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WELL-POSEDNESS AND ILL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE MAXWELL–DIRAC SYSTEM IN $1+1$ SPACE TIME DIMENSIONS

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Abstract. We completely determine the range of Sobolev regularity for the Maxwell–Dirac system in $1+1$ space time dimensions to be well-posed locally in the case that the initial data of the Dirac part regularity is of L^2. The well-posedness follows from the standard energy estimates. Outside the range for the well-posedness, we show either the flow map is not continuous or not twice differentiable at zero.

1. Introduction

In this paper we study the Cauchy problem of the Maxwell–Dirac (M–D) system in $1+1$ dimensions,

$$(-i\gamma^\mu \partial_\mu + m)\psi = A_\mu \gamma^\mu \psi, \tag{1.1}$$

$$\Box A_\mu = -\langle \gamma^0 \gamma_\mu \psi, \psi \rangle, \tag{1.2}$$

$$\partial^\mu A_\mu = 0, \tag{1.3}$$

$$\psi(0) = \psi_0,\ A_\mu(0) = a_\mu,\ \partial_t A_\mu(0) = \dot{a}_\mu, \tag{1.4}$$

where $\partial_0 = \partial_t$, $\partial_1 = \partial_x$, $\Box = -\partial_t^2 + \partial_x^2$, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^2$, $\psi = \psi(t,x)$ is a $\mathbb{C}^2$-valued unknown function, $A_\mu = A_\mu(t,x)$ are real-valued unknown functions, and m is a nonnegative constant. The summation convention is used for summing over repeated indices. Matrices γ^μ satisfy the conditions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{1.5}$$

$$(\gamma^0)^* = \gamma^0,\ (\gamma^1)^* = -\gamma^1, \tag{1.6}$$

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where $g^{\mu\nu} = \mathrm{diag}(1, -1)$. Constraint (1.3) is the Lorenz gauge condition. The M–D system describes an electron self-interacting with its own electromagnetic field. The system in $1+1$ dimensions is the prototype model in the quantum field theory.

We put $\alpha^0 = I_2$, $\alpha = \alpha^1 = \gamma^0\gamma^1$, and $\beta = \gamma^0$, where I_2 denotes the identity matrix of size 2. Matrices α^μ and β are Hermitian matrices and satisfy the conditions

$$(\alpha^\mu)^2 = \beta^2 = I_2, \quad \alpha^1\beta + \beta\alpha^1 = 0.$$

Then, (1.1) and (1.2) become

$$(-i\alpha^\mu\partial_\mu + m\beta)\psi = A_\mu\alpha^\mu\psi, \tag{1.7}$$

$$\Box A_\mu = -\langle \alpha_\mu\psi, \psi\rangle. \tag{1.8}$$

In the one-dimensional case, the equations (1.2) and (1.3) require the initial data to satisfy the following two compatibility conditions:

$$\partial_x\dot{a}_1(x) = |\psi_0(x)|^2 + \partial_x^2 a_0(x), \quad \dot{a}_0(x) = \partial_x a_1(x). \tag{1.9}$$

The Lorenz gauge condition (1.3) restricts the behavior of the solutions at spatial infinity. If $\partial_x a_0$ and $\dot{a}_1$ vanish at $x = \pm\infty$, then (1.9) implies that

$$\int_{-\infty}^{\infty} |\psi_0|^2 = \|\psi_0\|_{L^2}^2 = 0,$$

which excludes the nontrivial case. This was pointed out in [24], and it is a difficulty of the one-dimensional case. Let f be a real-valued function in $C^\infty(\mathbb{R})$ satisfying the following assumption:

$$f(x) = \frac{c_0}{2}x \text{ on } |x| \le \frac{2}{5}, \quad f(x) = \mathrm{sgn}\, x \cdot \frac{c_0}{2} \text{ on } |x| \ge \frac{3}{5}, \tag{1.10}$$

where $c_0 := \|\psi_0\|_{L^2}^2$.

Remark 1.1. The function f restricts the behavior of $\dot{a}_1$ at the spatial infinity. By the condition (1.10), f depends on the initial data ψ_0. The essential assumption of f is the profile at the spatial infinity.

In this paper, we consider the case that $\psi_0 \in L^2$ and the initial data $\dot{a}_1 - f$ vanishes at $\pm\infty$. Replacing $A_1(t,x)$ with $A_1(t,x) + tf(x)$, we rewrite (1.1)–(1.4) as follows:

$$(-i\alpha^\mu\partial_\mu + m\beta)\psi = A_\mu\alpha^\mu\psi + tf\alpha\psi, \tag{1.11}$$

$$\Box A_\mu = -\langle \alpha_\mu\psi, \psi\rangle - \mu t\partial_x^2 f, \tag{1.12}$$

$$\partial^\mu A_\mu = -t\partial_x f, \tag{1.13}$$

$$\psi(0)=\psi_0,\ A_\mu(0)=a_\mu,\ \partial_t A_\mu(0)=\dot{a}_0. \tag{1.14}$$

Remark 1.2. If (1.12) and (1.13) are satisfied by the initial datum, then the solution to M–D system also satisfies (1.13). Thus, we can remove (1.13) from the system.

The initial datum ψ_0, a_μ, and $\dot{a}_\mu$ of the Cauchy problem will be taken in a Sobolev space $H^s = H^s(\mathbb{R})$ defined by the norm

$$\|u\|_{H^s} := \|\langle\cdot\rangle^s \widehat{u}\|_{L^2},$$

where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\widehat{u}$ denotes the Fourier transform of u. For $1+n$ dimensions, the M–D system with $m=0$ is invariant under the scaling

$$\psi(t,x)\to\frac{1}{\lambda^{3/2}}\psi\left(\frac{t}{\lambda},\frac{x}{\lambda}\right),\ A_\mu(t,x)\to\frac{1}{\lambda}A_\mu\left(\frac{t}{\lambda},\frac{x}{\lambda}\right);$$

hence, the scaling-invariant data space is $\psi_0\in\dot{H}^{\frac{n}{2}-\frac{3}{2}}(\mathbb{R}^n)$, $a_\mu\in\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$, where $\dot{H}^s(\mathbb{R}^n)$ denotes a homogeneous Sobolev space. One does not expect the well-posedness below this regularity. This is still a gap between the results in this paper and the regularity suggested by the scaling.

There are not many results on the 1+1-dimensional case, unlike the higher-dimensional case. Chadam [5] obtained the global existence of a solution in $H^1(\mathbb{R})\times H^1(\mathbb{R})\times L^2(\mathbb{R})$. In the case $m=0$, Huh [12] proved the global well-posedness in $L^2(\mathbb{R})\times C_b(\mathbb{R})\times C_b(\mathbb{R})$. Note that the wave data a_μ and $\dot{a}_\mu$ are taken in the same space $C_b(\mathbb{R})$ and $\partial_t A_\mu\in C_b(\mathbb{R})$ is not proved in [12]. Usually, we assume that the regularity of $\dot{a}_\mu$ is one derivative less than a_μ, and for the well-posedness, we have to prove the solution stays in the same space as the initial data, which is called the "persistency." Recently, the well-posedness for the M–D system in $1+3$ and $1+2$ dimensions has intensively been studied by D'Ancona, Foschi, and Selberg [7] and D'Ancona and Selberg [9] (see also [6]). In particular, the three-dimensional result obtained by D'Ancona, Foschi, and Selberg [7] is optimal with respect to the scaling except for the critical case $L^2(\mathbb{R}^3)\times H^{1/2}((\mathbb{R}^3)$.

We describe two new ingredients of the proof by D'Ancona, Foschi, and Selberg [7] and the difference between the higher-dimensional and the one-dimensional cases. The first one is they have uncovered an additional null form in the Dirac equation. We here explain null forms and null form estimates. In the 3-dimension case, quadratic forms in the first derivatives

$$Q_0(f,g)=-\partial_t f\partial_t g+\sum_{j=1}^{3}\partial_j f\partial_j g,$$

$$Q_{\mu\nu}(f,g) = \partial_\mu f \partial_\nu g - \partial_\nu f \partial_\mu g,\ 0 \le \mu < \nu \le 3,$$

are said to be null forms. The space-time estimates for null forms were first proved in Klainerman and Machedon [13]. They were used to improve the classical local existence theorem for nonlinear wave equations with the null forms. Using the classical method, i.e., energy estimates and the embedding theorems, one can prove that the M–D system in $1+3$ dimensions is locally well-posed in $H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Roughly speaking, the use of the Strichartz inequality allows us to improve classical local existence theorems by $1/2$ derivative. However, the Strichartz inequality method does not take into account the special structure of the nonlinearities that come up in the equations. Using the null form estimates, Bournaveas [3] proved local well-posedness in $H^{1/2+\varepsilon}(\mathbb{R}^3) \times H^{1+\varepsilon}(\mathbb{R}^3)$ for $\varepsilon > 0$. D'Ancona, Foschi, and Selberg [6, 7] have uncovered the full null structure, which can not be seen directly. The null structure found in [6, 7] is not the usual bilinear null structure that may be seen in bilinear terms of each individual component equation of a system. But one can find the special property depends on the structure of the system as a whole. Hence, they call it system null structure. In the $1+1$-dimensional case, the regularity for the well-posedness is determined by the nonlinear term with no null structure.

The second one is the fact that the M–D system in Lorenz gauge with space being 3-dimensional or 2-dimensional can be rewritten as the following system:

$$\nabla \cdot \boldsymbol{E} = \rho,\ \nabla \cdot \boldsymbol{B} = 0,\ \nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0,\ \nabla \times \boldsymbol{B} - \partial_t \boldsymbol{E} = \boldsymbol{J},$$
$$(\alpha^\mu D_\mu + m\beta)\psi = 0,$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = \nabla A_0 - \partial_t \boldsymbol{A}$, $\boldsymbol{A} = (A_1, A_2, A_3)$, $D_\mu = \frac{1}{i}\partial_\mu - A_\mu$, $J^\mu = \langle \alpha^\mu \psi, \psi \rangle$, $\rho = J^0 = |\psi|^2$, and $\boldsymbol{J} = (J^1, J^2, J^3)$. From this expression, we may consider the M–D system in $1+n$ dimensions, $n \ge 2$ as the system of the fields $(\boldsymbol{B}, \boldsymbol{E})$ and the spinor ψ, instead of the potentials A_μ and the spinor ψ. In that case, the worst part of A_μ, that has no better structure, can be neglected. This observation plays a crucial role in the proof of [7] and [9]. More precisely, they take $a_0 = \dot{a}_0 = 0$ considering the fields $(\boldsymbol{B}, \boldsymbol{E})$. On the other hand, in $1+1$ dimensions this constraint is very strong. Indeed, by (1.9), a_1 and $\dot{a}_1$ are determined by ψ_0, and we can not choose such wave initial data. Since we deal with the potentials A_μ in this paper, we must estimate the worst part of A_0, the estimate of which determines the regularity for the well-posedness. In this respect, the null form estimate is not helpful for the proof of the well-posedness in the one-dimensional case, unlike the two- or three-dimensional case.

The M–D system has the charge conservation:

$$\int |\psi(t)|^2 \, dx = \text{constant}.$$

It is natural and important to ask whether or not the global existence of the solution to the M–D system follows the charge conservation. Using this conservation, the global existence of solution was proved by [5], [10], and [12] for $1+1$ dimensions and by [9] for $1+2$ dimensions. In view of the scaling, $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ is a natural charge class. The problem with initial data in $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ has been solved for the $1+2$-dimensional case, but it remains open in $1+1$ and $1+3$ dimensions.

We define the well-posedness in this paper as follows.

Definition 1.1. *The Cauchy problem* (1.11)–(1.14) *is said to be locally well-posed in $H^s \times H^r$ if for any radius R there exists a time $T = T(R) > 0$ and a continuous flow map from*

$$\{(\psi_0, a_\mu, \dot{a}_\mu) \in H^s \times H^r \times H^{r-1} : \|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq R\}$$

to

$$C([-T,T]; H^s) \times \big(C([-T,T]; H^r) \cap C^1([-T,T]; H^{r-1})\big).$$

Remark 1.3. The following assertion is equivalent to Definition 1.1: for every $\delta > 0$, there exists a $T > 0$ such that if $\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq \delta$ holds, the solution to the M–D system on $[-T,T]$ exists, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq \delta$ holds, $\|(\psi, A_\mu, \partial_t A_\mu)\|_{C([-T,T]; H^s \times H^r \times H^{r-1})} \leq \varepsilon$, where $(\psi, A_\mu, \partial_t A_\mu)$ is the solution to the M–D system with initial data $(\psi_0, a_\mu, \dot{a}_\mu)$.

The critical scaling regularity is $(-1, -1/2)$. We obtain the local well-posedness in $(0+, 1/2+)$.

Theorem 1.2. *If $s > 0$, $s \leq r \leq \min(2s + 1/2, s + 1)$, $r > 1/2$, and $(s, r) \neq (1/2, 3/2)$, then* (1.11)–(1.14) *is locally well-posed in $H^s \times H^r$.*

The part $\langle \psi, \psi \rangle$ has no better structure and many restrictions in Theorem 1.2; this comes from the estimate of this part. Thus, we may suppose the well-posedness is broken by this part. We analyze this part in detail and obtain the following theorems, which say Theorem 1.2 is optimal.

Theorem 1.3. *Suppose $0 \leq s < 1/2$ and $r > \max(2s + 1/2, 1/2)$. Then there exist sequences $\{u_N\} \subset \mathcal{S}(\mathbb{R})$ and $t_N \searrow 0$ such that $\|u_N\|_{H^s} \to 0$, as*

$N \to \infty$, and the corresponding solution $(\psi_N, A_{\mu,N})$ to (1.11)–(1.12) *t with initial data* $\left(\binom{u_N}{0}, 0, 0\right)$ *satisfies*

$$\|A_{0,N}(t_N)\|_{H^r} \to \infty, \text{ as } N \to \infty.$$

Remark 1.4. The ill-posedness appearing in Theorem 1.3 is referred to as norm inflation. It says that the flow map of (1.11)–(1.14) fails to be continuous at 0, and fails to be bounded in a neighborhood of 0. In the case of a nonlinear operator, the notions of boundedness and continuity are not equivalent.

Theorem 1.4. *Suppose $r < s$ or $r > s + 1$ or $r \leq 1/2$ or $s = 1/2$, $r \geq 3/2$. Then for any $T > 0$, the flow map of* (1.11)–(1.14), *as a map from the unit ball centered at* 0 *in $H^s \times H^r \times H^{r-1}$ to $C([-T,T]; H^s) \times \big(C([-T,T]; H^r) \cap C^1([-T,T]; H^{r-1})\big)$, fails to be C^2.*

Remark 1.5. *If $m = 0$, we can prove the norm inflation at $(s, r) = (0, 1/2)$.*

We note that the M–D system does not have better structure than the Dirac–Klein–Gordon (D–K–G) system,

$$(-i\gamma^\mu \partial_\mu + M)\psi = \varphi\psi,$$
$$(-\Box + m^2)\varphi = \langle \gamma^0\psi, \psi\rangle,$$

where $\psi = \psi(t,x)$ is a $\mathbb{C}^2$-valued unknown function, $\varphi = \varphi(t,x)$ is a real-valued unknown function, and m and M are nonnegative constants. Machihara, Nakanishi, and Tsugawa [16] proved the local well-posedness for D–K–G in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$, provided that s and r satisfy the conditions $s > -1/2$ and $|s| \leq r \leq s+1$. The difference between Theorem 1.2 and [16] comes from the structure of the right-hand side of each second equation. The right-hand side of (1.2) with $\mu = 0$ is the square of ψ, which is the worst part. This part has no null structure, and proving the local well-posedness for small (s, r) is a difficult problem. The part that breaks down the proof of the well-posedness may imply the ill-posedness. In our case, the norm inflation comes from this part.

The largest space in Theorem 1.2 is $H^{0+}(\mathbb{R}) \times H^{1/2+}(\mathbb{R})$. We also consider $B^0_{2,1}(\mathbb{R}) \times B^{1/2}_{2,1}(\mathbb{R})$, which is a subspace of $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ but larger than any of $H^{0+}(\mathbb{R}) \times H^{1/2+}(\mathbb{R})$. If $H^s(\mathbb{R})$ is a border between well-posedness and ill-posedness, the Besov space $B^s_{2,1}(\mathbb{R})$ might be useful. For instance, let us consider the wave equation

$$-\Box u + f(u)(\dot{u}^2 - |\nabla u|^2) = 0, \ (t,x) \in \mathbb{R}^{1+n}, \tag{1.15}$$

$$u(0) = u_0,\ \partial_t u(0) = u_1, \tag{1.16}$$

where f satisfies some conditions. The equation (1.15) is invariant under the scaling $u(t,x) \to u(t/\lambda, x/\lambda)$; hence, the scaling-invariant data space is $u_0 \in H^{n/2}(\mathbb{R}^n)$. In the case $s \le n/2$, the Cauchy problem (1.15)–(1.16) is ill-posed, since (1.15)–(1.16) has blow-up solutions ([21, 22]). Nakanishi and Ohta [18] proved that, when $n \ge 2$, (1.15)–(1.16) is well-posed in $B^{n/2}_{2,1}(\mathbb{R}^n)$, and ill-posed in $B^{n/2}_{2,q}(\mathbb{R}^n)$, for $q > 1$. More precisely, they construct blow-up solutions with small initial data in $B^{n/2}_{2,q}(\mathbb{R}^n)$ for $n \ge 2$ and $q > 1$; i.e., they show norm inflation (see Remark 1.4). Thus, in the higher-dimensional case, the Besov space is useful. On the other hand, when $n = 1$, Nakanishi [17] proved the ill-posedness in $B^{1/2}_{2,1}(\mathbb{R})$. In the one-dimensional case, the space $B^{1/2}_{2,1}(\mathbb{R})$ is not useful.

We show that the M–D system is also ill-posed in the Besov space. Recall that the critical regularity of the M–D system is $(-1, -1/2)$. Thus the ill-posedness occurs in a subcritical space, not the scaling-critical space. Because the nonlinearities of the M–D system do not have good structure, the ill-posedness in a subcritical space is caused. On the other hand, the ill-posedness for (1.15)–(1.16) in $B^{1/2}_{2,1}(\mathbb{R})$ is caused by the scaling-critical space. We note that the M–D system contains a first-order PDE, and the wave equation (1.15) is a second-order PDE. There is a difference between a first-order PDE and a second-order PDE. More precisely, in the wave equation, the inhomogeneous part gains one more differentiability, but in the Dirac equation, this is not the case.

Here, we give the definition of the Besov space for completeness.

Definition 1.5. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be*

$$0 \le \varphi(\xi) \le 1,\ \varphi(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 1, \\ 0, & \text{if } |\xi| \ge 3/2. \end{cases}$$

Define $\varphi_0(\xi) = \varphi(\xi)$, $\varphi_1(\xi) = \varphi(\xi/2) - \varphi(\xi)$, and $\varphi_j(\xi) = \varphi_1(2^{-j+1}\xi)$. Then we have

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$$

for any $\xi \in \mathbb{R}\backslash\{0\}$. We define the Besov space $B^s_{p,q} = B^s_{p,q}(\mathbb{R})$ as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|f\|_{B^s_{p,q}} := \|\{2^{sj}\|\mathcal{F}^{-1}[\varphi_j \widehat{f}]\|_{L^p}\}\|_{l^q}.$$

Theorem 1.6. *For any* $T>0$*, the flow map of* (1.1)–(1.4), *as a map from the unit ball centered at* 0 *in* $B^0_{2,1}\times B^{1/2}_{2,1}\times B^{-1/2}_{2,1}$ *to* $C([-T,T];B^0_{2,1})\times \big(C([-T,T];B^{1/2}_{2,1})\cap C^1([-T,T];B^{-1/2}_{2,1})\big)$*, fails to be* C^2*.*

Remark 1.6. Theorems 1.4 and 1.6 do not imply ill-posedness, but preclude proofs of well-posedness by the contraction argument. Indeed, if the contraction argument works, the flow map proves to be C^∞ in most cases.

We also study the Maxwell–Dirac system with the Thirring model,

$$(-i\alpha^\mu\partial_\mu+m\beta)\psi=A_\mu\alpha^\mu\psi+\langle\alpha_\mu\psi,\psi\rangle\alpha^\mu\psi+tf\alpha\psi,\tag{1.17}$$

$$\Box A_\mu=-\langle\alpha_\mu\psi,\psi\rangle-\mu t\partial_x^2 f,\tag{1.18}$$

$$\partial^\mu A_\mu=-t\partial_x f,\tag{1.19}$$

$$\psi(0)=\psi_0,\ A_\mu(0)=a_\mu,\ \partial_t A_\mu(0)=\dot a_\mu.\tag{1.20}$$

Selberg and Tesfahun [23] proved that the Dirac equation with the Thirring model is locally well-posed in $H^s(\mathbb{R})$, $s>0$. They use the good structure of the cubic term. Thanks to this structure, we can treat the cubic term as the quadratic term, so the local well-posedness of the M–D system with the Thirring model holds the same as in the case of the M–D system.

Theorem 1.7. *If* $s>0$*,* $s\le r\le\min(2s+1/2,s+1)$*,* $r>1/2$*, and* $(s,r)\ne(1/2,3/2)$*, then the Cauchy problem* (1.17)–(1.20) *is locally well-posed in* $H^s\times H^r$*.*

This paper is organized as follows. In Section 2, we prove the well-posedness results. In Section 3, we prove the ill-posedness results.

2. Local well-posedness

We mention the proof of Theorem 1.2. By the energy estimates and the contraction argument, it suffices to show that

$$\|A_\mu\alpha^\mu\psi\|_{L_t^\infty H^s}\lesssim\|A_\mu\|_{L_t^\infty H^r}\|\psi\|_{L_t^\infty H^s},\tag{2.1}$$

$$\|\langle\alpha^\mu\psi,\psi\rangle\|_{L_t^\infty H^{r-1}}\lesssim\|\psi\|^2_{L_t^\infty H^s}.\tag{2.2}$$

The estimates (2.1) and (2.2) follow from the Sobolev product estimates (see Proposition 2.1), which is well-known.

Propositon 2.1. *Let* s_0*,* s_1*, and* s_2 *be real numbers. If*

$$s_0+s_1+s_2\ge\max(s_0,s_1,s_2),\ \ s_0+s_1+s_2\ge 1/2,$$

and we do not allow both to be equalities, we then have

$$\|u_1u_2\|_{H^{-s_0}} \lesssim \|u_1\|_{H^{s_1}}\|u_2\|_{H^{s_2}}$$

for all $u_j \in H^{s_j}$, $j = 1, 2$.

In the Thirring model, we use the following spaces of Bourgain–Klainerman–Machedon type. For $s, b \in \mathbb{R}$, define $X_\pm^{s,b}$ and $H^{s,b}$ to be the completion of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norms

$$\|u\|_{X_\pm^{s,b}} = \|\langle\xi\rangle^s\langle\tau \pm |\xi|\rangle^b\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}, \quad \|u\|_{H^{s,b}} = \|\langle\xi\rangle^s\langle|\tau| - |\xi|\rangle^b\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

respectively. Spaces of these type were first used by Bourgain [2] and Klainerman and Machedon [13]. By a standard argument, the proof of Theorem 1.7 is reduced to the following estimate:

$$\|A_\mu\alpha^\mu\psi\|_{X_\pm^{s,-1/2+2\varepsilon}} \lesssim \|A_\mu\|_{H^{r,1/2+\varepsilon}}\|\psi\|_{X_\pm^{s,1/2+\varepsilon}}, \tag{2.3}$$

$$\|\langle\alpha^\mu\psi,\psi\rangle\|_{H^{r-1,-1/2+2\varepsilon}} \lesssim \|\psi\|^2_{X_\pm^{s,1/2+\varepsilon}}, \tag{2.4}$$

$$\|\langle\alpha_\mu\psi,\psi\rangle\alpha^\mu\psi\|_{X_\pm^{s,-1/2+2\varepsilon}} \lesssim \|\psi\|^3_{X_\pm^{s,1/2+\varepsilon}}, \tag{2.5}$$

for $\varepsilon > 0$ sufficiently small. The estimates (2.3) and (2.4) follow from an argument similar to that used for estimates (2.1) and (2.2). We can prove the estimate (2.5), using the same argument as used for estimates (4.2) and (4.3) in [23]. Combining the above estimates, we obtain the local well-posedness for (1.17)–(1.20)

3. ILL-POSEDNESS

Since all representations of operators satisfying (1.5) and (1.6) are unitary equivalent, we may choose

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.1}$$

for calculation.

The following statement follows from Theorem 1.2. Let s and r satisfy $0 < s < 1$ and $\max(s, 1/2) < r < \min(2s + 1/2, s + 1)$. For $(\psi_0, a_\mu, \dot{a}_\mu) \in H^s \times H^r \times H^{r-1}$, there exists $T = T(\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s\times H^r\times H^{r-1}}) \in (0, 1]$, and the solution of the M–D system with initial data $(\psi_0, a_\mu, \dot{a}_\mu)$ satisfies

$$\|\psi\|_{L_t^\infty H^s(S_T)} \le C\|\psi_0\|_{H^s}, \tag{3.2}$$

$$\|A_\mu\|_{L_t^\infty H^r(S_T)} \le C(\|(a_\mu, \dot{a}_\mu)\|_{H^r\times H^{r-1}} + \|\psi_0\|^2_{H^s}), \tag{3.3}$$

where $S_T = (-T, T) \times \mathbb{R}$.

3.1. **Proof of Theorem 1.3.** Let S_m be the free evolution operator of the Dirac equation expressed as $S_m(t) := \cos(\langle\partial_x\rangle_m t) + (\gamma^1\partial_x - im)\gamma^0 \frac{\sin(\langle\partial_x\rangle_m t)}{\langle\partial_x\rangle_m}$, where $\langle\partial_x\rangle_m := (m^2 - \partial_x^2)^{1/2}$. By (3.1), we get

$$S_m(t) = \begin{pmatrix} \cos(\langle\partial_x\rangle_m t) + \frac{\partial_x}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) & -\frac{im}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) \\ -\frac{im}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) & \cos(\langle\partial_x\rangle_m t) - \frac{\partial_x}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) \end{pmatrix}. \tag{3.4}$$

We put $W(t) := \frac{\sin(t\sqrt{-\partial_x^2})}{\sqrt{-\partial_x^2}}$, which is the free evolution operator of the wave equation. We set

$$\widehat{u}_N(\xi) = N^{-2s+r/2-3/4}(\chi_{[N,N+N^{2s-r+3/2}]}(\xi) + \chi_{[-N-N^{2s-r+3/2},-N]}(\xi)),$$

where χ_A is the characteristic function of A. Then we have, for $s' \in \mathbb{R}$,

$$\|u_N\|_{H^{s'}} \lesssim N^{-2s+r/2-3/4}N^{s'+s-r/2+3/4} = N^{s'-s}. \tag{3.5}$$

We split the proof into four steps.

Step 1. *We now prove*

$$\left\|\int_0^t W(t-s)|S_m(s)\psi_{0,N}|^2\,ds\right\|_{H^r} \gtrsim tN^\sigma,\ \ \sigma := -s + r/2 - 1/4 > 0$$

for $t \gtrsim 1/N$. *Thus the desired result holds provided* $u_{0,N}$ *is replaced by* $S_m(t)\psi_{0,N}$, *where* $\psi_{0,N} = \binom{u_N}{0}$.

Without loss of generality, we may consider the case $m = 1$.

$$\begin{aligned}\mathcal{F}[|S_1(t)\psi_{0,N}|^2](\xi) = &\int_{\xi_1+\xi_2=\xi}\Big(\cos(\langle\xi_1\rangle t) + i\frac{\xi_1}{\langle\xi_1\rangle}\sin(\langle\xi_1\rangle t)\Big)\\ &\times\Big(\cos(\langle\xi_2\rangle t) + i\frac{\xi_2}{\langle\xi_2\rangle}\sin(\langle\xi_2\rangle t)\Big)\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2)\\ &+\int_{\xi_1+\xi_2=\xi}\frac{\sin(\langle\xi_1\rangle t)}{\langle\xi_1\rangle}\frac{\sin(\langle\xi_2\rangle t)}{\langle\xi_2\rangle}\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2).\end{aligned}$$

We rewrite part of the integrand as follows:

$$\begin{aligned}&\Big(\cos(\langle\xi_1\rangle t) + i\frac{\xi_1}{\langle\xi_1\rangle}\sin(\langle\xi_1\rangle t)\Big)\Big(\cos(\langle\xi_2\rangle t) + i\frac{\xi_2}{\langle\xi_2\rangle}\sin(\langle\xi_2\rangle t)\Big)\\ &= \frac{1}{4}\Big(\Big(1+\frac{\xi_1}{\langle\xi_1\rangle}\Big)e^{i\langle\xi_1\rangle t} + \Big(1-\frac{\xi_1}{\langle\xi_1\rangle}\Big)e^{-i\langle\xi_1\rangle t}\Big)\\ &\quad\times\Big(\Big(1+\frac{\xi_2}{\langle\xi_2\rangle}\Big)e^{i\langle\xi_2\rangle t} + \Big(1-\frac{\xi_2}{\langle\xi_2\rangle}\Big)e^{-i\langle\xi_2\rangle t}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big\{\Big(1+\frac{\xi_1\xi_2}{\langle\xi_1\rangle\langle\xi_2\rangle}\Big)\cos((\langle\xi_1\rangle+\langle\xi_2\rangle)t)+\Big(1-\frac{\xi_1\xi_2}{\langle\xi_1\rangle\langle\xi_2\rangle}\Big)\cos((\langle\xi_1\rangle-\langle\xi_2\rangle)t)\Big\}\\
&+\frac{i}{2}\Big\{\Big(\frac{\xi_1}{\langle\xi_1\rangle}+\frac{\xi_2}{\langle\xi_2\rangle}\Big)\sin((\langle\xi_1\rangle+\langle\xi_2\rangle)t)+\Big(\frac{\xi_1}{\langle\xi_1\rangle}-\frac{\xi_2}{\langle\xi_2\rangle}\Big)\sin((\langle\xi_1\rangle-\langle\xi_2\rangle)t)\Big\}\\
&= e^{i(\langle\xi_1\rangle+\langle\xi_2\rangle)t}+\frac{1}{2}\Big\{\Big(-1+\frac{\xi_1\xi_2}{\langle\xi_1\rangle\langle\xi_2\rangle}\Big)\cos((\langle\xi_1\rangle+\langle\xi_2\rangle)t)\\
&+\Big(1-\frac{\xi_1\xi_2}{\langle\xi_1\rangle\langle\xi_2\rangle}\Big)\cos((\langle\xi_1\rangle-\langle\xi_2\rangle)t)\Big\}+\frac{i}{2}\Big\{\Big(-2+\frac{\xi_1}{\langle\xi_1\rangle}+\frac{\xi_2}{\langle\xi_2\rangle}\Big)\\
&\times\sin((\langle\xi_1\rangle+\langle\xi_2\rangle)t)+\Big(\frac{\xi_1}{\langle\xi_1\rangle}-\frac{\xi_2}{\langle\xi_2\rangle}\Big)\sin((\langle\xi_1\rangle-\langle\xi_2\rangle)t)\Big\}\\
&=: e^{i(\langle\xi_1\rangle+\langle\xi_2\rangle)t}+\frac{1}{2}M_1(t,\xi_1,\xi_2)+\frac{i}{2}M_2(t,\xi_1,\xi_2).
\end{aligned}
$$

We also divide $e^{i(\langle\xi_1\rangle+\langle\xi_2\rangle)t}$ as follows:

$$
e^{i(\langle\xi_1\rangle+\langle\xi_2\rangle)t}=e^{i(|\xi_1|+|\xi_2|)t}+e^{i(|\xi_1|+|\xi_2|)t}\left(e^{i(\langle\xi_1\rangle-|\xi_1|)t}e^{i(\langle\xi_2\rangle-|\xi_2|)t}-1\right).
$$

Restricting ξ to the region $2N\le|\xi|\le 2N+2N^{1-2\sigma}$, we can neglect the terms

$$
\int_{\xi_1+\xi_2=\xi}F_t(\xi_1)F_t(\xi_2)\chi_{[N,N+N^{1-2\sigma}]}(\xi_1)\chi_{[-N-N^{1-2\sigma},-N]}(\xi_2),
$$
$$
\int_{\xi_1+\xi_2=\xi}F_t(\xi_1)F_t(\xi_2)\chi_{[-N-N^{1-2\sigma},-N]}(\xi_1)\chi_{[N,N+N^{1-2\sigma}]}(\xi_2),
$$

where $F_t(\xi)=\cos(\langle\xi\rangle t)+i\frac{\xi}{\langle\xi\rangle}\sin(\langle\xi\rangle t)$. By symmetry, we only consider the case $\xi_1,\xi_2\in[N,N+N^{1-2\sigma}]$. Then we get $\widehat{|u_N|^2}(\xi)=N^{4\sigma-r-1/2}h(\xi)$, where

$$
h(\xi)=\begin{cases}\xi-2N, & \xi\in[2N,2N+N^{1-2\sigma}],\\ -\xi+2N+2N^{1-2\sigma}, & \xi\in[2N+N^{1-2\sigma},2N+2N^{1-2\sigma}],\\ 0, & \text{otherwise.}\end{cases}
$$

Thus we have

$$
\int_0^t\frac{\sin((t-s)\xi)}{\xi}e^{i\xi s}h(\xi)\,ds=-\frac{1}{4\xi^2}h(\xi)e^{it\xi}(e^{-2it\xi}-1+2it\xi).
$$

For $|t\xi|\gtrsim 1$, we get

$$
\left|\int_0^t\frac{\sin((t-s)\xi)}{\xi}e^{i\xi s}h(\xi)\,ds\right|\gtrsim\frac{h(\xi)}{\xi}t.
$$

We obtain

$$\Big(\int_{2N}^{2N+2N^{2s-r+3/2}} \langle\xi\rangle^{2r}\Big|\int_0^t \frac{\sin((t-s)\xi)}{\xi} e^{i\xi s}\widehat{|u_N|^2}(\xi)\,ds\Big|^2 d\xi\Big)^{1/2} \tag{3.6}$$
$$\gtrsim tN^{-4s+r-3/2}N^{r-1}\|h\|_{L^2} = tN^{\sigma}.$$

Since

$$|e^{i(\langle\xi_1\rangle-|\xi_1|)t}e^{i(\langle\xi_2\rangle-|\xi_2|)t}-1| \le \sum_{j=1}^{2}|e^{i(\langle\xi_j\rangle-|\xi_j|)t}-1| = \sum_{j=1}^{2}\frac{t}{\langle\xi_j\rangle+|\xi_j|} \lesssim \frac{t}{N},$$

for $\xi_1,\xi_2 \in [N, N+N^{1-2\sigma}]$, we have

$$\Big|\int_0^t \frac{\sin((t-s)\xi)}{\xi}\int_{\xi_1+\xi_2=\xi} e^{i(|\xi_1|+|\xi_2|)s} \tag{3.7}$$
$$\times\Big(e^{i(\langle\xi_1\rangle-|\xi_1|)s}e^{i(\langle\xi_2\rangle-|\xi_2|)s}-1\Big)\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2)\,dt'\Big| \lesssim t^2N^{4\sigma-r-5/2}h(\xi),$$

for $2N \le |\xi| \le 2N+2N^{1-2\sigma}$. Since $|M_j(t,\xi_1,\xi_2)| \lesssim 1/N^2$ for $\xi_1,\xi_2 \in [N, N+N^{1-2\sigma}]$, we get

$$\Big|\int_0^t \frac{\sin((t-s)\xi)}{\xi}\int_{\xi_1+\xi_2=\xi} M_j(t,\xi_1,\xi_2)\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2)\,ds\Big| \lesssim tN^{4\sigma-r-7/2}h(\xi). \tag{3.8}$$

By

$$\Big|\frac{\sin(\langle\xi_1\rangle t)}{\langle\xi_1\rangle}\frac{\sin(\langle\xi_2\rangle t)}{\langle\xi_2\rangle}\Big| \lesssim \frac{1}{N^2},$$

for $\xi_1,\xi_2 \in [N, N+N^{1-2\sigma}]$, we get

$$\Big|\int_0^t \frac{\sin((t-s)\xi)}{\xi}\int_{\xi_1+\xi_2=\xi}\frac{\sin(\langle\xi_1\rangle s)}{\langle\xi_1\rangle}\frac{\sin(\langle\xi_2\rangle s)}{\langle\xi_2\rangle}\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2)\,ds\Big|$$
$$\lesssim tN^{4\sigma-r-7/2}h(\xi). \tag{3.9}$$

From (3.6), (3.7), (3.8), and (3.9), we obtain

$$\Big\|\int_0^t W(t-s)|S_1\psi_{0,N}(s)|^2\,ds\Big\|_{H^r} \gtrsim t(N^{\sigma}-N^{\sigma-1}-N^{\sigma-2}) \gtrsim tN^{\sigma},$$

which completes the proof of Step 1.

Step 2. *When* $0 < s < 1/2$ *and* $2s+1/2 < r < \min(14s/11+19/22, 14s/3+1/2)$*, we prove*

$$\Big\|\int_0^t W(t-s)(|\psi_N(s)|^2 - |S_m(s)\psi_{0,N}|^2)\,ds\Big\|_{L_t^\infty H^r(S_T)} \lesssim N^{\sigma/2},$$

where $(\psi_N, A_{\mu,N})$ *are the corresponding solution to* (1.11)–(1.12) *with initial data* $\left(\binom{u_N}{0}, 0, 0\right)$ *and* f_N*. The function* f_N *is a smooth function satisfying the condition* (1.10) *with* $c_0 = \|\psi_{0,N}\|_{L^2}$*.*

Since

$$|\psi_N(t)|^2 - |S_m(t)\psi_{0,N}|^2$$
$$= |\psi_N(t) - S_m(t)\psi_{0,N}|^2 + 2\Re\langle\psi_N(t) - S_m(t)\psi_{0,N}, S_m(t)\psi_{0,N}\rangle,$$

it suffices to show that

$$\Big\|\int_0^t W(t-s)|\psi_N(s) - S_m(s)\psi_{0,N}|^2\,ds\Big\|_{L_t^\infty H^r(S_T)} \lesssim N^{\sigma/2}, \tag{3.10}$$

$$\Big\|\int_0^t W(t-s)\langle\psi_N(s) - S_m(s)\psi_{0,N}, S_m(s)\psi_{0,N}\rangle\,ds\Big\|_{L_t^\infty H^r(S_T)} \lesssim N^{\sigma/2}. \tag{3.11}$$

We prove only (3.10), because (3.11) can be handled similarly. We put

$$s_1 = 7s/4 - 3r/8 + 3/16,\ \ s_2 = -7s/4 + 11r/8 - 11/16.$$

From the conditions in Step 2, $0 < s_1 < s < s_2 < 1/2$. By Proposition 2.1,

$$\|\Box^{-1}\langle\alpha^\mu\psi,\psi\rangle\alpha_\mu\psi\|_{L_t^\infty H^{s_2}(S_T)} \lesssim \|\Box^{-1}\langle\alpha^\mu\psi,\psi\rangle\|_{L_t^\infty H^{1/2+}(S_T)}\|\psi\|_{L_t^\infty H^{s_2}(S_T)}$$
$$\lesssim \|\psi\|^2_{L_t^\infty H^{s_1}(S_T)}\|\psi\|_{L_t^\infty H^{s_2}(S_T)}. \tag{3.12}$$

Thanks to the energy estimate and (2.1), we have

$$\|\psi_N - S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)} \le CT\|(A_\mu\alpha^\mu + tf_N\alpha)\psi\|_{L_t^\infty H^{s_2}(S_T)}$$
$$\le CT\|\Box^{-1}\langle\alpha^\mu\psi_N,\psi_N\rangle\alpha_\mu\psi_N\|_{L_t^\infty H^{s_2}(S_T)} + \|W(t)[0,f_N]\psi_N\|_{L_t^\infty H^{s_2}(S_T)}$$
$$\le CT(\|\psi_N\|^2_{L_t^\infty H^{s_1}(S_T)} + \|f_N\|_{C^{1/2}})\|\psi_N\|_{L_t^\infty H^{s_2}(S_T)}$$
$$\le CT(\|\psi_N\|^2_{L_t^\infty H^{s_1}(S_T)} + \|f_N\|_{C^{1/2}})(\|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}$$
$$+ \|\varphi S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}).$$

We have $\|\psi_N\|_{L_t^\infty H^{s_1}(S_T)} \lesssim \|\psi_{0,N}\|_{H^{s_1}} \lesssim N^{s_1-s}$, by (3.5), and $\|f_N\|_{C^{1/2}} \lesssim N^{-s}$, provided N is taken large enough. Thus we get

$$\|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)} \le CN^{2(s_1-s)}\|\varphi S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}.$$

By the linear estimates and Proposition 2.1, we obtain

$$\text{L.H.S. of (3.10)} \lesssim \|\langle\psi_N - S_m(t)\psi_{0,N}, \psi_N - S_m(t)\psi_{0,N}\rangle\|_{L_t^\infty H^{r-1}(S_T)}$$
$$\lesssim \|\psi_N - S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_1}(S_T)}\|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}$$

$$\lesssim (\|\psi_N\|_{L_t^\infty H^{s_1}(S_T)} + \|S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}) N^{2(s_1-s)} \|S_m(t)\psi_{0,N}\|_{L_t^\infty H^{s_2}(S_T)}$$
$$\lesssim \|\psi_{0,N}\|_{H^{s_1}} N^{2(s_1-s)} \|\psi_{0,N}\|_{H^{s_2}} \lesssim N^{-4s+3s_1+s_2} = N^{\sigma/2}.$$

Step 3. *We obtain* $\|A_{0,N}(t)\|_{H^r} \gtrsim tN^\sigma$ *if* $0 < s < 1/2$ *and* $2s + 1/2 < r < \min(14s/11 + 19/22, 14s/3 + 1/2)$, *and* $t \gtrsim 1/N$.

For, by Steps 1 and 2, we have that

$$\|A_{0,N}(t)\|_{H^r} \geq \Big\| \int_0^t W(t-s)|S_m(s)\psi_{0,N}|^2\, ds \Big\|_{H^r}$$
$$- \Big\| \int_0^t W(t-s)(|\psi_N(s)|^2 - |\varphi(s)S_m(s)\psi_{0,N}|^2)\, ds \Big\|_{L_t^\infty H^r(S_T)} \gtrsim tN^\sigma.$$

Step 4. *When* $0 \leq s < 1/2$ *and* $r > 2s + 1/2$, *we have* $\|A_{0,N}(t)\|_{H^r} \geq CtN^\alpha$ *for some* $\alpha > 0$ *and* $t \gtrsim 1/N$.

Indeed, let r' be such that $r' \leq r$ and r' satisfies the conditions in Step 3, and let s' be such that $0 < s' < r'/2 - 1/4$; i.e., $r' > 2s' + 1/2$. From $\|\psi\|_{H^s} \leq \|\psi\|_{H^{s'}}$, appealing to the conclusion of Step 3 with s and r replaced by s' and r', we obtain

$$\|A_{0,N}\|_{H^r} \geq \|A_{0,N}\|_{H^{r'}} \gtrsim tN^{-s'+r'/2-1/2}.$$

3.2. **Proof of Theorem 1.3 in the massless case.** In the massless case, using the notation $\psi = \binom{\psi_-}{\psi_+}$ and $A_\pm = A_0 \pm A_1$, we can rewrite the M–D system as

$$(\partial_t \pm \partial_x)\psi_\pm = i(A_\pm \pm tf)\psi_\pm, \tag{3.13}$$
$$\Box A_\pm = -2|\psi_\mp|^2 \mp t\partial_x^2 f. \tag{3.14}$$

Equations (3.13) with initial data $\psi_0 = \binom{\psi_{-,0}}{\psi_{+,0}}$ have the explicit solutions

$$\psi_\pm(t,x) = \psi_{\pm,0}(x \mp t) \exp\Big(i \int_0^t (A_\pm(s, x \mp (t-s)) \pm sf(x \mp (t-s)))\, ds \Big).$$

By (3.14),

$$\Box A_\pm = -2|\psi_\mp|^2 \mp t\partial_x^2 f = -2|\psi_{\mp,0}(t \pm x)|^2 \mp t\partial_x^2 f.$$

Thus we obtain

$$A_\pm(t,x) = \partial_t W(t) a_\pm(x) + W(t)\dot{a}_\pm$$
$$+ \int_0^t W(t-s)(-2|\psi_{\mp,0}(s \pm x)|^2 \mp s\partial_x^2 f(x))\, ds,$$

where $a_\pm = a_0 \pm a_1$ and $\dot a_\pm = \dot a_0 \pm \dot a_1$. We take

$$\widehat{\psi}_{-,0,N} := (\log N)^{-1/2}|\xi|^{-1/2}(\chi_{[1,N]}(\xi) + \chi_{[-N,-1]}(\xi)),\ t_N := (\log N)^{-1/4},$$

and $\psi_{+,0} = a_\mu = \dot a_\mu = 0$. Then we get

$$\|\psi_{-,0,N}\|_{L^2} \lesssim (\log N)^{-1/2}\Big(\int_{1\le|\xi|\le N}\frac{d\xi}{|\xi|}\Big)^{1/2} \lesssim 1.$$

For $\xi \in [2, N-1]$, we get

$$\begin{aligned}
&\mathcal{F}[|\psi_{-,0,N}|^2](\xi)\\
&= \frac{1}{\log N}\Big(\int_1^{\xi-1}\frac{d\xi}{\sqrt{\xi_1(\xi-\xi_1)}} + \int_{\xi+1}^{N}\frac{d\xi_1}{\sqrt{\xi_1(\xi_1-\xi)}} + \int_{\xi-N}^{-1}\frac{d\xi_1}{\sqrt{\xi_1(\xi_1-\xi)}}\Big)\\
&= \frac{2}{\log N}\arcsin(1-2/\xi) + \frac{2}{\log N}\log\Big(\frac{2N}{\xi} - 1 + \sqrt{\Big(\frac{2N}{\xi}-1\Big)^2 - 1}\Big)\\
&\quad - \frac{2}{\log N}\log\Big(\frac{2}{\xi} + 1 + \sqrt{\Big(\frac{2}{\xi}+1\Big)^2 - 1}\Big).
\end{aligned}$$

By a direct calculation and $\partial_x f \in C_0^\infty$,

$$\begin{aligned}
\|A_+(t_N)\|_{H^{1/2}} &\gtrsim \Big(\int_{1/t_N}^{N-1}\xi|\widehat{A_+}(t_N,\xi)|^2\,d\xi\Big)^{1/2} - \|\psi_{\pm,0,N}\|_{L^2}^2\\
&\gtrsim \frac{t_N}{\log N}\Big(\int_{1/t_N}^{N-1}\xi^{-1}\log^2\Big(\frac{2N}{\xi} - 1 + \sqrt{\Big(\frac{2N}{\xi}-1\Big)^2-1}\Big)\,d\xi\Big)^{1/2}\\
&\quad - \frac{t_N}{\log N}\Big(\int_{1/t_N}^{N-1}\xi^{-1}\log^2\Big(\frac{2}{\xi}+1+\sqrt{\Big(\frac{2}{\xi}+1\Big)^2-1}\Big)\,d\xi\Big)^{1/2}\\
&\quad - \frac{t_N}{\log N}\Big(\int_{1/t_N}^{N-1}\xi^{-1}|\arcsin(1-2/\xi)|^2\,d\xi\Big)^{1/2} - \|\psi_{\pm,0,N}\|_{L^2}^2.
\end{aligned}$$

In the first term, changing variables using $\eta = 2N/\xi - 1$, we get

$$\int_{1/t_N}^{N-1}\xi^{-1}\log^2\Big(\frac{2N}{\xi} - 1 + \sqrt{\Big(\frac{2N}{\xi}-1\Big)^2-1}\Big)\,d\xi \sim \log^3(Nt_N).$$

In the second term, changing variables using $\eta = 2/\xi + 1$, we get

$$\int_{1/t_N}^{N-1}\xi^{-1}\log^2\Big(\frac{2}{\xi}+1+\sqrt{\Big(\frac{2}{\xi}+1\Big)^2-1}\Big)\,d\xi$$

$$= \int_{(N+1)/(N-1)}^{2t_N+1} \frac{\log^2(\eta + \sqrt{\eta^2 - 1})}{\eta - 1} \, d\eta \sim \log^3 t_N.$$

In the last term, we get

$$\int_{1/t_N}^{N-1} \xi^{-1} |\arcsin(1 - 2/\xi)|^2 \, d\xi \lesssim \log(N-1) + \log t_N \lesssim \log N.$$

Thus, we obtain

$$\begin{aligned} \|A_+(t_N)\|_{H^{1/2}} &\gtrsim \frac{t_N}{\log N} \Big(\log^{3/2}(Nt_N) - \log^{3/2} t_N - \log^{1/2} N \Big) - 1 \\ &\gtrsim \frac{t_N}{\log N} \log^{3/2}(Nt_N) \gtrsim (\log N)^{1/4} \to \infty, \ N \to \infty. \end{aligned}$$

3.3. **Proof of Theorem 1.4.** In the proof of Theorem 1.4, we can neglect the mass term (see, for instance, [16]). In this subsection, we abbreviate S_0 to S. We prove that if (s, r) satisfies the assumptions of Theorem 1.4, there exists a sequence $(\psi_{0,N}, a_{\mu,N}, \dot{a}_{\mu,N})$ satisfying

$$\|(\psi_{0,N}, a_{\mu,N}, \dot{a}_{\mu,N})\|_{H^s \times H^r \times H^{r-1}} \lesssim 1,$$

but $\|\psi_N^{(2)}\|_{H^s}$ or $\|A_{0,N}^{(2)}\|_{H^r}$ is unbounded. By the definition (3.4), setting $\psi_0 = \binom{u}{v}$, we have

$$\widehat{S(t)\psi_0}(t, \xi) = \begin{pmatrix} e^{it\xi} \widehat{u}(\xi) \\ e^{-it\xi} \widehat{v}(\xi) \end{pmatrix}, \quad \mathcal{F}[W(t)(a_\mu, \dot{a}_\mu)](t, \xi) = \cos(t\xi) + \frac{\sin(t\xi)}{\xi} \widehat{\dot{a}_\mu}(\xi).$$

We set

$$\begin{aligned} \psi^{(2)}(t) &= -i \int_0^t S(t-s)(A_\mu^{(1)}(s) \alpha^\mu \psi^{(1)}(s)) \, ds, \\ A_\mu^{(2)}(t) &= -i \int_0^t W(t-s) \langle \alpha_\mu \psi^{(1)}(s), \psi^{(1)}(s) \rangle \, ds. \end{aligned}$$

If $a_1 = \dot{a}_0 = \dot{a}_1 = 0$, a direct calculation then shows that

$$\widehat{\psi^{(2)}}(t, \xi) = \frac{1}{2} \begin{pmatrix} e^{it\xi} \left(t\widehat{a_0} * \widehat{u}(\xi) + \int e^{-it\xi_1} \frac{\sin(t\xi_1)}{\xi_1} \widehat{a_0}(\xi_1) \widehat{u}(\xi - \xi_1) \, d\xi_1 \right) \\ e^{-it\xi} \left(\int e^{it\xi_1} \frac{\sin(t\xi_1)}{\xi_1} \widehat{a_0}(\xi_1) \widehat{v}(\xi - \xi_1) \, d\xi_1 + t\widehat{a_0} * \widehat{v}(\xi) \right) \end{pmatrix}, \tag{3.15}$$

$$\widehat{A_0^{(2)}}(t, \xi) = -\frac{e^{it\xi}}{4\xi^2} (e^{-2it\xi} - 1 + 2it\xi) \widehat{|u|^2}(\xi) - \frac{e^{-it\xi}}{4\xi^2} (e^{2it\xi} - 1 - 2it\xi) \widehat{|v|^2}(\xi). \tag{3.16}$$

In the case $r < s$, we set

$$\widehat{u_N}(\xi) = \chi_{[-1/2,1/2]}(\xi),$$
$$\widehat{a_{0,N}}(\xi) = N^{-r}(\chi_{[-N-1/2,-N+1/2]}(\xi) + \chi_{[N-1/2,N+1/2]}(\xi))$$

and $v_N = a_{1,N} = \dot{a}_{0,N} = \dot{a}_{1,N} = 0$. Then $\|u_N\|_{H^s} \lesssim 1$, $\|a_{0,N}\|_{H^r} \lesssim 1$, and

$$\widehat{u_N} * \widehat{a_{0,N}}(\xi) = N^{-r}(h_{1,N}(\xi) + h_{1,-N}(\xi)),$$

where

$$h_{1,N}(\xi) = \begin{cases} \xi - N + 1, & N-1 \le \xi \le N, \\ -\xi + N + 1, & N \le \xi \le N+1, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \|\psi_N^{(2)}\|_{H^s} &\gtrsim N^{s-r}\Big(\int |h_{1,N}(\xi)|^2\,d\xi\Big)^{1/2} \\ &\quad - N^{s-r-1}\Big(\int_{N-1}^{N+1}\int \chi_{[N-1/2,N+1/2]}(\xi_1)\chi_{[-1/2,1/2]}(\xi-\xi_1)\,d\xi_1\Big)^{1/2} \\ &\gtrsim N^{s-r} - N^{s-r-1} = N^{s-r}(1-1/N). \end{aligned}$$

In the case $r > s+1$, we set

$$\widehat{u_N}(\xi) = \chi_{[-1/2,1,2]}(\xi) + N^{-s}(\chi_{[-N-1/2,-N+1/2]}(\xi) + \chi_{[N-1/2,N+1/2]}(\xi))$$

and $v_N = a_{0,N} = a_{1,N} = \dot{a}_{0,N} = \dot{a}_{1,N} = 0$. Then we have $\|u_N\|_{H^s} \lesssim 1$ and

$$\widehat{|u_N|^2}(\xi) \gtrsim N^{-s}\int \chi_{[-1/2,1/2]}(\xi_1)\chi_{[N-1/2,N+1/2]}(\xi-\xi_1)\,d\xi_1 = N^{-s}h_{1,N}(\xi).$$

Thus,

$$\begin{aligned} \|A_{0,N}^{(2)}\|_{H^r} &\gtrsim \Big(\int \langle\xi\rangle^{2r}|\widehat{A_{0,N}}(\xi)|^2\,d\xi\Big)^{1/2} \\ &\gtrsim tN^{-s}\Big(\int |\xi|^{2r-2}|h_{1,N}(\xi)|^2\,d\xi\Big)^{1/2} \gtrsim tN^{-s+r-1}. \end{aligned}$$

For $s \in \mathbb{R}$ and $r < 1/2$, we define $a_{1,N} = \dot{a}_{0,N} = \dot{a}_{1,N} = 0$,

$$\widehat{u_N}(\xi) = N^{-s}N^{-a/2}\chi_{[N,N+N^a]}(\xi), \quad \widehat{a_{0,N}}(\xi) = N^{-r}N^{-a/2}\chi_{[N,N+N^a]}(\xi),$$

where $a = r + 1/2 < 1$. Then we have $\|u_N\|_{H^s} \lesssim 1$, $\|a_{0,N}\|_{H^r} \lesssim 1$, and $\widehat{a_{0,N}} * \widehat{u_N}(\xi) = N^{-s-r}N^{-a}h(\xi)$, where

$$h_N(\xi) = \begin{cases} \xi - 2N, & \xi \in [2N, 2N + N^a], \\ -\xi + 2N + 2N^a, & \xi \in [2N + N^a, 2N + 2N^a], \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\Big| \int e^{-it\xi_1} \frac{\sin(t\xi_1)}{\xi_1} \widehat{a_{0,N}}(\xi_1)\widehat{u_N}(\xi - \xi_1)\, d\xi_1 \Big| \lesssim N^{-s-r}N^{-a}N^{-1}h(\xi),$$

we have

$$\begin{aligned} \|\psi_N^{(2)}\|_{H^s} &\gtrsim (tN^{-s-r}N^{-a}N^s - N^{-s-r-1}N^{-a}N^s)\|h_N\| \\ &\gtrsim tN^{-r+a/2} - N^{-r+a/2-1} \gtrsim tN^{1/4-r/2}. \end{aligned}$$

For $s \in \mathbb{R}$ and $r = 1/2$, we define $a_{1,N} = \dot{a}_{0,N} = \dot{a}_{1,N} = 0$,

$$\widehat{u_N}(\xi) = N^{-2s-1/2}\chi_{[N^2-N, N^2+N]}(\xi), \quad \widehat{a_{0,N}}(\xi) = (\log N)^{-1/2}\langle\xi\rangle^{-1}\chi_{[1,N]}(\xi).$$

Since

$$\begin{aligned} \int_{N^2}^{N^2+N} |\widehat{a_{0,N}} * \widehat{u_N}(\xi)|\, d\xi &\gtrsim N^{-2s-1/2}(\log N)^{-1/2} \int_{N^2}^{N^2+N} \int_1^N \frac{d\xi_1}{\xi_1}\, d\xi \\ &= N^{-2s+1/2}(\log N)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} &\int_{N^2}^{N^2+N} \Big| \int e^{-it\xi_1} \frac{\sin(t\xi_1)}{\xi_1} \widehat{a_{0,N}}(\xi_1)\widehat{u}(\xi - \xi_1)\, d\xi_1 \Big|\, d\xi \\ &\lesssim tN^{-2s-1/2}(\log N)^{-1/2} \int_{N^2}^{N^2+N} \int_1^N \frac{1}{\xi_1^2}\chi_{[N^2-N,N^2+N]}(\xi - \xi_1)\, d\xi_1\, d\xi \\ &\le tN^{-2s+1/2}(\log N)^{-1/2}, \end{aligned}$$

we get, by (3.15),

$$\begin{aligned} &\|\psi_N^{(2)}(t)\|_{H^s} \ge \|u_N^{(2)}\|_{H^s} \gtrsim N^{2s-1/2}\|\widehat{u_N^{(2)}}\|_{L^1_\xi(N^2<\xi<N^2+N)} \\ &\gtrsim N^{2s-\frac{1}{2}} \int_{N^2}^{N^2+N} \Big(t|\widehat{a_{0,N}} * \widehat{u_N}(\xi)| - \Big| \int e^{-it\xi} \frac{\sin(t\xi_1)}{\xi_1} \widehat{a_{0,N}}(\xi_1)\widehat{u_N}(\xi - \xi_1)\, d\xi_1 \Big| \Big)\, d\xi \\ &\gtrsim t(\log N)^{1/2} - t(\log N)^{-1/2} \gtrsim t(\log N)^{1/2}. \end{aligned}$$

We consider the case $(s,r)=(1/2,3/2)$. Put

$$\widehat{u_N}(\xi)=\langle\xi\rangle^{-1}(\log N)^{-1/2}\chi_{[1,N]}(\xi)+\langle\xi\rangle^{-1}(\log N)^{-1/2}\chi_{[-N,-1]}(\xi)\\ +N^{-3/2}(\chi_{[N^2-N,N^2+N]}(\xi)+\chi_{[-N^2-N,-N^2+N]}(\xi))$$

and $v_N=a_{0,N}=a_{1,N}=\dot a_{0,N}=\dot a_{1,N}=0$. Then we have $\|u_{0,N}\|_{H^{1/2}}\lesssim 1$. Since

$$\begin{aligned}\widehat{|u_N|^2}(\xi)&=\widehat{u_N}*\widehat{u_N}(\xi)\\ &\gtrsim N^{-3/2}(\log N)^{-1/2}\int\langle\xi_1\rangle^{-1}\chi_{[1,N]}(\xi_1)\chi_{[N^2-N,N^2+N]}(\xi-\xi_1)\,d\xi_1\\ &\gtrsim N^{-3/2}(\log N)^{1/2}\chi_{[N^2,N^2+N]}(\xi),\end{aligned}$$

where we have used $\xi-\xi_1\in[N^2-N,N^2+N-1]$ if $\xi\in[N^2,N^2+N]$ and $\xi_1\in[1,N]$, we get

$$\|A^{(2)}_{0,N}\|_{H^{3/2}}\gtrsim tN^3\Big(\int_{N^2}^{N^2+N}\frac{1}{\xi^2}|\widehat{|u_N|^2}(\xi)|^2\,d\xi\Big)^{1/2}=t\log^{1/2}N.$$

3.4. **Proof of Theorem 1.6.** We use the same notation as in Definition 1.5. We set

$$\widehat{u_N}(\xi)=\begin{cases}\frac{2^{N/2}}{|\xi|}, & \text{if } 2^{N-1}\le|\xi|\le 2^N,\\ 0, & \text{otherwise,}\end{cases}$$

and $v_N=a_{0,N}=a_{1,N}=\dot a_{0,N}=\dot a_{1,N}=0$. Then we have $\|u_N\|_{B^0_{2,1}}=\|u_N\|_{L^2}=1$, and

$$\|\varphi_k\widehat{u_N}\|_{L^2}\lesssim\begin{cases}1, & \text{if } k=N-1,N,\\ 0, & \text{otherwise.}\end{cases}$$

By direct calculation,

$$\begin{aligned}\widehat{|u_N|^2}(\xi)=\frac{2^N}{\xi}\Big\{&2\chi_{[2^N,3\cdot 2^{N-1}]}(\xi)(\log(\xi-2^{N-1})-(N-1)\log 2)\\ &+2\chi_{[3\cdot 2^{N-1},2^{N+1}]}(\xi)(N\log 2+2\log(\xi-2^N))\\ &+\chi_{[-2^{N-1},0]}(\xi)\Big(\log\frac{2^N}{\xi+2^N}-\log\frac{2^{N-1}-\xi}{2^{N-1}-1}\Big)\\ &+\chi_{[0,2^{N-1}]}(\xi)\Big(\log(1-\frac{\xi}{2^N})+\log(1+\frac{\xi}{2^{N-1}})\Big)\Big\}.\end{aligned}$$

Thanks to (3.16), for $l = 1, \dots, N-2$,

$$\|\varphi_l \widehat{A_0^{(2)}(t,\cdot)}\|_{L^2}\| \gtrsim t2^N \Big(\int_{2^{l-1}}^{3\cdot 2^{l-1}} \frac{1}{\xi^4} \Big| \log(1 - \frac{\xi}{2^N}) + \log(1 + \frac{\xi}{2^{N-1}}) \Big|^2 d\xi \Big)^{1/2}$$
$$\gtrsim t2^{N-3l/2} \Big| \log(1 - 3\cdot 2^{l-N-1}) + \log(1 + 3 \cdot 2^{l-N}) \Big|,$$

where we have used that the integrand is decreasing for $l = 1, 2, \dots, N-2$. Hence we have

$$\sum_{l=1}^{N} 2^{l/2} \|\varphi_l \widehat{A_0^{(2)}}(t,\cdot)\|_{L^2} = t \sum_{l=2}^{N-1} 2^l \Big| \log(1 - 3 \cdot 2^{-l-1}) + \log(1 + 3 \cdot 2^{-l}) \Big|.$$

Since $\big|\log(1 - 3\cdot 2^{-l-1}) + \log(1 + 3 \cdot 2^{-l})\big| \gtrsim 2^{-l}$, we obtain

$$\|A_0^{(2)}(t,\cdot)\|_{B^{1/2}_{2,1}} \geq \sum_{l=1}^{N} 2^{l/2} \|\varphi_l \widehat{A_0^{(2)}}(t,\cdot)\|_{L^2} \gtrsim tN.$$

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