{where} \quad \varepsilon_R \underset{R\to+\infty}{\to} 0.$$

We can now proceed with the central part of the proof of Theorem 4.

We consider the Euclidean Sobolev inequality and equation (E_t) viewed in the chart $\exp_{x_t}^{-1}$. Using the same computations as in subsection 4.3, we get

$$\int_{B(0,\delta)} \overline{h}_t(\eta\overline{u}_t)^2 dx \le \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}} \int_{B(0,\delta)} \overline{f}_t \eta^2 \overline{u}_t^{2^*} dx$$
$$- \frac{1}{K(n,2)^2}(\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx)^{\frac{2}{2^*}} + C\delta^{-2} \int_{B(0,\delta)\backslash B(0,\delta/2)} \overline{u}_t^2 dx + B_t + C_t$$

with

$$B_t = \frac{1}{2} \int_{B(0,\delta)} (\partial_k(\mathbf{g}_t^{ij}\Gamma(\mathbf{g}_t)_{ij}^k + \partial_{ij}\mathbf{g}_t^{ij})(\eta\overline{u}_t^2)dx,$$
$$C_t = \Big| \int_{B(0,\delta)} \eta^2(\mathbf{g}_t^{ij} - \delta^{ij})\partial_i\overline{u}_t\partial_j\overline{u}_t dx\Big|,$$
$$A_t = \frac{1}{K(n,2)^2(\sup_M f)^{\frac{n-2}{n}}} \int_{B(0,\delta)} \overline{f}_t\eta^2\overline{u}_t^{2^*} dx - \frac{1}{K(n,2)^2}(\int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx)^{\frac{2}{2^*}}.$$

We can write

$$A_t \le \frac{1}{K(n,2)^2(\sup_M f_t)^{\frac{n-2}{n}}}(A_t^1 + A_t^2),$$

where $A_t^1 = (\int_{B(0,\delta)} \overline{f}_t(\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}} - (\sup f_t. \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}}$;
from the computation of subsection 4.3,

$$\overline{\lim_{t\to 0}} \frac{K(n,2)^{-2}(\sup_M f_t)^{-\frac{n-2}{n}} A_t^2 + C\delta^{-2}\int_{B(0,\delta)\backslash B(0,\delta/2)} \overline{u}_t^2 dx + B_t + C_t}{\int_{B(0,\delta)} \overline{u}_t^2 dx}$$
$$\le \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) + \varepsilon_\delta,$$

where $\varepsilon_\delta \to 0$ when $\delta \to 0$. We now consider

$$\overline{\lim_{t\to 0}} \frac{A_t^1}{\int_{B(0,\delta)} \overline{u}_t^2 dx}.$$

We remark that from its definition, f_t is decreasing when $t \to 0$ in the sense that if $t \le t'$, then $f_t \le f_{t'}$. We fix a t_0. Then, for any $t \le t_0$

$$\int_{B(0,\delta)} \overline{f}_t(\eta\overline{u}_t)^{2^*} dx = \int_{B(x_t,\delta)} f_t.(\eta \circ \exp_{x_t}^{-1})^{2^*}.u_t^{2^*}.(\exp_{x_t}^{-1})^* dx$$

$$\leq \int_{B(x_t,\delta)} f_{t_0}.(\eta \circ \exp_{x_t}^{-1})^{2^*}.u_t^{2^*}.(\exp_{x_t}^{-1})^* dx = \int_{B(0,\delta)} (f_{t_0} \circ \exp_{x_t})(\eta \overline{u}_t)^{2^*} dx.$$

We note $\overline{f}_{t_0,t} = f_{t_0} \circ \exp_{x_t}$ and $\widetilde{f}_{t_0,t} = \overline{f}_{t_0,t} \circ \psi^{-1}_{\mu_t^{-1}}$. Then

$$A_t^1 \leq (\int_{B(0,\delta)} \overline{f}_{t_0,t}(\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}} - (\sup f_t. \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}}$$
$$\leq (\int_{B(0,\delta)} \overline{f}_{t_0,t}(\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}} - (\sup f_{t_0}. \int_{B(0,\delta)} (\eta\overline{u}_t)^{2^*} dx)^{\frac{n-2}{n}}$$

as $\sup f_t = \sup f_{t_0} = 1 = f_{t_0}(x_0)$ for all t.

We therefore obtain by the same method as that of section 4.3

$$\overline{\lim_{t\to 0}} \frac{A_t^1}{\int_{B(0,\delta)} \overline{u}_t^2 dv_{\mathbf{g}_t}} \leq -\frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)} + \varepsilon_\delta$$

and thus, after letting δ tend to 0, we obtain

$$h(x_0) \leq \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)}.$$

But

$$\triangle_{\mathbf{g}} f_t(x_0) \sim +\frac{c}{t^2} \underset{t\to 0}{\to} +\infty$$

so taking t_0 close to 0 we obtain a contradiction.

This proves that we can find in the sequence (f_t) functions with Laplacian in x_0, $\triangle_{\mathbf{g}} f_t(x_0)$ as large as we want such that the equations $\triangle_{\mathbf{g}} u + h.u = f_t.u^{\frac{n+2}{n-2}}$ do *not* have minimizing solutions and therefore such that h is weakly critical for f_t and $\mathbf{g}$.

Remark 1. We also have in this setting the analog of Theorem 6 on the speed of convergence of (x_t) to x_0.

Remark 2. This can be applied to $h = cste < B_0 K^{-2}$ or to $h = S_{\mathbf{g}}$ if M is not the sphere.

Second step. For our function h such that $(h, 1, \mathbf{g})$ is subcritical, we know now that there exists a function f, with a Laplacian as large as we want at its maximum points, such that $(h, f, \mathbf{g})$ is weakly critical. More precisely, we found a function f such that

1) $(h, f, \mathbf{g})$ is weakly critical,

2) $h(x_0) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$, where

a) $h(x_0) > 0$,

b) $\{x_0\} = \{x : f(x) = \sup_M f\}$ and $f(x_0) = 1$, $0 \leq f \leq 1$, $Supp\, f = B(x_0, r)$,

c) $\nabla^2 f(x_0) < 0$.

We now consider the path $t \to f_t = (1-t).1 + t.f$. Recall that for all t: $\triangle_{\mathbf{g}} f_t = t \triangle_{\mathbf{g}} f$ and $f_t(x_0) = 1 = \sup_M f_t$. We set $\lambda_t = Inf\, J_{h,f_t,\mathbf{g}}$. Then

$$\lambda_0 < K(n,2)^{-2} (\sup_M f_0)^{-\frac{n-2}{n}}$$

because $(h, 1, \mathbf{g})$ is sub-critical and

$$\lambda_1 = K(n,2)^{-2} (\sup_M f_1)^{-\frac{n-2}{n}}$$

as $(h, f, \mathbf{g})$ is weakly critical. Recall that $\sup_M f_t$ is always equal to 1. Let

$$t_0 = \sup\{t : \lambda_t < K(n,2)^{-2} (\sup_M f_t)^{-\frac{n-2}{n}}\}.$$

Then $0 < t_0 \le 1$ and $\lambda_{t_0} = K(n,2)^{-2} (\sup_M f_{t_0})^{-\frac{n-2}{n}}$. Before applying the method of section 4.3, we need to prove one more thing: as h is weakly critical for f_{t_0}, we know that at the maximum point x_0 we have

$$h(x_0) \ge \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)}$$

because $\frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)} = t_0 \frac{\triangle_{\mathbf{g}} f(x_0)}{f(x_0)}$ with $t_0 \le 1$, but we need a strict inequality.

We consider the sequence (f_i), that we can construct using the first step; f_i is such that $(h, f_i, \mathbf{g})$ is weakly critical with

$$f_i(x_0) = 1 = \sup f_i \ \ et \ \ \triangle_{\mathbf{g}} f_i(x_0) \to +\infty.$$

For each f_i, we denote by t_i the "t_0" built above. Therefore, for any i, h is weakly critical for $(1-t_i).1 + t_i.f_i$ and $\mathbf{g}$. Suppose that liminf $t_i = 0$, or, after extracting, that $t_i \to 0$. Then, $(1-t_i).1 + t_i.f_i \to 1$ uniformly on M as $0 \le f_i \le 1$. But $(h, 1, \mathbf{g})$ is sub-critical, thus there exists $u \in H_1^2(M)$ such that

$$\frac{\int |\nabla u|^2 + \int h u^2}{(\int u^{2^*})^{\frac{2}{2^*}}} < K(n,2)^{-2}.$$

But then

$$\frac{\int |\nabla u|^2 + \int h u^2}{(\int ((1-t_i).1 + t_i.f_i) u^{2^*})^{\frac{2}{2^*}}} \to \frac{\int |\nabla u|^2 + \int h u^2}{(\int u^{2^*})^{\frac{2}{2^*}}} < K(n,2)^{-2}$$

whereas

$$K(n,2)^{-2} = K(n,2)^{-2} (\sup_M ((1-t_i).1 + t_i.f_i))^{-\frac{n-2}{n}}$$

which contradicts the fact that $(h, (1-t_i).1 + t_i.f_i, \mathbf{g})$ is weakly critical.

Therefore, up to extraction, $t_i \to t_1 > 0$.

As $\triangle_{\mathbf{g}} f_i(x_0) \to +\infty$, we can find i large enough so that

$$\frac{(n-2)(n-4)}{8(n-1)} t_i \frac{\triangle_{\mathbf{g}} f_i(x_0)}{f_i(x_0)} > \frac{n-2}{4(n-1)} S_g(x_0) - h(x_0).$$

If we now denote by f this last function f_i and t_0 this t_i, we get a path $t \to f_t = (1-t).1 + t.f$ such that

a) for all $t < t_0$, $(h, f_t, \mathbf{g})$ is sub-critical,

b) $(h, f_{t_0}, \mathbf{g})$ is weakly critical with

b1) $\{x_0\} = \{x : f_t(x) = \sup_M f_t\}$ and $f_t(x_0) = 1$ for all t

b2) $h(x_0) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)}$

b3) $\nabla^2 f_{t_0}(x_0) < 0$.

For any $t < t_0$ there exists a minimizing solution u_t of the equation

$$\triangle_{\mathbf{g}} u_t + h.u_t = \lambda_t.f_t.u_t^{\frac{n+2}{n-2}}$$

with $\int f_t u_t^{2^*} = 1$. The sequence (u_t) is bounded in H_1^2, therefore

$$u_t \underset{t \to t_0}{\overset{H_1^2}{\rightharpoonup}} u$$

and we are once again in the situation where

- either $u > 0$ and then u is a minimizing solution of $\triangle_{\mathbf{g}} u + h.u = \lambda_{t_0} f_{t_0}.u^{\frac{n+2}{n-2}}$, and therefore $(h, f_{t_0}, \mathbf{g})$ is critical; or

- either $u \equiv 0$ and once again the sequence (u_t) concentrates. In this case, the study of the concentration phenomenon is easier than in the first step as the family (f_t) tends uniformly to f when $t \to t_0$ with $Supp\, f_t = B(x_0, r)$. We can find $\delta < r$ such that $f > 0$ on $B(x_0, \delta)$. Then there exists $c > 0$ such that for any t we have $0 < c \leq f_t \leq 1$ on $B(x_0, \delta)$, Furthermore, the f_t all reach their maximum at x_0, this maximum being always 1. We can then go over all the results and methods of section 4.3, the functions f_t bringing this time no changes. We finally obtain

$$h(x_0) \leq \frac{n-2}{4(n-1)} S_g(x_0) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f_{t_0}(x_0)}{f_{t_0}(x_0)},$$

thus, a contradiction. Therefore $(h, f_{t_0}, \mathbf{g})$ is critical with a minimizing solution.

This proof in fact shows the following result:

Theorem 4′. *If h is weakly critical for a function f and a metric $\mathbf{g}$, these data satisfy:*

1) $h(x) > \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x) - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_{\mathbf{g}} f(x)}{f(x)}$ *at the maximum points of* f;

2) $\nabla^2 f(x) < 0$ *at the maximum points of* f;

3) *there exists a sequence* $f_t \underset{t\to t_0}{\overset{C^2}{\to}} f$ *with* $\sup_M f_t = \sup_M f$ *such that* $(h, f_t, \mathbf{g})$ *is subcritical for* $t < t_0$, *then* $(h, f, \mathbf{g})$ *is critical and has minimizing solutions.*

As we said in the introduction, this leads to another, dual, definition of critical functions, definition 3. The natural question is then

Is f critical for h if and only if h is critical for f?

Recall that in both cases, if P is a point where f is maximum on M : $\frac{4(n-1)}{n-2}h(P) \geqslant S_{\mathbf{g}}(P) - \frac{n-4}{2}\frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$.

This problem seems difficult. We prove here the result we obtain, Theorem 5.

The proof starts with the following remark. We have seen that if h is weakly critical for f and $\mathbf{g}$ and that $\triangle_{\mathbf{g}} u + h.u = f.u^{\frac{n+2}{n-2}}$ has a minimizing solution, then h is critical for f and $\mathbf{g}$. In the same way, if f is weakly critical for h (in the sense that $\lambda_{h,f,\mathbf{g}} = K(n,2)^{-2}(\sup_M f)^{-\frac{n-2}{n}}$) and if $\triangle_{\mathbf{g}} u + h.u = f.u^{\frac{n+2}{n-2}}$ has a minimizing solution $u > 0$, then f is critical for h. Indeed, if f' is a function such that $\sup f = \sup f'$ and $f' \gneqq f$, we have

$$\int f' u^{2^*} > \int f u^{2^*}$$

because $u > 0$. Therefore,

$$J_{h,f',\mathbf{g}}(u) < J_{h,f,\mathbf{g}}(u) = K(n,2)^{-2}(\sup_M f)^{-\frac{n-2}{n}} = K(n,2)^{-2}(\sup_M f')^{-\frac{n-2}{n}}.$$

Using our work of section 4.3 and this section, the proof is now short:

-If h is critical for f, we apply Theorem 1, $\triangle_{\mathbf{g}} u + h.u = f.u^{\frac{n+2}{n-2}}$ has a minimizing solution, and therefore f is critical for h.

-If f is critical for h, these two functions (and the metric) satisfying the hypothesis of the theorem, we have $\lambda_{h,f,\mathbf{g}} = K(n,2)^{-2}(\sup_M f)^{-\frac{n-2}{n}}$, so h is weakly critical for f. We then consider, for $t \overset{<}{\to} 1$, the sequence

$$t \to f_t = (1-t)\sup f + t.f.$$

For all t, we have $\sup f_t = \sup f$ and, if $t < 1$, then $f_t \gneqq f$. Therefore, as f is critical for h, by definition

$$\lambda_{h,f_t,\mathbf{g}} < K(n,2)^{-2}(\sup_M f_t)^{-\frac{n-2}{n}}.$$

We then apply Theorem 4' above to obtain that h is critical for f with minimizing solutions.

8. The case of dimension 3; ending remarks

8.1. **The case of dimension 3.** We just state the results in the case of dimension 3, as they are immediate generalizations of results of O. Druet proved in the case where f is a constant; we refer to his article for the proofs [11]. The dimension 3 requires fundamentally the use of the Green's function. We refer to the proof of Proposition 8 in section 4.2 for the definition and the property of the Green's function. In dimension 3, for any point $x \in M$, and for y close to x, G_h can be written in the following way

$$G_h(x,y) = \frac{1}{\omega_2 d_{\mathbf{g}}(x,y)} + M_h(x) + o(1)$$

where $o(1)$ is to be taken for $y \to x$. We call $M_h(x)$ the mass of the Green's function at x.

The generalization of the results of O. Druet to the case of an arbitrary function f in $(E_{h,f,\mathbf{g}})$ gives the following:

Let $(M, \mathbf{g})$ be a compact manifold of dimension 3, and let $f \in C^\infty(M)$ be such that $\sup f > 0$. We have the following results:

- For any function h weakly critical for f and $\mathbf{g}$, and for any $x \in Maxf$, we have $M_h(x) \leq 0$.
- For any $h \in C^\infty(M)$, let $B(h) = \inf\{B : h + B$ is weakly critical for $f\}$. Then $h + B(h)$is a critical function for f.
- Let h be a critical function for f and $\mathbf{g}$. Then one of the following conditions is true:
  1. There exists $x \in Maxf$ such that $M_h(x) = 0$.
  2. $((E_{h,f,\mathbf{g}}))$ has minimizing solutions.

Remarks. Condition $M_h(x) \leqslant 0$ appears as the analog of the condition

$$\frac{4(n-1)}{n-2} h(P) \geqslant S_{\mathbf{g}}(P) - \frac{n-4}{2} \frac{\triangle_{\mathbf{g}} f(P)}{f(P)}$$

we had in dimension $\geqslant 4$. In the case $f = cst$, this condition must be satisfied on all of M.

The particular quality of dimension 3 is to offer critical functions of any shape; that is the meaning of the second point.

The main difference with the case $f = cst$ studied by O. Druet is that the conditions on the mass of the Green's function are to be considered only at the point of maximum of f.

8.2. **Degenerate Hessian at the point of maximum and fundamental estimate.** In Theorem 6, we made the hypothesis that the Hessian of f is non-degenerate at each of its points of maximum. We give here a counterexample to show that this hypothesis is necessary. Consider the n-dimensional sphere S^n with its standard metric $\mathbf{s}$. Rewriting known results (c.f. for example [18]), there exists a unique critical function for 1 and $\mathbf{s}$, which is

$$h = \frac{n-2}{4(n-1)} S_{\mathbf{s}} = \frac{n-2}{4(n-1)}$$

and this critical function has only two types of extremal functions, the constants and the functions of the form $u = a(b - \cos r)^{-\frac{n-2}{2}}$, where $a \neq 0$, $b > 1$, and r is the geodesic distance to some fixed point of S^n. Consider now on S^n a sequence of points x_t converging to a point x_0, and let

$$u_t = \mu_t^{\frac{n-2}{2}} (\mu_t^2 + 1 - \cos r_t)^{-\frac{n-2}{2}},$$

where $r_t(x) = d_{\mathbf{s}}(x, x_t)$ and μ_t is a sequence of real numbers converging to 0. Then

$$\int_M u_t^{2^*} dv_{\mathbf{s}} = 1$$

and we obtain in this way a sequence of solutions of the equation

$$\triangle_{\mathbf{s}} u_t + \frac{n-2}{4(n-1)} . u_t = K(n,2)^{-2} u_t^{\frac{n+2}{n-2}},$$

where obviously the function $f = K(n,2)^{-2}$ has degenerate Hessian at its maximum points! Furthermore,

$$\sup_M u_t = u_t(x_t) = \mu_t^{-\frac{n-2}{2}}.$$

This sequence concentrates and satisfies Propositions 2 through 9 seen in section 4.2, whatever the choice of the sequence $x_t \to x_0$ and of the sequence $\mu_t \to 0$. By spherical symmetry, we can easily find two sequences (x_t) and (μ_t) such that $\frac{d_{\mathbf{s}}(x_t, x_0)}{\mu_t} \to +\infty$ by taking for example $\mu_t = d_{\mathbf{s}}(x_t, x_0)^2$.

Once again, it seems that the hypothesis on the Hessian of f "fixes" the position of the concentration point, and so imposes a speed of convergence of the sequence (x_t).

8.3. **Further questions.** First we make a remark concerning the requirement of a strict inequality at the point of maximum of f in Theorem 1. An easy but somewhat artificial extension of a result of Hebey and Vaugon is the following:

Suppose that the manifold $(M, \mathbf{g})$ is of dimension ≥ 7 ,and let $(h, f, \mathbf{g})$ be a critical triple. Let $T_f = \{x \in M : f(x) = Max f$ and $h(x) = \frac{n-2}{4(n-1)} S_{\mathbf{g}}(x) -$

$\frac{(n-2)(n-4)}{8(n-1)}\frac{\triangle_{\mathbf{g}}f(P)}{f(P)}\}$. *We suppose that* T_f *is not dense in* M *and that for any point* x *of* T_f,

1) *the Weyl tensor vanishes on a neihbourhood of* x,

2) $\nabla^2(h-\frac{n-2}{4(n-1)}S_{\mathbf{g}})$ *is not degenerate in* x,

3) $\triangle_{\mathbf{g}}f(x)=0$ *if* $x\in T_f$, *and we suppose that* f *is non-degenerate at the points of maximum which are not in* T_f.

Then $(h,f,\mathbf{g})$ *has minimizing solutions.*

The main interest of this result is that we can expect existence of solutions in this case. Looking to our method, it seems that one needs to find some other intrinsic parameters, i.e., invariant by the exponential charts exp_{x_t}. See our thesis for more precision.

Another question is the following. We saw that the study of equations $\triangle_{\mathbf{g}}u+hu=fu^{2^*-1}$ is linked to the study of the best constants in the Sobolev inclusions of H_1^2 in $L^{\frac{2n}{n-2}}$. In the same way, the study of the Sobolev inclusions of H_1^p in $L^{\frac{pn}{n-p}}$, where $\frac{pn}{n-p}$ is the critical exponent, and of the associated best constants, goes through the study of equations of the form

$$\triangle_p u+hu=fu^{\frac{pn}{n-p}-1}$$

where $\triangle_p u=-\nabla(|\nabla u|_{\mathbf{g}}^{p-2}\nabla u)$ is the p-Laplacian; see for example O. Druet, E. Hebey and Z. Faget [15]. Here also variational methods are used. The functional used is

$$I(u)=\int|\nabla u|_{\mathbf{g}}^p+\int hu^p$$

from where we see the link with the Sobolev inclusion

$$(\int u^{\frac{pn}{n-p}})^{\frac{n-p}{n}}\leq K(n,p)\int|\nabla u|_{\mathbf{g}}^p+B\int u^p,$$

where $K(n,p)$ is the associated best constant. The starting point is again the following. If

$$\underset{\int u^{\frac{pn}{n-p}}=1}{Inf}\quad I(u)<K(n,p)^{-1}(\sup f)^{-\frac{n-p}{n}},$$

then the equation has a minimizing solution $u>0$ (knowing that the large inequality is always true). We therefore see that it is easy to extend the definition of critical functions to this case. It would therefore be interesting to know if our results can be extended to this setting.

Another question that can be asked after our work is the following:

f being given, are there constant critical functions?

This would give some kind of "best second constant $B_0(\mathbf{g}, f)$" linked to f.

At last, there is a question which emerges from our work:

For a given arbitrary function h on M, do there exist solutions (not minimizing) to the equation $\triangle_{\mathbf{g}} u + hu = fu^{2^*-1}$?

Indeed, we saw that this equation has (minimizing) solutions when h is subcritical and when h is critical with some hypothesis. However, variational methods do not give any answer is h larger and different than some critical function, or if $\triangle_{\mathbf{g}} + h$ is not coercive. In these cases, if solutions exist, they cannot be minimizing. One therefore needs other methods for these cases. See [4] and [5] who study the case $f = cst$ and $3 \leq dimM \leq 6$.

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