Advances in Differential Equations Volume 12, Number 1 (2007), 27–54

# PARTIALLY DELAYED STABILIZING FEEDBACKS FOR MAXWELL'S SYSTEM

Serge Nicaise
Université de Valenciennes et du Hainaut Cambrésis
MACS, Institut des Sciences et Techniques de Valenciennes
59313 Valenciennes Cedex 9 France

Cristina Pignotti
Dipartimento di Matematica Pura e Applicata, Università di L'Aquila
Via Vetoio, Loc. Coppito, 67010 L'Aquila, Italy

(Submitted by: Viorel Barbu)

**Abstract.** We consider, in a bounded and smooth domain, Maxwell's equations with a delay term in the boundary or in the internal feedbacks. Under suitable assumptions we obtain exponential stability results. Some instability examples are also given.

## 1. Introduction

In this paper we investigate the effect of time delay in boundary or internal stabilization of Maxwell's equations in domains in $\mathbb{R}^3$.

Let $\Omega \subset \mathbb{R}^3$ be an open and connected bounded set with a boundary $\Gamma$ of class $C^\infty$ consisting of a single connected component.

In this domain $\Omega$, we consider the initial-boundary-value problem

$$E_t(x,t) - \text{curl } H(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.1}$$

$$H_t(x,t) + \text{curl } E(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.2}$$

$$\text{div } E(x,t) = \text{div } H(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.3}$$

$$[(\mu_1 E(x,t) + \mu_2 E(x,t-\tau)) \times \nu + H(x,t)] \times \nu = 0 \quad \text{on} \quad \Gamma \times (0,+\infty) \tag{1.4}$$

$$E(x,0) = E_0(x) \quad \text{and} \quad H(x,0) = H_0(x) \quad \text{in} \quad \Omega \tag{1.5}$$

$$E(x,t-\tau) \times \nu(x) = F_0(x,t-\tau) \quad \text{in } \Gamma \times (0,\tau), \tag{1.6}$$

where $E, H$ are the electric and magnetic vector fields and $\nu(x)$ denotes the outward unit normal vector to the point $x \in \Gamma$. Moreover, the constant $\tau > 0$

Accepted for publication: August 2006.
AMS Subject Classifications: 93D15, 93C20.

is the time delay, $\mu_1$ and $\mu_2$ are positive real numbers and the initial datum $(E_0, H_0, F_0)$ belongs to a suitable space.

We are interested in giving an exponential stability result for such a problem. Let us denote by $\langle v, w\rangle$ or, equivalently, by $v \cdot w$ the Euclidean inner product between two vectors $v, w \in \mathbb{R}^n$. It is well known that if $\mu_2 = 0$, that is, in absence of delay, the energy of problem (1.1)–(1.6) is exponentially decaying to zero. See for instance [5, 4, 2, 9]. Assuming that

$$\mu_2 < \mu_1, \tag{1.7}$$

we will obtain a stabilization result, by using a suitable observability estimate. Moreover, we show that if $\mu_1 = \mu_2$ then there exists a sequence of arbitrarily small (and large) delays such that instabilities occur. In the case $\mu_2 > \mu_1$, we also obtain delays which destabilize the system.

Furthermore, we study the problem for Maxwell's equations with an internal feedback. In particular, we consider the system

$$E_t(x,t) - \text{curl } H(x,t) + \sigma(x)[\mu_1 E(x,t) + \mu_2 E(x,t-\tau)] = 0 \text{ in } \Omega \times (0,+\infty) \tag{1.8}$$

$$H_t(x,t) + \text{curl } E(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.9}$$

$$\text{div } H(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{1.10}$$

$$E(x,t) \times \nu = 0 \quad \text{and} \quad H(x,t) \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0,+\infty) \tag{1.11}$$

$$E(x,0) = E_0(x) \quad \text{and} \quad H(x,0) = H_0(x) \quad \text{in} \quad \Omega \tag{1.12}$$

$$E(x,t-\tau) = G_0(x,t-\tau) \quad \text{in } \Omega \times (0,\tau), \tag{1.13}$$

where $\sigma \in L^\infty(\Omega)$ is a function such that

$$\sigma(x) \geq 0 \quad \text{a. e.} \quad \text{in} \quad \Omega, \tag{1.14}$$

and

$$\sigma(x) > \sigma_0 > 0, \quad \text{a. e.} \quad \text{in} \quad \omega, \tag{1.15}$$

where $\omega \subset \overline{\Omega}$ is an open neighborhood of $\Gamma$.

Exponential stability results for the above problem in the case of $\mu_2 = 0$, that is, without delay, have been obtained in some papers. See for instance [9, 7].

In this paper, in the case $\mu_2 < \mu_1$, we show that the energy is exponentially decaying to zero. This is done, as for the problem with boundary feedback, by using a suitable observability estimate. If $\mu_2 \geq \mu_1$, we obtain an explicit sequence of arbitrarily small delays that destabilize the system.

Analogous analysis for scalar wave equations in both cases, boundary and internal feedbacks, has been recently carried out by the authors in [8].

This problem has been inspired by [10] where similar results are proved, after a careful spectral analysis, for the scalar wave equation with boundary feedback in one space dimension.

The paper is organized as follows. Well posedness of the problems is analyzed in section 2 using semigroup theory. In subsection 2.1 we study the well posedness of problem (1.1)–(1.6), while in subsection 2.2 we concentrate on problem (1.8)–(1.13). In section 3 and section 4 we prove the exponential stability of the problem with boundary and internal feedbacks respectively. Finally, in section 5 we give instability examples when $\mu_2 \geq \mu_1$ in the case of boundary feedback (subsection 5.1) and in the case of internal feedback (subsection 5.2).

## 2. Well posedness of the problems

In this section, assuming

$$\mu_2 \leq \mu_1, \tag{2.1}$$

we will give well–posedness results for problem (1.1)–(1.6) and for problem (1.8)–(1.13) using semigroup theory.

### 2.1. **Boundary feedback.** Let us set

$$z(x,\rho,t) = E(x,t-\tau\rho) \times \nu(x), \quad x \in \Gamma,\ \rho \in (0,1),\ t > 0. \tag{2.2}$$

Then, problem (1.1)–(1.6) is equivalent to

$$E_t(x,t) - \text{curl } H(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{2.3}$$

$$H_t(x,t) + \text{curl } E(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{2.4}$$

$$\text{div } E(x,t) = \text{div } H(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{2.5}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in} \quad \Gamma \times (0,1) \times (0,+\infty) \tag{2.6}$$

$$[\mu_1 E(x,t) \times \nu + \mu_2 z(x,1,t) + H(x,t)] \times \nu = 0 \quad \text{on} \quad \Gamma \times (0,+\infty) \tag{2.7}$$

$$z(x,0,t) = E(x,t) \times \nu(x) \quad \text{on} \quad \Gamma \times (0,\infty) \tag{2.8}$$

$$E(x,0) = E_0(x) \quad \text{and} \quad H(x,0) = H_0(x) \quad \text{in} \quad \Omega \tag{2.9}$$

$$z(x,\rho,0) = F_0(x,-\rho\tau) \quad \text{in } \Gamma \times (0,1). \tag{2.10}$$

We need to introduce suitable space functions. As usual, we denote by $\mathcal{H}(\text{div } 0, \Omega)$ the space

$$\mathcal{H}(\text{div } 0, \Omega) := \{\, \Phi \in L^2(\Omega)^3 \ :\ \text{div } \Phi(x) = 0 \,\} \tag{2.11}$$

and by $\mathcal{L}^2_\tau(\Gamma \times (0,1))$ the space

$$\mathcal{L}^2_\tau(\Gamma\times(0,1)) := \{\, \Psi \in L^2(\Gamma\times(0,1))^3 \;:\; \nu(x)\cdot\Psi(x,\rho) = 0,\ x \in \Gamma,\ \rho \in (0,1)\,\}. \tag{2.12}$$

Let us introduce the Hilbert space

$$\mathcal{H} := \mathcal{H}(\mathrm{div}\ 0, \Omega) \times \mathcal{H}(\mathrm{div}\ 0, \Omega) \times \mathcal{L}^2_\tau(\Gamma \times (0,1)), \tag{2.13}$$

equipped with the inner product

$$\left\langle \begin{pmatrix} E \\ H \\ z \end{pmatrix}, \begin{pmatrix} \tilde E \\ \tilde H \\ \tilde z \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega (E\cdot \tilde E + H\cdot \tilde H)dx + \xi \int_\Gamma \int_0^1 z\cdot \tilde z d\rho d\Gamma, \tag{2.14}$$

where $\xi$ is a positive real number such that

$$\tau\mu_2 \le \xi \le \tau(2\mu_1 - \mu_2). \tag{2.15}$$

Note that, from (2.1), such a constant $\xi$ exists. If we denote by $U := (E, H, z)^T$, then

$$U' := (E_t, H_t, z_t)^T = \left(\mathrm{curl}\ H, -\mathrm{curl}\ E, -\tau^{-1} z_\rho\right)^T.$$

Therefore, problem (2.3)–(2.10) can be rewritten as

$$U' = \mathcal{A}U, \qquad U(0) = (E_0, H_0, F_0(\cdot, -\cdot\tau))^T, \tag{2.16}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} E \\ H \\ z \end{pmatrix} := \begin{pmatrix} \mathrm{curl}\ H \\ -\mathrm{curl}\ E \\ -\tau^{-1} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}) := \Big\{ &(E, H, z)^T \in V \times V \times L^2(\Gamma; H^1(0,1))^3 \;: \\ &[\mu_1 E(x) \times \nu(x) + \mu_2 z(x,1) + H(x)] \times \nu(x) = 0 \text{ on } \Gamma; \\ &E(x) \times \nu(x) = z(x,0) \text{ on } \Gamma \Big\},\end{aligned} \tag{2.17}$$

where,

$$V := \{\, \Phi \in L^2(\Omega)^3 \;:\; \mathrm{curl}\ \Phi \in L^2(\Omega)^3,\ \mathrm{div}\ \Phi = 0,\ \Phi\times\nu \in \mathcal{L}^2_\tau(\Gamma)\,\}, \tag{2.18}$$

$$\|\Phi\|_V^2 = \int_\Omega (|\Phi|^2 + |\mathrm{curl}\ \Phi|^2)dx + \int_\Gamma |\nu\times\Phi|^2 d\Gamma.$$

**Theorem 2.1.** *For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C([0,+\infty), \mathcal{H})$ of problem (2.16). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

**Proof.** Take $U = (E, H, z)^T \in \mathcal{D}(\mathcal{A})$. Then,

$$(\mathcal{A}U, U) = \left\langle \begin{pmatrix} \text{curl } H \\ -\text{curl } E \\ -\tau^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} E \\ H \\ z \end{pmatrix} \right\rangle_{\mathcal{H}}$$
$$= \int_\Omega \{E \text{curl } H - H \text{curl } E\} dx - \xi\tau^{-1} \int_\Gamma \int_0^1 z_\rho(x,\rho) \cdot z(x,\rho) d\rho d\Gamma.$$

So, by Green's formula (cfr. [2], page 150),

$$(\mathcal{A}U, U) = \int_\Gamma (E \times \nu) \cdot H d\Gamma - \xi\tau^{-1} \int_\Gamma \int_0^1 z_\rho(x,\rho) \cdot z(x,\rho) d\rho d\Gamma. \tag{2.19}$$

Integrating by parts in $\rho$, we get

$$\int_\Gamma \int_0^1 z_\rho(x,\rho) \cdot z(x,\rho) d\rho d\Gamma$$
$$= -\int_\Gamma \int_0^1 z_\rho(x,\rho) \cdot z(x,\rho) d\rho d\Gamma + \int_\Gamma \{|z(x,1)|^2 - |z(x,0)|^2\} d\Gamma;$$

that is,

$$\int_\Gamma \int_0^1 z_\rho(x,\rho) \cdot z(x,\rho) d\rho d\Gamma = \frac{1}{2} \int_\Gamma \{|z(x,1)|^2 - |z(x,0)|^2\} d\Gamma. \tag{2.20}$$

Therefore, from (2.19) and (2.20),

$$(\mathcal{A}U, U) = -\int_\Gamma E \cdot (H \times \nu) d\Gamma - \frac{\xi\tau^{-1}}{2} \int_\Gamma \{|z(x,1)|^2 - |z(x,0)|^2\} d\Gamma$$
$$= \int_\Gamma E \cdot [(\mu_1 (E \times \nu) + \mu_2 z(x,1)) \times \nu] d\Gamma - \frac{\xi\tau^{-1}}{2} \int_\Gamma \{|z(x,1)|^2 - |z(x,0)|^2\} d\Gamma$$
$$= -\mu_1 \int_\Gamma |E \times \nu|^2 d\Gamma - \mu_2 \int_\Gamma (z(x,1) \times \nu) \cdot (E \times \nu) d\Gamma$$
$$- \frac{\xi\tau^{-1}}{2} \int_\Gamma |z(x,1)|^2 d\Gamma + \frac{\xi\tau^{-1}}{2} \int_\Gamma |E \times \nu|^2 d\Gamma,$$

from which follows, using Cauchy-Schwarz's inequality,

$$(\mathcal{A}U, U) \le \Big( -\mu_1 + \frac{\mu_2}{2} + \frac{\xi\tau^{-1}}{2} \Big) \int_\Gamma |E \times \nu|^2 d\Gamma + \Big( \frac{\mu_2}{2} - \frac{\xi\tau^{-1}}{2} \Big) \int_\Gamma |z(x,1)|^2 d\Gamma. \tag{2.21}$$

Then, from (2.15) we have that $(\mathcal{A}U, U) \leq 0$, which means that the operator $\mathcal{A}$ is dissipative.

Now, we want to prove the maximality of $\mathcal{A}$ showing that the operator $I - \mathcal{A}$ is surjective.

For any given $(F, G, w)^T \in \mathcal{H}$ we seek $U = (E, H, z)^T \in \mathcal{D}(\mathcal{A})$, a solution of

$$(I - \mathcal{A}) \begin{pmatrix} E \\ H \\ z \end{pmatrix} = \begin{pmatrix} F \\ G \\ w \end{pmatrix};$$

that is,

$$\begin{cases} E - \text{curl } H = F \\ H + \text{curl } E = G \\ z + \tau^{-1} z_\rho = w. \end{cases} \tag{2.22}$$

Suppose that we have found $E$ and $H$ with the appropriate regularity.

Then we can determine $z$. Indeed, from (2.17),

$$z(x, 0) = E(x) \times \nu(x), \quad \text{for} \quad x \in \Gamma, \tag{2.23}$$

and, from (2.22),

$$z(x, \rho) + \tau^{-1} z_\rho(x, \rho) = w(x, \rho), \quad \text{for} \quad x \in \Gamma, \ \rho \in (0, 1). \tag{2.24}$$

Then, by (2.23) and (2.24), we deduce

$$z(x, \rho) = e^{-\rho\tau} E(x) \times \nu(x) + \tau e^{-\rho\tau} \int_0^\rho w(x, \sigma) e^{\sigma\tau} d\sigma, \quad x \in \Gamma, \rho \in (0, 1), \tag{2.25}$$

and, in particular,

$$z(x, 1) = e^{-\tau} E(x) \times \nu(x) + \tau e^{-\tau} \int_0^1 w(x, \sigma) e^{\sigma\tau} d\sigma \quad \text{on} \quad \Gamma,$$

that we rewrite as

$$z(x, 1) = e^{-\tau} E(x) \times \nu(x) + z_0(x) \quad \text{on} \quad \Gamma, \tag{2.26}$$

with $z_0 \in \mathcal{L}^2_\tau(\Gamma)$ defined by

$$z_0(x) = \tau e^{-\tau} \int_0^1 w(x, \sigma) e^{\sigma\tau} d\sigma, \quad x \in \Gamma. \tag{2.27}$$

From (2.22), we get

$$\int_\Omega \{(E - \text{curl } H)\Phi + (H + \text{curl } E)\Psi\} dx = \int_\Omega (F\Phi + G\Psi) dx, \tag{2.28}$$

for all $(\Phi, \Psi) \in V \times L^2(\Omega)^3$. Integrating by parts, we obtain

$$\int_\Omega \{E\Phi - H\text{curl } \Phi + (H + \text{curl } E)\Psi\} dx + \int_\Gamma (H \times \nu) \cdot \Phi d\Gamma = \int_\Omega (F\Phi + G\Psi) dx,$$

and then,

$$\int_\Omega \{E\Phi - H\text{curl } \Phi + (H + \text{curl } E)\Psi\} dx \\ - \int_\Gamma [(\mu_1 E \times \nu + \mu_2 z(x,1)) \times \nu] \cdot \Phi d\Gamma = \int_\Omega (F\Phi + G\Psi) dx.$$

From this, putting $\Psi = \text{curl } \Phi$, we obtain

$$\int_\Omega (E\Phi + \text{curl } E\text{curl } \Phi) dx - \int_\Gamma [(\mu_1 E \times \nu + \mu_2 z(x,1)) \times \nu] \cdot \Phi d\Gamma \\ = \int_\Omega (F\Phi + G\text{curl } \Phi) dx, \quad \forall \Phi \in V. \tag{2.29}$$

Using (2.26) in (2.29) we have

$$\int_\Omega (E\Phi + \text{curl } E\text{curl } \Phi) dx + \int_\Gamma (\mu_1 + \mu_2 e^{-\tau})(E \times \nu) \cdot (\Phi \times \nu) d\Gamma \\ = \int_\Omega (F\Phi + G\text{curl } \Phi) dx - \int_\Gamma \mu_2 z_0 (\Phi \times \nu) d\Gamma, \quad \forall \Phi \in V. \tag{2.30}$$

Since the left-hand side of (2.30) is coercive on $V$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $E \in V$ of (2.30).

Now, we set $H = G - \text{curl } E \in L^2(\Omega)^3$. Then, (2.30) becomes

$$\int_\Omega (E\Phi - H\text{curl } \Phi) dx + \int_\Gamma (\mu_1 + \mu_2 e^{-\tau})(E \times \nu)(\Phi \times \nu) d\Gamma \\ = \int_\Omega F\Phi dx - \int_\Gamma \mu_2 z_0 (\Phi \times \nu) d\Gamma, \quad \forall \Phi \in V,$$

and, integrating by parts,

$$\int_\Omega (E - \text{curl } H)\Phi dx - \int_\Gamma (H \times \nu) \cdot \Phi d\Gamma + \int_\Gamma (\mu_1 + \mu_2 e^{-\tau})(E \times \nu)(\Phi \times \nu) d\Gamma \\ = \int_\Omega F\Phi dx - \int_\Gamma \mu_2 z_0 (\Phi \times \nu) d\Gamma, \quad \forall \Phi \in V. \tag{2.31}$$

From (2.31) we deduce $\text{curl } H \in L^2(\Omega)^3$ and $E - \text{curl } H = F$. Moreover,

$$\int_\Gamma (H \times \nu) \cdot \Phi d\Gamma - \int_\Gamma (\mu_1 + \mu_2 e^{-\tau})(E \times \nu)(\Phi \times \nu) d\Gamma = \int_\Gamma \mu_2 z_0 (\Phi \times \nu) d\Gamma;$$

that is,

$$\int_\Gamma (H\times\nu)\cdot\Phi d\Gamma+\int_\Gamma[(\mu_1+\mu_2 e^{-\tau})(E\times\nu)\times\nu]\Phi d\Gamma+\int_\Gamma \mu_2(z_0\times\nu)\Phi d\Gamma=0.$$

This implies

$$H(x)\times\nu(x)=-(\mu_1 E(x)\times\nu(x)+\mu_2 z(x,1))\times\nu(x),\quad x\in\Gamma.$$

This proves the existence of a solution $(E,H,z)^T$ in $\mathcal{D}(\mathcal{A})$ of (2.22) and consequently the maximality of $\mathcal{A}$.

The well-posedness result follows from the Hille-Yosida theorem. □

2.2. **Internal feedback.** Let us set

$$z(x,\rho,t)=E(x,t-\tau\rho),\quad x\in\Omega,\ \rho\in(0,1),\ t>0. \tag{2.32}$$

Then, problem (1.8)–(1.13) is equivalent to

$$E_t(x,t)-\text{curl } H(x,t)+\sigma[\mu_1 E(x,t)+\mu_2 z(x,1,t)]=0 \text{ in } \Omega\times(0,+\infty) \tag{2.33}$$

$$H_t(x,t)+\text{curl } E(x,t)=0 \quad\text{in}\quad \Omega\times(0,+\infty) \tag{2.34}$$

$$\text{div } H(x,t)=0 \quad\text{in}\quad \Omega\times(0,+\infty) \tag{2.35}$$

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad\text{in}\quad \Omega\times(0,1)\times(0,+\infty) \tag{2.36}$$

$$E(x,t)\times\nu=0 \quad\text{and}\quad H(x,t)\cdot\nu=0 \quad\text{on}\quad \Gamma\times(0,+\infty) \tag{2.37}$$

$$z(x,0,t)=E(x,t) \quad\text{in}\quad \Omega\times(0,\infty) \tag{2.38}$$

$$E(x,0)=E_0(x) \quad\text{and}\quad H(x,0)=H_0(x) \quad\text{in}\quad \Omega \tag{2.39}$$

$$z(x,\rho,0)=G_0(x,-\rho\tau) \quad\text{in } \Omega\times(0,1). \tag{2.40}$$

We need to introduce suitable space functions. As usual, we denote

$$\mathcal{H}_0(\text{curl },\Omega):=\{\,\Psi\in L^2(\Omega)^3:\text{curl }\Psi\in L^2(\Omega)^3;\Psi\times\nu=0 \text{ on } \Gamma\},\tag{2.41}$$

$$\mathcal{H}_0(\text{div } 0,\Omega):=\{\,\Phi\in\mathcal{H}(\text{div } 0,\Omega)\ :\Phi\cdot\nu=0 \text{ on } \Gamma\,\}.\tag{2.42}$$

Let us introduce the Hilbert space

$$\mathcal{H}^0:=L^2(\Omega)^3\times\mathcal{H}_0(\text{div } 0,\Omega)\times L^2(\Omega\times(0,1))^3,\tag{2.43}$$

equipped with the inner product

$$\left\langle \begin{pmatrix}E\\H\\z\end{pmatrix},\begin{pmatrix}\tilde E\\\tilde H\\\tilde z\end{pmatrix}\right\rangle_{\mathcal{H}^0}=\int_\Omega(E\cdot\tilde E+H\cdot\tilde H)dx+\xi\int_\Omega\int_0^1\sigma z\cdot\tilde z d\rho d\Gamma,\tag{2.44}$$

where $\xi$ is a positive real number satisfying (2.15).

If we denote $U := (E, H, z)^T$, then

$$U' := (E_t, H_t, z_t)^T = \big(\text{curl } H - \sigma\mu_1 E - \sigma\mu_2 z(\cdot, 1), -\text{curl } E, -\tau^{-1} z_\rho\big)^T.$$

Therefore, problem (2.33)–(2.40) can be rewritten as

$$U' = \mathcal{A}^0 U, \quad U(0) = (E_0, H_0, G_0(\cdot, -\cdot\,\tau))^T, \tag{2.45}$$

where the operator $\mathcal{A}^0$ is defined by

$$\mathcal{A}^0 \begin{pmatrix} E \\ H \\ z \end{pmatrix} := \begin{pmatrix} \text{curl } H - \sigma\mu_1 E - \sigma\mu_2 z(\cdot, 1) \\ -\text{curl } E \\ -\tau^{-1} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}^0) := \Big\{(E, H, z)^T \in \mathcal{H}_0(\text{curl }, \Omega) \times (\mathcal{H}_0(\text{div } 0, \Omega) \cap H^1(\Omega)^3) \times L^2(\Omega; H^1(0,1))^3 : E = z(\cdot, 0) \text{ in } \Omega\Big\}. \tag{2.46}$$

**Theorem 2.2.** *For any initial datum $U_0 \in \mathcal{H}^0$ there exists a unique solution $U \in C([0, +\infty), \mathcal{H}^0)$ of problem* (2.45). *Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}^0)$, then*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}^0)) \cap C^1([0, +\infty), \mathcal{H}^0).$$

**Proof.** Take $U = (E, H, z)^T \in \mathcal{D}(\mathcal{A}^0)$. Then

$$\begin{aligned}\big(\mathcal{A}^0 U, U\big) &= \left\langle \begin{pmatrix} \text{curl } H - \sigma\mu_1 E - \sigma\mu_2 z(\cdot, 1) \\ -\text{curl } E \\ -\tau^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} E \\ H \\ z \end{pmatrix} \right\rangle_{\mathcal{H}^0} \\ &= \int_\Omega \{E\text{curl } H - H\text{curl } E - \sigma\mu_1 |E|^2 - \sigma\mu_2 z(x,1)E\} dx \\ &\quad -\xi\tau^{-1} \int_\Omega \int_0^1 \sigma z_\rho(x, \rho) \cdot z(x, \rho) d\rho d\Gamma.\end{aligned}$$

So, by Green's formula,

$$\begin{aligned}\big(\mathcal{A}^0 U, U\big) = &- \int_\Omega \sigma(x)[\mu_1 |E|^2 + \mu_2 z(x,1)E] dx \\ &- \xi\tau^{-1} \int_\Omega \int_0^1 \sigma(x) z_\rho(x, \rho) \cdot z(x, \rho) d\rho dx.\end{aligned} \tag{2.47}$$

Integrating by parts in $\rho$, we have

$$\begin{aligned}\int_\Omega \int_0^1 \sigma(x) z_\rho(x, \rho) \cdot z(x, \rho) d\rho dx = &- \int_\Omega \int_0^1 \sigma(x) z_\rho(x, \rho) \cdot z(x, \rho) d\rho dx \\ &+ \int_\Omega \sigma(x)(|z(x,1)|^2 - |z(x,0)|^2) dx;\end{aligned}$$

that is,

$$\int_\Omega \int_0^1 \sigma(x) z_\rho(x,\rho)\cdot z(x,\rho) d\rho dx = \frac{1}{2}\int_\Omega \sigma(x)(|z(x,1)|^2 - |z(x,0)|^2) dx. \quad (2.48)$$

Therefore, from (2.47) and (2.48),

$$\begin{aligned}\left(\mathcal{A}^0 U, U\right) &= -\int_\Omega \sigma(x)[\mu_1 |E|^2 + \mu_2 z(x,1)E] dx \\ &\quad - \frac{\xi\tau^{-1}}{2}\int_\Omega \sigma(x)(|z(x,1)|^2 - |z(x,0)|^2) dx \\ &= -\int_\Omega \sigma(x)\mu_1 |E|^2 dx + \int_\Omega \frac{\sigma(x)\mu_2}{2}(|z(x,1)|^2 + |E|^2) dx \\ &\quad - \frac{\xi\tau^{-1}}{2}\int_\Omega \sigma(x)|z(x,1)|^2 dx + \frac{\xi\tau^{-1}}{2}\int_\Omega \sigma(x)|E|^2 dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\mathcal{A}^0 U, U\right) \le &\left(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi\tau^{-1}}{2}\right)\int_\Omega \sigma(x)|E|^2 dx \\ &+ \left(\frac{\mu_2}{2} - \frac{\xi\tau^{-1}}{2}\right)\int_\Omega \sigma(x)|z(x,1)|^2 dx.\end{aligned} \quad (2.49)$$

Then, from (2.15) we deduce that the operator $\mathcal{A}^0$ is dissipative.

Now, we want to prove the maximality of $\mathcal{A}^0$ showing that the operator $I - \mathcal{A}^0$ is surjective.

For any given $(F, G, w)^T \in \mathcal{H}^0$ we seek a solution $U = (E, H, z)^T \in \mathcal{D}(\mathcal{A}^0)$ of

$$(I - \mathcal{A}^0)\begin{pmatrix} E \\ H \\ z \end{pmatrix} = \begin{pmatrix} F \\ G \\ w \end{pmatrix};$$

that is,

$$\begin{cases} E - \text{curl } H + \sigma\mu_1 E + \sigma\mu_2 z(\cdot,1) = F \\ H + \text{curl } E = G \\ z + \tau^{-1} z_\rho = w. \end{cases} \quad (2.50)$$

Suppose that we have found $E$ and $H$ with the appropriate regularity.

Then we can determine $z$. Indeed, from (2.46),

$$z(x,0) = E(x), \quad \text{for} \quad x \in \Omega, \quad (2.51)$$

and, from (2.50),

$$z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = w(x,\rho), \quad \text{for} \quad x \in \Omega,\ \rho \in (0,1). \quad (2.52)$$

Then, by (2.51) and (2.52), we deduce

$$z(x,\rho) = e^{-\rho\tau}E(x) + \tau e^{-\rho\tau}\int_0^\rho w(x,\sigma)e^{\sigma\tau}d\sigma, \quad x\in\Omega,\ \rho\in(0,1), \tag{2.53}$$

and, in particular,

$$z(x,1) = e^{-\tau}E(x) + \tau e^{-\tau}\int_0^1 w(x,\sigma)e^{\sigma\tau}d\sigma \quad \text{in} \ \ \Omega,$$

that we rewrite as

$$z(x,1) = e^{-\tau}E(x) + z_0(x) \quad \text{in} \ \ \Omega, \tag{2.54}$$

with $z_0 \in L^2(\Omega)^3$ defined by

$$z_0(x) = \tau e^{-\tau}\int_0^1 w(x,\sigma)e^{\sigma\tau}d\sigma, \quad x\in\Omega. \tag{2.55}$$

From (2.50), we see that

$$\begin{aligned}&\int_\Omega\{(E - \text{curl } H + \sigma\mu_1 E + \sigma\mu_2 z(\cdot,1))\Phi + (H + \text{curl } E)\Psi\}dx\\ &= \int_\Omega(F\Phi + G\Psi)dx, \quad \forall\ (\Phi,\Psi)\in\mathcal{H}_0(\text{curl },\Omega)\times L^2(\Omega)^3.\end{aligned} \tag{2.56}$$

Integrating by parts and using the boundary condition satisfied by $\Phi$, we obtain

$$\begin{aligned}\int_\Omega\{E\Phi - H\text{curl } \Phi + \sigma\mu_1 E\Phi + \sigma\mu_2 z(\cdot,1)\Phi + (H + \text{curl } E)\Psi\}dx&\\ = \int_\Omega(F\Phi + G\Psi)dx.&\end{aligned}$$

From this, putting $\Psi = \text{curl } \Phi$, we obtain

$$\begin{aligned}&\int_\Omega\{E\Phi + \text{curl } E\text{curl } \Phi + \sigma\mu_1 E\Phi + \sigma\mu_2 z(\cdot,1)\Phi\}dx\\ &= \int_\Omega(F\Phi + G\text{curl } \Phi)dx, \quad \forall\ \Phi\in\mathcal{H}_0(\text{curl },\Omega).\end{aligned} \tag{2.57}$$

Using (2.54) in (2.57) we have

$$\begin{aligned}&\int_\Omega(E\Phi + \text{curl } E\text{curl } \Phi + \sigma\mu_1 E\Phi + \sigma\mu_2 e^{-\tau}E\Phi)dx\\ &= \int_\Omega(F\Phi + G\text{curl } \Phi)dx - \int_\Omega\sigma\mu_2 z_0\Phi dx, \quad \forall\Phi\in\mathcal{H}_0(\text{curl },\Omega).\end{aligned} \tag{2.58}$$

Since the left-hand side of (2.58) is coercive on $\mathcal{H}_0(\text{curl },\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $E\in\mathcal{H}_0(\text{curl },\Omega)$.

Now, put

$$H = G - \text{curl } E \in L^2(\Omega)^3. \tag{2.59}$$

Then, from (2.58) we obtain

$$\text{curl } H \in L^2(\Omega)^3 \quad \text{and} \quad \text{curl } H = E + \sigma\mu_1 E + \sigma\mu_2 z(\cdot, 1) - F.$$

Moreover, since $G \in \mathcal{H}_0(\text{div } 0, \Omega)$, from (2.59) we have

$$\text{div } H = \text{div } G = 0,$$

and

$$H \cdot \nu = -\text{curl } E \cdot \nu = 0 \text{ on } \Gamma.$$

Thus, $H$ belongs to $\mathcal{H}_0(\text{div } 0, \Omega)$. Moreover, $H$ belongs to the space

$$\mathcal{H}_T(\text{curl }, \text{div }, \Omega) := \{\Phi \in L^2(\Omega)^3 : \text{div } \Phi \in L^2(\Omega), \text{curl } \Phi \in L^2(\Omega)^3, \Phi \cdot \nu = 0 \text{ on } \Gamma\},$$

which is injected in $H^1(\Omega)^3$ since the boundary of $\Omega$ is smooth.

Therefore, $(E, H, z)^T$ belongs to $\mathcal{D}(\mathcal{A}^0)$. This concludes the proof of the maximality of $\mathcal{A}^0$. □

## 3. Boundary stability result

In this section, we will prove the exponential stability of problem (1.1)–(1.6) assuming (1.7).

Let us define the energy as

$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega \{|E(x,t)|^2 + |H(x,t)|^2\}dx + \frac{\xi}{2}\int_\Gamma \int_0^1 |E(x, t - \tau\rho) \times \nu|^2 d\rho d\Gamma, \tag{3.1}$$

where $\xi$ is a positive constant which satisfies

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2). \tag{3.2}$$

We have the following result.

**Proposition 3.1.** *For any strong solution of problem* (1.1)−(1.6) *the energy is decreasing and there exist two positive constants $C$ and $C'$ such that*

$$\mathcal{E}'(t) \le -C\int_\Gamma \{|E(x,t) \times \nu|^2 + |E(x, t-\tau) \times \nu|^2\}d\Gamma \tag{3.3}$$

*and*

$$\mathcal{E}'(t) \ge -C'\int_\Gamma \{|E(x,t) \times \nu|^2 + |E(x, t-\tau) \times \nu|^2\}d\Gamma. \tag{3.4}$$

**Proof.** Differentiating (3.1) we obtain

$$\mathcal{E}'(t) = \int_\Omega \{E\cdot E_t + H\cdot H_t\}dx + \xi\int_\Gamma\int_0^1 (E(x,t-\tau\rho)\times\nu)\cdot(E_t(x,t-\tau\rho)\times\nu)d\rho d\Gamma,$$

and then, using equations (1.1), (1.2) and integrating by parts,

$$\begin{aligned}\mathcal{E}'(t) = &-\int_\Gamma (H(x,t)\times\nu)\cdot E(x,t)d\Gamma \\ &+\xi\int_\Gamma\int_0^1 (E(x,t-\tau\rho)\times\nu)\cdot(E_t(x,t-\tau\rho)\times\nu)d\rho d\Gamma.\end{aligned} \tag{3.5}$$

Now, observe that

$$E_t(x,t-\tau\rho) = -\tau^{-1}E_\rho(x,t-\tau\rho).$$

Then, we can rewrite

$$\begin{aligned}&\int_\Gamma\int_0^1 (E(x,t-\tau\rho)\times\nu)\cdot(E_t(x,t-\tau\rho)\times\nu)d\rho d\Gamma \\ &= -\tau^{-1}\int_\Gamma\int_0^1 (E(x,t-\tau\rho)\times\nu)\cdot(E_\rho(x,t-\tau\rho)\times\nu)d\rho d\Gamma \\ &= -\frac{\tau^{-1}}{2}\int_\Gamma \{|E(x,t-\tau)\times\nu|^2 - |E(x,t)\times\nu|^2\}d\Gamma.\end{aligned} \tag{3.6}$$

Thus, from (3.6), we obtain

$$\begin{aligned}\mathcal{E}'(t) = &-\int_\Gamma (H(x,t)\times\nu)\cdot E(x,t)d\Gamma \\ &+\frac{\xi\tau^{-1}}{2}\int_\Gamma \{|E(x,t)\times\nu|^2 - |E(x,t-\tau)\times\nu|^2\}d\Gamma.\end{aligned} \tag{3.7}$$

From the boundary condition (1.4), we have

$$\begin{aligned}&\int_\Gamma (H(x,t)\times\nu)\cdot E(x,t)d\Gamma \\ &= \int_\Gamma (\mu_1 E(x,t)\times\nu + \mu_2 E(x,t-\tau)\times\nu)\cdot(E(x,t)\times\nu)d\Gamma \\ &= \mu_1\int_\Gamma |E(x,t)\times\nu|^2 d\Gamma + \mu_2\int_\Gamma (E(x,t-\tau)\times\nu)\cdot(E(x,t)\times\nu)d\Gamma.\end{aligned} \tag{3.8}$$

So, by (3.7) and (3.8), we can estimate

$$\mathcal{E}'(t) = -\Big(\mu_1 - \frac{\xi\tau^{-1}}{2}\Big)\int_\Gamma |E(x,t)\times\nu|^2 d\Gamma \tag{3.9}$$

$$- \mu_2 \int_\Gamma (E(x,t-\tau)\times\nu)\cdot(E(x,t)\times\nu)d\Gamma - \frac{\xi\tau^{-1}}{2}\int_\Gamma |E(x,t-\tau)\times\nu|^2 d\Gamma.$$

Therefore, by Schwarz's inequality,

$$\mathcal{E}'(t) \le -\Big(\mu_1 - \frac{\mu_2}{2} - \frac{\xi\tau^{-1}}{2}\Big)\int_\Gamma |E(x,t)\times\nu|^2 d\Gamma$$
$$- \Big(\frac{\xi\tau^{-1}}{2} - \frac{\mu_2}{2}\Big)\int_\Gamma |E(x,t-\tau)\times\nu|^2 d\Gamma.$$

Since the constant $\xi$ satisfies assumption (3.2), the upper bound (3.3) follows.

The lower bound (3.4) directly follows from (3.7) and (3.8), and the well-known estimate

$$2ab \ge -a^2 - b^2,$$

valid for any real numbers $a, b$. □

Now we give a boundary observability inequality which will be useful to deduce the exponential decay of the energy.

**Proposition 3.2.** *There exists a time* $\overline{T} > 0$ *such that for all times* $T > \overline{T}$ *there exists a positive constant* $C_0$ *(depending on* $T$*) for which*

$$\mathcal{E}(T) \le C_0 \int_0^T \int_\Gamma \{|E(x,t)\times\nu|^2 + |E(x,t-\tau)\times\nu|^2\} d\Gamma\, dt, \tag{3.10}$$

*for any regular solution of problem* (1.1) − (1.6).

**Proof.** From [3] (Corollary 3.4), there exists a positive time $T_0$ such that for any $\delta, \epsilon > 0$ and any time $T > T_0$, there exists a constant $\hat{C} = \hat{C}(T, \delta, \epsilon)$ for which

$$\begin{aligned} &\int_\delta^{T-\delta} \mathcal{E}_S(t)dt - \frac{T_0}{2}(\mathcal{E}_S(\delta) + \mathcal{E}_S(T-\delta)) \\ &\qquad \le \hat{C}\left\{\int_0^T \int_\Gamma |E(t)\times\nu|^2 + |H(t)\times\nu|^2 d\Gamma dt + l.o.t.(E,H)\right\}, \end{aligned} \tag{3.11}$$

where $\mathcal{E}_S$ denotes the standard energy for Maxwell's equations

$$\mathcal{E}_S(t) := \frac{1}{2}\int_\Omega (|E(t)|^2 + |H(t)|^2)dx$$

and $l.o.t.(E,H)$ denotes lower order terms, namely

$$l.o.t.(E,H) = \|(E,H)\|^2_{(H^{1/2-\epsilon}(\Omega\times(0,T))^3)'} + \|(E,H)\|^2_{H^{-1+\epsilon}(\Gamma\times(0,T))^3}.$$

The proof of this estimate is based on the multiplier method and requires some arguments from microlocal analysis.

Now, observe that, for $\delta > \tau$ and $T > 2\delta$,

$$\begin{aligned}
&\int_\delta^{T-\delta}\int_\Gamma\int_0^1 |E(x,t-\tau\rho)\times\nu|^2 d\rho d\Gamma dt\\
&\le \tau^{-1}\int_\delta^{T-\delta}\int_\Gamma\int_{t-\tau}^{t} |E(x,s)\times\nu|^2 ds d\Gamma dt\\
&\le \tau^{-1}\int_\delta^{T-\delta}\int_\Gamma\int_{t}^{t+\tau} |E(x,\sigma-\tau)\times\nu|^2 d\sigma d\Gamma dt\\
&\le \tau^{-1} T\int_\Gamma\int_0^T |E(x,t-\tau)\times\nu|^2 d\Gamma dt.
\end{aligned}$$

So, from (3.11), adding to both sides

$$\frac{\xi}{2}\int_\delta^{T-\delta}\int_\Gamma\int_0^1 |E(x,t-\tau\rho)\times\nu|^2 d\rho d\Gamma dt,$$

and using the boundary condition (1.4), we obtain

$$\begin{aligned}
&\int_\delta^{T-\delta}\mathcal{E}(t)dt - \frac{T_0}{2}(\mathcal{E}(\delta)+\mathcal{E}(T-\delta))\\
&\le \overline{C}\left\{\int_0^T\int_\Gamma (|E(t)\times\nu|^2 + |E(t-\tau)\times\nu|^2)d\Gamma dt + l.o.t.(E,H)\right\},
\end{aligned}\tag{3.12}$$

for a suitable positive constant $\overline{C} > 0$ depending on $T$.

Note that, since the energy $\mathcal{E}$ is decreasing, for $T > 2\delta$, we have

$$\mathcal{E}(T-\delta) \le \mathcal{E}(\delta) \quad \text{and} \quad \int_\delta^{T-\delta}\mathcal{E}(t)dt \ge (T-2\delta)\mathcal{E}(T).$$

Then, from (3.12), we obtain

$$\begin{aligned}
&(T-2\delta)\mathcal{E}(T)\\
&\le \overline{C}\left\{\int_0^T\int_\Gamma (|E(t)\times\nu|^2 + |E(t-\tau)\times\nu|^2)d\Gamma dt + l.o.t.(E,H)\right\} + T_0\mathcal{E}(\delta).
\end{aligned}\tag{3.13}$$

Adding $-T_0\mathcal{E}(T)$ to both sides of (3.13) and recalling (3.3), we obtain for $T > T_0 + 2\delta$,

$$\begin{aligned}
\mathcal{E}(T) \le \frac{\overline{C}}{T-2\delta-T_0}\Big\{&\int_0^T\int_\Gamma (|E(t)\times\nu|^2 + |E(t-\tau)\times\nu|^2)d\Gamma dt\\
&+ l.o.t.(E,H)\Big\} + \frac{C'T_0}{T-2\delta-T_0}\int_0^T\int_\Gamma (|E(t)\times\nu|^2 + |E(t-\tau)\times\nu|^2)d\Gamma dt.
\end{aligned}\tag{3.14}$$

Now, applying a compactness-uniqueness result as in [2] (page 156) we obtain, from (3.14), the observability inequality (3.10). □

We are ready to give the exponential stability result.

**Theorem 3.3.** *Let the assumption* (1.7) *be satisfied. Then, there exist positive constants* $C_1$ *and* $C_2$ *such that, for any regular solution of problem* (1.1) – (1.6),

$$\mathcal{E}(t) \leq C_1 \mathcal{E}(0) e^{-C_2 t}, \quad \forall t \geq 0. \tag{3.15}$$

**Proof.** From (3.3),

$$\mathcal{E}(T) - \mathcal{E}(0) \leq -C \int_0^T \int_\Gamma (|E(t) \times \nu|^2 + |E(t-\tau) \times \nu|^2) d\Gamma dt. \tag{3.16}$$

From (3.16) and the observability inequality (3.10), we obtain

$$\mathcal{E}(T) \leq C_0 \int_0^T \int_\Gamma (|E(t) \times \nu|^2 + |E(t-\tau) \times \nu|^2) d\Gamma dt \leq C_0 C^{-1} (\mathcal{E}(0) - \mathcal{E}(T)).$$

Then

$$\mathcal{E}(T) \leq \tilde{C} \mathcal{E}(0),$$

with $\tilde{C} < 1$. This easily implies the exponential stability estimate (3.15) since the system (1.1) – (1.6) is invariant under translation and the energy $\mathcal{E}$ is decreasing. □

## 4. Internal stability result

Here, under the assumption (1.7), we will prove that problem (1.8)–(1.13) is exponentially stable.

Let us define the energy as

$$\mathcal{F}(t) := \frac{1}{2} \int_\Omega \{|E(x,t)|^2 + |H(x,t)|^2\} dx + \frac{\xi}{2} \int_\Omega \sigma(x) \int_0^1 |E(x, t - \tau\rho)|^2 d\rho dx, \tag{4.1}$$

where $\xi$ is a positive real number satisfying (3.2).

We can give a first estimate on the derivative of the energy $\mathcal{F}$.

**Proposition 4.1.** *For any strong solution of problem* (1.8) – (1.13) *the energy is decreasing and there exists a positive constant* $C$ *such that*

$$\mathcal{F}'(t) \leq -C \int_\Omega \sigma(x) \{|E(x,t)|^2 + |E(x, t-\tau)|^2\} dx. \tag{4.2}$$

**Proof.** Differentiating (4.1) we obtain

$$\mathcal{F}'(t) = \int_\Omega \{E \cdot E_t + H \cdot H_t\} dx + \xi \int_\Omega \sigma(x) \int_0^1 E(x, t-\tau\rho) \cdot E_t(x, t-\tau\rho) d\rho dx,$$

and then, using (1.8), (1.9), the boundary conditions (1.11), and integrating by parts,

$$\begin{aligned} \mathcal{F}'(t) = &- \int_\Omega \mu_1 \sigma(x) |E(x,t)|^2 dx - \int_\Omega \mu_2 \sigma(x) E(x,t) \cdot E(x, t-\tau) dx \\ &+\xi \int_\Omega \sigma(x) \int_0^1 E(x, t-\tau\rho) \cdot E_t(x, t-\tau\rho) d\rho dx. \end{aligned} \tag{4.3}$$

Now, as in the proof of Proposition 3.1, we can compute

$$\begin{aligned} &\int_\Omega \sigma(x) \int_0^1 E(x, t-\tau\rho) \cdot E_t(x, t-\tau\rho) d\rho dx \\ &\quad = -\tau^{-1} \int_\Omega \sigma(x) \int_0^1 E(x, t-\tau\rho) \cdot E_\rho(x, t-\tau\rho) d\rho dx \\ &\quad = -\frac{\tau^{-1}}{2} \int_\Omega \sigma(x) \{|E(x, t-\rho)|^2 - |E(x,t)|^2\} dx. \end{aligned} \tag{4.4}$$

Using (4.4) in identity (4.3), we have

$$\begin{aligned} \mathcal{F}'(t) = &-\mu_1 \int_\Omega \sigma(x) |E(x,t)|^2 dx - \mu_2 \int_\Omega \sigma(x) E(x,t) \cdot E(x, t-\tau) dx \\ &+\frac{\xi\tau^{-1}}{2} \int_\Omega \sigma(x) |E(x,t)|^2 dx - \frac{\xi\tau^{-1}}{2} \int_\Omega \sigma(x) |E(x, t-\tau)|^2 dx. \end{aligned} \tag{4.5}$$

From (4.5), applying Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} \mathcal{F}'(t) = &-\Big(\mu_1 - \frac{\mu_2}{2} - \frac{\xi\tau^{-1}}{2}\Big) \int_\Omega \sigma(x) |E(x,t)|^2 dx \\ &-\Big(\frac{\xi\tau^{-1}}{2} - \frac{\mu_2}{2}\Big) \int_\Omega \sigma(x) |E(x, t-\tau)|^2 dx. \end{aligned} \tag{4.6}$$

Therefore, recalling the assumption (3.2) on the constant $\xi$, estimate (4.2) immediately follows. □

Consider the homogeneous problem

$$E_{ht}(x,t) - \text{curl } H_h(x,t) = 0 \text{ in } \Omega \times (0, +\infty) \tag{4.7}$$

$$H_{ht}(x,t) + \text{curl } E_h(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.8}$$

$$\text{div } E_h(x,t) = \text{div } H_h(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.9}$$

$$E_h(x,t) \times \nu = 0 \quad \text{and} \quad H_h(x,t) \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0, +\infty) \tag{4.10}$$

$$E_h(x,0) = E_0(x) \quad \text{and} \quad H_h(x,0) = H_0(x) \quad \text{in} \quad \Omega. \tag{4.11}$$

We recall the following result from [7].

**Proposition 4.2.** *Let $(E_h, H_h)$ be the solution of the homogeneous problem $(4.7) - (4.11)$ and let $\omega \subset \overline{\Omega}$ be an open neighborhood of $\Gamma$. Then, there exists a time $\overline{T} > 0$ such that for every time $T > \overline{T}$,*

$$\int_\Omega (|E_h(x,0)|^2 + |H_h(x,0)|^2) dx \le C_0 \int_0^T \int_\omega |E_h(x,t)|^2 dx dt, \tag{4.12}$$

*for a suitable positive constant $C_0$ depending on $T$ and independent of the initial datum $(E_0, H_0)$.*

Using Proposition 4.2, we can prove an internal observability estimate for problem (1.8)–(1.13).

**Proposition 4.3.** *There exists a time $T_0 > 0$ such that for all times $T > \overline{T}_0$ there exists a positive constant $\hat{C}$ (depending on $T$) for which*

$$\mathcal{F}(0) \le \hat{C} \int_0^T \int_\Omega \sigma(x) \{ |E(x,t)|^2 + |E(x,t-\tau)|^2 \} dx dt, \tag{4.13}$$

*for any regular solution $(E, H)$ of problem $(1.8) - (1.13)$.*

**Proof.** As in [7] (cfr. Zuazua [11]), we can decompose the solution $(E, H)$ of problem (1.8)–(1.13) as

$$(E, H) = (E_h, H_h) + (\tilde{E}, \tilde{H}),$$

where $(E_h, H_h)$ solves the homogeneous problem (4.7)–(4.10) with initial condition

$$E_h(x,0) = E_0(x), \quad H_h(x,0) = H_0(x), \quad \text{in } \Omega,$$

and $(\tilde{E}, \tilde{H})$ satisfies

$$\tilde{E}_t(x,t) - \text{curl } \tilde{H}(x,t) = -\sigma(x)[\mu_1 E(x,t) + \mu_2 E(x,t-\tau)] \text{ in } \Omega \times (0, +\infty) \tag{4.14}$$

$$\tilde{H}_t(x,t) + \text{curl } \tilde{E}(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.15}$$

$$\text{div } \tilde{H}(x,t) = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \tag{4.16}$$

$$\tilde{E}(x,t) \times \nu = 0 \quad \text{and} \quad \tilde{H}(x,t) \cdot \nu = 0 \quad \text{on} \quad \Gamma \times (0, +\infty) \tag{4.17}$$

$$\tilde{E}(x,0) = 0 \quad \text{and} \quad \tilde{H}(x,0) = 0 \quad \text{in} \quad \Omega. \tag{4.18}$$

By definition (4.1),

$$\begin{aligned}\mathcal{F}(0) &= \frac{1}{2}\int_{\Omega}\{|E(x,0)|^2+|H(x,0)|^2\}dx+\frac{\xi}{2}\int_{\Omega}\sigma(x)\int_0^1|E(x,-\tau\rho)|^2d\rho dx\\ &= \frac{1}{2}\int_{\Omega}\{|E_h(x,0)|^2+|H_h(x,0)|^2\}dx+\frac{\xi}{2}\int_{\Omega}\sigma(x)\int_0^1|E(x,-\tau\rho)|^2d\rho dx.\end{aligned}$$

Then, for $T>\tau$, using a change of variable in the last integral of the above identity, we obtain

$$\mathcal{F}(0)\leq\frac{1}{2}\int_{\Omega}\{|E_h(x,0)|^2+|H_h(x,0)|^2\}dx+c\int_{\Omega}\sigma(x)\int_0^T|E(x,t-\tau)|^2dtdx,$$

for some positive constant $c$.

Now, let $\overline{T}$ be as in Proposition 4.2. Therefore, using the observability estimate (4.12) for the homogeneous problem, we have for $T>T_0=\max\{\tau,\overline{T}\}$,

$$\begin{aligned}\mathcal{F}(0)&\leq C_0\int_0^T\int_{\omega}|E_h(x,t)|^2dxdt+c\int_{\Omega}\sigma(x)\int_0^T|E(x,t-\tau)|^2dtdx\\ &\leq C'\Big\{\int_0^T\int_{\Omega}\sigma(x)(|E(x,t)|^2+|\tilde{E}(x,t)|^2)dxdt \qquad (4.19)\\ &\qquad +\int_{\Omega}\sigma(x)\int_0^T|E(x,t-\tau)|^2dtdx\Big\},\end{aligned}$$

for a suitable positive constant $C'$.

From (4.19) and standard energy estimates for problem (4.14)–(4.18), we easily obtain

$$\mathcal{F}(0)\leq\hat{C}\int_0^T\int_{\Omega}\sigma(x)\{|E(x,t)|^2+|E(x,t-\tau)|^2\}dxdt. \qquad \square$$

Now, using estimate (4.13), as in the case of boundary feedback we can deduce the exponential stability result.

**Theorem 4.4.** *Let the assumption* (1.7) *be satisfied. Then, there exist positive constants* $C_1$ *and* $C_2$ *such that , for any regular solution of problem* (1.8) – (1.13),

$$\mathcal{F}(t)\leq C_1\mathcal{F}(0)e^{-C_2t},\quad\forall t\geq 0. \qquad (4.20)$$

## 5. Some instability examples

In this section we will give some instability examples in the case $\mu_2\geq\mu_1$.

5.1. **Boundary feedback.** In this subsection we consider the problem with boundary feedback (1.1)–(1.6), and prove the following result.

**Theorem 5.1.** *If* (1.7) *does not hold, then there exist a sequence of delays and solutions of problem* (1.1) – (1.6)*, corresponding to these delays, such that their standard energy is larger than a positive constant.*

**Proof.** Let us consider the spectral problem for the system (1.1)–(1.4) by seeking a solution in the form

$$E(x,t) = e^{\lambda t} e(x), \quad H(x,t) = e^{\lambda t} h(x), \quad \lambda \in \mathbb{C}. \tag{5.1}$$

Then, $(e,h)$ has to be a solution of the eigenvalue problem

$$\begin{cases} \lambda e(x) - \text{curl } h(x) = 0 \quad \text{in} \quad \Omega \\ \lambda h(x) + \text{curl } e(x) = 0 \quad \text{in} \quad \Omega \\ \text{div } e(x) = \text{div } h(x) = 0 \quad \text{in} \quad \Omega \\ ((\mu_1 + \mu_2 e^{-\lambda\tau}) e \times \nu + h) \times \nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.2}$$

Assuming that $\lambda$ is different from 0, we can eliminate $h$, by the second equation, namely $h = -\lambda^{-1}\text{curl } e$ and consequently $e$ is a solution of

$$\begin{cases} \lambda^2 e(x) + \text{curl curl } e(x) = 0 \quad \text{in} \quad \Omega \\ \text{div } e(x) = 0 \quad \text{in} \quad \Omega \\ (\lambda(\mu_1 + \mu_2 e^{-\lambda\tau}) e \times \nu - \text{curl } e) \times \nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.3}$$

This problem can be reformulated, in a variational form, as

$$\int_\Omega \text{curl } e \cdot \text{curl } \overline{v} dx + \lambda^2 \int_\Omega e \cdot \overline{v} dx + (\mu_1 + \mu_2 e^{-\lambda\tau})\lambda \int_\Gamma e \times \nu \cdot \overline{v} \times \nu d\Gamma = 0, \tag{5.4}$$

for all $v \in V$, where $V$ is defined in (2.18).

As in [8] we want to find a solution for $\lambda := ib$, with $b \in \mathbb{R}$. For this choice of $\lambda$ the problem (5.4) can be rewritten as

$$\int_\Omega \text{curl } e \cdot \text{curl } \overline{v} dx - b^2 \int_\Omega e \cdot \overline{v} dx + (\mu_1 + \mu_2 e^{-ib\tau}) ib \int_\Gamma e \times \nu \cdot \overline{v} \times \nu d\Gamma = 0, \tag{5.5}$$

for all $v \in V$. Assume that

$$\cos(b\tau) = -\frac{\mu_1}{\mu_2}. \tag{5.6}$$

Note that, since we are considering the case $\mu_2 \geq \mu_1$, there exist $b, \tau$ such that (5.6) holds. Then, we choose

$$\mu_2 \sin(b\tau) = \sqrt{\mu_2^2 - \mu_1^2}. \tag{5.7}$$

Under these assumptions, (5.5) becomes

$$\int_\Omega \text{curl } e \cdot \text{curl } \overline{v} dx - b^2 \int_\Omega e \cdot \overline{v} dx + b\sqrt{\mu_2^2 - \mu_1^2} \int_\Gamma e \times \nu \cdot \overline{v} \times \nu d\Gamma = 0, \quad (5.8)$$

for all $v \in V$. In particular, for $v = e$, (5.8) gives

$$\int_\Omega |\text{curl } e|^2 dx - b^2 \int_\Omega |e|^2 dx + b\sqrt{\mu_2^2 - \mu_1^2} \int_\Gamma |e \times \nu|^2 d\Gamma = 0. \quad (5.9)$$

Without loss of generality we can assume

$$\|e\|_2^2 := \int_\Omega |e|^2 dx = 1 \quad (5.10)$$

and then, the identity (5.9) can be rewritten as

$$b^2 - b\sqrt{\mu_2^2 - \mu_1^2} q_0(e) - q_1(e) = 0, \quad (5.11)$$

where

$$q_0(\varphi) := \int_\Gamma |\varphi \times \nu|^2 d\Gamma, \quad q_1(\varphi) := \int_\Omega |\text{curl } \varphi|^2 dx. \quad (5.12)$$

Now we distinguish two cases.

**Case (a)** $\mu_2 > \mu_1$. In this case, from (5.11) we have

$$b = \frac{1}{2}\left(\sqrt{\mu_2^2 - \mu_1^2} q_0(e) \pm \sqrt{(\mu_2^2 - \mu_1^2) q_0^2(e) + 4q_1(e)}\right).$$

Write for brevity

$$l(w) = \sqrt{\mu_2^2 - \mu_1^2} q_0(w) + \sqrt{(\mu_2^2 - \mu_1^2) q_0^2(w) + 4q_1(w)}.$$

Define

$$b := \frac{1}{2} \min_{\substack{w \in V \\ \|w\|_2 = 1}} l(w). \quad (5.13)$$

Let us first show that this minimum is positive. Indeed if this is not the case, there exists a sequence of $w_n \in V, n \in \mathbb{N}$, such that

$$l(w_n) \leq \frac{1}{n}, \quad (5.14)$$

$$\|w_n\|_2 = 1, \ \forall n \in \mathbb{N}. \quad (5.15)$$

As

$$\sqrt{\mu_2^2 - \mu_1^2} q_0(w) \leq l(w) \quad \text{and } 4q_1(w) \leq l(w)^2, \quad (5.16)$$

the above properties imply that there exists $C > 0$ such that

$$\|w_n\|_V \leq C, \ \forall n \in \mathbb{N}.$$

Consequently the sequence $(w_n)_n$ is bounded in $V$. Since $V$ is embedded into $H^{1/2}(\Omega)^3$ (see Theorem 2 of [1]), by the Rellich-Kondracov theorem $V$ is compactly embedded into $L^2(\Omega)^3$. Therefore there exists a subsequence, still denoted by $(w_n)_n$ for the sake of simplicity, and $w \in V$ such that

$$w_n \to w \text{ weakly in } V \text{ and } w_n \to w \text{ strongly in } L^2(\Omega)^3. \tag{5.17}$$

Coming back to (5.14)–(5.15) and using again (5.16), we deduce that

$$w_n \to w \text{ strongly in } V,$$

with

$$q_0(w) = q_1(w) = 0, \tag{5.18}$$

$$\|w\|_2 = 1, \ \forall n \in \mathbb{N}. \tag{5.19}$$

This implies that $w \in V$ satisfies

$$\text{curl } w = 0, \ \text{div } w = 0 \text{ in } \Omega,$$
$$w \times \nu = 0 \text{ on } \Gamma,$$

and due to our assumptions on $\Omega$ and its boundary we deduce that $w = 0$, which contradicts (5.19). This proves that $b > 0$.

Let us now show that the minimum in (5.13) is hit at an element $e \in V$. Let us consider a minimizing sequence $(w_n)_n$, namely a sequence of $w_n \in V$ satisfying (5.15) and such that

$$l(w_n) \to 2b \text{ as } n \to \infty. \tag{5.20}$$

As before, using (5.16), the sequence $(w_n)_n$ will be bounded in $V$. Therefore, there exist a subsequence, still denoted by $(w_n)_n$ for the sake of simplicity, and $e \in V$ such that

$$w_n \to e \text{ weakly in } V \text{ and } w_n \to e \text{ strongly in } L^2(\Omega)^3. \tag{5.21}$$

Now we remark that from the definition of $b$ and (5.20), we have

$$b^2 = \lim_{n\to\infty} \left( b\sqrt{\mu_2^2 - \mu_1^2} q_0(w_n) + q_1(w_n) \right). \tag{5.22}$$

Indeed, if we denote

$$b_n := \frac{l(w_n)}{2},$$

then $b_n$ satisfies

$$b_n^2 - b_n\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) - q_1(w_n) = 0;$$

that is,

$$b_n^2 = b_n\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) + q_1(w_n).$$

So, since $b_n \to b$ as $n \to \infty$, we have that

$$b^2 = \lim_{n\to\infty}[b_n\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) + q_1(w_n)]. \tag{5.23}$$

From (5.16) and (5.20) the sequence $(q_0(w_n))_n$ is bounded and therefore, since $b_n \to b$,

$$(b_n - b)\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) \to 0. \tag{5.24}$$

By (5.23) and (5.24) we obtain (5.22).

Moreover, the quantity

$$\int_\Omega w\cdot w'dx + b\sqrt{\mu_2^2 - \mu_1^2}\int_\Gamma w\times\nu\cdot w'\times\nu d\Gamma + \int_\Omega \text{curl } w\cdot\text{curl } w'dx,\ \forall w, w' \in V$$

defines an inner product in $V$, whose associated norm is equivalent to the natural one. Consequently, from (5.21), we have

$$\|e\|_2^2 + b\sqrt{\mu_2^2 - \mu_1^2}q_0(e) + q_1(e) \le \lim_{n\to\infty}\left(\|w_n\|_2^2 + b\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) + q_1(w_n)\right).$$

The strong convergence of $w_n$ to $e$ in $L^2(\Omega)^3$ and (5.15) further imply that $\|e\|_2 = 1$, and the above property becomes

$$b\sqrt{\mu_2^2 - \mu_1^2}q_0(e) + q_1(e) \le \lim_{n\to\infty}\left(b\sqrt{\mu_2^2 - \mu_1^2}q_0(w_n) + q_1(w_n)\right).$$

Due to (5.22), we conclude that $b\sqrt{\mu_2^2 - \mu_1^2}q_0(e) + q_1(e) \le b^2$. This prove that $l(e) \le 2b$, because $b > 0$ and $\sqrt{\mu_2^2 - \mu_1^2}q_0(e) - \sqrt{(\mu_2^2 - \mu_1^2)q_0^2(e) + 4q_1(e)} \le 0$. Since $2b \le l(e)$, we finally obtain $l(e) = 2b$.

As in [8] we prove that if the minimum in the right–hand side of (5.13) is attained at $e$; that is,

$$\sqrt{\mu_2^2 - \mu_1^2}q_0(e) + \sqrt{(\mu_2^2 - \mu_1^2)q_0^2(e) + 4q_1(e)} := \min_{\substack{w \in V \\ \|w\|_2 = 1}} \left(\sqrt{\mu_2^2 - \mu_1^2}q_0(w) + \sqrt{(\mu_2^2 - \mu_1^2)q_0^2(w) + 4q_1(w)}\right), \tag{5.25}$$

then $e$ is a solution of (5.8) with $b$ as in (5.13). So, for such positive $b$,

$$b\tau = \arccos\left(-\frac{\mu_1}{\mu_2}\right) + 2l\pi, \quad l \in \mathbb{N},$$

defines a sequence of time delays for which the problem (1.1)–(1.4) is not asymptotically stable, since the standard energy

$$\int_\Omega (|E(x,t)|^2 + |H(x,t)|^2)dx \geq 1, \quad \forall t \geq 0.$$

**Case (b)** $\mu_1 = \mu_2$. In this case, under our assumptions, (5.11) becomes

$$b^2 = q_1(e). \tag{5.26}$$

But in this case the quantity

$$\min_{\substack{w \in V \\ \|w\|_2 = 1}} \quad q_1(w) \tag{5.27}$$

is equal to zero and therefore we cannot conclude as before.

Consequently, we come back to (5.2) and remark that with the choice $\lambda = ib$, $\cos(b\tau) = -1$, $\sin(b\tau) = 0$, the factor $(\mu_1 + \mu_2 e^{-\lambda\tau})$ in front of $e \times \nu$ is zero. This means that the boundary condition on $\Gamma$ is simply

$$h \times \nu = 0 \text{ on } \Gamma,$$

which is the standard "electric" boundary condition. We then eliminate $e$ instead of $h$, in other words, we use the identity $e = \lambda^{-1}\text{curl } h$ and consequently $h$ is a solution of

$$\begin{cases} \lambda^2 h(x) + \text{curl curl } h(x) = 0 \quad \text{in} \quad \Omega \\ \text{div } h(x) = 0 \quad \text{in} \quad \Omega \\ h \times \nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.28}$$

This problem is the classical eigenvalue problem for the Maxwell system with electric boundary condition. So, we can take a sequence $\{b_n\}_n$ of positive real numbers defined by

$$b_n^2 = \Lambda_n^2, \quad n \in \mathbb{N},$$

where $\Lambda_n^2$, $n \in \mathbb{N}$, are the eigenvalues for the above problem [6]. Then, putting

$$b_n\tau = (2l+1)\pi, \quad l \in \mathbb{N},$$

we obtain a sequence of delays

$$\tau_{n,l} = \frac{(2l+1)\pi}{b_n}, \quad l, n \in \mathbb{N},$$

which become arbitrarily small (or large) for suitable choices of the indices $n, l \in \mathbb{N}$. Therefore, in the case $\mu_1 = \mu_2$, we have found a set of time delays for which problem (1.1)–(1.4) is not asymptotically stable. Indeed, if $h$ is a solution of (5.28) with the above choice of $\lambda, b$ and $\tau$, then,

$$E(x,t) := -ib^{-1}e^{ibt}\text{curl } h(x) \quad H(x,t) := e^{ibt}h(x) \tag{5.29}$$

is a solution of problem (1.1)–(1.4). Therefore, we have found a solution of our boundary-value problem whose energy is constant. Indeed, an easy computation shows that, for the pair $(E,H)$ defined in (5.29),

$$\int_\Omega (|E(x,t)|^2 + |H(x,t)|^2)dx = 2\int_\Omega |h(x)|^2 dx > 0, \quad \forall t \geq 0.$$

The above examples prove Theorem 5.1. □

5.2. **Internal feedback.** In this subsection we will give instability examples for the problem with internal feedback (1.8)–(1.13), proving the following result.

**Theorem 5.2.** *If* (1.7) *does not hold, there exist a sequence of arbitrarily small* (*or large*) *delays and solutions of problem* (1.8) − (1.13), *corresponding to these delays, such that their standard energy does not tend to* 0.

**Proof.** Let us consider the spectral problem for the system (1.8)–(1.11) in the case $\sigma(x) \equiv 1$ in $\Omega$. Namely we seek a solution of (1.8)–(1.11) in the form (5.1). Then, the pair $(e,h)$ has to solve the eigenvalue problem

$$\begin{cases} \lambda e(x) - \text{curl } h(x) + [\mu_1 e(x) + \mu_2 e^{-\lambda\tau} e(x)] = 0 \text{ in } \Omega \\ \lambda h(x) + \text{curl } e(x) = 0 \quad \text{in} \quad \Omega \\ \text{div } e(x) = \text{div } h(x) = 0 \quad \text{in} \quad \Omega \\ e(x) \times \nu = 0 \quad \text{and} \quad h(x)\cdot\nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.30}$$

As before assuming that $\lambda$ is different from 0, we can eliminate $h$, by the second equation, namely $h = -\lambda^{-1}\text{curl } e$ and consequently $e$ is a solution of

$$\begin{cases} \text{curl curl } e = -\lambda(\lambda + \mu_1 + \mu_2 e^{-\lambda\tau})e \text{ in } \Omega \\ \text{div } e = 0 \quad \text{in} \quad \Omega \\ e \times \nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.31}$$

Let us consider the standard problem for the Maxwell system with electric boundary condition [6]

$$\begin{cases} \text{curl curl } e = -\mu^2 e \quad \text{in } \Omega \\ \text{div } e = 0 \quad \text{in} \quad \Omega \\ e \times \nu = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{5.32}$$

Following [8], we show that for any $\Lambda^2$ eigenvalue of problem (5.32), there exists a solution $\lambda \in \mathbb{C}$ of the equation

$$\lambda^2 + (\mu_1 + \mu_2 e^{-\lambda\tau})\lambda = -\Lambda^2. \tag{5.33}$$

We seek a solution $\lambda = \alpha + i\beta,\ \alpha, \beta \in \mathbb{R}$, with

$$\beta\tau = (2l+1)\pi, \quad l \in \mathbb{N}. \tag{5.34}$$

Under this assumption the equation (5.33) becomes

$$\begin{cases} \alpha^2 + \beta^2 = \Lambda^2 \\ \mu_2 e^{-\alpha\tau} = 2\alpha + \mu_1. \end{cases} \tag{5.35}$$

Now, we distinguish two cases.

**Case (a)** $\mu_1 = \mu_2$. In this case, from (5.35) we have

$$\alpha = 0, \quad \beta^2 = \Lambda^2.$$

Therefore, for any $\Lambda_n^2$ eigenvalue of problem (5.32), if $\beta_n \in \mathbb{R}$ satisfies

$$\beta_n^2 = \Lambda_n^2,$$

then for $\lambda = i\beta_n$ problem (5.30) admits a non–zero solution.

Take $\beta_n$ positive. From our assumption (5.34)

$$\tau_{n,l} = \frac{(2l+1)\pi}{\beta_n}, \quad n, l \in \mathbb{N},$$

is a set of time delays that become arbitrarily small (or large) for suitable choices of the indices $n, l \in \mathbb{N}$. For such delays the problem (1.8)–(1.13) admits solutions in the form

$$E(x,t) = e^{i\beta t} e(x), \quad H(x,t) = e^{i\beta t} h(x),$$

whose standard energy is constant and strictly positive. So, system (1.8)–(1.13) is not asymptotically stable.

**Case (b)** $\mu_2 > \mu_1$. For a fixed $\alpha > 0$, from the second equation of (5.35), we obtain

$$\tau(\alpha) = \frac{1}{\alpha} \ln\left(\frac{\mu_2}{\mu_1 + 2\alpha}\right), \tag{5.36}$$

and so, in order to have $\tau(\alpha) > 0$, we consider $0 < \alpha < \frac{1}{2}(\mu_2 - \mu_1)$. From (5.34), the first equation of (5.35) becomes

$$\alpha^2 + \frac{(2l+1)^2\pi^2}{\tau^2(\alpha)} = \Lambda^2, \tag{5.37}$$

where $\tau(\alpha)$ is given by (5.36).

It is easy to verify (see [8]) that for any fixed $\Lambda^2$ eigenvalue of problem (5.32) there exists $\alpha$ $(0 < \alpha < (\mu_2 - \mu_1)/2)$ such that (5.37) is satisfied. Therefore, for such $\alpha$ there exists a delay $\tau(\alpha)$ (defined by (5.36)) such that a function of the form $e^{\alpha+i\beta}(e(x), h(x))$ solves problem (1.8)–(1.13). Since $\alpha \geq 0$ the energy of such a solution is not decaying to zero. So, this solution is not asymptotically stable.

Note that, for any $\Lambda_n^2$ eigenvalue of problem (5.32) and for any $l \in \mathbb{N}$ there exist $\alpha_{n,l}$ and a delay $\tau_{n,l} = \tau(\alpha_{n,l})$ such that (5.35) is satisfied with

$$\beta_{n,l} = \frac{(2l+1)\pi}{\tau_{n,l}}.$$

From the first equation of (5.35),

$$\frac{(2l+1)^2\pi^2}{\tau_{n,l}^2} \leq \Lambda_n^2.$$

Then, for a fixed $l \in \mathbb{N}$, if $n \to +\infty$, then $\tau_{n,l} \to 0^+$. On the contrary, for a fixed $n \in \mathbb{N}$, for $l \to +\infty$, then $\tau_{n,l} \to +\infty$. Therefore, we have instability phenomena for a sequence of arbitrarily small or large time delays.

The examples of case (a) and case (b) prove Theorem 5.2.

## REFERENCES


[1] M. Costabel, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains,* Math. Methods Appl. Sc., 12 (1990), 365–368.

[2] M. Eller, J. Lagnese, and S. Nicaise, *Decay rates for solutions of Maxwell system with nonlinear boundary damping,* Comput. Appl. Math., 21 (2002), 135–165.

[3] M. Eller and J. Masters, *Exact boundary controllability of electromagnetic fields in a general region*, Appl. Math. Optim., 45 (2002), 99–123.

[4] B. V. Kapitonov, *Stabilization and exact boundary controllability for Maxwell's equations*, SIAM J. Control Optim., 32 (1994), 408–420.

[5] V. Komornik, *Boundary stabilization, observation and control of Maxwell's equations*, PanAm. Math. J., 4 (1994), 47–61.

[6] P. Monk, "Finite Element Methods for Maxwell's Equations," Numer. Math. Scientific Comp., Oxford Univ. Press, New York, 2003.

[7] S. Nicaise and C. Pignotti, *Internal stabilization of Maxwell's equations in heterogeneous media*, Abstr. Appl. Anal., 7 92005), 791–811.

[8] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, Siam J. Control Opt., to appear.

[9] K. D. Phung, *Contrôle et stabilisation d'ondes électromagnétiques*, ESAIM: Control Optim. Calc. Var., 5 (2000), 87–137.

[10] G. Q. Xu, S. P. Yung, and L. K. Li. *Stabilization of wave systems with input delay in the boundary control*, ESAIM: Control Optim. Calc. Var., to appear.

[11] E. Zuazua, *Exponential decay for the semi-linear wave equation with locally distributed damping,* Comm. Partial Differential Equations, 15 (1990), 205–235.