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ON PARTIAL REGULARITY OF THE BORDERLINE SOLUTION OF SEMILINEAR PARABOLIC EQUATION WITH CRITICAL GROWTH

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Abstract. In this paper, we consider the borderline solution to the semilinear equations with critical growth. A concentration phenomenon of the solution when the time goes to infinity is proved. First, we show that a ε-regularity property holds for an H^1 solution to the related elliptic equation, and then give a precise description of the formation of the bubbles. A similar bubbling description is also derived for the harmonic maps on surface. (Cf. Struwe [22], Qing [19], Qing-Tian [20], Chen-Tian [6], Lin-Wang [13], and Parker [16]).

Introduction

Consider the following problem:

$$\begin{cases} u_t = \triangle u + |u|^{p-1}u & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) \text{ given}, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 3$, with smooth boundary and $p>1$. When $u(0)$ is equal to $\lambda\varphi$ where φ is a fixed positive smooth function and λ is a positive number, it is well-known that (0.1) has a global classical solution when λ is small and the solution blows up in finite time when λ is large. The number

$$\lambda^* = \sup\{\lambda : (0.1) \text{ has a global classical solution for } u(0)=\lambda\varphi\}$$

is positive. The solution starting at $\lambda^*\varphi$, which we call the borderline solution of (0.1) at φ, was introduced in Ni–Sacks–Tavantzis [15], where it is shown that it can be extended beyond possible blow-up points to become a global weak solution. In fact, the solution is classical when p is subcritical, that is, when it is less than the critical exponent $p^*=(n+2)/(n-2)_+$; see,

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for instance, Cazenave–Haraux [5]. The regularity properties of the borderline solution in the critical and supercritical cases were studied by several authors including Galaktionov–Vazquez [10] and Mizoguchi [14]. (One may consult the introduction in Chou–Du–Zheng [3] and the survey by Fila [7] for results in this topic.) In the supercritical case where radially symmetric borderline solutions are considered in a ball, it is shown that these solutions blow up in finite time and then become regular, decaying to zero as time goes to infinity (One may refer to [8].) On the other hand, in the critical case, the radial symmetric solution remains regular for all time and blow up at infinity (refer to [10]). The same result is proved in [24], when Ω is convex and symmetric around the original point, and furthermore, when the solution u is symmetric and decreasing along the positive coordinate axis. The blow-up rates for radial symmetric borderline solution in $L^\infty(\Omega)$ are discussed in Galaktionov–King [9]. In [3] we develop a partial regularity theory for (0.1). As an application we show that these properties remain valid for borderline solution in a convex domain in the supercritical case. It is the purpose of this paper to consider the long-time behavior of the borderline in the critical case.

Our starting point is the observation that some scaling invariance properties of the borderline solutions of problem (0.1) are in striking similarity with the problem of harmonic heat flows on surfaces; see Struwe [22], Qing [19], Qing-Tian [20], Chen-Tian [6], Lin-Wang [13], and Parker [16]. Utilizing the ideas in these works, we shall establish the following result concerning the bubbles formation of the borderline solution at infinity.

Main Theorem. *Let u be a positive borderline solution of* (0.1) *in the critical case. Assume that there is no sub-convergence for $u(t)$ strongly in $H^1(\Omega)$ as t goes to infinity. Then there exist $\{t_k\}$, $t_k \to \infty$, and $u_\infty \in H^1_0(\Omega)$, a steady state of* (0.1), *such that $u(t_k) \rightharpoonup u_\infty$ weakly in $H^1(\Omega)$. Furthermore, there exist $x^1, \dots, x^N \in \overline{\Omega}$, $x^j_{i,k} \to x^j$, $\lambda^j_{i,k} \to 0$ as $k \to \infty$ for $i = 1, 2, \dots, I_j$ and $j = 1, \dots, N$, such that the following hold:*

$$u(t_k) - \sum_{j=1}^{N} \sum_{i=1}^{I_j} \big(\lambda^j_{i,k}\big)^{\frac{-2}{p-1}} \omega\Big(\frac{x - x^j_{i,k}}{\lambda^j_{i,k}}\Big) \to u_\infty \quad \text{in } H^1(\Omega), \quad \text{as } k \to \infty,$$

and the energy identity

$$\lim_{k\to\infty} E(u(t_k)) = E(u_\infty) + \Big(\sum_{j=1}^{N} I_j\Big)\mathcal{E}.$$

Here the energy of u is given by

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p^*+1}$$

and ω is the standard bubble $\omega(x) = \left(\frac{\sqrt{n(n-2)}}{1+|x|^2}\right)^{\frac{n-2}{2}}$, with energy $\mathcal{E}$ given by

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \omega|^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} \omega^{p^*+1}.$$

Compared to harmonic heat flow, the main difficulty is that the energy density is not always positive. However, we can overcome the difficulty thanks to Nash–Moser iteration.

As an application of the above theorem, we have the following proposition:

Main Proposition. *Let u be a positive borderline solution of* (0.1) *in the critical case and β be the essential limit of energy defined in Section* 1. *Assume that Ω is star-shaped; then either $\beta = 0$ or $\beta = \kappa\mathcal{E}$, $\kappa \in \mathbb{Z}^+$, where $\mathcal{E}$ is the energy of the standard bubble defined above. Moreover, we have the following equivalences:*

(1) *$\beta = 0$ is equivalent to $u(t)$ blowing up in finite time, then eventually becoming smooth and tending to* 0 *uniformly as t goes to infinity.*

(2) *$\beta = \kappa\mathcal{E}$, $\kappa \in \mathbb{Z}^+$ is equivalent to there being no sub-convergence of $u(t)$ strongly in $H^1(\Omega)$ to* 0 *as t goes to infinity.*

It is notable that for a radial symmetric solution u, $\beta = 0$ has been excluded as mentioned above. For a global smooth solution, discussed in [11], $\beta = 0$ corresponds to the case in which the Palais–Smale condition holds along a time sequence; $\beta = \kappa\mathcal{E}$, $k \in \mathbb{Z}^+$, corresponds to the case in which the Palais–Smale condition is false along a time sequence.

The method in this paper can be applicable for more general nonlinearity. Since most of the argument of the proof is essentially elliptic, it would be useful to relate the results and techniques in this paper with the large existing literature devoting to spike or bubble solutions for critical or nearly-critical elliptic problems of the form

$$-\triangle u = u^p + f(x, u).$$

(See M. Del Pino and J. Wei [18], O. Rey [21], etc.)

1. Some known results

Here we review some known results which will be used later. First, consider the problem

$$\begin{cases} u_t = \triangle u + |u|^{p-1}u & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = \lambda\varphi, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$, $\varphi \in C^\infty(\overline{\Omega})$ is a positive function in Ω, and $\lambda > 0$. A function $u \in C([0,\infty); L^2(\Omega))$ is called an H^1 solution to (1.1) if $\nabla u, u_t \in L^2(Q_T)$ and

$$\int_s^t \int_\Omega (u_t\phi + \nabla u \cdot \nabla\phi - |u|^{p-1}u\phi)\, dx\, dt = 0, \quad 0 < s < t,$$

for all $\phi \in C([0,\infty), H_0^1(\Omega))$ and $\|u(t) - \lambda\varphi\|_{L^2(\Omega)} \to 0$ as $t \downarrow 0$. For any given φ, (1.1) admits a global classical solution when λ is small and has a classical solution which blows up in finite time when λ is large. Thus there exists a critical λ^* such that global solvability holds for (1.1) when $\lambda < \lambda^*$ and finite-time blow-up holds when $\lambda > \lambda^*$. When $\lambda = \lambda^*$, it is known that (1.1) admits a global H^1 solution whose energy is decreasing and nonnegative. Therefore, the essential limit of energy

$$\beta \equiv \operatorname{ess} \lim_{t\to\infty} E(u(t))$$

is well defined and nonnegative. Moreover, this solution satisfies the following further regularity properties: There exist constants C_1 and C_2 depending only on n, $|\Omega|$, and the initial energy such that

$$\int_0^T \Big(\int_\Omega u^{p+1} \Big)^2 \leq C_1(1+T), \quad \forall T > 0, \tag{1.2}$$

and

$$\int_0^\infty \int_\Omega u_t^2 \leq C_2; \tag{1.3}$$

see, e.g., Chou–Du–Zheng [3].

Next, we consider entire solutions to the elliptic equation with critical exponent $p = (n+2)/(n-2)$

$$\triangle u + |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

A function u with finite Dirichlet energy is called a weak solution to (1.4) if

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla\varphi = \int_{\mathbb{R}^n} |u|^{p-1}u\varphi, \quad \forall\varphi \in C_c^\infty(\mathbb{R}^n).$$

According to [4], any positive weak solution of (1.4) must be a bubble. More precisely, it is of the form

$$U_{x_0,\lambda}(x) = \Big(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x-x_0|^2}\Big)^{\frac{n-2}{2}},$$

where $x_0 \in \mathbb{R}^n$ and $\lambda > 0$. $U_{x_0,\lambda}$ is a radially decreasing function centered at x_0. A direct computation shows that

$$\int_{\mathbb{R}^n} |\nabla U_{x_0,\lambda}|^2 = \int_{\mathbb{R}^n} |\nabla U_{0,1}|^2, \qquad \int_{\mathbb{R}^n} U_{x_0,\lambda}^{p+1} = \int_{\mathbb{R}^n} U_{0,1}^{p+1},$$

so all bubbles have the same energy.

Last, let us look at the scaling-invariance property of the nonlinear operator $\triangle u + |u|^{p-1}u$. Specifically, letting, $\widetilde{u}(z) = \lambda^{\frac{2}{p-1}} u(\lambda z + x_0)$, where $p > 1$, we have

$$\triangle \widetilde{u}(z) + |\widetilde{u}|^{p-1}\widetilde{u}(z) = \lambda^{\frac{2}{p-1}}(\triangle u(x) + |u|^{p-1}u(x)), \quad x = \lambda z + x_0$$

and, when p is critical,

$$\int_{T\Omega} |\nabla \widetilde{u}|^2\, dz = \int_\Omega |\nabla u|^2 \quad \text{and} \quad \int_{T\Omega} \widetilde{u}^{p+1}\, dz = \int_\Omega u^{p+1}\, dx,$$

where $Tx \equiv (x - x_0)/\lambda$. With these invariance properties, we have the following lemma concerning the splitting in L^{p+1}-norm and energy of a bounded sequence in H^1 when a bubble develops.

Lemma 1.1. *Let $\{u_k\}$ be a bounded sequence in $H^1(\Omega)$. Suppose that there exist $\{x_k\}$ in $\overline{\Omega}$, and $\{\lambda_k\}$, $\lambda_k \to 0$, $\operatorname{dist}(\{x_k\}, \partial\Omega)/\lambda_k \to \infty$ as $k \to \infty$ such that the rescaled function*

$$\widetilde{u}_k(z) = \lambda_k^{\frac{2}{p-1}} u_k(\lambda_k z + x_k) \to \omega \quad \text{in } H^1_{loc}(\mathbb{R}^n),$$

where ω is a bubble. Then

$$\int_\Omega |u_k|^{p+1} = \int_\Omega \Big| u_k - \lambda_k^{-\frac{2}{p-1}} \omega\big(\frac{x - x_k}{\lambda_k}\big)\Big|^{p+1} + \int_{\mathbb{R}^n} |\omega|^{p+1} + o(1), \tag{1.5}$$

and

$$\int_\Omega |\nabla u_k|^2 = \int_\Omega \Big|\nabla\Big(u_k - \lambda_k^{-\frac{2}{p-1}} \omega\big(\frac{x - x_k}{\lambda_k}\big)\Big)\Big|^2 + \int_{\mathbb{R}^n} |\nabla \omega|^2 + o(1), \tag{1.6}$$

as $k \to \infty$.

Proof. Let $z = T_k x \equiv (x - x_k)/\lambda_k$. Then $T_k(\Omega)$ exhausts $\mathbb{R}^n$ as $k \to \infty$. Apply the formula $(a, b > 0)$

$$\begin{aligned}(a+b)^{p+1} &= a^{p+1} + (p+1)\int_0^1 b(a+tb)^p\,dt \\ &= a^{p+1} + (p+1)\int_0^1 t^p b^{p+1}\,dt + \int_0^b \int_0^1 b\frac{d}{ds}(sa+tb)^p\,ds\,dt \\ &= a^{p+1} + b^{p+1} + p(p+1)ab\int_0^1\int_0^1 (sa+tb)^{p-1}\,ds\,dt\end{aligned}$$

to $\left|u_k - \lambda_k^{\frac{-2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\right|$ and $\lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)$ and then integrate over Ω. As a result, we have

$$\begin{aligned}\int_\Omega |u_k|^{p+1} &\le \int_\Omega \Big|u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big|^{p+1} + \int_\Omega \lambda_k^{-\frac{2(p+1)}{p-1}}\omega^{p+1}\big(\frac{x-x_k}{\lambda_k}\big) \\ &\quad + p(p+1)\int_\Omega \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big(u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big) \\ &\quad \times \Big[\int_0^1\int_0^1 \Big|s\lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big) + t\Big|u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big|\Big|^{p-1} ds\,dt\Big]dx \\ &\le \int_\Omega \Big|u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big|^{p+1} + \int_{T_k(\Omega)} |\omega|^{p+1} \\ &\quad + C\Big(\int_\Omega \lambda_k^{-\frac{p+1}{p-1}}|\omega|^{\frac{p+1}{2}}\big(\frac{x-x_k}{\lambda_k}\big)\Big|u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big|^{\frac{p+1}{2}}\Big)^{\frac{2}{p+1}} \\ &= \int_\Omega \Big|u_k - \lambda_k^{-\frac{2}{p-1}}\omega\big(\frac{x-x_k}{\lambda_k}\big)\Big|^{p+1} + \int_{T_k(\Omega)} |\omega|^{p+1} \\ &\quad + C\Big(\int_{T_k(\Omega)} |\omega|^{\frac{p+1}{2}}|\widetilde{u}_k - \omega|^{\frac{p+1}{2}}\Big)^{\frac{2}{p+1}},\end{aligned}$$

after using Hölder's inequality and the H^1-boundedness of $\{u_k\}$. For any fixed $L > 0$, as $B_L \subset T_k(\Omega)$ for all large k, we have

$$\begin{aligned}\int_{T_k(\Omega)} |\omega|^{\frac{p+1}{2}}|\widetilde{u}_k - \omega|^{\frac{p+1}{2}} &\le \Big(\int_{B_L}|\omega|^{p+1}\Big)^{\frac12}\Big(\int_{B_L}|\widetilde{u}_k - \omega|^{p+1}\Big)^{\frac12} \\ &\quad + \Big(\int_{T_k(\Omega)\setminus B_L}|\omega|^{p+1}\Big)^{\frac12}\Big(\int_{T_k(\Omega)\setminus B_L}|\widetilde{u}_k - \omega|^{p+1}\Big)^{\frac12} \\ &\le C\Big[\Big(\int_{B_L}|\widetilde{u}_k - \omega|^{p+1}\Big)^{\frac12} + \Big(\int_{\mathbb{R}^n\setminus B_L}|\omega|^{p+1}\Big)^{\frac12}\Big].\end{aligned}$$

As $\widetilde{u}_k \to \omega$ in $H^1_{loc}(\mathbb{R}^n)$, it is clear that

$$\int_{T_k(\Omega)} |\omega|^{\frac{p+1}{2}} |\widetilde{u}_k - \omega|^{\frac{p+1}{2}} = o(1),$$

so (1.5) holds. (1.6) can be proved in a similar way. □

Remark 1.1. Since every bubble is strictly radially decreasing from its center, the bubble ω described in Lemma 1.1 is centered at x_0 under either one of the following conditions: Suppose $x_k \to x_0$ and there exists a small r_0 such that $\forall x \in B_{r_0}(x_0)$,

$$\int_{B_{\lambda_k}(x_k)} |u_k|^{p+1} \geq \int_{B_{\lambda_k}(x)} |u_k|^{p+1}, \quad \text{or}$$

$$\int_{B_{\lambda_k}(x_k)} (|\nabla u_k|^2 + |u_k|^{p+1}) \geq \int_{B_{\lambda_k}(x)} (|\nabla u_k|^2 + |u_k|^{p+1})$$

holds.

As a direct consequence of some well-known results ([26] and [1]), we have the following implication on the size of the singular times for a positive borderline solution of (1).

Proposition 1.1. *For any positive borderline solution u of* (0.1) *the set* $\mathrm{S} = \mathbb{R}\backslash\{t > 0 : u$ *is smooth near* $\overline{\Omega} \times \{t\}\}$ *is of measure zero.*

Proof. Let u be a positive borderline solution starting at $\lambda^*\phi$. The solution u_λ starting at $\lambda\phi, 0 < \lambda < \lambda*$, is a global classical solution, and it converges monotonically to u from below. By (1.2) and (1.3), there exists a subset $\Gamma \subset (0, \infty)$ with $|(0,\infty)\backslash\Gamma| = 0$ such that

$$\int_\Omega u^{p+1}(t) < \infty, \ \forall t \in \Gamma. \tag{1.7}$$

By Theorem 1 in [1], for each $t_0 \in \Gamma$, there exists $\delta(t_0) > 0$ and a classical solution v of (1) on $(t_0, t_0 + \delta(t_0)]$, with $v \in C([t_0, t_0 + \delta(t_0)], L^{p+1}(\Omega))$ and $v(t_0) = u(t_0)$. By a comparison theorem, the $u_\lambda(x,t)$ lie uniformly bounded in a Hölder space of any compact subset K of $\overline{\Omega} \times (t_0, t_0 + \delta(t_0))$. Therefore, $u = \lim_{\lambda \to \lambda^*} u_\lambda$ is also smooth in $\Omega \times (t_0, \delta(t_0))$.

Setting $\Gamma_k = \{t \in \Gamma : u$ is smooth in $\Omega \times (t, t + \frac{1}{k})\}$, we have $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. Since u is smooth in $\Omega \times (t_0, t_0 + \frac{1}{k})$ for all $t_0 \in \mathrm{S} \cap \Gamma_k$, $\mathrm{S} \cap \Gamma_k$ is a countable set. It follows that $\mathrm{S} \cap \Gamma$ is countable. Hence $\mathrm{S} \subset (\mathrm{S} \cap \Gamma) \cup \mathbb{R}\backslash\Gamma$ is of measure zero. □

2. ε-REGULARITY

In this section, we consider the following problem:

$$\begin{cases} \triangle u + |u|^{p-1}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where p is critical and f is in $L^2(\Omega)$. A function $u \in H_0^1(\Omega)$ is called an H^1-weak solution of (2.1) if

$$-\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega |u|^{p-1}u\phi = \int_\Omega f\phi, \quad \forall \phi \in H_0^1(\Omega). \tag{2.2}$$

We have the following ε-regularity result.

Theorem 2.1. *Let u be an H^1-weak solution of (2.1), where $f \in L^2(\Omega)$ and Ω is an n-dimensional bounded domain ($n \geq 3$). Then there exist ε_0 and δ_0 in $(0,1)$, and C_0 depending only on n and $\|f\|_{L^2(\Omega)}$, such that if*

$$\int_{B_R(x_0)\cap\Omega} |u|^{p+1} \leq \varepsilon_0$$

for some $x_0 \in \overline{\Omega}$ and $R > 0$, then

$$\int_{B_{\delta_0 R}(x_0)\cap\Omega} |\triangle u|^2 \leq C_0 R^{-2}$$

holds.

Using elliptic L^2 theory and a covering argument, we immediately deduce

Corollary 2.1. *Under the assumptions of Theorem 2.1, suppose that there exists $R_0 > 0$ such that for each x in the closure of a subdomain Ω',*

$$\int_{B_{R_0}(x)\cap\Omega} |u|^{p+1} \leq \varepsilon_0.$$

Then $\|u\|_{H^2(\Omega')} \leq C$ for some constant C depending on n, $\|f\|_{L^2(\Omega)}$, Ω', and R_0.

We also have

Corollary 2.2. *Let u be an H^1-weak solution of (2.1), where $f \in L^2(\Omega)$ and $B_{3R}(x_0) \subset \Omega$. Suppose there exist $\varepsilon \in (0, \varepsilon_0]$ and $\rho \in (0, R)$ such that for all $x \in \partial B_{2\rho}(x_0)$*

$$\int_{B_\rho(x)} |u|^{p+1} \leq \varepsilon.$$

Then

$$\int_{\partial B_{2\rho}(x_0)} |u|^{p+1} \le \frac{C\varepsilon^{\frac{1}{2}}}{\rho},$$

where C depends on n and $\|f\|_{L^2(\Omega)}$.

Corollary 2.2 will not be used until Section 4. The reader may wish to skip its proof at this moment.

Proof. By the Sobolev inequality and elliptic regularity, for any $B_r(x) \subset \Omega$,

$$\begin{aligned}\|\nabla u\|_{L^{p+1}(B_r(x))} &\le C\Big(\|\triangle u\|_{L^2(B_r(x))} + \frac{1}{r^2}\|u\|_{L^2(B_r(x))}\Big)\\ &\le C\Big(\|\triangle u\|_{L^2(B_r(x))} + \frac{1}{r}\|u\|_{L^{p+1}(B_r(x))}\Big),\end{aligned}$$

where C depends only on n. Taking $r = \delta_0\rho$ and $x \in \partial B_{2\rho}(x_0)$ in this estimate and applying Theorem 2.1, we have

$$\|\nabla u\|_{L^{p+1}(B_{\delta_0\rho}(x))} \le \frac{C}{\rho}.$$

On the other hand, applying the trace theorem to $\widetilde{u}^2$ in $B_{\delta_0}(\frac{x-x_0}{\rho}) \cap B_2$ where $\widetilde{u} = \rho^{\frac{2}{p-1}} u(\rho z + x_0)$, we have

$$\begin{aligned}&\int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap\partial B_2} |\widetilde{u}|^{p+1} \le C\Big[\int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap B_2} |\nabla\widetilde{u}^2|^{\frac{p+1}{2}} + \int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap B_2} |\widetilde{u}|^{p+1}\Big]\\ &\le C\Big[\Big(\int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap B_2} |\widetilde{u}|^{p+1}\Big)^{\frac{1}{2}}\Big(\int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap B_2} |\nabla\widetilde{u}|^{p+1}\Big)^{\frac{1}{2}} + \int_{B_{\delta_0}(\frac{x-x_0}{\rho})\cap B_2} |\widetilde{u}|^{p+1}\Big].\end{aligned}$$

Scaling the domain back to $B_{\delta_0\rho}(x) \cap B_{2\rho}(x_0)$, we obtain

$$\begin{aligned}&\rho\int_{B_{\delta_0\rho}(x)\cap\partial B_{2\rho}(x_0)} |u|^{p+1}\\ &\le C\Big(\int_{B_{\delta_0\rho}(x)\cap B_{2\rho}(x_0)} |u|^{p+1}\Big)^{\frac{1}{2}}\Big(\rho^{p+1}\int_{B_{\delta_0\rho}(x)\cap B_{2\rho}(x_0)} |\nabla u|^{p+1}\Big)^{\frac{1}{2}}\\ &+ C\int_{B_{\delta_0\rho}(x)\cap B_{2\rho}(x_0)} |u|^{p+1} \le C(\varepsilon^{1/2} + \varepsilon). \qquad \square\end{aligned}$$

Theorem 2.1 will be proved by De Giorgi's method. To this end we need the following elementary iteration lemma taken from Ladyzhenskaya–Uraltseva [Lemma 2.4.7, 12].

Lemma 2.1. *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence of nonnegative numbers. Suppose it satisfies* $a_{k+1} \le b\mu^k a_k^{1+\varepsilon}$, $\forall k = 0,1,2,\dots$ *for some positive* b, ε, *and* $\mu > 1$. *Then* $\lim_{k\to\infty} a_k = 0$ *provided* $a_0 \le b^{-\frac{1}{\varepsilon}}\mu^{-\frac{1}{\varepsilon^2}}$.

Proof of Theorem 2.1. We shall prove the theorem by taking $B_R(x_0)$ to be B_1, the unit ball centered at the origin. The general case follows from translating and scaling. For two numbers r and R, $0 < r < R \le 1$, let $\phi \in C^\infty(B_R)$ be a cut-off function equal to 1 in B_r, vanishing outside B_R, and satisfying

$$0 \le \phi \le 1, \quad |\nabla\phi| \le \frac{2}{R-r}.$$

Setting $u_k \equiv (u-k)_+$, $A_k \equiv \{x \in \Omega : u(x) > k\}$, and $A_k^R \equiv A_k \cap B_R$, and taking $u_k\phi^2$ as the test function in (2.2), we have

$$\begin{aligned}\int_{A_k} |\nabla u_k|^2\phi^2 &= \int_{A_k} (u_k+k)^p u_k\phi^2 - \int_{A_k} u_k \nabla u_k \cdot \nabla\phi^2 - \int_{A_k} f u_k \phi^2 \\ &\le C\int_{A_k} u_k^{p+1}\phi^2 + Ck^p \int_{A_k} u_k\phi^2 + \frac{1}{2}\int_{A_k} |\nabla u_k|^2\phi^2 \\ &+ \frac{1}{2}\int_{A_k} u_k^2|\nabla\phi|^2 + \|f\|_{L^2}\Big(\int_{A_k} u_k^2\phi^4\Big)^{\frac{1}{2}} \\ &\le C\Big[\int_{A_k} u_k^{p+1}\phi^2 + k^p\int_{A_k} u_k\phi^2 + \int_{A_k} u_k^2|\nabla\phi|^2 + \Big(\int_{A_k} u_k^2\phi^4\Big)^{\frac{1}{2}}\Big],\end{aligned}$$

where the constant C in the last line depends only on n and $\|f\|_{L^2}$.

By the Sobolev inequality in the critical case, we have, for $0 < k' < k$,

$$\begin{aligned}\Big(\int_{A_k} u_k^{p+1}\phi^{p+1}\Big)^{\frac{2}{p+1}} &\le C\int_{A_k} |\nabla(u_k\phi)|^2 \le C\Big[\int_{A_k} |\nabla u_k|^2\phi^2 + \int_{A_k} u_k^2|\nabla\phi|^2\Big] \\ &\le C\Big[\int_{A_k} u_k^{p+1}\phi^2 + k^p\int_{A_k} u_k\phi^2 + \int_{A_k} u_k^2|\nabla\phi|^2 + \Big(\int_{A_k} u_k^2\phi^4\Big)^{\frac{1}{2}}\Big] \\ &\le C\Big[\int_{A_{k'}^R} u_{k'}^{p+1} + \frac{k^p}{(k-k')^p}\int_{A_{k'}^R} u_{k'}^{p+1} + \frac{1}{(k-k')^{p-1}(R-r)^2}\int_{A_{k'}^R} u_{k'}^{p+1} \\ &\quad + \frac{1}{(k-k')^{\frac{p-1}{2}}}\Big(\int_{A_{k'}^R} u_{k'}^{p+1}\Big)^{\frac{1}{2}}\Big], \qquad (2.3)\end{aligned}$$

where we have used $u_{k'} \geq k - k'$ on A_k. When $n = 3$, we take $k = 1 - \frac{1}{2^{i+1}}$, $k' = 1 - \frac{1}{2^i}$, $R = \frac{1}{2} + \frac{1}{2^{i+1}}$, and $r = \frac{1}{2} + \frac{1}{2^{i+2}}$, for $i = 0, 1, 2, \dots$, and set

$$a_i = \int_{A_{1-\frac{1}{2^i}}^{\frac{1}{2}+\frac{1}{2^{i+1}}}} u_{1-\frac{1}{2^i}}^{p+1}.$$

From (2.3), we deduce

$$\begin{aligned} a_{i+1}^{\frac{2}{p+1}} &\leq C\Big[a_i + (2^p)^i a_i + (2^{p+1})^i a_i + (2^{\frac{p-1}{2}})^i a_i^{\frac{1}{2}}\Big] \\ &\leq C\Big[(2^{p+1})^i a_i + (2^{\frac{p-1}{2}})^i a_i^{\frac{1}{2}}\Big] \leq C\Big[(2^{p+1})^i a_0^{\frac{1}{2}} + (2^{\frac{p-1}{2}})^i\Big] a_i^{\frac{1}{2}} \end{aligned}$$

after using $a_i \leq a_0$ for all i and $a_0 \leq 1$. It follows that $a_{i+1} \leq C 2^{\frac{(p+1)^2}{2} i} a_i^{\frac{p+1}{4}}$. Noting that $\frac{p+1}{4} > 1$ when $n = 3$, by Lemma 2.1 we conclude $\lim_{i\to\infty} a_i = 0$, provided $a_0 \leq \varepsilon_0$, where ε_0 is a small number depending on n and C. Hence the upper bound $\sup_{B_{\frac{1}{2}} \cap \Omega} u \leq 1$ is derived. A similar lower bound can be established by the same argument. Now the conclusion follows from elliptic regularity theory.

When $n \geq 4$, we take $k = 2^{i+1}$, $k' = 2^i$, $R = \frac{1}{2} + \frac{\alpha^i}{2}$, and $r = \frac{1}{2} + \frac{\alpha^{i+1}}{2}$, where $\alpha \in (0, 1)$ will be chosen later, and set

$$a_i = \int_{A_{2^i}^{\frac{1}{2}+\frac{\alpha^i}{2}}} u_{2^i}^{p+1}.$$

From (2.3),

$$a_{i+1}^{\frac{2}{p+1}} \leq C a_i + \frac{C}{(1-\alpha)^2}\Big(\frac{1}{2^{p-1}\alpha^2}\Big)^i a_i + C\Big(\frac{1}{2^{\frac{p-1}{2}}}\Big)^i a_i^{\frac{1}{2}}.$$

Letting $\beta \equiv 2^{p-1}\alpha^2 \in (0, 1]$, we have

$$a_{i+1}^{\frac{2}{p+1}} \leq C\frac{a_i}{\beta^i} + C\Big(\frac{1}{2^{\frac{p-1}{2}}}\Big)^i a_i^{\frac{1}{2}} \leq C_\varepsilon \frac{a_i}{\beta^i} + \varepsilon\Big(\frac{\beta}{2^{p-1}}\Big)^i. \tag{2.4}$$

First we take $\beta = 1$. From (2.4), we have

$$a_{i+1}^{\frac{2}{p+1}} \leq \gamma a_i^{\frac{2}{p+1}} + \varepsilon\Big(\frac{1}{2^{p-1}}\Big)^i,$$

where $\gamma = C_\varepsilon a_0^{\frac{p-1}{p+1}}$. By iteration,

$$a_i^{\frac{2}{p+1}} \leq \gamma^i a_0^{\frac{2}{p+1}} + \varepsilon\Big(\frac{1}{2^{p-1}}\Big)^{i-1} \sum_0^{j-1} \Big(\gamma 2^{p-1}\Big)^k$$

$$= \gamma^i a_0^{\frac{2}{p+1}} + \varepsilon\Big(\frac{1}{2^{p-1}}\Big)^{i-1} \frac{1-(\gamma\cdot 2^{p-1})^i}{1-\gamma\cdot 2^{p-1}}.$$

It follows that for any $\varepsilon_1 \in (0,1]$, there exists a small ε_0 such that $a_i \le \varepsilon_1\big(\frac{1}{2^{\frac{1}{2}(p^2-1)}}\big)^i$, or equivalently,

$$\int_{A_{2^i}\cap B_{\frac{1}{2}}} u_{2^i}^{p+1} \le \varepsilon_1\Big(\frac{1}{2^{\frac{1}{2}(p^2-1)}}\Big)^i,$$

whenever $a_0 \le \varepsilon_0$. Rescaling u by $\widetilde{u}(x) = (\frac{1}{2})^{\frac{2}{p-1}} u(\frac{x}{2})$, the new function $\widetilde{u}$ satisfies

$$\triangle\widetilde{u} + |\widetilde{u}|^{p-1}\widetilde{u} = \widetilde{f} \equiv 2^{\frac{-2p}{p-1}} f(x/2), \quad \int_{B_1} \widetilde{f}^2 = 2^{\frac{4p}{1-p}+n} \int_{B_{\frac{1}{2}}} f^2 \le \int_{B_1} f^2$$

and

$$\int_{\widetilde{A}_{2^i}\cap B_1} \widetilde{u}_{2^i}^{p+1} \le \int_{A_{2^i}\cap B_{\frac{1}{2}}} u_{2^i}^{p+1} \le \varepsilon_1\Big(\frac{1}{2^{\frac{1}{2}(p^2-1)}}\Big)^i,$$

where $\widetilde{A}_k = \{x\in\Omega : \widetilde{u} \ge k\}$. Let $\widetilde{A}_k^R = \widetilde{A}_k \cap B_R$ and

$$\widetilde{a}_i = \int_{\widetilde{A}_{2^i}^{\frac{1}{2}+\frac{\alpha^i}{2}}} \widetilde{u}_{2^i}^{p+1}.$$

Applying (2.4) to $\widetilde{u}$ with $\beta = \frac{1}{2^{\frac{1}{2}(p-1)^2}}$ and using $\widetilde{a}_i \le \varepsilon_1\big(\frac{1}{2^{\frac{1}{2}(p^2-1)}}\big)^i$, we get

$$\widetilde{a}_{i+1}^{\frac{2}{p+1}} \le \gamma \widetilde{a}_i^{\frac{2}{p+1}} + \varepsilon\Big(\frac{1}{2^{\frac{1}{2}(p^2-1)}}\Big)^i,$$

where now $\gamma = C_\varepsilon \varepsilon_1^{\frac{p-1}{p+1}}$. Iterating as before, for any $\varepsilon_2 \in (0,1]$, there exists ε_0 such that

$$\widetilde{a}_i \le \varepsilon_2\Big(\frac{1}{2^{\frac{p+2}{2}(p-1)(1+\frac{p-1}{2})}}\Big)^i,$$

or equivalently,

$$\int_{A_{2^i}\cap B_{\frac{1}{4}}} u_{2^i}^{p+1} \le \int_{A_{2^{i-n}}\cap B_{\frac{1}{2}}} \widetilde{u}_{2^{i-n}}^{p+1} \le \varepsilon_2\Big(\frac{1}{2^{\frac{p+2}{2}(p-1)(1+\frac{p-1}{2})}}\Big)^i,$$

whenever $a_0 \le \varepsilon_0$. By repeating this argument we arrive at the fact that, for any ε_k, there exists a small ε_0 such that, for $a_0 \le \varepsilon_0$,

$$\int_{A_{2^i}\cap B_{\frac{1}{2^k}}} u_{2^i}^{p+1} \le \varepsilon_k\Big(\frac{1}{2^{\nu_k}}\Big)^i, \quad \text{for } i \ge 0,$$

where

$$\nu_k = \frac{p+1}{2}(p-1)\Big[1 + \frac{p-1}{2} + \cdots + \Big(\frac{p-1}{2}\Big)^{k-1}\Big].$$

Therefore, for all $\lambda \geq 1$, we have

$$\int_{A_\lambda \cap B_{\frac{1}{2^k}}} u_\lambda^{p+1} \leq \varepsilon_k \frac{1}{\lambda^{\nu_k}}.$$

So

$$|A_\lambda \cap B_{\frac{1}{2^k}}| \leq \Big(\frac{\lambda}{2}\Big)^{-(p+1)} \int_{A_{\frac{\lambda}{2}} \cap B_{\frac{1}{2^k}}} u_{\frac{\lambda}{2}}^{p+1} \leq \varepsilon_k \frac{1}{(\frac{\lambda}{2})^{\nu_k+p+1}}.$$

Consequently,

$$\begin{aligned}
&\int_{B_{\frac{1}{2^{k+1}}} \cap \Omega} u_+^{\nu_k+p+1}\,dx \\
&= (\nu_k+p+1)\Big(\int_1^\infty t^{\nu_k+p}|A_t \cap B_{\frac{1}{2^{k+1}}}|\,dt + \int_0^1 t^{\nu_k+p}|A_t \cap B_{\frac{1}{2^{k+1}}}|\,dt\Big) \\
&\leq C\Big(\varepsilon_k \int_1^\infty t^{\nu_k-\nu_{k+1}-1}\,dt + |B_{\frac{1}{2^{k+1}}}|\Big) \leq C\Big(\varepsilon_k + \frac{1}{2^{nk}}\Big).
\end{aligned}$$

Similarly, one proves that

$$\int_{B_{\frac{1}{2^{k+1}}} \cap \Omega} u_-^{\nu_k+p+1}\,dx \leq C\Big(\varepsilon_k + \frac{1}{2^{nk}}\Big).$$

Observing that ν_k tends to ∞ for $n=4$ and to $4n/(n-2)(n-4)$ for $n>4$ as $k \to \infty$, we can fix a large k_0 such that $\nu_{k_0} + p + 1 \geq 2(n+2)/(n-2)$. By taking $\varepsilon_{k_0} = 1$, we find a small ε_0 such that

$$\int_{B_{2\delta_0} \cap \Omega} |u|^{2p}\,dx \leq C, \quad \delta_0 = \frac{1}{2^{k_0+2}},$$

whenever $a_0 \leq \varepsilon_0$. Now the desired conclusion follows from elliptic L^2 theory. □

3. Extracting bubbles

Let us start with a more general situation than that stated in Main Theorem. Assume that we are given a sequence of weak H^1 solutions $\{u_k\}$ of the following problem:

$$\begin{cases} \triangle u_k + |u_k|^{p-1}u_k = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where for some constants C_0 and C_1,

$$\int_\Omega |\nabla u_k|^2 + |u_k|^{p+1} \le C_0, \quad \int_\Omega f_k^2 \le C_1. \tag{3.2}$$

We may also assume

$$u_k \rightharpoonup u_\infty \quad \text{in } H^1(\Omega), \qquad f_k \rightharpoonup f_\infty \quad \text{in } L^2(\Omega), \text{ as } k \to \infty. \tag{3.3}$$

Under the assumption that $\{u_k\}$ does not contain any convergent subsequence in $H^1(\Omega)$, we shall describe how to extract finitely many bubbles so that there is a subsequent converging to u_∞ in $H^1(\Omega)$ when these bubbles are removed. To this end we define the "blow-up set" of $\{u_k\}$ to be

$$\mathcal{S} = \{x \in \overline{\Omega} : \lim_{R\to 0} \liminf_{k\to\infty} \int_{B_R(x)\cap\Omega} |u_k|^{p+1} \ge \varepsilon_0\},$$

where ε_0 is given in Theorem 2.1 and it depends on C_1. By Corollary 2.1, $\mathcal{S}$ is a non-empty finite set $\{x^1, \dots, x^N\}$.

We shall show that there are bubbles ω_i^j and sequences $\{x_{i,k}^j\} \to x^j$, $\lambda_{i,k}^j \to 0$, $i = 1, 2, \dots, I_j$, $j = 1, \dots, N$, such that for a subsequence of $\{u_k\}$ (still denoted by $\{u_k\}$)

$$u_k - \sum_{j=1}^{N} \sum_{i=1}^{I_j} (\lambda_{i,k}^j)^{\frac{-2}{p-1}} \omega_i^j \Big(\frac{x - x_{i,k}^j}{\lambda_{i,k}^j} \Big) \to u_\infty \quad \text{in } H^1(\Omega). \tag{3.4}$$

In the following discussion we shall pass to a subsequence of the original sequence $\{u_k\}$ many times to get the subsequence with the desired property. To simplify notation we shall not distinguish the chosen subsequence with its preceding one.

Step 1. Extracting the first bubble.

For simplicity, we shall take x^1 to be the origin and assume $x^2, \dots, x^N$ do not belong to $\overline{B_1}$. We also assume that x^1 is an interior blow-up point and $\overline{B_1} \subset \Omega$. We will discuss boundary blow-up points at the end of Step 1.

For each $x \in \overline{B_1}$, the quantity

$$\int_{B_R(x)\cap\Omega} |u_k|^{p+1}$$

increases strictly from 0 to $\|u_k\|_{L^{p+1}}^{p+1}$ as R increases from 0 to $\operatorname{diam}\Omega$. Thus there exists a unique R_k depending on x, such that

$$\int_{B_{R_k}(x)\cap\Omega} u_k^{p+1} = \frac{\varepsilon_0}{2}.$$

It is readily checked that for each k, the function $R_k(x)$ attains a minimum at $x_{1,k} \in \overline{B_1 \cap \Omega}$. Letting $\lambda_{1,k} = R_k(x_{1,k})$, we have

$$\int_{B_{\lambda_{1,k}}(x_{1,k})\cap\Omega} u_k^{p+1} = \frac{\varepsilon_0}{2}, \quad \int_{B_{\lambda_{1,k}}(x)\cap\Omega} u_k^{p+1} \le \frac{\varepsilon_0}{2}, \quad \forall x \in \overline{B_1}. \tag{3.5}$$

It is easy to see that $\lambda_{1,k} \to 0$ and $x_{1,k} \to 0$ as $k \to \infty$. In fact, if not, then for a subsequence, either $\lambda_{1,k} \ge \delta > 0$ or $x_{1,k} \to y \ne 0$. For the former case, by Corollary 2.1, we have $\{u_k\}$ is pre-compact in $H^1(B_1)$, which contradicts the assumption. For the latter case, noting that $u_k \to u_\infty$ locally in $\overline{B_1} \setminus 0$, we have

$$\begin{aligned} \int_{B_{\lambda_{1,k}}(x_{1,k})\cap\Omega} u_k^{p+1} &= \int_{B_{\lambda_{1,k}}(x_{1,k})\cap\Omega} u_\infty^{p+1} + o(1) \\ &= \int_{B_{\lambda_{1,k}}(y)\cap\Omega} u_\infty^{p+1} + o(1) = o(1) \end{aligned}$$

since $\lim_{k\to\infty} \lambda_{1,k} = 0$. This implies a contradiction with (3.5).

Consider the rescaled function $\widetilde{u}_k(z) = \lambda_{1,k}^{\frac{2}{p-1}} u_k(\lambda_{1,k} z + x_{1,k})$, which satisfies

$$\triangle \widetilde{u}_k + \widetilde{u}_k^p = \widetilde{f}_k \equiv \lambda_{1,k}^{\frac{2p}{p-1}} f_k.$$

We have

$$\int_{B_M} (|\nabla \widetilde{u}_k|^2 + \widetilde{u}_k^{p+1}) = \int_{B_{M\lambda_{1,k}}(x_{1,k})} (|\nabla u_k|^2 + u_k^{p+1}) \le C_0$$

for each fixed M and

$$\int_{T_k(\Omega)} \widetilde{f}_k^2 \to 0 \quad \text{as } k \to \infty,$$

where $T_k x = (x - x_{1,k})/\lambda_{1,k}$. Also, (3.5) becomes

$$\int_{B_1} \widetilde{u}_k^{p+1} = \frac{\varepsilon_0}{2} \ge \int_{B_1(z)} \widetilde{u}_k^{p+1}, \quad \forall z \in \overline{T_k(B_1)}. \tag{3.5$'$}$$

By Corollary 2.1 and Remark 1.1 we conclude that $\{\widetilde{u}_k\}$ subconverges to a bubble ω_1 centered at the origin in $H^1_{loc}(\mathbb{R}^n)$. In other words, for any $L > 0$,

$$u_k - \lambda_1^{\frac{-2}{p-1}} \omega_1\Big(\frac{x - x_{1,k}}{\lambda_{1,k}}\Big) \to 0 \quad \text{in } H^1(B_{L\lambda_{1,k}}(x_{1,k})), \qquad \text{as } k \to \infty,$$

holds.

So far we have found the first bubble when the blow-up point 0 is in Ω. When 0 lies on the boundary, we still have $\{x_{1,k}\} \to 0$ and $\{\lambda_{1,k}\} \to 0$ as before, but now $x_{1,k}$ may be on the boundary of Ω. Thus (3.5′) should be replaced by

$$\int_{B_1'} \widetilde{u}_k^{p+1} = \frac{\varepsilon_0}{2} \geq \int_{B_1'(z)} \widetilde{u}_k^{p+1}, \quad \forall z \in \overline{T_k(B_1 \cap \Omega)}, \tag{3.5''}$$

where $B_1'(z) = B_1(z) \cap T_k(B_1 \cap \Omega)$. If one can show that

$$\frac{\operatorname{dist}(x_{1,k}, \partial\Omega)}{\lambda_{1,k}} \to \infty,$$

then the above argument works also for a boundary blow-up point and shows that $\{\widetilde{u}_k\}$ subconverges to a bubble in $H^1_{loc}(\mathbb{R}^n)$. To establish this we assume to the contrary that

$$\frac{\operatorname{dist}(x_{1,k}, \partial\Omega)}{\lambda_{1,k}} \to d_0$$

for some number d_0 (for a subsequence). Let x_k' be a point on the boundary realizing $\operatorname{dist}(x_{1,k}, \partial\Omega)$. One may assume that

$$z_k' \equiv \frac{x_k' - x_{1,k}}{\lambda_{1,k}} \to z^*, \qquad \nu_k \to \nu_0, \quad \text{as } k \to \infty,$$

where ν_k is the unit outer normal of $T_k(\Omega)$ at z_k'. From (3.5″) we conclude as before that $\{\widetilde{u}_k\}$ subconverges to a nontrivial, positive classical solution w of $\triangle w + w^p = 0$ in the half space $\{x : x \cdot \nu_0 \leq 0\}$ and vanishes on its boundary. However, this is impossible by the Pohožaev identity; see Struwe [23, Lemma 3.1.4].

Since there is a little difference between treating interior and boundary blow-up points, in the following steps we shall assume the blow-up point to be an interior one.

Step 2. Extracting the second bubble. Let

$$v_k(x) = u_k(x) - \lambda_{1,k}^{-\frac{2}{p-1}} \omega_1\Big(\frac{x - x_{1,k}}{\lambda_{1,k}}\Big).$$

If $\{v_k\}$ subconverges to u_∞ in $H^1(B_1)$, we stop and pass to the next blow-up point. Otherwise, we would have $v_k \to u_\infty$ in $H^1_{loc}(B_1 \setminus \{0\})$ and there would exist $\nu_0 > 0$ such that

$$\lim_{\delta \to 0} \limsup_{k \to \infty} \int_{B_\delta} (|\nabla v_k|^2 + |v_k|^{p+1}) \geq \nu_0.$$

Arguing as in Step 1, we then obtain $\{x_{2,k}\} \subset \overline{B_1}$, $x_{2,k} \to 0$, and $\lambda_{2,k} \to 0$ such that

$$\int_{B_{\lambda_{2,k}}(x_{2,k})} (|\nabla v_k|^2 + |v_k|^{p+1}) = \sup_{x\in\overline{B_1}} \int_{B_{\lambda_{2,k}}(x)} (|\nabla v_k|^2 + |v_k|^{p+1}) = \frac{1}{2}\min\{\varepsilon_0, \nu_0\}. \tag{3.6}$$

We claim that

$$\frac{\lambda_{2,k}}{\lambda_{1,k}} + \frac{|x_{1,k} - x_{2,k}|}{\lambda_{1,k}} \to \infty \quad \text{as } k \to \infty. \tag{3.7}$$

For, if, after passing to a subsequence,

$$\frac{\lambda_{2,k}}{\lambda_{1,k}} + \frac{|x_{1,k} - x_{2,k}|}{\lambda_{1,k}} \le M$$

for some M, then the ball $B_{\lambda_{2,k}/\lambda_{1,k}}((x_{2,k} - x_{1,k})/\lambda_{1,k})$ is contained in B_M for all large k. However, as $\widetilde{u}_k \to \omega_1$ in $H^1(B_M)$,

$$\begin{aligned}
&\int_{B_{\lambda_{2,k}}(x_{2,k})} (|\nabla v_k|^2 + |v_k|^{p+1}) \\
&= \int_{B_{\frac{\lambda_{2,k}}{\lambda_{1,k}}}\left(\frac{x_{2,k}-x_{1,k}}{\lambda_{1,k}}\right)} (|\nabla(\widetilde{u}_k - \omega_1)|^2 + |\widetilde{u}_k - \omega_1|^{p+1}) \\
&\le \int_{B_M} (|\nabla(\widetilde{u}_k - \omega_1)|^2 + |\widetilde{u}_k - \omega_1|^{p+1}) \to 0
\end{aligned}$$

as $k \to \infty$, contradicting (3.6), so, (3.7) must hold.

Noting that (3.7) is equivalent to

$$\frac{\lambda_{2,k}}{\lambda_{1,k}} + \frac{|x_{1,k} - x_{2,k}|}{\lambda_{1,k} + \lambda_{2,k}} \to \infty, \quad \text{as } k \to \infty,$$

we consider the following two cases separately:
Case (i)

$$\frac{|x_{1,k} - x_{2,k}|}{\lambda_{1,k} + \lambda_{2,k}} \to \infty,$$

and
Case (ii)

$$\frac{\lambda_{2,k}}{\lambda_{1,k}} \to \infty \quad \text{and} \quad \frac{|x_{1,k} - x_{2,k}|}{\lambda_{1,k} + \lambda_{2,k}} \le M_0 \qquad \text{as } k \to \infty,$$

for some M_0 after passing to a subsequence. Our goal is to show that in both cases,

$$\widetilde{v}_k(z) \equiv \lambda_{2,k}^{\frac{2}{p-1}} v_k(\lambda_{2,k} z + x_{2,k}) \equiv \widehat{u}_k(z) - \Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2}{p-1}} \omega_1\Big(\frac{\lambda_{2,k} z + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big)$$

tends to another bubble ω_2 centered at the origin in $H^1_{loc}(\mathbb{R}^n)$. To establish this fact we first note that for all $x \in \overline{B_1}$ and $L > 0$,

$$\begin{aligned}
&\int_{B_{\lambda_{2,k}}(x)\setminus B_{L\lambda_{1,k}}(x_{1,k})} (|\nabla u_k|^2 + |u_k|^{p+1}) \\
&= \int_{B_{\lambda_{2,k}}(x)\setminus B_{L\lambda_{1,k}}(x_{1,k})} (|\nabla v_k|^2 + |v_k|^{p+1}) \\
&\quad + \int_{B_{\lambda_{2,k}}(x)\setminus B_{L\lambda_{1,k}}(x_{1,k})} \Big(2\nabla v_k + \lambda_{1,k}^{-\frac{p+1}{p-1}} \nabla\omega_1\big(\frac{x - x_{1,k}}{\lambda_{1,k}}\big)\Big) \\
&\qquad\qquad \cdot \lambda_{1,k}^{-\frac{p+1}{p-1}} \nabla\omega_1\big(\frac{x - x_{1,k}}{\lambda_{1,k}}\big) \\
&\quad + (p+1) \int_0^1 \int_{B_{\lambda_{2,k}}(x)\setminus B_{L\lambda_{1,k}}(x_{1,k})} \Big(v_k + t\lambda_{1,k}^{-\frac{2}{p-2}} \omega_1\big(\frac{x - x_{1,k}}{\lambda_{1,k}}\big)\Big)^p \\
&\qquad\qquad \cdot \lambda_{1,k}^{-\frac{2}{p-1}} \omega_1\big(\frac{x - x_{1,k}}{\lambda_{1,k}}\big) dx\, dt \\
&\qquad \le \frac{1}{2}\min\{\varepsilon_0, \nu_0\} + C\Big[\big(\int_{\mathbb{R}^n\setminus B_L} |\nabla\omega_1|^2\big)^{\frac{1}{2}} + \big(\int_{\mathbb{R}^n\setminus B_L} |\omega_1|^{p+1}\big)^{\frac{1}{p+1}}\Big].
\end{aligned}$$

Therefore, there exists a large L_0 such that

$$\int_{B_{\lambda_{2,k}}(x)\setminus B_{L\lambda_{1,k}}(x_{1,k})} (|\nabla u_k|^2 + |u_k|^{p+1}) \le \varepsilon_0, \quad \forall x \in \overline{B_1},\ \forall L \ge L_0. \tag{3.8}$$

Let us consider Case (i) first. In this case, as

$$|x_{1,k} - x_{2,k}|/\lambda_{2,k} \to \infty \quad \text{and} \quad |x_{1,k} - x_{2,k}|/\lambda_{1,k} \to \infty,$$

for any $L > 0$, $B_{\lambda_{1,k}/\lambda_{2,k}}((x_{1,k} - x_{2,k})/\lambda_{2,k})$ becomes disjoint from B_L for all large k. In other words, $\forall z \in B_L$, $B_{\lambda_{2,k}}(x) \cap B_{L\lambda_{1,k}}(x_{1,k}) = \emptyset$, where $x = \lambda_{2,k} z + x_{2,k}$. From (3.8) we have

$$\int_{B_1(z)} (|\nabla \widehat{u}_k|^2 + |\widehat{u}_k|^{p+1}) = \int_{B_{\lambda_{2,k}}(x)} (|\nabla u_k|^2 + |u_k|^{p+1}) \le \varepsilon_0,$$

so by Corollary 2.1 and Remark 1.1, $\widehat{u}_k$ subconverges to a bubble ω_2 centered at the origin, locally in $H^1(\mathbb{R}^n)$. We also have

$$\begin{aligned}
&\int_{B_L} (|\nabla(\widetilde{v}_k - \omega_2)|^2 + |\widetilde{v}_k - \omega_2|^{p+1}) \\
&\quad \le C\bigg\{ \int_{B_L} \big(|\nabla \widehat{u}_k - \nabla\omega_2|^2 + |\widehat{u}_k - \omega_2|^{p+1}\big) \\
&\qquad + \int_{B_L} \Big[\Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2(p+1)}{p-1}} |\nabla\omega_1|^2\Big(\frac{\lambda_{2,k}x + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big) \\
&\qquad\qquad + \Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2(p+1)}{p-1}} |\omega_1|^{p+1}\Big(\frac{\lambda_{2,k}x + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big)\Big]\bigg\} \\
&\quad \le C\bigg\{ \int_{B_L} \big(|\nabla \widehat{u}_k - \nabla\omega_2|^2 + |\widehat{u}_k - \omega_2|^{p+1}\big) \\
&\qquad + \int_{B_{L\frac{\lambda_{2,k}}{\lambda_{1,k}}}\left(\frac{x_{2,k}-x_{1,k}}{\lambda_{1,k}}\right)} \big(|\nabla\omega_1|^2 + |\omega_1|^{p+1}\big)\bigg\}.
\end{aligned}$$

Since $B_{L\lambda_{2,k}/\lambda_{1,k}}((x_{2,k} - x_{1,k})/\lambda_{1,k})$ moves to infinity as k gets large, we see that

$$\int_{B_L} \big(|\nabla(\widetilde{v}_k - \omega_2)|^2 + |\widetilde{v}_k - \omega_2|^{p+1}\big) = o(1)$$

as $k \to \infty$ for every fixed L. This shows that $\widetilde{v}_k$ tends to ω_2 locally in $H^1(\mathbb{R}^n)$.

Next we consider Case (ii). As $\{\widetilde{v}_k\}$ is bounded in $H^1(B_L)$ for any L, we can find some ω_2 in $H^1(\mathbb{R}^n)$ so that $\widetilde{v}_k$ subconverges to ω_2 weakly in $H^1_{loc}(\mathbb{R}^n)$. It is readily checked that

$$\Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2}{p-1}}\omega_1\Big(\frac{\lambda_{2,k}z + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big) \rightharpoonup 0 \quad \text{in } H^1_{loc}(\mathbb{R}^n).$$

Therefore, $\widehat{u}_k \rightharpoonup \omega_2$ in $H^1_{loc}(\mathbb{R}^n)$. In particular, ω_2 must be a bubble. Now, noting that from (3.8) we have

$$\int_{B_1(z)\setminus B_{L\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} (|\nabla\widehat{u}_k|^2 + |\widehat{u}_k|^{p+1}) \le \varepsilon_0, \quad p_k = \frac{x_{1,k} - x_{2,k}}{\lambda_{2,k}},$$

as $\lambda_{1,k}/\lambda_{2,k} \to 0$, $\widehat{u}_k \to \omega_2$ in $H^1_{loc}(\mathbb{R}^n \setminus \{p^*\})$, where $p^* = \lim_{k\to\infty} p_k$ (after passing to a subsequence).

Given any $L \geq M_0 + 2$, we want to show that $\widetilde{v}_k \to \omega_2$ in $H^1(B_L)$. Under the assumptions in Case (ii), $p^* \in B_L$ and for any (small) ε and (large) M, $B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k) \subset B_\varepsilon(p_k) \subset B_L$, for all large k. We shall consider the convergence of $\widetilde{v}_k$ in the following three regions separately: $B_L \setminus B_\varepsilon(p_k)$, $B_{M\lambda_{1,k}/\lambda_{2,k}}(p_k)$, and the neck $B_\varepsilon(p_k) \setminus B_{M\lambda_{1,k}/\lambda_{2,k}}(p_k)$. First,

$$\begin{aligned}\int_{B_L\setminus B_\varepsilon(p_k)} |\widetilde{v}_k - \omega_2|^{p+1} &\leq C\int_{B_L\setminus B_\varepsilon(p_k)} |\widehat{u}_k - \omega_2|^{p+1} \\ &+ C\int_{B_L\setminus B_\varepsilon(p_k)} \Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2(p+1)}{p-1}} |\omega_1|^{p+1}\Big(\frac{\lambda_{2,k}z + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big) \\ &\leq C\Big(\int_{B_L\setminus B_\varepsilon(p_k)} |\widehat{u}_k - \omega_2|^{p+1} + \int_{\mathbb{R}^n\setminus B_{\varepsilon\frac{\lambda_{2,k}}{\lambda_{1,k}}}(0)} |\omega_1|^{p+1}\Big) = o(1),\end{aligned} \tag{3.9}$$

as $k \to \infty$. Next,

$$\begin{aligned}\int_{B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\widetilde{v}_k - \omega_2|^{p+1} &\leq C\Big(\int_{B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\widetilde{v}_k|^{p+1} + \int_{B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\omega_2|^{p+1}\Big) \\ &= C\Big(\int_{B_M} |\widetilde{u}_k - \omega_1|^{p+1} + \int_{B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\omega_2|^{p+1}\Big) = o(1),\end{aligned} \tag{3.10}$$

as $k \to \infty$. Similarly one can show that

$$\Big(\int_{B_L\setminus B_\varepsilon(p_k)} + \int_{B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)}\Big)|\nabla(\widetilde{v}_k - \omega_2)|^2 = o(1) \tag{3.11}$$

as $k \to \infty$ for every ε and M. In the next section we will show that

$$\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} (|\nabla(\widetilde{v}_k - \omega_2)|^2 + |\widetilde{v}_k - \omega_2|^{p+1}) = o_M(1) + o_\varepsilon(1) + o(1), \tag{3.12}$$

where $o_M(1)$ (respectively $o_\varepsilon(1)$) is a quantity which tends to 0 as $M \to \infty$ (respectively $\varepsilon \to 0$). (3.9)–(3.12) together show that $\widetilde{v}_k \to \omega_2$ in $H^1_{loc}(\mathbb{R}^n)$. Note that (3.6) implies that ω_2 is centered at the origin. It follows that

$$u_k - \lambda_{1,k}^{\frac{-2}{p-1}}\omega_1\Big(\frac{x - x_{1,k}}{\lambda_{1,k}}\Big) - \lambda_{2,k}^{\frac{-2}{p-1}}\omega_2\Big(\frac{x - x_{2,k}}{\lambda_{2,k}}\Big) \to 0$$

in $H^1(B_{L\lambda_{1,k}} \cup B_{L\lambda_{2,k}})$.

Step 3. Extracting the remaining bubbles.

The procedure of bubble extraction in the last step can be repeated to obtain $m-1$ many bubbles ω_j, $j=1,2,\dots,m-1$ if necessary. Specifically, assume that there are $x_{j,k}\to 0$ and $\lambda_{j,k}\to 0$ such that

$$v_k(x)=u_k(x)-\sum_{j=1}^{m-1}\lambda_{j,k}^{-\frac{2}{p-1}}\omega_j\Big(\frac{x-x_{j,k}}{\lambda_{j,k}}\Big)\to 0 \qquad (3.13)_{m-1}$$

in $H^1(\cup_{j=1}^{m-1}B_{L\lambda_{j,k}}(x_{j,k}))$ for every $L>0$. In case $\{v_k\}$ does not subconverge to u_∞ in $H^1(B_1)$, a new bubble can be extracted as described in the previous step. In brief, first of all, there exists some $\nu_0>0$ such that

$$\lim_{\delta\to 0}\limsup_{k\to\infty}\int_{B_\delta}(|\nabla v_k|^2+|v_k|^{p+1})=\nu_0.$$

Consequently, there are $x_{m,k}\to 0$ and $\lambda_{m,k}\to 0$ such that

$$\begin{aligned}\int_{B_{\lambda_{m,k}}(x_{m,k})}(|\nabla v_k|^2+|v_k|^{p+1}) &= \sup_{x\in\overline{B_1}}\int_{B_{\lambda_{m,k}}(x)}(|\nabla v_k|^2+|v_k|^{p+1})\\ &=\frac{1}{2}\min\{\varepsilon_0,\nu_0\}, \end{aligned}\qquad (3.14)$$

and

$$\frac{\lambda_{m,k}}{\lambda_{j,k}}+\frac{|x_{m,k}-x_{j,k}|}{\lambda_{m,k}+\lambda_{j,k}}\to\infty,\quad j=1,2,\dots,m-1 \qquad (3.15)$$

as $k\to\infty$. By passing to a subsequence, one may assume also the rescaled function

$$\widetilde{v}_k(z)=\lambda_{m,k}^{\frac{2}{p-1}}v_k(\lambda_{m,k}z+x_{m,k})\rightharpoonup\omega_m\quad\text{in } H^1_{loc}(\mathbb{R}^n) \qquad (3.16)$$

for some bubble ω_m.

The bubbles found in this way display a tree-like structure which can be described as follows. Two bubbles ω_i and ω_j are called separated, denoted by $\omega_i\diamond\omega_j$, if

$$\frac{|x_{i,k}-x_{j,k}|}{\lambda_{i,k}+\lambda_{j,k}}\to\infty,\quad \text{as } k\to\infty,$$

and ω_i is contained in ω_j, denoted by $\omega_i<\omega_j$, if

$$\frac{|x_{i,k}-x_{j,k}|}{\lambda_{i,k}+\lambda_{j,k}}\le M_0,\quad \frac{\lambda_{j,k}}{\lambda_{i,k}}\to\infty,\quad\text{as } k\to\infty$$

for some M_0. The relations $\diamond$ and $<$ enjoy the following three properties:

(P_1) If $\omega_i<\omega_j$ and $\omega_j<\omega_l$, then $\omega_i<\omega_l$.

(P_2) If $\omega_i < \omega_j$ and $\omega_j \diamond \omega_l$, then $\omega_i \diamond \omega_l$.

(P_3) For any distinct indices i and j, either $\omega_i < \omega_j$ or $\omega_i \diamond \omega_j$.

(P_1) and (P_2) are obvious from the definitions; (P_3) will be established below.

Assuming $(3.13)_{m-1}$ and (P_3) for $(m-1)$ bubbles, we shall establish $(3.13)_m$ and (P_3) for m bubbles when (3.14) holds. Indeed, from (3.15) we have either Case (i),

$$\frac{|x_{m,k}-x_{j,k}|}{\lambda_{m,k}+\lambda_{j,k}} \to \infty, \quad \text{as } k\to\infty, \quad \text{for } j=1,2,\dots,m-1,$$

or Case (ii),

$$\frac{|x_{m,k}-x_{j,k}|}{\lambda_{m,k}+\lambda_{j,k}} \le M_0, \quad \frac{\lambda_{m,k}}{\lambda_{j,k}} \to \infty, \quad \text{for } j\in J_1,$$

and

$$\frac{|x_{m,k}-x_{j,k}|}{\lambda_{m,k}+\lambda_{j,k}} \to \infty, \quad \text{as } k\to\infty, \quad \text{for } j\in J_2,\ J_2=\{1,2,\dots,m-1\}\setminus J_1,$$

where J_1 and J_2 are non-empty subsets of $\{1,2,\dots,m-1\}$ and $J_1\cup J_2 = \{1,2,\dots,m-1\}$. In Case (i) we have $\omega_m \diamond \omega_j$, $j=1,\dots,m-1$, and in Case (ii), $\omega_i < \omega_m$, $i\in J_1$, and $\omega_i \diamond \omega_m$, $i \in J_2$, so (P_3) holds for m bubbles.

Next, we want to establish $(3.13)_m$ for every $L>0$. The proof has essentially been carried out in Step 2, and we shall be sketchy. Let's consider Case (ii) only. The bubble ω_j, $j\in J_1$, may further be grouped into maximal branches of bubbles. By a branch of bubbles we mean a subset of bubbles with the properties that either $\omega_i < \omega_j$ or $\omega_i > \omega_j$ must hold for any two bubbles inside. It is maximal if it cannot be enlarged as a branch. Clearly every maximal branch has a unique maximal bubble. Suppose there are N many maximal branches contained inside ω_m, and let their maximal bubbles be $\omega_{i_1},\dots,\omega_{i_N}$. To prove $(3.13)_m$ it suffices to show that

$$\begin{aligned}\widetilde{v}_k(z) &= \lambda_{m,k}^{\frac{2}{p-1}} v_k(\lambda_{m,k}z+x_{m,k})\\ &= \widehat{u}_k(z) - \sum_{j=1}^{m-1}\Big(\frac{\lambda_{m,k}}{\lambda_{j,k}}\Big)^{\frac{2}{p-1}}\omega_j\Big(\frac{\lambda_{m,k}z+x_{m,k}-x_{j,k}}{\lambda_{j,k}}\Big) \to \omega_m, \qquad (3.17)\end{aligned}$$

strongly in $H^1_{loc}(\mathbb{R}^n)$. (This, together with (3.14), also shows that ω_m is centered at 0.) First of all, we have

$$\widetilde{v}_k(z) = \widehat{u}_k(z) - \sum{}'\Big(\frac{\lambda_{m,k}}{\lambda_{j,k}}\Big)^{\frac{2}{p-1}}\omega_j\Big(\frac{\lambda_{m,k}z+x_{m,k}-x_{j,k}}{\lambda_{j,k}}\Big)$$

$$-\sum\nolimits''\Big(\frac{\lambda_{m,k}}{\lambda_{j,k}}\Big)^{\frac{2}{p-1}}\omega_j\Big(\frac{\lambda_{m,k}z+x_{m,k}-x_{j,k}}{\lambda_{j,k}}\Big),$$

where $\sum'$ denotes the summation of all bubbles contained in ω_m and $\sum''$ the summation of all bubbles separated from ω_m. It is clear that for each $L>0$, the second summation tends to 0 in $H^1(B_L)$. Then, to estimate

$$\widehat{v}_k(z)\equiv\widehat{u}_k(z)-\sum\nolimits'\Big(\frac{\lambda_{m,k}}{\lambda_{j,k}}\Big)^{\frac{2}{p-1}}\omega_j\Big(\frac{\lambda_{m,k}z+x_{m,k}-x_{j,k}}{\lambda_{j,k}}\Big)$$

on B_L, we consider the convergence on different regions:

$$B_L\setminus B_\varepsilon(p_{l,k}),\quad B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k}),\quad \text{and the necks}\quad B_\varepsilon(p_{l,k})\setminus B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k}),$$

where $\varepsilon>0$ is a fixed number, $p_{l,k}=\frac{x_{i_l,k}-x_{m,k}}{\lambda_{m,k}}$, $l=1,\dots,N$. We may assume for each l, $p_{l,k}\to p_l^*$ as $k\to\infty$. We have $\widehat{u}_k\to\omega_m$ in $H^1_{loc}(\mathbb{R}^n\setminus\{p_1^*,\dots,p_N^*\})$ (see Step 2). In particular, $\widehat{u}_k\to\omega_m$ in $H^1(B_L\setminus B_\varepsilon(p_{l,k}))$. It follows easily that

$$\begin{aligned}\int_{B_L\setminus B_\varepsilon(p_{l,k})}&(|\nabla(\widehat{v}_k-\omega_m)|^2+|\widehat{v}_k-\omega_m|^{p+1})\\&\le C\int_{B_L\setminus B_\varepsilon(p_{l,k})}(|\nabla(\widehat{u}_k-\omega_m)|^2+|\widehat{u}_k-\omega_m|^{p+1})+o(1).\end{aligned}$$

Next,

$$\begin{aligned}&\int_{B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k})}(|\nabla(\widehat{v}_k-\omega_m)|^2+|\widehat{v}_k-\omega_m|^{p+1})\\&\le C\int_{B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k})}(|\nabla\widehat{v}_k|^2+|\widehat{v}_k|^{p+1})+\int_{B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k})}(|\nabla\omega_m|^2+\omega_m^{p+1})\\&\le C\int_{B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k})}(|\nabla\widetilde{v}_k|^2+|\widetilde{v}_k|^{p+1})+o(1)\\&= C\int_{B_{M\lambda_{j_l,k}}(x_{\lambda_{j_l,k}})}(|\nabla v_k|^2+|v_k|^{p+1})+o(1)=o(1)\end{aligned}$$

by $(3.13)_{m-1}$. Finally, performing similar estimates at the necks as in Section 4 one can show that $\widetilde{v}_k$ converges to ω_m in $B_\varepsilon(p_{l,k})\setminus B_{M\frac{\lambda_{j_l,k}}{\lambda_{m,k}}}(p_{l,k})$, so $(3.13)_m$ holds.

In the above discussion we have shown how to extract bubbles. Now we observe that the procedure of extraction must stop in finitely many steps. For, in case there are m bubbles around the blow-up point x^1, we have the splitting

$$u_k(x) - \sum_{j=1}^{m} \lambda_{j,k}^{-\frac{2}{p-1}} \omega_j\Big(\frac{x - x_{j,k}}{\lambda_{j,k}}\Big) \to 0 \quad \text{in } H^1(\cup_{j=1}^m B_{L\lambda_{j,k}}(x_{j,k})),$$

or (3.17) holds. By Lemma 1.1, we have

$$\begin{aligned}
\|u_k\|_{L^{p+1}(\Omega)} &= \Big\| u_k(x) - \lambda_{1,k}^{-\frac{2}{p-1}} \omega_1\Big(\frac{x - x_{1,k}}{\lambda_{1,k}}\Big)\Big\|_{L^{p+1}(\Omega)} + \|\omega\|_{L^{p+1}(\mathbb{R}^n)} + o(1) \\
&= \Big\| u_k(x) - \sum_{j=1}^{2} \lambda_{j,k}^{-\frac{2}{p-1}} \omega_j\Big(\frac{x - x_{j,k}}{\lambda_{j,k}}\Big)\Big\|_{L^{p+1}(\Omega)} + 2\|\omega\|_{L^{p+1}(\mathbb{R}^n)} + o(1) \\
&\vdots \\
&= \Big\| |u_k(x) - \sum_{j=1}^{m} \lambda_{j,k}^{-\frac{2}{p-1}} \omega_j\Big(\frac{x - x_{j,k}}{\lambda_{j,k}}\Big)\Big\|_{L^{p+1}(\Omega)} + m\|\omega\|_{L^{p+1}(\mathbb{R}^n)} + o(1),
\end{aligned}$$

so $m \le C_0^{1/(p+1)}/\|\omega\|_{L^{p+1}(\mathbb{R}^n)}$. Now we have, after passing to a subsequence,

$$u_k(x) - \sum_{j=1}^{m} \lambda_{j,k}^{-\frac{2}{p-1}} \omega_j\Big(\frac{x - x_{j,k}}{\lambda_{j,k}}\Big) \to u_\infty(x) \quad \text{in } H^1(B_1).$$

Performing the same analysis to other blow-up points, after finitely many steps we find a subsequence and u_∞ in $H^1(\Omega)$ satisfying (3.4).

Remark 3.1. Tree-like structures for bubbles were first found in harmonic heat flows and approximate harmonic maps; see Parker–Wolfson [17], Qing [19], and Parker [16].

Proof of Main Theorem. By the properties of the borderline solution it is easy to see that one can always extract a sequence of $\{t_k\} \to \infty$ such that $\{u(t_k)\}$ satisfies (3.1)–(3.3), where $f_k = u_t(t_k)$ in fact tends to 0 in $L^2(\Omega)$. Hence Main Theorem follows from the above discussion. □

Proof of Main Proposition. Note that for a positive borderline solution, $\beta \ge 0$, and for a star-shaped domain Ω, any nonnegative steady state of (0.1) must be 0.

When $\beta = 0$, the energy of a borderline solution tends to 0. Applying Theorem 2 in [2] to Proposition 3 in [3], we obtain the conclusion of Proposition 3′ in [3] for a general bounded domain. Since a positive borderline solution must be globally unbounded ([15]), we have finite-time blow-up taking place. Therefore, one side in statement (1) holds true. The other side in statement (1) is trivial.

When $\beta > 0$, there is no subsequence $t_k \to \infty$ so that $u(t_k)$ tends to 0 strongly in $H^1(\Omega)$; otherwise, the energy of $u(t_k)$ would tend to 0 as k gets large, contradicting the assumption $\beta > 0$. Using Main Theorem, we conclude that $\lim_{k\to\infty} E(u(t_k)) = \kappa\mathcal{E}$ for some $\kappa \in \mathbb{Z}^+$. Hence, we have $\beta = \kappa\mathcal{E}$, and statement (2) is proved. □

4. ESTIMATES AT THE NECK

In this section we prove (3.12). Our proof is inspired by Lin–Wang [13]. First of all,

$$\begin{aligned}
&\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\widetilde{v}_k - \omega_2|^{p+1} \\
&\le C \int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} \Big(\lambda_{2,k}^{\frac{2(p+1)}{p-1}} |u_k|^{p+1}(\lambda_{2,k} z + x_{2,k}) \\
&+ \Big(\frac{\lambda_{2,k}}{\lambda_{1,k}}\Big)^{\frac{2(p+1)}{p-1}} |\omega_1|^{p+1}\Big(\frac{\lambda_{2,k}x + x_{2,k} - x_{1,k}}{\lambda_{1,k}}\Big) + |\omega_2|^{p+1}\Big) \\
&\le C\Big(\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |u_k|^{p+1} + \int_{\mathbb{R}^n\setminus B_M} |\omega_1|^{p+1} + \int_{B_\varepsilon(p_k)} |\omega_2|^{p+1}\Big) \\
&= C \int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |u_k|^{p+1} + o(1)
\end{aligned}$$

as $k \to \infty$. Similarly, we have

$$\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\nabla\widetilde{v}_k - \nabla\omega_2|^2 \le C \int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} |\nabla u_k|^2 + o(1)$$

as $k \to \infty$. To establish (3.12) it suffices to show

$$\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} (|\nabla u_k|^2 + |u_k|^{p+1}) = o(1), \quad \text{as } k \to \infty. \tag{4.1}$$

To establish (4.1) we need to derive some preliminary information. First, as

$$\int_{B_{M\lambda_{1,k}}(x_{1,k})} \Big|\nabla\Big(u_k - \lambda_{1,k}^{-\frac{2}{p-1}}\omega_1\big(\frac{x-x_{1,k}}{\lambda_{1,k}}\big)\Big)\Big|^2 + \Big|u_k - \lambda_{1,k}^{-\frac{2}{p-1}}\omega_1\big(\frac{x-x_{1,k}}{\lambda_{1,k}}\big)\Big|^{p+1} = o(1),$$

by the mean-value theorem, there is some $M' \in (M/2, M)$ such that

$$M'\lambda_{1,k}\int_{\partial B_{M'\lambda_{1,k}}(x_{1,k})} \Big|\nabla\Big(u_k - \lambda_{1,k}^{-\frac{2}{p-1}}\omega_1\big(\frac{x-x_{1,k}}{\lambda_{1,k}}\big)\Big)\Big|^2 + \Big|u_k - \lambda_{1,k}^{-\frac{2}{p-1}}\omega_1\big(\frac{x-x_{1,k}}{\lambda_{1,k}}\big)\Big|^{p+1} = o(1).$$

After replacing M' by M,

$$\begin{aligned} M\lambda_{1,k}\int_{\partial B_{M\lambda_{1,k}}(x_{1,k})}(|\nabla u_k|^2 + |u_k|^{p+1}) &\le M\int_{\partial B_M}(|\nabla\omega_1|^2 + |\omega_1|^{p+1}) + o(1) \\ &= o_M(1) + o(1), \end{aligned} \tag{4.2}$$

where $o_M(1)$ stands for a quantity which tends to 0 as $M \to \infty$.

Similarly, from

$$\int_{B_{\lambda_{2,k}}(x_{1,k})\setminus B_{\varepsilon\lambda_{2,k}}(x_{1,k})} \Big|\nabla\Big(u_k - \lambda_{2,k}^{-\frac{2}{p-1}}\omega_2\big(\frac{x-x_{2,k}}{\lambda_{2,k}}\big)\Big)\Big|^2 + \Big|u_k - \lambda_{2,k}^{-\frac{2}{p-1}}\omega_2\big(\frac{x-x_{2,k}}{\lambda_{2,k}}\big)\Big|^{p+1} = o(1),$$

we deduce there exists some $\varepsilon' \in (\varepsilon, 2\varepsilon)$ such that

$$\varepsilon'\lambda_{2,k}\int_{\partial B_{\varepsilon'\lambda_{2,k}}(x_{1,k})}(|\nabla u_k|^2 + |u_k|^{p+1}) = o_\varepsilon(1) + o(1), \tag{4.3}$$

where $o_\varepsilon(1)$ stands for a quantity which tends to 0 as $\varepsilon \to 0$. For simplicity we shall replace ε' by ε.

We need the following last piece of information on the annulus. In view of (3.8), we can use Corollary 2.2 to show that for each $r \in [M\lambda_{1,k}, \lambda_{2,k}/2]$,

$$\int_{\partial B_r(x_{1,k})} |u_k|^{p+1} \le \frac{(\varepsilon)^{1/2}}{r}, \tag{4.4}$$

where $\varepsilon \le \varepsilon_0$.

Rescale u_k by $w_k(z) = (\varepsilon\lambda_{2,k})^{\frac{2}{p-1}} u_k(\varepsilon\lambda_{2,k} z + x_{1,k})$ for $z \in B_1 \setminus B_{r_k}$, where $r_k = M\lambda_{1,k}/\varepsilon\lambda_{2,k} \to 0$ as $k \to \infty$. The function $w_k \in H^2(B_1 \setminus B_{r_k})$ and satisfies

$$\triangle w_k + |w_k|^{p-1} w_k = g_k \equiv (\varepsilon\lambda_{2,k})^{\frac{2p}{p-1}} f_\varepsilon(\varepsilon\lambda_{2,k} z + x_{1,k}),$$

where $g_k \to 0$ in $L^2(B_1)$. Also, (4.2), (4.3), and (4.4) turn into

$$\int_{\partial B_1} (|\nabla w_k|^2 + |w_k|^{p+1}) = o_\varepsilon(1) + o(1),$$

$$r_k \int_{\partial B_{r_k}} (|\nabla w_k|^2 + |w_k|^{p+1}) = o_M(1) + o(1),$$

and

$$\int_{\partial B_r} |w_k|^{p+1} \leq \frac{(\varepsilon)^{1/2}}{r}, \quad \forall r \in [r_k, 1].$$

An application of Lemma 4.1 below shows that

$$\begin{aligned}
&\int_{B_\varepsilon(p_k)\setminus B_{M\frac{\lambda_{1,k}}{\lambda_{2,k}}}(p_k)} (|\nabla u_k|^2 + |u_k|^{p+1}) = \int_{B_1\setminus B_{r_k}} (|\nabla w_k|^2 + |w_k|^{p+1}) \\
&\leq C\Big(\int_{\partial B_1} (|\nabla w_k|^2 + |w_k|^{p+1}) + r_k \int_{\partial B_{r_k}} (|\nabla w_k|^2 + |w_k|^{p+1}) + \int_{B_1\setminus B_{r_k}} g_k^2 \Big) \\
&= o_\varepsilon(1) + o_M(1) + o(1),
\end{aligned}$$

and (3.12) follows from (4.1).

Lemma 4.1. *Let $v \in H^2(B_1 \setminus B_{r_0})$ satisfy*

$$\triangle v + |v|^{p-1} v = g \quad \text{in } B_1 \setminus B_{r_0},$$

where $g \in L^2(B_1 \setminus B_{r_0})$. Then there exists a small ε_* depending on p and n such that if

$$\int_{\partial B_r} |v|^{p+1} \leq \frac{\varepsilon_*}{r}, \quad \forall r \in [r_0, 1],$$

then

$$\int_{B_1\setminus B_{r_0}} (|\nabla v|^2 + |v|^{p+1}) \leq C(A + B + \int_{B_1\setminus B_{r_0}} g^2), \tag{4.5}$$

where

$$A = \int_{\partial B_1} (|\nabla v|^2 + |v|^{p+1}), \quad \text{and} \quad B = r_0 \int_{\partial B_{r_0}} (|\nabla v|^2 + |v|^{p+1}).$$

In order to use the lemma we replace ε_0 by a smaller one if necessary so that $(\varepsilon_0)^{1/2} \le \varepsilon_*$.

Proof. Setting $\widetilde{v}(t,x) = e^{-\frac{2}{p-1}t} v(e^{-t}x),\ t > 0$, and $x \in S^{n-1}$, $\widetilde{v}$ satisfies

$$\widetilde{v}_{tt} - \frac{(n-2)^2}{4}\widetilde{v} + \triangle_{S^{n-1}}\widetilde{v} = -|\widetilde{v}|^{p-1}\widetilde{v} - \widetilde{g}$$

in $[0,T] \times S^{n-1},\ T = |\log r_0|$, where $\widetilde{g}(t,x) = e^{-\frac{2p}{p-1}t} g(e^{-t}x)$. Note that we have

$$\int_{S^{n-1}} |\widetilde{v}|^{p+1}(t) \le \varepsilon_*, \quad \forall t \in [0,T],$$
$$\int_{S^{n-1}} (|\nabla \widetilde{v}|^2 + |\widetilde{v}|^{p+1})(0) = A, \quad \int_{S^{n-1}} (|\nabla \widetilde{v}|^2 + |\widetilde{v}|^{p+1})(T) = B,$$

and

$$\int_0^T \int_{S^{n-1}} \widetilde{g}^2 e^{2t}\, dx\, dt = \int_{B_1 \setminus B_{r_0}} g^2.$$

Assuming for the moment that v is smooth, we compute

$$\frac{d^2}{dt^2}\int_{S^{n-1}} |\widetilde{v}|^{p+1} = p(p+1)\int_{S^{n-1}} |\widetilde{v}|^{p+1}\widetilde{v}_t^2 + p(p+1)\int_{S^{n-2}} |\widetilde{v}|^{p-1}|\nabla_{S^{n-1}}\widetilde{v}|^2$$
$$+ (p+1)\frac{(n-2)^2}{4}\int_{S^{n-1}} |\widetilde{v}|^{p+1} - (p+1)\int_{S^{n-1}} |\widetilde{v}|^{2p} - (p+1)\int_{S^{n-1}} \widetilde{v}^p\widetilde{g}.$$

Using an interpolation inequality, we have

$$\int_{S^{n-1}}|\widetilde{v}|^{2p} \le C\Big(\int_{S^{n-1}}|\widetilde{v}|^{p+1}\Big)^{\frac{3}{n}}\Big(\int_{S^{n-1}}|\nabla_{S^{n-1}}\widetilde{v}^{\frac{p+1}{2}}|^2\Big)^{\frac{n-1}{n}} + C\Big(\int_{S^{n-1}}|\widetilde{v}|^{p+1}\Big)^{\frac{n+2}{n}}$$
$$\le \frac{1}{2}\int_{S^{n-1}} |\widetilde{v}|^{p-1}|\nabla_{S^{n-1}}\widetilde{v}|^2 + C\Big(\int_{S^{n-1}} |\widetilde{v}|^{p+1}\Big)^3 + C\Big(\int_{S^{n-1}} |\widetilde{v}|^{p+1}\Big)^{\frac{n+2}{n}}$$
$$\le \frac{1}{2}\int_{S^{n-1}} |\widetilde{v}|^{p-1}|\nabla_{S^{n-1}}\widetilde{v}|^2 + C(\varepsilon_*^2 + \varepsilon_*^{\frac{2}{n}})\int_{S^{n-1}} |\widetilde{v}|^{p+1}$$

and also

$$\Big|\int_{S^{n-1}} \widetilde{v}^p\widetilde{g}\Big| \le \int_{S^{n-1}} |\widetilde{v}|^{2p} + \int_{S^{n-1}} \widetilde{g}^2.$$

Therefore, if ε_* is sufficiently small,

$$\frac{d^2}{dt^2}\int_{S^{n-1}} |\widetilde{v}|^{p+1} \ge \frac{1}{4}\int_{S^{n-1}} |\widetilde{v}|^{p+1} - C_0\int_{S^{n-1}} \widetilde{g}^2$$

holds on $[0,T]$.

Consider the boundary-value problem

$$y'' - \frac{1}{4}y = G(t), \quad y(0) = A, \quad y(T) = B,$$

where

$$G(t) = C_0 \int_{S^{n-1}} \widetilde{g}^2.$$

Its solution is given explicitly by

$$y(t) = e^{\frac{t}{2}}\Big(\int_t^T G(\tau)e^{-\frac{\tau}{2}}\,d\tau + d_1\Big) + e^{-\frac{t}{2}}\Big(-\int_t^T G(\tau)e^{\frac{\tau}{2}}\,d\tau + d_2\Big),$$

where

$$d_1 = \frac{Be^{\frac{T}{2}} - A - \int_0^T G(\tau)(e^{\frac{\tau}{2}} - e^{-\frac{\tau}{2}})\,d\tau}{e^T - 1},$$

and

$$d_2 = \frac{Ae^T - Be^{\frac{T}{2}} + e^T\int_0^T G(\tau)(e^{\frac{\tau}{2}} - e^{-\frac{\tau}{2}})\,d\tau}{e^T - 1}.$$

By comparing y with $\|\widetilde{v}\|_{L^{p+1}}^{p+1}$, we see that

$$\begin{aligned}
&\int_0^T \int_{S^{n-1}} |\widetilde{v}|^{p+1} \le \int_0^T y(\tau)\,d\tau \\
&\le 2d_1(e^{\frac{T}{2}} - 1) + \int_0^T e^{-\frac{t}{2}} \int_t^T G(\tau)e^{\frac{\tau}{2}}\,d\tau\,dt + 2d_2(1 - e^{-\frac{\tau}{2}}) \\
&\le C(A+B) + Ce^{-\frac{T}{2}} \int_0^T G(\tau)e^{\frac{\tau}{2}}\,d\tau + \int_0^T G(\tau)e^{\frac{\tau}{2}}(1 - e^{-\frac{\tau}{2}})\,d\tau \\
&\le C(A+B) + C\int_0^T G(\tau)e^{2\tau}\,d\tau.
\end{aligned}$$

Back to the rectangular coordinates, we have

$$\int_{B_1\setminus B_{r_0}} |v|^{p+1} \le C\Big(A + B + \int_{B_1\setminus B_{r_0}} g^2\Big). \tag{4.6}$$

Applying the Pohozaev identity to v, we have

$$\begin{aligned}
&\frac{2-n}{2}\int_{B_1\setminus B_{r_0}} |\nabla v|^2 - \frac{1}{2}\int_{\partial B_1} |\frac{\partial v}{\partial r}|^2 + \frac{1}{2}r_0\int_{\partial B_{r_0}} |\frac{\partial v}{\partial r}|^2 \qquad (4.7)\\
&= -\frac{n}{p+1}\int_{B_1\setminus B_{r_0}} |v|^{p+1} + \int_{\partial B_1} |v|^{p+1} - r_0\int_{\partial B_{r_0}} |v|^{p+1} - \int_{B_1\setminus B_{r_0}} (x\cdot\nabla v)g.
\end{aligned}$$

So (4.5) follows from (4.6) and (4.7). For the general case, approximating v in $H^2(B_1 \setminus B_{r_0})$ by $v_k \in C^\infty(B_1 \setminus B_{r_0})$, we have

$$g_k \equiv \triangle v_k + |v_k|^{p-1} v_k \to g \quad \text{in } L^2(B_1 \setminus B_{r_0}).$$

Note that by the trace theorem,

$$\int_{\partial B_r} |v_k|^{p+1} \leq \varepsilon_* + o(1), \quad \forall r \in [r_0, 1].$$

So, we get

$$\int_{B_1 \setminus B_{r_0}} (|\nabla v_k|^2 + |v_k|^{p+1}) \leq C(A_k + B_k + \int_{B_1 \setminus B_{r_0}} g_k^2), \tag{4.8}$$

where

$$A_k = \int_{\partial B_1} (|\nabla v_k|^2 + |v_k|^{p+1}) \quad \text{and} \quad B_k = r_0 \int_{\partial B_{r_0}} (|\nabla v_k|^2 + |v_k|^{p+1}).$$

Passing to the limits in (4.7) and using the trace theorem again, (4.5) holds. □

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REFERENCES

[1] H. Brezis and T. Cazenave, *A nonlinear heat equation with singular initial data*, Journal d'Analyse Mathématique, 68 (1996), 277–304.

[2] K.S. Chou and S.Z. Du, "On Global Partial Regularity for Borderline Solutions of Semilinear Parabolic Problems," Trends in Partial Differential Equations, 2009.

[3] K.S. Chou, S.Z. Du, and G.F. Zheng, *On partial regularity of the borderline solution of semilinear parabolic problems*, Calc. Var. and PDEs, 30 (2007), 251–275.

[4] L. Caffarelli, B. Gidas, and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math., 42 (1989), 271–297.

[5] T. Cazenave and A. Haraux, "An Introduction to Semilinear Evolution Equations," In: Oxford Lecture Series in Mathematics and Its Applications, vol. 13. Clarendon Press, Oxford University Press, New York 1998.

[6] J.Y. Chen and G. Tian, *Compactification of moduli space of harmonic mappings*, Comment. Math. Helv., 74 (1999), 201–237.

[7] M. Fila, *Blow-up solutions of supercritical parabolic equations*, Evolutionary Equations, vol. II, Handb. Differ. Equ., (2005), 105–158.
[8] M. Fila, H. Matano, and P. Poláčik, *Immediate regularization after blow-up*, SIAM J. Math. Anal., 37 (2005), 752–776.
[9] V.A. Galaktionov and J.R. King, *Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents*, J. Differential Equations, 189 (2003), 199–233.
[10] V.A. Galaktionov and J.L. Vazquez, *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math., 50 (1997), 1–67.
[11] M. Ishiwata, *On bounds for global solutions of semilinear parabolic equations with critical and subcritical Sobolev exponent*, Differential Integral Equations, 20 (2007), 1021–1034.
[12] O.A. Ladyzhenskaya and N.N. Uraltseva, "Linear and Quasilinear Equations of Elliptic Type," Academic Press, New York, 1968.
[13] F.H. Lin and C.Y. Wang, *Energy identity of harmonic map flows from surfaces at finite singular time*, Calc. Var., 6 (1998), 369–380.
[14] N. Mizoguchi, *Boundedness of global solutions for a supercritical semilinear heat equation and its application*, Indiana Univ. Math. J., 54 (2005), 1047–1059.
[15] W.M. Ni, P. Sacks, and J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations, 54 (1984), 97–120.
[16] T. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom., 44 (1996), 595–633.
[17] T. Parker and J. Wolfson, *Pseudo-holomorphic maps and bubble trees*, J. Geom. Anal., 3 (1993), 63–98.
[18] M. del Pino and J.C. Wei, *Supercritical elliptic problems in domains with small holes*, Ann. I. H. Poincaré-AN, 24 (2007), 507–520.
[19] J. Qing, *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. Anal. Geom., 3 (1995), 297–315.
[20] J. Qing and G. Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math., 50 (1997), 295–310.
[21] O. Rey, *On a variational problem with lack of compactness: the effect of small holes in the domain*, C.R. Acad. Sci. Paris Ser. I Math., 308 (1989), 349–352.
[22] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom., 28 (1988), 485–502.
[23] M. Struwe, "Variational Methods," 2^{nd} edition, vol. 34, A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin 1996.
[24] T. Suzuki, *Semilinear parabolic equation on bounded domain with critical Sobolev exponent*, Indiana Univ. Math. J, 57 (2008), 3365–3396.
[25] Z. Tan, *Global solution and blowup of semilinear heat equation with critical Sobolev exponent*, Comm. Partial Differential Equations, 26 (2001), 717–741.
[26] F.B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in* L^p, Indiana Univ Math. J., 29 (1980), 79–102.