Advances in Differential Equations Volume 12, Number 1 (2007), 1–26

GROW-UP RATE OF SOLUTIONS OF A SEMILINEAR PARABOLIC EQUATION WITH A CRITICAL EXPONENT

Marek Fila
Department of Applied Mathematics and Statistics
Comenius University, 84248 Bratislava, Slovakia

John R. King
Division of Theoretical Mechanics, University of Nottingham
Nottingham NG7 2RD, UK

Michael Winkler
Department of Mathematics I, RWTH Aachen, 52056 Aachen, Germany

Eiji Yanagida
Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

(Submitted by: Juan Luis Vazquez)

Abstract. We consider the Cauchy problem for a semilinear parabolic equation with a nonlinearity which is critical in the Joseph-Lundgren sense. We find the grow-up rate of solutions that approach a singular steady state from below as $t \to \infty$. The grow-up rate in the critical case contains a logarithmic term which does not appear in the Joseph-Lundgren supercritical case, making the calculations more delicate.

1. Introduction

In this paper we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplace operator with respect to x, and u_0 is a nonnegative continuous function on $\mathbb{R}^N$. It is known that if

$$p \geq p_c := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, \qquad N > 10,$$

Accepted for publication: October 2006.
AMS Subject Classifications: 35K57, 35B40, 35B33.

then there is a completely ordered family of positive radial steady states. Namely, the elliptic equation

$$\Delta\varphi + \varphi^p = 0 \quad \text{on } \mathbb{R}^N$$

has a family of positive radial solutions $\varphi = \varphi_\alpha(|x|)$, where $\varphi = \varphi_\alpha(r)$, $r = |x|$, is a positive decreasing function satisfying

$$\begin{cases} \varphi_{rr} + \frac{N-1}{r}\varphi_r + \varphi^p = 0, & r > 0, \\ \varphi(0) = \alpha, \qquad \varphi'(0) = 0, \end{cases}$$

and moreover, the solution φ_α is monotone increasing in α for each $r \geq 0$ (see [6, 7, 10]). We note that φ_α satisfies

$$\lim_{\alpha\to 0} \varphi_\alpha(|x|) = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \varphi_\alpha(|x|) = L|x|^{-m},$$

where $\varphi = L|x|^{-m}$ ($x \neq 0$) is a singular steady state with $m := \frac{2}{p-1}$, $L := \{m\,(N-2-m)\}^{\frac{1}{p-1}}$. If u_0 satisfies

$$0 \leq u_0(x) \leq L|x|^{-m} \quad \text{for } x \neq 0, \tag{1.2}$$

then the solution of (1.1) exists globally in time (see [9]) and, by comparison, the solution remains between the trivial steady state and the singular steady state for all $t > 0$. A remarkable feature of (1.1) in the case of $p \geq p_c$ is the existence of global unbounded solutions. Indeed, [9] gave a sufficient condition on u_0 for which the solution of (1.1) approaches the singular steady state from below. In our previous papers [2, 3], we determined the grow-up rate of such solutions in the supercritical case $p > p_c$ when $u_0(x)$ is close to $L|x|^{-m}$ near infinity with some algebraic order. To describe the result, we introduce a quadratic equation

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0, \tag{1.3}$$

which has two distinct positive roots $\lambda_1 < \lambda_2$ for $p > p_c$.

The following theorem was obtained in [2, 3].

Theorem A. *Let $p > p_c$. Assume that u_0 satisfies* (1.2) *and*

$$L|x|^{-m} - b_-|x|^{-l} \leq u_0(x) \leq L|x|^{-m} - b_+|x|^{-l} \qquad \text{for } |x| > R$$

with some $l \in (m+\lambda_1, m+\lambda_2+2)$, $0 < b_+ < b_-$ and $R > 0$. Then there exist positive constants c, C and T such that the solution of (1.1) *satisfies*

$$ct^{\frac{m(l-m-\lambda_1)}{2\lambda_1}} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > T$.

Next, let us consider the critical case $p = p_c$. In this case, (1.3) has a double root $\lambda := \frac{N-2-2m}{2}$, from which it follows that $m + \lambda = \frac{N-2}{2}$, $m + \lambda + 2 = \frac{N+2}{2}$. It turns out that Theorem A cannot be extended as it stands to the critical case $p = p_c$. Our first result shows that a logarithmic term appears in the precise grow-up rate.

Theorem 1.1. *Let $p = p_c$. Assume that u_0 satisfies* (1.2) *and*

$$L|x|^{-m} - b_-|x|^{-l} \le u_0(x) \le L|x|^{-m} - b_+|x|^{-l} \qquad \text{for } |x| > R$$

with some $l \in (m+\lambda, m+\lambda+2)$, $0 < b_+ < b_-$ and $R > 0$. Then there exist positive constants c, C and T such that the solution of (1.1) *satisfies*

$$ct^{\frac{m(l-m-\lambda)}{2\lambda}}(\ln t)^{\frac{m}{\lambda}} \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le Ct^{\frac{m(l-m-\lambda)}{2\lambda}}(\ln t)^{\frac{m}{\lambda}} \tag{1.4}$$

for all $t > T$.

One reason for the difference between the supercritical case and the critical case is that, for $p > p_c$, the asymptotic behavior of regular steady states is

$$\varphi_\alpha(|x|) = L|x|^{-m} - a|x|^{-m-\lambda_1} + h.o.t. \quad \text{as } |x| \to \infty, \tag{1.5}$$

whereas for $p = p_c$,

$$\varphi_\alpha(|x|) = L|x|^{-m} - a|x|^{-m-\lambda} \ln|x| + h.o.t. \quad \text{as } |x| \to \infty, \tag{1.6}$$

where $a = a(\alpha, N, p)$ is a positive number that is monotone decreasing in α in a manner that can be determined from the scaling properties of the ordinary differential equation for φ (see [5, 7]). In the previous paper [3], we did not take into account the precise asymptotics of regular steady states in the critical case $p = p_c$, and were led to a wrong conclusion. Namely, the lower bound in (1.4) implies that Proposition 3.1 in [3] does not hold for $p = p_c$. It is one purpose of this paper to fix this error (see Lemma 4.3 for the correct analogue of Proposition 3.1 in [3] when $p = p_c$) and give a precise grow-up rate in the critical case.

Theorem 1.1 cannot be extended to $l > m+\lambda+2$. In fact, the next result gives a universal upper bound on solutions which shows that the lower bound in (1.4) cannot be valid if $l > m + \lambda + 2$.

Theorem 1.2. *Let $p = p_c$. Assume that u_0 satisfies* (1.2). *Then, for any $\varepsilon > 0$, there exist positive constants C and T such that the solution of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le Ct^{\frac{m}{\lambda}+\varepsilon} \quad \text{for all } t > T.$$

As for $p > p_c$, a sharp universal upper bound was found by Mizoguchi [8] by using a spectral property of the linearized operator at the singular steady state. This method is not applicable to the critical case $p = p_c$, because the required spectral properties are only known to hold for $p > p_c$. In this paper, we employ a different technique to derive the universal upper bound as above.

Finally, we have the following corollary as a direct consequence of Theorems 1.1 and 1.2.

Corollary 1.3. *Let $p = p_c$. Assume that u_0 satisfies* (1.2) *and*

$$u_0(x) \geq L|x|^{-m} - b|x|^{-(m+\lambda+2)} \qquad \text{for } x \neq 0$$

with some $b > 0$. Then, for any $\varepsilon > 0$, there exist positive constants c, C and T such that the solution of (1.1) *satisfies*

$$ct^{\frac{m}{\lambda}-\varepsilon} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{\frac{m}{\lambda}+\varepsilon} \qquad \text{for all } t > T.$$

This paper is organized as follows. In Section 2, we carry out a formal asymptotic analysis for radially symmetric solutions and derive formally the expected grow-up rate for all $l \geq m + \lambda$. We mention here that for $l = m + \lambda + 2$ there is no logarithmic term in the rate, cf. (2.19). In Section 3, taking the formal expansion of solutions into account, we construct a suitable subsolution to derive a lower bound. In Section 4, we construct a supersolution to derive an upper bound. In Section 5, we give a universal upper bound. Proofs of the main results will be given in Section 6. In the following sections, we assume $p = p_c$ and $\lambda = (N - 2 - 2m)/2$ throughout.

Acknowledgment. M. Fila, J. R. King and M. Winkler were partially supported by the European Community's Human Potential Programme under contract HPRN-CT-2002-00274, Fronts - Singularities. M. Fila acknowledges also the support of the VEGA Grant 1/3021/06. E. Yanagida was partially supported by the Grant-in-Aid for Scientific Research (B) (No. 15340052) from the Japan Society for the Promotion of Science. Part of this work was done while M. Fila visited the Tohoku University. He is grateful for the warm hospitality.

2. Formal asymptotics

Let us consider radial solutions $u = u(r,t)$, $r = |x|$, of (1.1); that is, u is a solution of

$$u_t = u_{rr} + \tfrac{N-1}{r}u_r + u^p, \quad r > 0,\ t > 0, \tag{2.1}$$

with the initial condition

$$u(r,0) = u_0(r), \quad r \geq 0,$$

where u_0 is nonnegative and continuous in $r \geq 0$.

Following the idea of the previous papers [2, 3, 4], we first consider the inner expansion. We write $u(r,t)$ as

$$u(r,t) = \sigma(t)\Big\{\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Phi(\xi,t)\Big\}, \tag{2.2}$$

where $\sigma(t) := u(0,t)$, $\xi := \sigma^{1/m} r$, and $\psi := \varphi_1(\xi)$ satisfies

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \psi^p = 0, & \xi > 0, \\ \psi(0) = 1, \quad \psi_\xi(0) = 0. \end{cases} \tag{2.3}$$

Substituting (2.2) into (2.1), we have

$$\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \frac{\sigma_t}{\sigma^p}\Big(\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi\Big) + \Big(\psi + \frac{\sigma_t}{\sigma^p}\Phi\Big)^p \sim \frac{\sigma_t}{\sigma^p}\Big(\psi + \frac{1}{m}\,\xi\psi_\xi\Big)$$

under some hypotheses on σ and Φ. In view of (2.3), we may put $\Phi = \Psi(\xi) + h.o.t.$, where Ψ satisfies

$$\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \psi + \frac{1}{m}\,\xi\psi_\xi, \qquad \xi > 0. \tag{2.4}$$

Thus we obtain the two-term inner expansion

$$u(r,t) \sim \sigma(t)\Big\{\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Psi(\xi)\Big\}.$$

In the inner region, where $\psi(\xi)$ dominates $(\sigma_t/\sigma^p)\Psi$, (1.6) yields (setting $a = a(1, N, p_c)$ in the earlier notation and including the next correction term)

$$\begin{aligned} u &\sim \sigma(L\xi^{-m} - a\xi^{-m-\lambda}\ln\xi) - d\xi^{-m-\lambda} \\ &= Lr^{-m} - a\sigma^{-\frac{\lambda}{m}} r^{-m-\lambda}\ln(\sigma^{\frac{1}{m}} r) - d\sigma^{-\frac{\lambda}{m}} r^{-m-\lambda}; \end{aligned} \tag{2.5}$$

here $a = a(N)$ is a positive constant and $d = d(N)$ is a constant whose values can be determined numerically from the initial-value problem (2.3).

In the outer region we set $u \sim Lr^{-m} + U(r,t)$ to obtain the linearized equation

$$U_t = U_{rr} + \frac{N-1}{r}U_r + \frac{pL^{p-1}}{r^2}U;$$

now setting $V(r,t) := r^{m+\lambda}U(r,t)$ furnishes the two-dimensional heat equation

$$V_t = \frac{1}{r}(rV_r)_r, \tag{2.6}$$

and matching with (2.5) requires that

$$V \sim -\frac{a}{m}\sigma^{-\frac{\lambda}{m}} \ln \sigma - \sigma^{-\frac{\lambda}{m}}(a \ln r + d), \tag{2.7}$$

as $r \to 0$. Since $\sigma \gg 1$, the leading-order solution to (2.6) must match as $r \to 0$ to the first term on the right-hand side of (2.7)[1], so the former is required to have no $\ln r$ term as $r \to 0$; i.e., to satisfy

$$\lim_{r \to 0} rV_r = 0. \tag{2.8}$$

Now we specialize to initial data satisfying $u_0(r) \sim Lr^{-m} - br^{-l}$ as $r \to \infty$.

If $l > m + \lambda + 2$, then it follows from (2.6), (2.8) that the leading-order solution satisfies

$$\frac{d}{dt}\int_0^\infty rV(r,t)\,dr = 0 \tag{2.9}$$

and we infer in the usual way that (generically)

$$V(r,t) \sim \frac{1}{t}G(\eta), \qquad \eta = \frac{r}{\sqrt{t}} \tag{2.10}$$

as $t \to \infty$ when $r = O(t^{1/2})$ wherein $G(\eta) = -Ae^{-\eta^2/4}$ for some positive constant A which will depend on the initial data in a way that cannot be extracted explicitly. We remark that were the constant obtained on integrating (2.9) positive, the matching that follows would not be possible, this being associated with finite-time blow-up; were it zero, the next mode in the sequence

$$V(r,t) \sim \frac{1}{t^2}G(\eta), \qquad \text{as } t \to \infty \quad \text{with } r = O(t^{1/2})$$

would arise but we shall not discuss such borderline cases. For initial data that satisfies (1.2) it in any case follows that A is necessarily positive.

Before matching with (2.7) we first need to rewrite (2.7) in terms of η in the form

$$V \sim -a\sigma^{-\frac{\lambda}{m}}\left(\frac{1}{m}\ln \sigma + \frac{1}{2}\ln t\right) - \sigma^{-\frac{\lambda}{m}}(a \ln \eta + d); \tag{2.11}$$

as we shall now see, the first two terms on the right-hand side of (2.11) turn out to be of the same order, so the comment following (2.7) is slightly misleading, though the conclusion (2.8) is not. Matching (2.10) with (2.11) requires

$$a\sigma^{-\frac{\lambda}{m}}\left(\frac{1}{m}\ln \sigma + \frac{1}{2}\ln t\right) \sim \frac{A}{t},$$

[1]The formal matching procedure prescribes that one take the limits in the order $\sigma \to \infty$ (small parameter tends to zero), *then* $r \to 0$ (matching).

so that for $l > m + \lambda + 2$ we have

$$\sigma \sim \left(\frac{(\lambda+2)a}{2\lambda A}\, t \ln t\right)^{\frac{m}{\lambda}} \qquad \text{as } t \to \infty. \tag{2.12}$$

This result is of course consistent with Theorem 1.2, though there is a subtlety associated with its dependence upon A and we shall not pursue a rigorous derivation of the precise asymptotics.

For $m + \lambda \le l < m + \lambda + 2$ the "mass" $\int_0^\infty rV\,dr$ is unbounded and we instead have

$$V(r,t) \sim t^{-\frac{l-m-\lambda}{2}} G(\eta), \quad \eta = \frac{r}{\sqrt{t}} \quad \text{as } t \to \infty \text{ with } r = O(t^{1/2}), \tag{2.13}$$

wherein G is subject to

$$G(\eta) \sim -b\eta^{-l+m+\lambda} \qquad \text{as } \eta \to \infty \tag{2.14}$$

and

$$G(0) = -bB \tag{2.15}$$

for some positive constant $B(N,l)$ that is determined by the boundary-value problem for G which follows from (2.6), (2.13)-(2.15). Matching with (2.11) then yields

$$a\sigma^{-\frac{\lambda}{m}}\left(\frac{1}{m}\ln\sigma + \frac{1}{2}\ln t\right) \sim bBt^{-\frac{l-m-\lambda}{2}}$$

so that for $m + \lambda \le l < m + \lambda + 2$ we have

$$\sigma \sim \left(\frac{(l-m)a}{2\lambda bB}\, t^{\frac{l-m-\lambda}{2}} \ln t\right)^{\frac{m}{\lambda}} \qquad \text{as } t \to \infty, \tag{2.16}$$

in keeping with Theorem 1.1. It is worth emphasizing that the result (2.16) holds for $l = m + \lambda$, in which case σ grows only logarithmically with t.

The other borderline case $l = m + \lambda + 2$ also warrants mention because of an interplay which then arises between logarithmic terms arising from different sources. In this case the outer expansion reads

$$V \sim -\frac{\ln t}{t} A e^{-\eta^2/4} + \frac{1}{t} G_1(\eta)\,, \tag{2.17}$$

with

$$-\frac{1}{2}b + 2Ae^{-\eta^2/4} - \frac{1}{2}\eta^2 G_1 = \eta \frac{dG_1}{d\eta}\,,$$

so that

$$G_1 \sim \left(2A - \frac{1}{2}b\right) \ln \eta \qquad \text{as } \eta \searrow 0. \tag{2.18}$$

In other such circumstances one would infer from (2.18) that A is given by $A = b/4$, but here one must instead account also for the $\ln \eta$ term in (2.11), giving the matching conditions

$$a\sigma^{-\frac{\lambda}{m}}\Big(\frac{1}{m}\ln\sigma + \frac{1}{2}\ln t\Big) \sim \frac{A\ln t}{t}\,, \qquad a\sigma^{-\frac{\lambda}{m}} \sim \Big(\frac{1}{2}b - 2A\Big)\frac{1}{t}\,,$$

from which we obtain $A = \frac{(\lambda+2)b}{8(\lambda+1)}$ and for $l = m + \lambda + 2$ that

$$\sigma \sim \Big(\frac{4(\lambda+1)a}{\lambda}\,t\Big)^{\frac{m}{\lambda}} \qquad \text{as } t \to \infty\,; \tag{2.19}$$

we emphasize that in this case there is no logarithmic dependence upon t.

3. Lower bound

In this section we rigorously derive the lower bound claimed by Theorem 1.1. We will restrict ourselves mostly to radially symmetric solutions $u = u(r,t)$. Throughout this section we assume that u_0 satisfies

$$\begin{aligned} &0 < u_0(r) \le Lr^{-m} \quad \text{for } r \ge 0,\\ &u_0(r) \ge Lr^{-m} - b_- r^{-l} \quad \text{for } r > 0, \end{aligned} \tag{3.1}$$

with some $b_- > 0$ and $l \in (m+\lambda, m+\lambda+2)$.

As to an estimate for large r, we recall from [2, Lemma 3.3] the following result, which was formulated there only for $p > p_c$, but as the proof shows, it still holds in the critical case $p = p_c$.

Lemma 3.1. *Suppose that u_0 satisfies* (3.1)*. Then for any $B_0 > 0$ there exists $b_0 > 0$ such that the solution of* (2.1) *satisfies*

$$u(r,t) \ge Lr^{-m} - b_0 r^{-l}$$

for all $t \ge 0$ and $r \ge B_0(t+1)^{\frac{1}{2}}$.

In order to obtain an appropriate lower estimate in an inner region, we let $\psi = \psi(\xi)$ be defined by (2.3). For technical purposes, we slightly modify the definition (2.4) of Ψ as follows. Fixing an arbitrary positive decreasing function $\chi \in C^\infty([0,\infty))$ such that

$$\chi(\xi) = o(\xi^{-m-\lambda}\ln\xi) \quad \text{as } \xi \to \infty, \tag{3.2}$$

we then define Ψ to be a solution of the initial-value problem

$$\begin{cases} \Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \psi + \frac{1}{m}\xi\psi_\xi + \chi(\xi), & \xi > 0,\\ \Psi(0) = -1, \qquad \Psi_\xi(0) = 0. \end{cases} \tag{3.3}$$

Since, by (1.6), ψ satisfies

$$\psi(\xi) = L\xi^{-m} - a_1\xi^{-m-\lambda}\ln\xi + h.o.t. \quad \text{as } \xi \to \infty, \tag{3.4}$$

it is easy to show that a corresponding expansion holds for ψ_ξ (see, e.g., the proof of Lemma 4.1 in [3]). Thus (3.2) implies that the right-hand side of (3.3) satisfies

$$\psi(\xi) + \frac{1}{m}\xi\psi_\xi(\xi) + \chi(\xi) = \frac{a_1\lambda}{m}\xi^{-m-\lambda}\ln\xi + o(\xi^{-m-\lambda}\ln\xi) \quad \text{as } \xi \to \infty.$$

As a consequence, the following lemma can be shown by obvious modifications of the proofs of Lemmas 4.1 and 4.2 in [3].

Lemma 3.2. *There exists a constant $K > 0$ such that*

$$\left.\begin{aligned} &\Psi(\xi) = K\xi^{2-m-\lambda}\ln\xi + o(\xi^{2-m-\lambda}\ln\xi),\\ &\Psi_\xi(\xi) = K(2-m-\lambda)\xi^{1-m-\lambda}\ln\xi + o(\xi^{1-m-\lambda}\ln\xi), \end{aligned}\right\} \quad \textit{as } \xi \to \infty.$$

Let us now fix a constant κ satisfying

$$\frac{2\lambda}{m(l-m)} < \kappa < \frac{2\lambda}{m(l-m-\lambda)}, \tag{3.5}$$

and define

$$\sigma(t) := \varepsilon(t+\varepsilon^{-\kappa})^k\{\ln(t+\varepsilon^{-\kappa})\}^{m/\lambda}, \qquad k := \frac{m(l-m-\lambda)}{2\lambda},$$

with a small parameter $\varepsilon \in (0,1)$. We note here that there exists $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$, then

$$\frac{\sigma_t}{\sigma^p} \le \frac{1}{2} \quad \text{for all } t \ge 0. \tag{3.6}$$

Indeed, by

$$\sigma_t = k\varepsilon(t+\varepsilon^{-\kappa})^{k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{m}{\lambda}} + \tfrac{m}{\lambda}\varepsilon(t+\varepsilon^{-\kappa})^{k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{m}{\lambda}-1}, \tag{3.7}$$

we have

$$\sigma_t \le \left(k + \frac{m}{\lambda}\right)\varepsilon(t+\varepsilon^{-\kappa})^{k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{m}{\lambda}} \quad \text{for all } t \ge 0 \tag{3.8}$$

if ε is sufficiently small, so that

$$\begin{aligned} \frac{\sigma_t}{\sigma^p} &\le \left(k + \frac{m}{\lambda}\right)\varepsilon^{1-p}(t+\varepsilon^{-\kappa})^{-(p-1)k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{-\frac{(p-1)m}{\lambda}}\\ &\le \left(k + \frac{m}{\lambda}\right)\varepsilon^{1-p}\varepsilon^{\{(p-1)k+1\}\kappa}(\ln\varepsilon^{-\kappa})^{-\frac{(p-1)m}{\lambda}}\\ &= \left(k + \frac{m}{\lambda}\right)\varepsilon^{-\frac{2}{m}+\frac{l-m}{\lambda}\kappa}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}} \end{aligned}$$

for such ε. By (3.5), the latter term becomes arbitrarily small as $\varepsilon \to 0$ so that (3.6) holds.

Let us now give a subsolution of (2.1) which is defined for all $r, t > 0$.

Lemma 3.3. *There exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then*

$$u_{in}(r,t) := \max\Big\{0,\ \sigma\Big(\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Psi(\xi)\Big)\Big\}, \qquad \xi = \xi(r,t) := \sigma^{\frac{1}{m}} r, \tag{3.9}$$

is a subsolution of (2.1) *for all $r, t > 0$.*

Proof. Since $u \equiv 0$ is a subsolution of (2.1), we only need to verify the subsolution property at those points where u_{in} is positive. Using

$$\sigma\xi_t = \frac{p-1}{2}\xi\sigma_t$$

and $p - 1 = 2/m$, as well as the equations (2.3) and (3.3) defining ψ and Ψ, we directly compute at these points

$$\begin{aligned}
u_{in,t} &- u_{in,rr} - \frac{N-1}{r}u_{in,r} - u_{in}^p \\
&= \Big\{\sigma_t\psi + \sigma\psi_\xi\xi_t + \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi\Big)_t\Big\} - \Big\{\sigma^{1+\frac{2}{m}}\psi_{\xi\xi} + \frac{\sigma_t}{\sigma^{p-1}}\sigma^{\frac{2}{m}}\Psi_{\xi\xi}\Big\} \\
&\quad -\frac{N-1}{r}\Big\{\sigma^{1+\frac{1}{m}}\psi_\xi + \frac{\sigma_t}{\sigma^{p-1}}\sigma^{\frac{1}{m}}\Psi_\xi\Big\} - \Big\{\sigma\Big(\psi + \frac{\sigma_t}{\sigma^p}\Psi\Big)\Big\}^p \\
&= \sigma_t\psi + \frac{p-1}{2}\xi\sigma_t\psi_\xi + \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi\Big)_t - \sigma^{1+\frac{2}{m}}\psi_{\xi\xi} - \sigma_t\Psi_{\xi\xi} \\
&\quad -\sigma^{1+\frac{2}{m}}\frac{N-1}{\xi}\psi_\xi - \sigma_t\frac{N-1}{\xi}\Psi_\xi - \sigma^p\Big(\psi + \frac{\sigma_t}{\sigma^p}\Psi\Big)^p \\
&= \sigma^p\Big\{\psi^p - \Big(\psi + \frac{\sigma_t}{\sigma^p}\Psi\Big)^p + \frac{\sigma_t}{\sigma^p}p\psi^{p-1}\Psi\Big\} + \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi\Big)_t - \chi(\xi)\sigma_t \\
&\le \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi\Big)_t - \chi(\xi)\sigma_t \quad =: I(r,t),
\end{aligned} \tag{3.10}$$

due to the convexity of $0 \le s \mapsto s^p$. Again in view of $\xi_t = \frac{1}{m}\frac{\sigma_t}{\sigma}\xi$, we obtain

$$\Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi(\xi)\Big)_t = \Big(\frac{\sigma_t}{\sigma^{p-1}}\Big)_t\Psi(\xi) + \frac{1}{m}\frac{\sigma_t^2}{\sigma^p}\,\xi\Psi_\xi(\xi).$$

Here, by (3.7), we have

$$\begin{aligned}
\frac{\sigma_t}{\sigma^{p-1}} &= k\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}} \\
&\quad + \tfrac{m}{\lambda}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}-1},
\end{aligned}$$

$$\begin{aligned}\Big(\frac{\sigma_t}{\sigma^{p-1}}\Big)_t &= k\{(2-p)k-1\}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}}\\ &\quad+\tfrac{m}{\lambda}\{2(2-p)k-1\}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}-1}\\ &\quad+\tfrac{m}{\lambda}\{\tfrac{(2-p)m}{\lambda}-1\}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}-2},\\ \frac{1}{m}\cdot\frac{\sigma_t^2}{\sigma^p} &= \frac{k^2}{m}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}}\\ &\quad+\frac{2k}{\lambda}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}-1}\\ &\quad+\frac{m}{\lambda^2}\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}-2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big(\tfrac{\sigma_t}{\sigma^{p-1}}\Psi(\xi)\Big)_t &= \varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}}\\ &\qquad\cdot\Big\{k\{(2-p)k-1\}\Psi(\xi)+\tfrac{k^2}{m}\xi\Psi_\xi(\xi)+J(r,t)\Big\}\end{aligned}\tag{3.11}$$

holds, with

$$\begin{aligned}J(r,t) &= \Big\{\frac{m}{\lambda}\{2(2-p)k-1\}\{\ln(t+\varepsilon^{-\kappa})\}^{-1}\\ &\qquad+\frac{m}{\lambda}\Big\{\frac{(2-p)m}{\lambda}-1\Big\}\{\ln(t+\varepsilon^{-\kappa})\}^{-2}\Big\}\Psi(\xi)\\ &\quad+\Big\{\frac{2k}{\lambda}\{\ln(t+\varepsilon^{-\kappa})\}^{-1}+\frac{m}{\lambda^2}\{\ln(t+\varepsilon^{-\kappa})\}^{-2}\Big\}\xi\Psi_\xi(\xi)\\ &=: J_1(r,t)+J_2(r,t).\end{aligned}$$

Let us fix a small number $\nu\in(0,1)$ such that

$$C_0 := \frac{k^2}{m}\frac{1-\nu}{1+\nu}(m+\lambda-2)-k\{(2-p)k-1\}>0.$$

Note that this is possible because

$$\begin{aligned}&\frac{k^2}{m}(m+\lambda-2)-k\{(2-p)k-1\} = k\Big(\frac{m+\lambda-2}{m}k+(p-2)k+1\Big)\\ &= k\Big(\frac{m+\lambda-2}{m}k+\frac{2-m}{m}k+1\Big) = k\Big(\frac{\lambda}{m}k+1\Big)>0.\end{aligned}$$

Then, by Lemma 3.2, we can take $\xi_1>1$ so large that

$$\left.\begin{aligned}&0\le\Psi(\xi)\le(1+\nu)K\xi^{2-m-\lambda}\ln\xi,\\ &\Psi_\xi(\xi)\le-(1-\nu)K(m+\lambda-2)\xi^{1-m-\lambda}\ln\xi,\end{aligned}\right\}\quad\text{for } \xi\ge\xi_1.\tag{3.12}$$

From now on we require ε to be so small that

$$\frac{m}{\lambda}\{2(2-p)k-1\}(\ln\varepsilon^{-\kappa})^{-1}+\frac{m}{\lambda}\Big\{\frac{(2-p)m}{\lambda}-1\Big\}(\ln\varepsilon^{-\kappa})^{-2}\le C_0. \quad (3.13)$$

Then for such ε, and all r,t satisfying $\xi(r,t)\ge\xi_1$, we obtain in particular that $\Psi(\xi)\ge 0$ and $\xi\Psi_\xi(\xi)\le 0$, and hence $J(r,t)\le J_1(r,t)\le C_0\Psi(\xi)$, by (3.13). Therefore (3.11) and (3.12) yield

$$\begin{aligned} I(r,t) &\le \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi\Big)_t \le \varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}} \\ &\quad \cdot\Big\{k\{(2-p)k-1\}\Psi(\xi)-\frac{k^2}{m}\cdot\frac{1-\nu}{1+\nu}(m+\lambda-2)\Psi(\xi)+C_0\Psi(\xi)\Big\} \\ &= 0 \qquad \text{when } \xi\ge\xi_1 \end{aligned} \quad (3.14)$$

for such ε.

In the region where $\xi<\xi_1$, however, we must use the fact that the uniform estimates

$$|\Psi(\xi)|\le C_1 \quad \text{and} \quad |\xi\Psi_\xi(\xi)|\le C_2 \qquad \text{for } \xi\in[0,\xi_1]$$

are valid for some $C_1>0$ and $C_2>0$. Thanks to these estimates, we have

$$\begin{aligned} |J(r,t)| &\le \Big\{\frac{m}{\lambda}\,|2(2-p)k-1|\,(\ln\varepsilon^{-\kappa})^{-1}+\frac{m}{\lambda}\Big|\frac{(2-p)m}{\lambda}-1\Big|\,(\ln\varepsilon^{-\kappa})^{-2}\Big\}C_1 \\ &\quad +\Big\{\frac{2k}{\lambda}(\ln\varepsilon^{-\kappa})^{-1}+\frac{m}{\lambda^2}(\ln\varepsilon^{-\kappa})^{-2}\Big\}C_2\le C_3 \end{aligned}$$

if $\varepsilon<1/2$ with some $C_3>0$, and thus

$$\begin{aligned} \Big(\frac{\sigma_t}{\sigma^{p-1}}\Psi(\xi)\Big)_t &\le \varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}} \\ &\qquad \cdot\Big\{k|(2-p)k-1|C_1+\tfrac{k^2}{m}C_2+C_3\Big\} \\ &=: C_4\varepsilon^{2-p}(t+\varepsilon^{-\kappa})^{(2-p)k-2}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{(2-p)m}{\lambda}} \end{aligned}$$

for such ε. Since χ is decreasing and σ_t is positive,

$$\chi(\xi)\sigma_t\ge\chi(\xi_1)\sigma_t\ge\chi(\xi_1)k\varepsilon(t+\varepsilon^{-\kappa})^{k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{m}{\lambda}} \qquad \text{for } \xi<\xi_1,$$

by (3.7). Hence, combining these results, we obtain

$$\begin{aligned} \frac{\big(\frac{\sigma_t}{\sigma^{p-1}}\Psi(\xi)\big)_t}{\chi(\xi)\sigma_t} &\le \frac{C_4}{k\chi(\xi_1)}\varepsilon^{-(p-1)}(t+\varepsilon^{-\kappa})^{-(p-1)k-1}\{\ln(t+\varepsilon^{-\kappa})\}^{-\frac{(p-1)m}{\lambda}} \\ &\le \frac{C_4}{k\chi(\xi_1)}\varepsilon^{-(p-1)}\varepsilon^{[(p-1)k+1]\kappa}(\ln\varepsilon^{-\kappa})^{-\frac{(p-1)m}{\lambda}} \end{aligned}$$

$$= \quad \frac{C_4}{k\chi(\xi_1)}\varepsilon^{-\frac{2}{m}+\frac{l-m}{\lambda}\kappa}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}} \qquad \text{if } \xi<\xi_1. \tag{3.15}$$

Here, since

$$-\frac{2}{m}+\frac{l-m}{\lambda}\kappa>0$$

by (3.5), we see that if ε is sufficiently small, then the right-hand side of (3.15) is less than one, and hence, $I(r,t)\le 0$ if $\xi<\xi_1$. Combined with (3.14) and (3.10), this proves the subsolution property. □

For the proof of an upper estimate for u_{in} on some matching boundary in Lemma 3.6 below, we need two elementary calculus lemmas.

Lemma 3.4. *For any $\theta>0$ and $B_0>0$, there exists $\varepsilon_\theta>0$ such that if $0<\varepsilon<\varepsilon_\theta$, then the inequality*

$$\ln\left\{2+B_0\varepsilon^{\frac{1}{m}}(t+\varepsilon^{-\kappa})^{\frac{k}{m}}\{\ln(t+\varepsilon^{-\kappa})\}^{\frac{1}{\lambda}}\left(t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}\right)^{\frac{1}{2}}\right\} \ge \varepsilon^{\theta}\ln(t+\varepsilon^{-\kappa})$$

holds for all $t\ge 0$.

Proof. Fix $\theta>0$ arbitrarily and set $\tau_1:=\varepsilon^{-\kappa}$, $\tau_2:=\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}$. Then it is easy to see that the desired inequality is equivalent to the estimate

$$\zeta(t):=2(t+\tau_1)^{-\varepsilon^{-\theta}}+B_0\varepsilon^{\frac{1}{m}}(t+\tau_1)^{\frac{k}{m}-\varepsilon^{\theta}}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}}(t+\tau_2)^{\frac{1}{2}}\ge 1$$

for all $t\ge 0$. We calculate

$$\begin{aligned}\zeta'(t)&=-2\varepsilon^{\theta}(t+\tau_1)^{-\varepsilon^{\theta}-1}\\&\quad+B_0\varepsilon^{\frac{1}{m}}(\tfrac{k}{m}-\varepsilon^{\theta})(t+\tau_1)^{\frac{k}{m}-\varepsilon^{\theta}-1}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}}(t+\tau_2)^{\frac{1}{2}}\\&\quad+\frac{B_0}{\lambda}\varepsilon^{\frac{1}{m}}(t+\tau_1)^{\frac{k}{m}-\varepsilon^{\theta}-1}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}-1}(t+\tau_2)^{\frac{1}{2}}\\&\quad+\frac{B_0}{2}\varepsilon^{\frac{1}{m}}(t+\tau_1)^{\frac{k}{m}-\varepsilon^{\theta}}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}}(t+\tau_2)^{-\frac{1}{2}}.\end{aligned}$$

Here we omit the last two positive terms to show that, if $\varepsilon^{\theta}<k/(2m)$, then

$$\begin{aligned}\zeta'(t)&\ge(t+\tau_1)^{-\varepsilon^{\theta}-1}\Big\{-2\varepsilon^{\theta}+B_0\varepsilon^{\frac{1}{m}}(\tfrac{k}{m}-\varepsilon^{\theta})(t+\tau_1)^{\frac{k}{m}}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}}(t+\tau_2)^{\frac{1}{2}}\Big\}\\&\ge(t+\tau_1)^{-\varepsilon^{\theta}-1}\Big\{-2\varepsilon^{\theta}+\tfrac{B_0k}{2m}\varepsilon^{\frac{1}{m}}(t+\tau_1)^{\frac{k}{m}}\{\ln(t+\tau_1)\}^{\frac{1}{\lambda}}(t+\tau_2)^{\frac{1}{2}}\Big\}\\&\ge(t+\tau_1)^{-\varepsilon^{\theta}-1}\Big\{-2\varepsilon^{\theta}+\tfrac{B_0k}{2m}\varepsilon^{\frac{1}{m}}\tau_1^{\frac{k}{m}}(\ln\tau_1)^{\frac{1}{\lambda}}\tau_2^{\frac{1}{2}}\Big\}\\&=(t+\tau_1)^{-\varepsilon^{\theta}-1}\Big\{-2\varepsilon^{\theta}+\tfrac{B_0k}{2m}\varepsilon^{\frac{1}{m}}\varepsilon^{-\frac{k\kappa}{m}}(\ln\varepsilon^{-\kappa})^{\frac{1}{\lambda}}\varepsilon^{-\frac{1-k\kappa}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{1}{\lambda}}\Big\}\end{aligned}$$

$$= (t+\tau_1)^{-\varepsilon^\theta-1}\Big\{-2\varepsilon^\theta + \tfrac{B_0 k}{2m}\Big\}$$

for all $t>0$. Hence if

$$\varepsilon < \min\Big\{\Big(\frac{k}{2m}\Big)^{\frac{1}{\theta}}, \Big(\frac{B_0 k}{4m}\Big)^{\frac{1}{\theta}}\Big\},$$

then ζ is increasing on $(0,\infty)$ and, in particular,

$$\inf_{t\ge 0}\zeta(t) = \zeta(0) \ge 2\tau_1^{-\varepsilon^\theta} = 2\varepsilon^{\kappa\varepsilon^\theta}.$$

Since $\varepsilon^{\kappa\varepsilon^\theta} \to 1$ as $\varepsilon \to 0$, it follows that $\inf_{t\ge0}\zeta(t) \ge 1$ for all suitably small $\varepsilon > 0$. □

Lemma 3.5. *There is $\varepsilon^* > 0$ such that, if $0 < \varepsilon < \varepsilon^*$, then*

$$\varepsilon^{\kappa-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}} \le \frac{t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}}{t+\varepsilon^{-\kappa}} \le 1 \quad \textit{for all } t\ge 0.$$

Proof. From assumption (3.5) we have $\kappa > \frac{2\lambda}{m(l-m)} = \frac{2}{m+2k}$ and hence

$$\kappa - \frac{2(1-k\kappa)}{m} = \frac{2k+m}{m}\kappa - \frac{2}{m} > 0.$$

Therefore, we can choose $\varepsilon^* > 0$ such that $\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}} < \varepsilon^{-\kappa}$ for any $\varepsilon < \varepsilon^*$. Since the function $t \mapsto \frac{t+\tau_2}{t+\tau_1}$ is increasing on $[0,\infty)$ if $0<\tau_2<\tau_1$, the claimed inequalities are obtained by evaluating the limits of this function as $t\to 0$ and $t\to\infty$. □

We now choose some constants for later reference. First, according to Lemma 3.2 there exist positive numbers c_1 and c_2 such that

$$\Psi(\xi) \le c_1 \quad \text{for } \xi \ge 0 \tag{3.16}$$

and

$$\Psi(\xi) \le c_2\xi^{2-m-\lambda}\ln(2+\xi) \quad \text{for } \xi > 0. \tag{3.17}$$

Picking $\xi_0 > 0$ such that

$$(1+2kc_1)\xi_0^m \le \frac{L}{2}, \tag{3.18}$$

thanks to the expansion (3.4) we can find $\delta > 0$ such that

$$\psi(\xi) \le L\xi^{-m} - \delta\xi^{-m-\lambda}\ln(2+\xi) \quad \text{for } \xi \ge \xi_0. \tag{3.19}$$

We finally fix $B_0 > 0$ sufficiently small that

$$\Psi(\xi) \le 0 \quad \text{for } \xi \le B_0 \tag{3.20}$$

and

$$B_0 \le \xi_0, \qquad 2c_2 k B_0^2 \le \frac{\delta}{2}. \tag{3.21}$$

We fix these constants in the remainder of this section, and proceed to derive an upper estimate for u_{in} on some parabola in the (r,t) plane.

Lemma 3.6. *For any $b_0 > 0$, there is $\tilde{\varepsilon} > 0$ such that if $0 < \varepsilon < \tilde{\varepsilon}$, then*

$$u_{in}(r,t) \le Lr^{-m} - b_0 r^{-l} \quad \text{for all } (r,t) \in \Lambda,$$

where Λ is the parabola in the (r,t) plane defined by

$$\Lambda := \Big\{ (r,t) \in [0,\infty)^2 : r = B_0 \Big\{ t + \varepsilon^{-\frac{2(1-k\kappa)}{m}} (\ln \varepsilon^{-\kappa})^{-\frac{2}{\lambda}} \Big\}^{\frac{1}{2}} \Big\}.$$

Proof. Setting $r_0 := (2b_0/L)^{1/(l-m)}$, we first restrict $\varepsilon \in (0,1)$ such that

$$B_0 \varepsilon^{-\frac{1-k\kappa}{m}} (\ln \varepsilon^{-\kappa})^{-\frac{1}{\lambda}} \ge r_0,$$

so that we have $r \ge r_0$ for all $(r,t) \in \Lambda$. Next, recalling (3.7), we see that if we diminish ε suitably, then (3.8) holds. Due to (3.5), we therefore see from (3.16) that the global estimate

$$\begin{aligned} u_{in} &= \sigma \Big\{ \psi(\xi) + \frac{\sigma_t}{\sigma^p} \Psi(\xi) \Big\} \\ &\le \sigma \Big\{ 1 + c_1 \Big(k + \frac{m}{\lambda} \Big) \frac{\varepsilon (t+\varepsilon^{-\kappa})^{k-1} \{\ln(t+\varepsilon^{-\kappa})\}^{\frac{m}{\lambda}}}{\varepsilon^p (t+\varepsilon^{-\kappa})^{pk} \{\ln(t+\varepsilon^{-\kappa})\}^{\frac{pm}{\lambda}}} \Big\} \\ &= \sigma \Big\{ 1 + c_1 \Big(k + \frac{m}{\lambda} \Big) \varepsilon^{1-p} (t+\varepsilon^{-\kappa})^{-(p-1)k-1} \{\ln(t+\varepsilon^{-\kappa})\}^{-\frac{(p-1)m}{\lambda}} \Big\} \\ &\le \sigma \Big\{ 1 + c_1 \Big(k + \frac{m}{\lambda} \Big) \varepsilon^{\frac{l-m}{\lambda}\kappa - \frac{2}{m}} \Big\} \le \sigma \Big\{ 1 + c_1 \Big(k + \frac{m}{\lambda} \Big) \Big\} \end{aligned}$$

is valid for all $r, t \ge 0$, provided that ε additionally satisfies $\varepsilon \le e^{-1/\kappa}$ (so that $\ln(t + \varepsilon^{-\kappa}) \ge 1$ for all t).

Whenever $\xi \le \xi_0$; that is, $\sigma = \xi^m r^{-m} \le \xi_0^m r^{-m}$, this and (3.18) show that

$$u_{in} \le \xi_0^m r^{-m} (1 + 2c_1 k) \le \frac{L}{2} r^{-m} \le Lr^{-m} - \frac{L}{2} r_0^{l-m} r^{-l} = Lr^{-m} - b_0 r^{-l}$$

by definition of r_0. Conversely, if $\xi > \xi_0$, then we use (3.19) and (3.17) to obtain

$$\begin{aligned} u_{in} &= \sigma \Big(\psi(\xi) + \frac{\sigma_t}{\sigma^p} \Psi(\xi) \Big) \\ &\le \sigma \Big(L\xi^{-m} - \delta \xi^{-m-\lambda} \ln(2+\xi) + c_2 \frac{\sigma_t}{\sigma^p} \xi^{2-m-\lambda} \ln(2+\xi) \Big) \end{aligned}$$

$$\begin{aligned} &= Lr^{-m} - \sigma^{-\frac{\lambda}{m}} r^{-m-\lambda}\Big(\delta - c_2\frac{\sigma_t}{\sigma}r^2\Big)\ln(2+\sigma^{\frac{1}{m}}r) \\ &\le Lr^{-m} - \sigma^{-\frac{\lambda}{m}} r^{-m-\lambda}\Big(\delta - \frac{2kc_2r^2}{t+\varepsilon^{-\kappa}}\Big)\ln(2+\sigma^{\frac{1}{m}}r). \end{aligned} \tag{3.22}$$

Now if additionally $(r,t)\in\Lambda$, then for sufficiently small ε we have

$$\delta - \frac{2kc_2r^2}{t+\varepsilon^{-\kappa}} = \delta - 2kc_2B_0^2\frac{t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}}{t+\varepsilon^{-\kappa}} \ge \delta - 2kc_2B_0^2,$$

in view of the auxiliary Lemma 3.2. According to (3.21), $\delta - 2kc_2B_0^2 \ge \delta/2$ and thus

$$\begin{aligned} u_{in} &\le Lr^{-m} - \tfrac{\delta}{2}\sigma^{-\frac{\lambda}{m}}r^{-\lambda}\ln(2+\sigma^{\frac{1}{m}}r) \\ &= Lr^{-m} - \tfrac{\delta}{2}r^{-l}\Big\{\sigma^{-\frac{\lambda}{m}}r^{l-m-\lambda}\ln(2+\sigma^{\frac{1}{m}}r)\Big\} \end{aligned} \tag{3.23}$$

for all $(r,t)\in\Lambda$ satisfying $\xi(r,t)>\xi_0$. However, on Λ we have

$$\begin{aligned} \sigma^{-\frac{\lambda}{m}}r^{l-m-\lambda}\ln(2+\sigma^{\frac{1}{m}}r) &= \varepsilon^{-\frac{\lambda}{m}}(t+\varepsilon^{-\kappa})^{-\frac{\lambda k}{m}}\{\ln(t+\varepsilon^{-\kappa})\}^{-1} \\ &\cdot B_0^{l-m-\lambda}\Big\{t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}\Big\}^{\frac{l-m-\lambda}{2}}\ln(2+\sigma^{\frac{1}{m}}r), \end{aligned}$$

where by Lemma 3.4,

$$\begin{aligned} &\{\ln(t+\varepsilon^{-\kappa})\}^{-1}\ln(2+\sigma^{\frac{1}{m}}r) = \{\ln(t+\varepsilon^{-\kappa})\}^{-1} \\ &\times\ln\Big\{2+B_0\varepsilon^{\frac{1}{m}}(t+\varepsilon^{-\kappa})^{\frac{k}{m}}\ \{\ln(t+\varepsilon^{-\kappa})\}^{\frac{1}{\lambda}}\Big\{t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}\Big\}^{\frac{1}{2}}\Big\} \\ &\ge \varepsilon^{\theta} \end{aligned}$$

for arbitrary $\theta>0$ and $\varepsilon<\varepsilon_\theta$. Here, since $\kappa<1/k$, it is possible to fix $\theta>0$ such that $\theta<(\lambda/m)(1-k\kappa)(m+2k)$. Adopting this choice, we obtain from Lemma 3.5, observing $(l-m-\lambda)/2=\lambda k/m$, that

$$\begin{aligned} \sigma^{-\frac{\lambda}{m}}r^{l-m-\lambda}\ln(2+\sigma^{\frac{1}{m}}r) &\ge B_0^{l-m-\lambda}\varepsilon^{-\frac{\lambda}{m}+\theta}\Big\{\tfrac{t+\varepsilon^{-\frac{2(1-k\kappa)}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2}{\lambda}}}{t+\varepsilon^{-\kappa}}\Big\}^{\frac{\lambda k}{m}} \\ &\ge B_-^{l-m-\lambda}\varepsilon^{-\frac{\lambda}{m}+\theta}\varepsilon^{-\frac{2(1-k\kappa)}{m}\frac{\lambda k}{m}+\frac{\lambda k\kappa}{m}}(\ln\varepsilon^{-\kappa})^{-\frac{2k}{m}} \\ &= B_0^{l-m-\lambda}\varepsilon^{-\frac{\lambda}{m}(1-k\kappa)(m+2k)+\theta}(\ln\varepsilon^{-\kappa})^{-\frac{2k}{m}} \end{aligned}$$

for all $(r,t)\in\Lambda$. As the term on the right-hand side tends to infinity as $\varepsilon\to0$ due to our restriction on θ, (3.23) shows that $u_{in}(r,t)\le Lr^{-m}-b_0r^{-l}$ holds for those $(r,t)\in\Lambda$ with $\xi>\xi_0$. □

In order to prepare a comparison argument for the proof of the next proposition, we now show that u_{in} is dominated by u_0 initially.

Lemma 3.7. *For any u_0 satisfying* (3.1)*, there exists $\hat{\varepsilon} > 0$ such that, if $0 < \varepsilon < \hat{\varepsilon}$, then*

$$u_{in}(r,0) \le u_0(r) \quad \textit{for all} \;\; r \in \left[0, B_0 \varepsilon^{-\frac{1-k\kappa}{m}} (\ln \varepsilon^{-\kappa})^{-\frac{1}{\lambda}}\right].$$

Proof. Since

$$\xi(r,0) = \sigma^{\frac{1}{m}}(0) r = \varepsilon^{\frac{1-k\kappa}{m}} (\ln \varepsilon^{-\kappa})^{\frac{1}{\lambda}} r,$$

the restriction $r \le B_0 \varepsilon^{-(1-k\kappa)/m} (\ln \varepsilon^{-\kappa})^{-1/\lambda}$ guarantees $\xi(r,0) \le B_0 \le \xi_0$. Therefore $\Psi(\xi(r,0)) \le 0$ for such r, so that, if additionally $r \ge r_- := (2b_-/L)^{1/(l-m)}$, then (3.18) yields

$$\begin{aligned} u_{in}(r,0) &= \sigma(0)\Big\{\psi(\xi(r,0)) + \frac{\sigma_t}{\sigma^p}(0)\Psi(\xi(r,0))\Big\} \le \sigma(0) = \varepsilon^{1-k\kappa}(\ln \varepsilon^{-\kappa})^{\frac{m}{\lambda}} \\ &\le \frac{L}{2\xi_0^m} \varepsilon^{1-k\kappa} (\ln \varepsilon^{-\kappa})^{\frac{m}{\lambda}} = \frac{L}{2}\Big\{\xi_0 \varepsilon^{-\frac{1-k\kappa}{m}} (\ln \varepsilon^{-\kappa})^{-\frac{1}{\lambda}}\Big\}^{-m} \\ &\le \frac{L}{2} r^{-m} \le L r^{-m} - b_- r_-^{-(l-m)} r^{-m} \le L r^{-m} - b_- r^{-l} \le u_0(r) \end{aligned}$$

for any $\varepsilon \in (0,1)$.

On the other hand, for $r \le r_-$, u_0 is bounded below by a positive constant, and hence

$$u_{in}(r,0) \le \sigma(0) = \varepsilon^{1-k\kappa}(\ln \varepsilon^{-\kappa})^{\frac{m}{\lambda}} \le u_0(r),$$

provided that ε is small enough. □

We are now ready to prove that u grows at least as fast as asserted by Theorem 1.1.

Proposition 3.8. *Suppose that u_0 satisfies* (1.2) *and*

$$u_0(x) \ge L|x|^{-m} - b_-|x|^{-l} \quad \textit{for } |x| > 0$$

with some $l \in (m+\lambda, m+\lambda+2)$ and $b_- > 0$. Then there exists a positive constant c such that the solution of (1.1) *satisfies*

$$u(0,t) \ge c t^{\frac{m(l-m-\lambda)}{2\lambda}} (\ln t)^{\frac{m}{\lambda}} \quad \textit{for all } t > 1.$$

Proof. Due to the comparison principle and the positivity property of the heat semigroup, it is sufficient to consider radially symmetric and positive initial data only. For these, thus satisfying (3.1), Lemma 3.1 applies to yield the estimate

$$u(r,t) \ge L r^{-m} - b_0 r^{-l} \quad \text{for } r \ge B_0 (t+1)^{\frac{1}{2}} \tag{3.24}$$

with some $b_0 \in (0, b_-)$ depending on b_- and B_0, where B_0 is the fixed constant that satisfies (3.20) and (3.21). The assumptions (3.1) and (3.24) imply that, for all sufficiently small $\varepsilon > 0$,

$$u \geq Lr^{-m} - b_0 r^{-l} \quad \text{for all} \quad (r,t) \in \partial_p Q_\varepsilon$$

holds on the parabolic boundary $\partial_p Q_\varepsilon$ of

$$Q_\varepsilon := \Big\{ (r,t) \in [0,\infty)^2 : r \leq B_0 \big[t + \varepsilon^{-\frac{2(1-k\kappa)}{m}} (\ln \varepsilon^{-\kappa})^{-\frac{2}{\lambda}} \big]^{\frac{1}{2}} \Big\}.$$

Now, fixing $\varepsilon > 0$ sufficiently small, we obtain from Lemma 3.3 a subsolution on Q_ε of the form (3.9) with the additional property (3.6). According to Lemmas 3.6 and 3.7, we can further show that $u_{in} \leq u$ on $\partial_p Q_\varepsilon$. Hence the comparison principle ensures that $u_{in} \leq u$ in Q_ε. By an explicit evaluation at $r = 0$, this implies in particular that

$$\begin{aligned} u(0,t) &\geq \sigma(t) \Big\{ \psi(0) + \frac{\sigma_t}{\sigma^p}(t) \Psi(0) \Big\} = \sigma(t) \Big\{ 1 - \frac{\sigma_t}{\sigma^p}(t) \Big\} \\ &\geq \frac{1}{2} \sigma(t) = \frac{1}{2} \varepsilon (t + \varepsilon^{-\kappa})^k \{ \ln(t + \varepsilon^{-\kappa}) \}^{\frac{m}{\lambda}} \geq \frac{\varepsilon}{2} t^k (\ln t)^{\frac{m}{\lambda}} \end{aligned}$$

for all $t > 1$ and thereby completes the proof, given the definition of k. □

4. Upper bound

This section is devoted to the proof of the upper bound for the grow-up rate. Again, we first consider radially symmetric initial data $u_0 = u_0(r)$ satisfying

$$\begin{aligned} &0 \leq u_0(r) \leq Lr^{-m} \quad \text{for } r > 0, \\ &u_0(r) \leq Lr^{-m} - b_+ r^{-l} \quad \text{for } r > R \end{aligned} \tag{4.1}$$

with some positive constants $b_+, R > 0$ and $l > m + \lambda$.

The following lemma was proved in [3, Lemma 3.2].

Lemma 4.1. *Suppose that u_0 satisfies (4.1). Then there exist positive constants B and T_0 such that the solution $u = u(r,t)$ of (2.1) satisfies*

$$u(r,t) \leq Lr^{-m} - \frac{b_+}{2} r^{-l}$$

for all $t > T_0$ and $r \geq Bt^{1/2}$.

We need one more statement from elementary analysis. Roughly speaking, it provides an a posteriori justification for neglecting a certain logarithmic term (see (4.2) and (4.8) below).

Lemma 4.2. *Suppose that* $\zeta : (0,\infty) \to (0,\infty)$ *is a function satisfying*

$$\zeta(t) \le c_1 t^\alpha \Big\{ \ln(c_2 t \zeta^\beta(t)) \Big\}^\gamma \qquad \textit{for } t \ge t_0 \tag{4.2}$$

with some positive constants $c_1, c_2, \alpha, \beta, \gamma$ *and* t_0*. Then there exists* $t_1 \ge t_0$ *such that*

$$\zeta(t) \le c_1(\alpha\beta+2)^\gamma t^\alpha (\ln t)^\gamma \qquad \textit{for } t \ge t_1. \tag{4.3}$$

Proof. Set $w(t) := c_2 t \zeta^\beta(t)$. Then (4.2) implies

$$w(t) \le c_2 t \Big\{ c_1 t^\alpha \{\ln(c_2 t\zeta^\beta(t))\}^\gamma \Big\}^\beta = c_1^\beta c_2 t^{\alpha\beta+1} (\ln w(t))^{\beta\gamma}.$$

Writing $\phi(s) := s/(\ln s)^{\beta\gamma}$ for $s > 1$, we thus have

$$\phi(w(t)) \le c_1^\beta c_2 t^{\alpha\beta+1} \qquad \text{if } w(t) > 1. \tag{4.4}$$

To derive from this an inequality for w itself, we set $c := (\alpha\beta+2)^\gamma c_1$ and fix $t_1 \ge t_0$ so large that

$$\ln(c_2 c^\beta) + \ln(\ln t)^{\beta\gamma} \le \ln t \qquad \text{for } t \ge t_1 \tag{4.5}$$

and

$$c_2 c^\beta t^{\alpha\beta+1} (\ln t)^{\beta\gamma} \ge e^{\beta\gamma} \qquad \text{for } t \ge t_1. \tag{4.6}$$

Using (4.5), we see that if $w(t) > 1$, then

$$\begin{aligned} \phi\Big(c_2 c^\beta t^{\alpha\beta+1} (\ln t)^{\beta\gamma}\Big) &= \frac{c_2 c^\beta t^{\alpha\beta+1}(\ln t)^{\beta\gamma}}{\Big\{ \ln(c_2 c^\beta) + (\alpha\beta+1)\ln t + \ln(\ln t)^{\beta\gamma} \Big\}^{\beta\gamma}} \\ &\ge \frac{c_2 c^\beta}{(\alpha\beta+2)^{\beta\gamma}}\, t^{\alpha\beta+1} = c_1^\beta c_2 t^{\alpha\beta+1} \ge \phi(w(t)). \end{aligned}$$

However, since

$$\phi'(s) = \frac{\ln s - \beta\gamma}{(\ln s)^{\beta\gamma+1}} > 0 \qquad \text{for } s > e^{\beta\gamma},$$

it follows from (4.6) that if $w(t)$ is even larger than $e^{\beta\gamma}$, then

$$c_2 c^\beta t^{\alpha\beta+1} (\ln t)^{\beta\gamma} \ge w(t) = c_2 t \zeta^\beta(t).$$

Solving this with respect to $\zeta(t)$, we obtain (4.3) in the case $w(t) > e^{\beta\gamma}$. Otherwise, we have $c_2 t \zeta^\beta(t) \le e^{\beta\gamma}$ and hence

$$\zeta(t) \le \Big(\frac{e^{\beta\gamma}}{c_2 t_0}\Big)^{\frac{1}{\beta}},$$

which implies (4.3) in this case also. □

With this at hand, we can utilize an intersection number argument to obtain the desired upper bound for the grow-up rate of radial solutions with much less technical difficulty than in the previous section.

Lemma 4.3. *Suppose that u_0 satisfies* (4.1) *and that, for each $\alpha > u_0(0)$, $u_0(r)$ has exactly one intersection point with $\varphi_\alpha(r)$. Then there exist positive constants C and T such that the solution of* (2.1) *satisfies*

$$u(0,t) \le Ct^{\frac{m(l-m-\lambda)}{2\lambda}} (\ln t)^{\frac{m}{\lambda}} \quad \text{for all } t > T.$$

Proof. Proceeding as in [3, Proposition 3.1] (see also [1, Lemma 3.2]), we first prove that, for large t,

$$u(r,t) \ge \varphi_{\sigma(t)}(r) \quad \text{for } r > 0, \tag{4.7}$$

where we abbreviate $\sigma(t) := u(0,t)$. Indeed, since (3.1) implies that $\sigma(t) \to \infty$ as $t \to \infty$ by the outcome of the previous section, we have $\sigma(t) > \sigma(0)$ for all $t > t_0$ with some $t_0 > 0$. For fixed $t_1 > t_0$, since u_0 is assumed to have exactly one intersection point with $\varphi_\alpha(r)$, we see that $u(\cdot,t)$ does not intersect $\varphi_{\sigma(t_1)}$ for $t > t_1$, because the number of intersections of $u(\cdot,t)$ with the stationary solution $\varphi_{\sigma(t_1)}$ does not increase with time and drops at $t = t_1$. Thus, $u(\cdot,t) > \varphi_{\sigma(t_1)}$ for all $t > t_1$, because the alternative $u(\cdot,t) < \varphi_{\sigma(t_1)}$ for all $t > t_1$ would mean that u is bounded. Taking $t \searrow t_1$, we see that (4.7) holds for all $t > t_0$.

Now, from (1.6) and scaling invariance, we have

$$\begin{aligned} \varphi_\alpha(r)) \equiv \alpha\varphi_1(\alpha^{1/m} r) &\ge \alpha\Big\{ L(\alpha^{\frac{1}{m}} r)^{-m} - 2a_1(\alpha^{\frac{1}{m}} r)^{-m-\lambda} \ln(\alpha^{\frac{1}{m}} r)\Big\} \\ &= Lr^{-m} - 2a_1\alpha^{-\frac{\lambda}{m}} r^{-m-\lambda} \ln(\alpha^{\frac{1}{m}} r) \end{aligned}$$

if $\alpha^{1/m} r$ is sufficiently large. Therefore, along the curve $r = Bt^{1/2}$, with B as provided by Lemma 4.1, we obtain from Lemma 4.1 and (4.7) that

$$Lr^{-m} - \frac{b_+}{2} r^{-l} \ge u(r,t) \ge Lr^{-m} - 2a_1\sigma^{-\frac{\lambda}{m}} r^{-m-\lambda} \ln(\sigma^{\frac{1}{m}} r)$$

at $r = Bt^{1/2}$ with large t. This can be reorganized to yield

$$\begin{aligned} \sigma(t) &\le \Big(\frac{4a_1}{b_+}\Big)^{\frac{m}{\lambda}} r^{\frac{m(l-m-\lambda)}{\lambda}} \Big\{\ln(\sigma^{\frac{1}{m}} r)\Big\}^{\frac{m}{\lambda}} \\ &= \Big(\frac{4a_1}{b_+}\Big)^{\frac{m}{\lambda}} B^{\frac{m(l-m-\lambda)}{\lambda}} 2^{-\frac{m}{\lambda}} t^{\frac{m(l-m-\lambda)}{2\lambda}} \Big\{\ln(B^2 t\sigma^{\frac{2}{m}})\Big\}^{\frac{m}{\lambda}}. \end{aligned} \tag{4.8}$$

Hence, by Lemma 4.2, we conclude that

$$\sigma(t) \le Ct^{\frac{m(l-m-\lambda)}{2\lambda}} (\ln t)^{\frac{m}{\lambda}}$$

for large t with some constant $C > 0$. □

Now we are in a position to prove the upper bound on the grow-up rate as in Theorem 1.1.

Proposition 4.4. *Suppose that u_0 satisfies (1.2) and that*

$$u_0(x) \le L|x|^{-m} - b_+|x|^{-l} \qquad \text{for } |x| > R$$

with some $l > m + \lambda$, $b_+ > 0$ and $R > 0$. Then the solution of (1.1) satisfies

$$u(0,t) \le Ct^{\frac{m(l-m-\lambda)}{2\lambda}}(\ln t)^{\frac{m}{\lambda}}$$

for all $t > T$ with some positive constants C and T.

Proof. Since any initial data with the assumed properties are dominated by some radial function that is decreasing in r and satisfies the requirements of Lemma 4.3, the claim immediately follows from this lemma and the comparison principle. □.

5. Universal upper bound

In this section we derive an upper bound for radially symmetric solutions of (2.1). In fact, we will construct a supersolution of the form $u^+ = Lr^{-m} - v$ with a compactly supported function v. For this purpose, we first consider the linearized equation of (2.1) at the singular steady state

$$v_t = v_{rr} + \frac{N-1}{r}v_r + \frac{pL^{p-1}}{r^2}v.$$

We can find a solution which behaves in a self-similar way

$$v(r,t) = t^{-\frac{l}{2}}F(\eta), \qquad \eta = t^{-\frac{1}{2}}r,$$

where F satisfies

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{l}{2}F + \frac{pL^{p-1}}{\eta^2}F = 0, \qquad \eta > 0. \tag{5.1}$$

We can show by direct substitution that $\rho(\eta) := \eta^{-m-\lambda}e^{-\frac{\eta^2}{4}}$ is a solution of (5.1) with $l = m + \lambda + 2$; that is, $\rho(\eta)$ satisfies

$$\rho_{\eta\eta} + \frac{N-1}{\eta}\rho_\eta + \frac{\eta}{2}\rho_\eta + \frac{m+\lambda+2}{2}\rho + \frac{pL^{p-1}}{\eta^2}\rho = 0, \qquad \eta > 0. \tag{5.2}$$

Now we impose the following initial condition for (5.1):

$$F(1) = \rho(1), \qquad F_\eta(1) = \rho_\eta(1). \tag{5.3}$$

Lemma 5.1. *Let F be the solution of* (5.1) *with the initial condition* (5.3). *If $l > m + \lambda + 2$, then there exist $\eta_1 \in (0,1)$ and $\eta_2 \in (1,\infty)$ such that $F(\eta_1) = F(\eta_2) = 0$ and $F(\eta) > 0$ for $\eta \in (\eta_1, \eta_2)$.*

Proof. We write (5.1) and (5.2) as

$$(\eta^{N-1} e^{\frac{\eta^2}{4}} F_\eta)_\eta + \eta^{N-1} e^{\frac{\eta^2}{4}} \Big(\frac{l}{2} + \frac{pL^{p-1}}{\eta^2}\Big) F = 0 \tag{5.4}$$

and

$$(\eta^{N-1} e^{\frac{\eta^2}{4}} \rho_\eta)_\eta + \eta^{N-1} e^{\frac{\eta^2}{4}} \Big(\frac{m+\lambda+2}{2} + \frac{pL^{p-1}}{\eta^2}\Big) \rho = 0, \tag{5.5}$$

respectively. Multiplying (5.4) by ρ and (5.5) by F, taking the difference, and integrating it on $[1, \eta]$, we obtain

$$\eta^{N-1} e^{\frac{\eta^2}{4}} (F_\eta \rho - F \rho_\eta) = -\frac{l - (m+\lambda+2)}{2} \int_1^\eta \eta^{N-1} e^{\frac{\eta^2}{4}} F \rho \, d\eta. \tag{5.6}$$

Suppose here that $F(\eta) > 0$ for all $\eta \in (1,\infty)$. Then, from (5.6), we have $(F/\eta)_\eta < 0$ for $\eta \in (1,\infty)$ so that $0 < F(\eta) < \rho(\eta)$ for $\eta \in (1,\infty)$. Then, by (5.4), we have

$$\begin{aligned} 0 \le -\eta^{N-1} e^{\frac{\eta^2}{4}} F_\eta &= -e^{\frac{1}{4}} F_\eta(1) + \int_1^\eta \eta^{N-1} e^{\frac{\eta^2}{4}} \Big(\frac{l}{2} + \frac{pL^{p-1}}{\eta^2}\Big) F d\eta \\ &\le -e^{\frac{1}{4}} F_\eta(1) + \int_1^\eta \eta^{N-1-m-\lambda} \Big(\frac{l}{2} + \frac{pL^{p-1}}{\eta^2}\Big) d\eta = O(\eta^{N-m-\lambda}) \quad \text{as } \eta \to \infty. \end{aligned}$$

Hence the left-hand side of (5.6) tends to 0 as $\eta \to \infty$, while the right-hand side converges to a negative constant. This is a contradiction.

Next, suppose that $F > 0$ for all $\eta \in (0,1)$. Then from

$$\eta^{N-1} e^{\frac{\eta^2}{4}} (F_\eta \rho - F \rho_\eta) = \frac{l - (m+\lambda+2)}{2} \int_\eta^1 \eta^{N-1} e^{\frac{\eta^2}{4}} F \rho \, d\eta, \tag{5.7}$$

we have $(F/\rho)_\eta > 0$ so that $0 < F(\eta) < \rho(\eta)$ for $\eta \in (0,1)$. Again, this leads to a contradiction with (5.7) letting $\eta \to 0$. □

Using F as in Lemma 5.1, we construct a supersolution as follows.

Lemma 5.2. *Let $l > m + \lambda + 2$, and $F(\eta)$ be the solution of* (5.1) *and* (5.3). *Define v by*

$$v(r,t) := \begin{cases} \delta (t+1)^{-\frac{\hat{l}}{2}} F((t+1)^{-\frac{1}{2}} r), & (t+1)^{-\frac{1}{2}} r \in [\eta_1, \eta_2], \\ 0, & (t+1)^{-\frac{1}{2}} r \in [0,\infty) \setminus [\eta_1, \eta_2]. \end{cases}$$

Then for any $\hat{l} > l$ the function $u^+(r,t) := Lr^{-m} - v(r,t)$ is a supersolution of (2.1) *provided $\delta > 0$ is sufficiently small.*

Proof. For $\eta \in [\eta_1, \eta_2]$, we have

$$\begin{aligned} &u_t^+ - u_{rr}^+ - \tfrac{N-1}{r} u_r^+ - (u^+)^p \\ &\quad = -v_t - (Lr^{-m} - v)_{rr} - \tfrac{N-1}{r}(Lr^{-m} - v)_r - (Lr^{-m} - v)^p \\ &\quad = -v_t + v_{rr} + \tfrac{N-1}{r} v_r + (Lr^{-m})^p - (Lr^{-m} - v)^p. \end{aligned}$$

We estimate the right-hand side as follows. By (5.1), v satisfies

$$\begin{aligned} -v_t + v_{rr} + \tfrac{N-1}{r} v_r &= (t+1)^{-\hat{l}/2-1}\Big(\tfrac{\hat{l}}{2}F + \tfrac{\eta}{2}F_\eta + F_{\eta\eta} + \tfrac{N-1}{\eta}F_\eta\Big) \\ &= (t+1)^{-\hat{l}/2-1}\Big(\tfrac{\hat{l}}{2}F - \tfrac{l}{2}F - \tfrac{pL^{p-1}}{\eta^2}F\Big) = \Big\{\tfrac{(\hat{l}-l)\eta^2}{2} - pL^{p-1}\Big\}\tfrac{v}{r^2}. \end{aligned}$$

Since $\eta \geq \eta_1$, we obtain

$$-v_t + v_{rr} + \frac{N-1}{r} v_r \geq \Big\{\frac{(\hat{l}-l)\eta_1^2}{2} - pL^{p-1}\Big\}\frac{v}{r^2}. \tag{5.8}$$

Next, we estimate $(Lr^{-m})^p - (Lr^{-m} - v)^p$. Set

$$w(r,t) := \max\{0, (t+1)^{-\frac{\hat{l}}{2}} F(\eta)\}, \quad \eta = (t+1)^{-\frac{1}{2}} r.$$

Then there exists a constant $C > 0$ (independent of r and t) such that

$$w(r,t) < Cr^{-m} \tag{5.9}$$

for all (r,t). Indeed, we have

$$\begin{aligned} w &= (t+1)^{-\frac{l}{2}} F(\eta) \leq (t+1)^{-\frac{l}{2}} \rho(\eta) \\ &\leq (t+1)^{-\frac{l}{2}} \eta^{-m-\lambda} e^{-\frac{\eta^2}{4}} \leq (t+1)^{-\frac{l-m-\lambda}{2}} r^{-m-\lambda} \leq r^{-m} \end{aligned}$$

for $r \in [1, \infty)$, and $w = (t+1)^{-l/2} F(\eta) \leq F_{\max} r^{-m}$ for $r \in (0,1]$, where

$$F_{\max} := \max_{\eta \in (\eta_1, \eta_2)} F(\eta) < \infty.$$

By (5.9), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} (Lr^{-m} - v)^p = (Lr^{-m} - \delta w)^p &= (Lr^{-m})^p \Big(1 - \frac{\delta w}{Lr^{-m}}\Big)^p \\ &\leq (Lr^{-m})^p \Big(1 - (p-\varepsilon)\frac{\delta w}{Lr^{-m}}\Big) \end{aligned}$$

for $\eta \in [\eta_1, \eta_2]$. Hence we obtain

$$(Lr^{-m})^p - (Lr^{-m} - v)^p \geq (p-\varepsilon)(Lr^{-m})^{p-1}\delta v = (p-\varepsilon)L^{p-1}\frac{v}{r^2}. \tag{5.10}$$

Consequently, by (5.8) and (5.10), we have the estimate

$$u_t^+ - u_{rr}^+ - \frac{N-1}{r}u_r^+ - (u^+)^p \geq \Big\{\frac{(\hat{l}-l)\eta_1^2}{2} - \varepsilon L^{p-1}\Big\}\frac{v}{r^2}.$$

Hence, if we take ε so small that the right-hand side becomes positive, then u^+ is a supersolution. □

Using the supersolution constructed as above, we prove the following result concerning the upper bound of $u(0,t)$.

Proposition 5.3. *Suppose that u_0 satisfies $0 \leq u_0(r) \leq u^+(r,0)$ for $r > 0$ and that, for each $\alpha > u_0(0)$, $u_0(r)$ has exactly one intersection point with $\varphi_\alpha(r)$. Then, for any $\varepsilon > 0$, there exist constants $c, \tau > 0$ such that the solution of* (2.1) *satisfies $u(0,t) \leq ct^{\frac{m}{\lambda}+\varepsilon}$ for all $t > \tau$.*

Proof. Let $\varepsilon > 0$ be arbitrarily fixed, and set $l = m + \lambda + 2 + \varepsilon/2$ and $\hat{l} = m + \lambda + 2 + \varepsilon$. By the comparison principle, $0 \leq u(r,t) \leq u^+(r,t)$ for all $r > 0$ and $t > 0$. In particular, at $r = (t+1)^{1/2}$ we have

$$u(r,t) \leq u^+(r,t) = Lr^{-m} - \delta(t+1)^{-\frac{\hat{l}}{2}}F(1). \tag{5.11}$$

Since the number of intersection points is nonincreasing in time, the solution $u(r,t)$ intersects φ_α exactly once for each $\alpha > \sigma(t) := u(0,t)$. This implies that $u(r,t) > \varphi_\sigma(r)$ for all $t > 0$ and $r > 0$. Here, by (1.6) with $\alpha = 1$, for any $\varepsilon > 0$ there exists $b > 0$ such that

$$\varphi_1(r) \geq Lr^{-m} - br^{-m-\lambda+\varepsilon} \qquad \text{for all } r > 0,$$

and hence

$$\varphi_\alpha(r) \equiv \alpha\varphi_1(\alpha^{\frac{1}{m}}r) \geq Lr^{-m} - b\alpha^{-\frac{\lambda-\varepsilon}{m}}r^{-m-\lambda+\varepsilon} \qquad \text{for all } r > 0.$$

Thus, by $u(r,t) \geq \varphi_\sigma(r)$ and (5.11), the inequality

$$Lr^{-m} - b\sigma^{-\frac{\lambda-\varepsilon}{m}}r^{-m-\lambda+\varepsilon} \leq Lr^{-m} - \delta(t+1)^{-\frac{\hat{l}}{2}}F(1)$$

holds at $r = (t+1)^{1/2}$. Solving this with respect to σ, we obtain

$$\sigma(t) \leq \Big\{\frac{\delta}{b}(t+1)^{-\frac{\hat{l}}{2}+(m+\lambda-\varepsilon)/2}\Big\}^{-\frac{m}{\lambda-\varepsilon}} = \Big(\frac{\delta}{b}\Big)^{-\frac{m}{\lambda-\varepsilon}}(t+1)^{\frac{(1+\varepsilon)m}{\lambda-\varepsilon}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired upper bound for $\sigma(t) = u(0,t)$. □

6. Proofs of the main results

Proof of Theorem 1.1. Given any initial data satisfying the conditions in Theorem 1.1, we can take $t_0 > 0$ and radially symmetric initial data $u_1(r)$ and $u_2(r)$ satisfying the hypotheses of Propositions 3.8 and 4.4, respectively, such that

$$u_1(|x|) \leq u(x, t_0) \leq u_2(|x|)$$

for all $x \in \mathbb{R}^N$. We denote by $u_1(r, t)$ and $u_2(r, t)$ solutions of (2.1) with the initial data $u_1(r)$ and $u_2(r)$, respectively. Then, by the comparison principle, we have

$$u_1(|x|, t) \leq u(x, t) \leq u_2(|x|, t)$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$. We also assume that $u_1(r)$ and $u_2(r)$ are decreasing functions of $r \geq 0$. Then $u_1(r, t)$ and $u_2(r, t)$ also are decreasing functions of r for each $t > 0$ so that

$$u_1(0, t) \leq \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq u_2(0, t)$$

for all $t > 0$. By Propositions 3.8 and 4.4, the proof is complete. □

Proof of Theorem 1.2. Let $\tilde{u}(r, t)$ be the solution of (2.1) with the initial condition

$$\tilde{u}_0(r) = \begin{cases} LR^{-m} & \text{for } r \in [0, R), \\ Lr^{-m} & \text{for } r \in [R, \infty). \end{cases}$$

If we take $R > 0$ sufficiently large, then we can apply Proposition 5.3 to show that the solution $\tilde{u}$ satisfies $\tilde{u}(0, t) \leq ct^{m/\lambda+\varepsilon}$ for all $t \geq \tau$. Moreover, since $\tilde{u}_0(r)$ is nonincreasing in r, the solution $\tilde{u}(r, t)$ also is nonincreasing in r for every $t > 0$. Hence we obtain

$$\tilde{u}(r, t) \leq ct^{\frac{m}{\lambda}+\varepsilon} \tag{6.1}$$

for all $t \geq \tau$ and $r \geq 0$.

Now, for any initial data $u_0(x)$ satisfying the conditions in Theorem 1.2, we take μ so large that $0 \leq u_0(x) \leq \mu^m \tilde{u}_0(\mu|x|)$ for all $|x| > 0$. By the scaling invariance, if $\tilde{u}(r, t)$ satisfies (2.1), then, for any $\mu > 0$,

$$u(r, t) := \mu^m \tilde{u}(\mu r, \mu^2 t)$$

is a solution of (2.1) with the initial data $u_0(r) = \mu^m \tilde{u}_0(\mu r)$. Then, by the comparison principle and (6.1), we obtain

$$0 \leq u(x, t) \leq \mu^m \tilde{u}(\mu r, \mu^2 t) \leq c\mu^m (\mu^2 t)^{\frac{m}{\lambda}+\varepsilon}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Proof of Corollary 1.3. Let $\varepsilon > 0$ be an arbitrary small number and set $\tilde{l} = m+\lambda+2-\varepsilon$. Let $\tilde{u}(x,t)$ be the solution of (1.1) with the initial condition

$$\tilde{u}_0(x) = \min\{u_0(x), L|x|^{-m} - d(1+|x|)^{-\tilde{l}}\},$$

where $d > 0$ is chosen suitably so that $\tilde{u}_0(x)$ is nonnegative. Then, by the comparison principle and Theorem 1.1, we have

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq \|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq ct^{\frac{m(2-\varepsilon)}{2\lambda}}(\ln t)^{\frac{m}{\lambda}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the lower bound. The upper bound immediately follows from Theorem 1.2. □

REFERENCES

[1] J.W. Dold, V.A. Galaktionov, A.A. Lacey, and J.L. Vazquez, *Rate of approach to a singular steady state in quasilinear reaction-diffusion equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 26 (1998), 663–687.

[2] M. Fila, J.R. King, M. Winkler, and E. Yanagida, *Optimal lower bound of the grow-up rate for a supercritical parabolic equation*, J. Differ. Equations, 228 (2006), 339–356.

[3] M. Fila, M. Winkler, and E. Yanagida, *Grow-up rate of solutions for a supercritical semilinear diffusion equation*, J. Differ. Equations, 205 (2004), 365–389.

[4] V.A. Galaktionov and J.R. King, *Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents*, J. Differ. Equations, 189 (2003), 199–233.

[5] C. Gui, W.-M. Ni, and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in* $\mathbb{R}^n$, Comm. Pure Appl. Math., 45 (1992), 1153–1181.

[6] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rat. Mech. Anal., 49 (1973), 241–269.

[7] Y. Li, *Asymptotic behavior of positive solutions of equation* $\Delta u + K(x)u^p = 0$ *in* $\mathbf{R}^n$, J. Differ. Equations, 95 (1992), 304–330.

[8] N. Mizoguchi, *Growup of solutions for a semilinear heat equation with supercritical nonlinearity*, J. Differ. Equations, 227 (2006), 652–669.

[9] P. Poláčik and E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann., 327 (2003), 745–771.

[10] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc., 337 (1993), 549–590.