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SOLUTION FORMULA FOR THE VORTICITY EQUATIONS IN THE HALF PLANE WITH APPLICATION TO HIGH VORTICITY CREATION AT ZERO VISCOSITY LIMIT

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Abstract. We consider the Navier–Stokes equations for viscous incompressible flows in the half plane under the no-slip boundary condition. In this paper we first establish a solution formula for the vorticity equations through the appropriate vorticity formulation. The formula is then applied to establish the asymptotic expansion of vorticity fields at $\nu \to 0$ that holds at least up to the time $c\nu^{1/3}$, where ν is the viscosity coefficient and c is a constant. As a consequence, we get a natural sufficient condition on the initial data for the vorticity to blow up at the inviscid limit, together with explicit estimates.

1. Introduction

In this paper we consider the two-dimensional Navier–Stokes equations for viscous incompressible flows under the no-slip boundary conditions:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \quad t>0, \quad x\in\Omega, \\ \operatorname{div} u = 0, \quad t\geq 0, \quad x\in\Omega, \\ u=0, \quad t\geq 0, \quad x\in\partial\Omega, \\ u|_{t=0} = a, \quad x\in\Omega. \end{cases} \tag{NS}$$

Here $u = u(t,x) = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ denote the velocity field and the pressure field, and $\nu > 0$ is the viscosity coefficient. We will use the standard notation for derivatives,

$$\partial_t = \partial/\partial t, \quad \partial_j = \partial/\partial x_j,$$

$$\Delta = \sum_{j=1}^{2} \partial_j^2, \quad \operatorname{div} u = \sum_{j=1}^{2} \partial_j u_j, \quad u\cdot\nabla u = \sum_{j=1}^{2} u_j\partial_j u.$$

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For the most part of this paper we deal with the case when Ω is the half plane (Sections 3 and 4), but in Section 2 the case of bounded domains with smooth boundary is also treated.

The system (NS) has been studied quite extensively in various settings. In particular, it is well known that (NS) admits a unique smooth solution globally in time, for example, in the energy class [21, 14]; see also [32, 35] and references therein. When $\Omega = \mathbb{R}^2$ the alternative approach using vorticity fields is also useful and has been well developed by now. Here the vorticity ω of the velocity u is defined by $\omega = \operatorname{Rot} u := \partial_1 u_2 - \partial_2 u_1$, and the equation for ω is then formally obtained by applying the Rot operator to the first equation of (NS):

$$\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = 0, \quad t > 0, \quad x \in \Omega. \tag{1.1}$$

When $\Omega = \mathbb{R}^2$ the vorticity equation (1.1) ensures the uniform bound of vorticity fields by the maximum principle, which is essentially used to show the global existence of smooth solutions to (NS) in the infinite energy class; see [12, 11, 10] and references therein. However, in the presence of boundaries, a serious difficulty arises in the study of vorticity fields. Indeed, under the no-slip boundary condition on velocity fields the vorticity fields have to be subject to nonlocal and nonlinear boundary conditions, and these complicated boundary conditions have been a crucial obstacle for the direct mathematical study of (1.1). As is observed in [1], the boundary conditions for vorticity fields are derived from a simple mathematical consideration through the Biot–Savart law. When Ω is a simply-connected bounded domain or is the half plane, they consist of the normal derivative, a Dirichlet–Neumann map, and the nonlocal nonlinear term. Under the compatibility conditions on the initial data a such that $\operatorname{div} a = 0$ in Ω and $a = 0$ on $\partial\Omega$, the equation (1.1) equipped with these boundary conditions is shown to be equivalent with (NS). Strictly speaking, the arguments in [1] are verified only when Ω is simply connected, but it is easy to modify them so as to work also for the multi-connected bounded domains by using the Helmholtz–Weyl decomposition of vector fields. For the reader's convenience we will briefly describe this vorticity formulation in Section 2.

The aim of this paper is to present a detailed analysis of the vorticity equations in the case of the half plane by using the vorticity formulation introduced in Section 2. In this case the boundary conditions on vorticity fields will read

$$\nu\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\omega = -\partial_2(-\Delta_D)^{-1}(u \cdot \nabla \omega), \quad t > 0, \quad x \in \partial\mathbb{R}^2_+. \tag{1.2}$$

Here $h = (-\Delta_D)^{-1} f$ denotes the solution to the Poisson equation with the homogeneous Dirichlet condition:

$$-\Delta h = f \quad \text{in } \mathbb{R}^2_+ \text{ and } h = 0 \text{ on } \partial\mathbb{R}^2_+.$$

From the calculation of the evolution of the enstrophy $\|\omega(t)\|^2_{L^2}$ based on integration by parts, one can see that the term $(-\partial_1^2)^{1/2}\omega$ plays a role of the linear creation of vorticity on the boundary. Although (1.2) is not a familiar condition due to the presence of the term $(-\partial_1^2)^{1/2}\omega$, we can derive a solution formula to (1.1)–(1.2) through the Fourier–Laplace transform. This is the first contribution of this paper and will be stated in Section 3. We note that a solution formula for the (Navier–) Stokes equations is obtained by [33, 37] for $\mathbb{R}^n_+$ with any $n \geq 2$, and it is a basic tool in the study of (NS) in the half space. Our solution formula leads to L^p-L^q estimates of the propagator to the linear vorticity equations when the initial velocity satisfies the no-slip boundary condition. From these estimates the mathematical validity of (1.1)–(1.2), i.e., the (local-in-time) solvability of (1.1)–(1.2) in suitable function spaces, is confirmed; see Theorem 3.6. We will see that the compatibility condition $a = 0$ on the boundary is essential to ensure the smoothing effect near the boundary. The generator of the propagator for the linearized problem is also studied in Appendix 5.3.

The solution formula helps to carry out a detailed analysis of vorticity fields even in the region near the boundary. Making use of this advantage, as a second contribution of this paper, we investigate in Section 4 the behavior of vorticity at the zero viscosity limit $\nu \to 0$. The precise statement of the result will be given in Theorem 4.1. Roughly speaking, we will see that the following asymptotic expansion holds at $\nu \to 0$ near the initial time:

$$\omega(t) \ \sim \ \omega_E(t) \ + \ \omega_{BL}(t) \quad \text{for} \ \ 0 < t \leq c_0 \nu^{\frac{1}{3}}. \tag{1.3}$$

Here ω_E is the vorticity field for the solution to the Euler equations with the initial velocity a, ω_{BL} is the function describing the boundary layer up to the time $c_0\nu^{1/3}$, and c_0 is a constant independent of $0 < \nu \ll 1$. The function ω_{BL} is written rather explicitly in terms of the initial data (see (4.3)), and it is a nontrivial function if and only if

$$\partial_2(-\Delta_D)^{-1}(a \cdot \nabla \operatorname{Rot} a) \not\equiv 0 \ \text{ on } \partial\mathbb{R}^2_+. \tag{1.4}$$

When (1.4) holds ω_{BL} will be shown to satisfy

$$c\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \leq \|\omega_{BL}(t)\|_{L^p(\Omega_{\nu t})} \leq \|\omega_{BL}(t)\|_{L^p} \leq C\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \tag{1.5}$$

for all $\nu, t > 0$ and $1 \le p \le \infty$, where

$$\Omega_{\nu t} = \{x \in \mathbb{R}^2_+ : 0 \le x_2 \le (\nu t)^{1/2}\}$$

is the region of the boundary layer. In particular, (1.3) and (1.5) imply the high creation of vorticity near the boundary in L^p for $2 < p \le \infty$ as follows:

$$\|\omega(c_0\nu^{\frac{1}{3}})\|_{L^p(\{0 \le x_2 \le c_0^{\frac{1}{2}}\nu^{\frac{2}{3}}\})} \ge c'\nu^{-\frac{1}{3}(1-\frac{2}{p})} \to \infty \ (\nu \to 0) \text{ if } 2 < p \le \infty; \quad (1.6)$$

see Corollary 4.2 for details.

The high creation of vorticity at the zero viscosity limit, which arises due to the nonlinearity of (1.1)–(1.2), is naturally expected from the boundary-layer theory. However, this phenomenon with explicit estimates has been mathematically observed only under some restricted situations. In [13] the nonlinear instability of the Prandtl boundary layer is proved around linearly unstable stationary solutions to the Euler equations. As a product of the calculations based on the spectral analysis and the energy argument for velocity fields, it is also shown that there exist a sequence of solutions $\{u^{(\nu)}\}$ to (NS) and $\{T_\nu\}$ such that $\|\operatorname{Rot} u^{(\nu)}(T_\nu)\|_{L^\infty} \to \infty$ and $T_\nu \to 0$ as $\nu \to 0$. So in [13] the high vorticity creation in L^∞ is observed around a certain class of stationary solutions to the Euler equations. On the other hand, in [2, 29, 30] the asymptotic expansion for solutions to (NS) of the form $u(t,x) = u_E(t,x) + u_P(t,x_1,x_2/\sqrt{\nu}) + O(\sqrt{\nu})$ at $\nu \to 0$ is established for analytic initial data. Here u_E is the solution to the Euler equations and u_P is the solution to the modified Prandtl equations. Hence, the results of [2, 29, 30] imply the high vorticity creation in L^p_{loc} for any $p > 1$, but under the regularity condition of analyticity on initial data.

In Theorem 4.1 the expansion (1.3) is proved for the time period depending on the viscosity, and thus general initial data in some Sobolev class are allowed there. Furthermore, the function ω_{BL} has a simple representation, and at least up to the time $c_0\nu^{1/3}$ we do not need the approximation using the Prandtl-type equations in the boundary layer. This observation of the order $\nu^{1/3}$ is newly obtained by the present paper, though it is not clear whether or not the power $1/3$ can be improved for sufficiently general initial data in some Sobolev class. Note that we cannot expect the expansion (1.3) up to the time $O(1)$ in general, because the function ω_{BL} in Theorem 4.1 does not take into account the nonlinear interaction in the boundary layer region.

The condition (1.4) is necessary and sufficient for the vorticity to exhibit an unbounded growth at $T_\nu = c_0\nu^{1/3}$ as $\nu \to 0$. The meaning of (1.4) is explained as follows. If we recall the Biot–Savart law in $\mathbb{R}^2_+$ and the vorticity equations associated with the Euler equations, (1.4) asserts nothing but

$$\partial_t u_{E,1}|_{\,t=0} \not\equiv 0 \quad \text{on } \partial\mathbb{R}^2_+.$$

Here $u_E = (u_{E,1}, u_{E,2})$ denotes the solution to the Euler equations with the initial data a. Hence (1.4) represents the nondegenerate condition for u_E to be a nonzero velocity field on the boundary right after the initial time. In such situations it is natural that the boundary layer immediately appears and the high vorticity is created near the initial time. In Theorem 4.1 it is also proved that the L^∞ norm of u is uniformly bounded in $0 < \nu \ll 1$ for the time period $(0, c_0\nu^{1/3})$. Although it depends on the viscosity, this time period is not trivial since our initial data are assumed to possess only Sobolev regularities. As a final remark of this section, the boundary conditions on vorticity fields can be derived also for the three-dimensional flows from the similar spirit as in Section 2, and consequently, a solution formula to the three-dimensional vorticity equations is obtained in the case of the half space. However, their representations become more complicated, for the vorticity fields have three components in three-dimensional flows and they interact with each other intricately.

The rest of this paper is organized as follows. In Section 2, we extend the argument in [1] to the case of multi-connected bounded domains and formulate the initial-/boundary-value problems of the vorticity equations by deriving the boundary conditions on vorticity fields. In Section 3 we establish a solution formula for the vorticity equations in the half plane; see Theorem 3.1 and Corollary 3.3. The L^p-L^q estimates for the associated propagator are given in Lemma 3.4. In Section 4 we study the behavior of vorticity fields at the zero viscosity limit and give the asymptotic expansion near the initial time in Theorem 4.1. For the proof we need to introduce a suitable decomposition of the vorticity field, and each term has to be estimated carefully. This step requires long calculations, and in order to tidy up the arguments and computations we will set out several subsections. Finally, in appendices we prove some results on the propagator of the linear vorticity equations which are used in the previous sections.

2. Vorticity formulation

In this section we derive an equivalent formulation to (NS) based on vorticity fields. The vorticity formulation to (NS) itself has a long history, and has been studied mostly from the numerical point of view. The key idea for the derivation of the formulation in this section is seen in [1]; the reader is also referred to [28] for another treatment of vorticity equations. For simplicity, in this section we deal with the case of multi-connected bounded domains only. But under suitable spatial decay conditions on velocity fields it is easy to see that the similar argument works also for $\Omega = \mathbb{R}^2_+$ (half plane) or $\Omega = \mathbb{R}^2 \backslash \Omega_{bdd}$ (exterior domain), where Ω_{bdd} is a simply-connected bounded domain. The assumptions on Ω are stated as follows: Ω is a bounded domain and $\partial\Omega$ has connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_L$ which are disjoint C^∞ closed curves, and each $\Gamma_i,\ 1 \leq i \leq L$, lies in Ω_0, where Ω_0 is a simply-connected bounded domain with $\partial\Omega_0 = \Gamma_0$. Before stating the results, let us introduce some function spaces. $C_0^\infty(\Omega)$ is the set of smooth functions with compact support in Ω; $W_0^{l,p}(\Omega)$, $l \in \mathbb{N}$, $1 \leq p \leq \infty$, is the closure of $C_0^\infty(\Omega)$ with respect to the norm of the Sobolev space $W^{l,p}(\Omega)$; $C_{0,\sigma}^\infty(\Omega)$ denotes the set of all C^∞ vector functions $u = (u_1, u_2)$ with compact support in Ω such that $\operatorname{div} u = 0$; $L^p_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the norm in $(L^p(\Omega))^2$.

2.1. **Helmholtz–Weyl decomposition.** Hereafter n denotes the outward unit normal to $\partial\Omega$. The boundary conditions on vorticity fields are closely related to the Helmholtz–Weyl decomposition of vector fields. In particular, we need a decomposition of tangential flows (i.e., vector fields u such that $\operatorname{div} u = 0$ in Ω and $u \cdot n = 0$ on $\partial\Omega$) into the irrotational ones and the rotational ones (see [15, 20] and references therein), which is a refinement of the classical decomposition theorem of vector fields [40, 34, 8, 31]. For convenience of reference we follow the notation in [20, Theorem 3.20]. Set

$$H_{har}(\Omega) = \{h \in C^\infty(\bar{\Omega}) : \operatorname{div} h = \operatorname{Rot} h = 0 \text{ in } \Omega,\ n \cdot h = 0 \text{ on } \Omega\},$$
$$\operatorname{Rot} h = \partial_1 h_2 - \partial_2 h_1. \tag{2.1}$$

Theorem 2.1. (i) *The dimension of $H_{har}(\Omega)$ is L and a basis $\{\varphi_1, \ldots, \varphi_L\}$ of $H_{har}(\Omega)$ is given by*

$$\varphi_j = \nabla^\perp q_j,\ 1 \leq j \leq L, \tag{2.2}$$

where $\nabla^\perp = (\partial_2, -\partial_1)$ and q_j is the solution to the Dirichlet boundary-value problem

$$\begin{cases} \Delta q_j = 0, \text{ in } \Omega, \\ q_j = \delta_{ij}, \text{ on } \Gamma_i,\ 0 \leq i \leq L. \end{cases} \tag{2.3}$$

(ii) *For any $u \in L^2(\Omega)^2$ there exist $h \in H_{har}(\Omega)$, $\psi \in W_0^{1,2}(\Omega)$, and $p \in W^{1,2}(\Omega)$ such that*

$$u = h + \nabla^\perp \psi + \nabla p. \tag{2.4}$$

This decomposition is unique, and h and ψ are given by

$$h = \sum_{j=1}^{L} (u, \nabla^\perp \tilde{q}_j)_{L^2} \nabla^\perp \tilde{q}_j, \ \tilde{q}_j = c_j q_j, \ \psi = (-\Delta_D)^{-1} \operatorname{Rot} u, \tag{2.5}$$

where $g = (-\Delta_D)^{-1} f$ denotes the solution to the Dirichlet boundary-value problem: $-\Delta g = f$ in Ω, $g = 0$ on $\partial\Omega$. Each c_j is a positive constant which normalizes the norm of $\|c_j \nabla q_j\|_{L^2}$.

Proof. See, for example, [20, Theorem 3.20].

As a corollary of Theorem 2.1, we get

Corollary 2.2. *Let $u \in L^2_\sigma(\Omega) \cap W^{1,2}(\Omega)^2$. Then $u \in (W_0^{1,2}(\Omega))^2$ if and only if*

$$\partial_n (-\Delta_D)^{-1} \operatorname{Rot} u + \sum_{j=1}^{L} \big(u, \nabla^\perp \tilde{q}_j\big)_{L^2} \partial_n \tilde{q}_j = 0 \text{ on } \partial\Omega. \tag{2.6}$$

In this case we have

$$u = (-\Delta_D)^{-1} \nabla^\perp \omega = \nabla^\perp (-\Delta_D)^{-1} \omega + \sum_{j=1}^{L} \big((-\Delta_D)^{-1} \nabla^\perp \omega, \nabla^\perp \tilde{q}_j\big)_{L^2} \nabla^\perp \tilde{q}_j. \tag{2.7}$$

Here $\omega = \operatorname{Rot}\ u$.

In (2.7) the operator $(-\Delta_D)^{-1}\nabla^\perp$ is regarded as a bounded operator from $L^2(\Omega)$ to $W_0^{1.2}(\Omega)$. We note that (2.7) gives the Biot–Savart law in the multi-connected bounded domain; the velocity field u is recovered from the vorticity field ω through the formula (2.7), if ω satisfies the integral condition

$$\partial_n (-\Delta_D)^{-1} \omega + \sum_{j=1}^{L} \big((-\Delta_D)^{-1} \nabla^\perp \omega, \nabla^\perp \tilde{q}_j\big)_{L^2} \partial_n \tilde{q}_j = 0 \text{ on } \partial\Omega. \tag{2.8}$$

2.2. **Vorticity equations.** Equation (2.8) gives the integral condition on the vorticity field by which the associated velocity field satisfies the no-slip boundary condition. But it is not so useful for the analysis of vorticity fields even when the topology of the domain is quite simple, for (2.8) is highly non-local. Following [1], we do not deal with (2.8) directly, but instead, we consider the boundary conditions so that (2.8) is preserved under the vorticity equations. Based on this idea we obtain the boundary conditions on vorticity fields which are more local than (2.8). As is pointed out by [7], these boundary conditions are not necessarily a drastic prescription to overcome difficulties in the numerical analysis. However, as will be seen in Sections 3 and 4, they indeed provide useful information for the mathematical analysis of vorticity fields.

Let a be a vector field which at least belongs to $L^2_\sigma(\Omega)\cap(W^{1,2}_0(\Omega))^2$. We consider the equation

$$\begin{cases}\partial_t\omega-\nu\Delta\omega+u\cdot\nabla\omega=0 & t>0,\ x\in\Omega,\\ u=\nabla^\perp(-\Delta_D)^{-1}\omega+\sum_{j=1}^L\big((-\Delta_D)^{-1}\nabla^\perp\omega,\nabla^\perp\tilde{q}_j\big)_{L^2}\nabla^\perp\tilde{q}_j,\ t>0,\ x\in\Omega,\\ \omega|_{t=0}=b:=\operatorname{Rot}a,\ x\in\Omega,\end{cases}\tag{V}$$

together with the boundary condition

$$\begin{aligned}&\nu\big\{\partial_n\omega-\Lambda_{DN}\omega+\sum_{j=1}^L\big(\nabla^\perp\omega,\nabla^\perp\tilde{q}_j\big)_{L^2}\partial_n\tilde{q}_j\big\}\\ &=-\partial_n(-\Delta_D)^{-1}(u\cdot\nabla\omega)+\sum_{j=1}^L\big(\omega u,\nabla\tilde{q}_j\big)_{L^2}\partial_n\tilde{q}_j,\end{aligned}\tag{BC}$$

for $t>0$ and $x\in\partial\Omega$. Here $\tilde{q}_j$ is the function in Theorem 2.1 and Λ_{DN} is the Dirichlet–Neumann map defined by $\Lambda_{DN}\omega=\partial_n\omega_{har}$, where ω_{har} is the solution to the Dirichlet boundary-value problem: $\Delta\omega_{har}=0$ in Ω, $\omega_{har}=\omega$ on $\partial\Omega$. In the next theorem (V)–(BC) is shown to be equivalent to (NS). For simplicity we assume that the solution (and the initial data) is smooth enough to ensure the formal calculations. We also note that the initial velocity a is always assumed to satisfy the compatibility condition $a=0$ on $\partial\Omega$, but no additional compatibility condition on the boundary is required in the argument.

Theorem 2.3. *The equation (NS) is equivalent to* (V)–(BC) *in the sense that if (u,p) is a smooth solution to (NS), then $\omega=\operatorname{Rot}u$ solves (V)–(BC), and conversely, if ω is a smooth solution to (V)–(BC), then u defined by (V) solves (NS) for some p.*

Proof. For simplicity of notation we write q_j for $\tilde{q}_j$. Let ω be a smooth function satisfying (V). Set $u\times\omega=(\omega u_2,-\omega u_1)$, and set

$$\tilde{u}=\nabla^\perp(-\Delta_D)^{-1}\omega+\sum_{j=1}^L\big(a-\nu\int_0^t\nabla^\perp\omega\,ds+\int_0^t u\times\omega\,ds,\nabla^\perp q_j\big)_{L^2}\nabla^\perp q_j.\tag{2.9}$$

Then from Theorem 2.1 we see $\tilde{u}|_{t=0} = a$, $\operatorname{div}\tilde{u} = 0$, and $\operatorname{Rot}\tilde{u} = \omega$ for $t \geq 0$, $x \in \Omega$, and $n \cdot \tilde{u} = 0$ for $t \geq 0$, $x \in \partial\Omega$. Let $\tau = (n_2, -n_1)$, where $n = (n_1, n_2)$ is the outward unit normal to $\partial\Omega$. Then, noting that the time derivative commutes with $\partial_n(-\Delta_D)^{-1}$, we have on $\partial\Omega$ and for $t > 0$,

$$\begin{aligned}\partial_t\tau\cdot\tilde{u} &= \partial_n(-\Delta_D)^{-1}\partial_t\omega + \sum_{j=1}^{L}\big(-\nu\nabla^{\perp}\omega + u\times\omega, \nabla^{\perp}q_j\big)_{L^2}\partial_n q_j\\ &= \partial_n(-\Delta_D)^{-1}\big(\Delta(\omega-\omega_{har}) - u\cdot\nabla\omega\big)\\ &\quad + \sum_{j=1}^{L}\big(-\nu\nabla^{\perp}\omega + u\times\omega, \nabla^{\perp}q_j\big)_{L^2}\partial_n q_j\\ &= -\partial_n\omega + \partial_n\omega_{har} - \partial_n(-\Delta_D)^{-1}(u\cdot\nabla\omega)\\ &\quad + \sum_{j=1}^{L}\big(-\nu\nabla^{\perp}\omega + u\times\omega, \nabla^{\perp}q_j\big)_{L^2}\partial_n q_j.\end{aligned}$$

Thus, ω satisfies (BC) if and only if

$$(\tau\cdot\tilde{u})(t,x) = (\tau\cdot\tilde{u})(0,x)\quad\text{for all } t>0 \text{ and } x\in\partial\Omega.$$

Assume that ω is a smooth solution to (V)–(BC). Then

$$(\tau\cdot\tilde{u})(t,x) = (\tau\cdot a)(x) = 0\quad\text{for } t\geq 0,\ x\in\partial\Omega$$

by the above argument, which implies $\tilde{u}(t) = 0$ on $\partial\Omega$. In particular, we have from Corollary 2.2 and (V),

$$\tilde{u}=(-\Delta_D)^{-1}\nabla^{\perp}\omega=\nabla^{\perp}(-\Delta_D)^{-1}\omega+\sum_{j=1}^{L}\big((-\Delta_D)^{-1}\nabla^{\perp}\omega, \nabla^{\perp}q_j\big)_{L^2}\nabla^{\perp}q_j = u,$$

which gives $u(t) = 0$ on $\partial\Omega$. Next we show that u solves (NS) with some p. Take any $v \in (C_0^\infty((0,T)\times\Omega))^2$ satisfying $\operatorname{div} v = 0$ and set $w = \operatorname{Rot} v$. Let us write

$$v = \nabla^{\perp}\psi + \sum_{j=1}^{L} d_j\nabla^{\perp}q_j,$$

where $\psi = (-\Delta_D)^{-1}w$ and $d_j = \big(v, \nabla^{\perp}q_j\big)_{L^2}$. Then by $-\Delta u = \nabla^{\perp}\omega$ and $\big(\nabla^{\perp}\omega, \nabla^{\perp}q_j\big)_{L^2} = \big(\nabla^{\perp}\omega_{har}, \nabla^{\perp}q_j\big)_{L^2}$ we have from the integration by parts,

$$\int_0^T \big(\partial_t u - \nu\Delta u + u\cdot\nabla u, v\big)_{L^2}\,dt$$

$$
\begin{aligned}
&= \int_0^T \big(\partial_t u - \nu \Delta u + u\cdot\nabla u, \nabla^\perp \psi + \sum_{j=1} d_j \nabla^\perp q_j\big)_{L^2}\, dt \\
&= \int_0^T \big(\partial_t \omega - \nu \Delta \omega + u\cdot\nabla \omega, \psi + \sum_{j=1} d_j q_j\big)_{L^2}\, dt + \int_0^T \int_{\partial\Omega} \nu \partial_n \omega \sum_{j=1} d_j q_j \, dS\, dt \\
&= \nu \int_0^T \int_{\partial\Omega} \{\partial_n \omega_{har} - \sum_{j=1}^L \big(\nabla^\perp \omega, \nabla^\perp q_j\big)_{L^2} \partial_n q_j\} \sum_{j=1}^L d_j q_j \, dS\, dt \\
&- \int_0^T \int_{\partial\Omega} \{\partial_n (-\Delta_D)^{-1}(u\cdot\nabla\omega) - \sum_{j=1}^L \big(u\times\omega, \nabla^\perp q_j\big)_{L^2} \partial_n q_j\} \sum_{j=1}^L d_j q_j \, dS\, dt \\
&= \nu \sum_{j=1}^L d_j \int_0^T \big(\nabla^\perp \omega_{har}, \nabla^\perp q_j\big)_{L^2}\, dt - \nu \sum_{j=1}^L d_j \int_0^T \big(\nabla^\perp \omega_{har}, \nabla^\perp q_j\big)_{L^2}\, dt \\
&- \int_0^T \int_{\partial\Omega} \{\partial_n (-\Delta_D)^{-1}(u\cdot\nabla\omega) - \sum_{j=1}^L \big(u\times\omega, \nabla^\perp q_j\big)_{L^2} \partial_n q_j\} \sum_{j=1}^L d_j q_j \, dS\, dt \\
&= -\sum_{j=1}^L d_j \int_0^T \big(\nabla(-\Delta_D)^{-1}(u\cdot\nabla\omega), \nabla q_j\big)_{L^2}\, dt \\
&+ \sum_{j=1}^L d_j \int_0^T \big(u\cdot\nabla\omega, q_j\big)_{L^2}\, dt + \sum_{j=1}^L d_j \int_0^T \big(u\times\omega, \nabla^\perp q_j\big)_{L^2}\, dt \\
&= -\sum_{j=1}^L d_j \int_0^T \big(\omega u, \nabla q_j\big)_{L^2}\, dt + \sum_{j=1}^L d_j \int_0^T \big(u\times\omega, \nabla^\perp q_j\big)_{L^2}\, dt = 0.
\end{aligned}
$$

Since $v \in (C_0^\infty((0,T)\times\Omega))^2$ with $\operatorname{div} v = 0$ is arbitrary, u is a solution to (NS) for some p.

Conversely, let (u,p) be a smooth solution to (NS). Then by Corollary 2.2 the velocity u is recovered from $\omega = \operatorname{Rot} u$ as in (V); hence, ω solves (V). Furthermore, (NS) implies

$$
\begin{aligned}
u(t) &= a + \nu \int_0^t \Delta u \, ds + \int_0^t u\times\omega \, ds - \int_0^t \nabla p \, ds \\
&= a - \nu \int_0^t \nabla^\perp \omega \, ds + \int_0^t u\times\omega \, ds - \int_0^t \nabla p \, ds.
\end{aligned}
$$

Let $\tilde{u}$ be the vector defined by (2.9). Then we have

$$\tilde{u} = \nabla^\perp(-\Delta_D)^{-1}\omega + \sum_{j=1}^{L} \big(u, \nabla^\perp q_j\big)_{L^2} \nabla^\perp q_j = u.$$

Thus, $(\tau \cdot \tilde{u})(t,x) = (\tau \cdot u)(t,x) = 0$ for $t \geq 0$, $x \in \partial\Omega$, which shows that ω satisfies (BC). This completes the proof.

3. Solution formula for the half-plane case

The argument in Section 2 is clearly valid also when $\Omega = \mathbb{R}^2_+$ if the velocity and the vorticity decay fast enough at spatial infinity. In the case of the half plane we have the explicit formulas $\partial_n = -\partial_2$ and $\Lambda_{DN}\omega = (-\partial_1^2)^{1/2}\omega$. Here the fact $\Lambda_{DN} f = (-\partial_1^2)^{1/2} f$ is derived from the formula $\hat{f}_{har}(\xi_1, x_2) = e^{-x_2|\xi_1|}\hat{f}(\xi_1)$, where f_{har} denotes the harmonic extension of $f \in C_0^\infty(\partial\mathbb{R}^2_+)$ to $\mathbb{R}^2_+$ and $\hat{f}$ is the (partial) Fourier transform of f in the x_1 direction. Then, for (V)–(BC) the method of the Fourier–Laplace transform is directly applied to derive a solution formula. This formula is considered as a vorticity counterpart of the well-known formula for solutions to (NS) by [33, 37]. For the moment let us consider the linear problem

$$\begin{cases} \partial_t \omega - \nu\Delta\omega = f & t > 0,\ x \in \mathbb{R}^2_+, \\ \omega|_{t=0} = b & x \in \mathbb{R}^2_+, \end{cases} \tag{LV}$$

subject to the boundary condition

$$\nu\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\omega = g, \qquad t > 0,\ x \in \partial\mathbb{R}^2_+. \tag{LBC}$$

Here f, g, and b are assumed to be smooth and decay fast enough at spatial infinity. The integral equation for the vorticity equations will be obtained by taking

$$f = -u \cdot \nabla\omega, \qquad g = \{\partial_2(-\Delta_D)^{-1} f\}|_{\partial\mathbb{R}^2_+}, \tag{3.1}$$

and $u = \nabla^\perp(-\Delta_D)^{-1}\omega$, $\nabla^\perp = (\partial_2, -\partial_1)$. We set

$$\Xi = 2\big(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}}\partial_2\big), \tag{3.2}$$

$$G(t,x) = \frac{1}{4\pi t}\exp(-\frac{|x|^2}{4t}),\ E(x) = -\frac{1}{2\pi}\log|x|, \tag{3.3}$$

$$\Gamma(t,x) = \big(\Xi E * G(t)\big)(x) = \big(\Xi(-\Delta_{\mathbb{R}^2})^{-1}G(t)\big)(x), \tag{3.4}$$

$$(h_1 \star h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y^*)h_2(y)\,dy,\ y^* = (y_1, -y_2). \tag{3.5}$$

Theorem 3.1. *The integral equation for (LV)–(LBC) is given by*

$$\omega(t) = e^{\nu t\Delta_N} b + \Gamma(\nu t)\star b - \Gamma(0)\star b + \int_0^t e^{\nu(t-s)\Delta_N}\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds$$
$$+ \int_0^t \Gamma(\nu(t-s))\star\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds$$
$$- \int_0^t \Gamma(0)\star\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds. \tag{3.6}$$

Here $e^{t\Delta_N}$ is the semigroup for the heat equation (with the unit viscosity) in $\mathbb{R}^2_+$ subject to the homogeneous Neumann boundary condition, $\Gamma(0)\star := \lim_{t\downarrow 0}\Gamma(t)\star$, and $g\mathcal{H}^1_{\{x_2=0\}}$ is a one-dimensional Hausdorff measure on $\partial\mathbb{R}^2_+$ with density g defined by

$$\langle h, g\mathcal{H}^1_{\{x_2=0\}}\rangle = \int_{\mathbb{R}} h(x_1,0)g(x_1)\,dx_1 \text{ for } h\in C_0(\mathbb{R}^2). \tag{3.7}$$

The proof of Theorem 3.1 is given in the appendix. We note that $\Gamma(0)\star h = \Xi E\star h$ in $\mathbb{R}^2_+$. In (3.6) the terms $\Gamma(0)\star$ seem to cause trouble when solving the vorticity equations, for apparently they could give rise to a derivative loss near the boundary. In fact, these terms do not appear in the vorticity equations, due to the following cancellation property.

Proposition 3.2. *If $g = \{\partial_2(-\Delta_D)^{-1}f\}|_{\partial\mathbb{R}^2_+}$, then $\Xi E\star\big(f - g\mathcal{H}^1_{\{x_2=0\}}\big) = 0$ in $\mathbb{R}^2_+$. In particular, we have*

$$\Xi E\star b = 0 \text{ in } \mathbb{R}^2_+ \text{ if } \partial_2(-\Delta_D)^{-1}b = 0 \text{ on } \partial\mathbb{R}^2_+. \tag{3.8}$$

Proof. Since $g = -\partial_n(-\Delta_D)^{-1}f$ on $\partial\mathbb{R}^2_+$ and $f = -\Delta(-\Delta_D)^{-1}f$ in $\mathbb{R}^2_+$, by integration by parts we have

$$\Xi E\star\big(f - g\mathcal{H}^1_{\{x_2=0\}}\big)(x) = \int_{\mathbb{R}^2_+}\nabla_y\big((\Xi E)(x-y^*)\big)\cdot\nabla_y(-\Delta_D)^{-1}f(y)\,dy$$
$$= -\int_{\mathbb{R}^2_+}\{\Delta_y\big((\Xi E)(x-y^*)\big)\}(-\Delta_D)^{-1}f(y)\,dy = 0,$$

for $x\in\mathbb{R}^2_+$. Here we have used $(-\Delta_D)^{-1}f = 0$ on the boundary,

$$\Delta_y\big((\Xi E)(x-y^*)\big) = (\Delta\Xi E)(x-y^*) = (\Xi\Delta E)(x-y^*),$$

and $\Delta E(x-y^*) = 0$ for $x_2, y_2 > 0$. The integration by parts in the calculations can be verified when f is smooth and decay fast enough. The proof is complete.

We note that the condition in (3.8) is nothing but (2.8). Thus, recalling also (3.1), we do not have the problematic terms $\Gamma(0)\star$ in (3.6) for the vorticity equations. It will be useful to rewrite the result of Theorem 3.1 under the conditions in Proposition 3.2.

Corollary 3.3. *Assume that*

$$\{\partial_2(-\Delta_D)^{-1}b\}|_{\partial\mathbb{R}^2_+} = 0 \quad \textit{and} \quad g = \{\partial_2(-\Delta_D)^{-1}f\}|_{\partial\mathbb{R}^2_+}.$$

Then the integral equation for (LV)–(LBC) is given by

$$\omega(t) = e^{\nu tB}b + \int_0^t e^{\nu(t-s)B}\big(f(s) - g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\, ds, \tag{3.9}$$

where

$$e^{tB}h = e^{t\Delta_N}h + \Gamma(t)\star h. \tag{3.10}$$

Corollary 3.3 shows that the integral equation for the vorticity equation is written as

$$\omega(t) = e^{\nu tB}b - \int_0^t e^{\nu(t-s)B}\Big(u\cdot\nabla\omega(s) - \partial_2(-\Delta_D)^{-1}\big(u\cdot\nabla\omega\big)(s)\mathcal{H}^1_{\{x_2=0\}}\Big)\, ds,$$
$$u = \nabla^{\perp}(-\Delta_D)^{-1}\omega. \tag{3.11}$$

The one-parameter family of linear operators $\{e^{tB}\}_{t\geq 0}$ defined by (3.10) is a C_0-analytic semigroup in the Banach space

$$X_q = \{\mathrm{Rot}\, u : u \in \dot{W}^{1,q}_{0,\sigma}(\mathbb{R}^2_+)\} \subset L^q(\mathbb{R}^2_+)$$

for $1 < q < \infty$. Here $\dot{W}^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$ is the completion with respect to the homogeneous norm $\|\nabla u\|_{L^q}$ of the space of all smooth, divergence-free vector fields with compact support in $\mathbb{R}^2_+$. The associated generator $B = B_q$ is given by

$$D(B_q) = \{f \in X_q \cap W^{2,q}(\mathbb{R}^2_+) : \partial_2 f + (-\partial_1^2)^{1/2}f = 0 \text{ on } \partial\mathbb{R}^2_+\},$$

$B_q f = \Delta f$ for $f \in D(B_q)$. The details of these properties are discussed in Appendix 5.3.

It is possible to show the local-in-time solvability of (3.11) in suitable function spaces. Indeed, from L^p-L^q estimates for e^{tB} in Lemma 3.4 below we can construct solutions to (3.11) at least locally in time if $b \in L^p(\mathbb{R}^2_+)$ for some $p \in (1,2)$ by the contraction-mapping theorem. Since its proof is rather standard we only state the result in Theorem 3.6 with a brief sketch of the proof, and the details are omitted in this paper.

Lemma 3.4. (i) *Let* $1 \le q < p \le \infty$ *or* $1 < q \le p < \infty$. *Then we have*

$$\|e^{tB}f\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}}\|f\|_{L^q},\ t>0. \tag{3.12}$$

(ii) *Let* $1 \le q \le p \le \infty$ *and* $p > 1$. *Then we have*

$$\|e^{tB}(g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \le Ct^{-\frac{1}{2}(1+\frac{1}{q}-\frac{2}{p})}\|g\|_{L^q_{x_1}},\ t>0. \tag{3.13}$$

(iii) *Let* $1 \le q \le p \le \infty$ *and* $k \in \mathbb{N}$. *Then we have*

$$\|\nabla^k e^{tB}f\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}-\frac{k}{2}}\|f\|_{L^q}\ t>0. \tag{3.14}$$

(iv) *Let* $1 \le q \le p \le \infty$. *Assume that* $g = \{\partial_2(-\Delta D)^{-1}f\}|_{\partial\mathbb{R}^2_+}$. *Then we have*

$$\|e^{tB}\big(f - g\mathcal{H}^1_{\{x_2=0\}}\big)\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}-\frac{1}{2}}\|\nabla^\perp(-\Delta_D)^{-1}f\|_{L^q}\ t>0. \tag{3.15}$$

Proof. (i) Since it is straightforward to get

$$\|e^{t\Delta_N}f\|_{L^p} \le Ct^{-1/p+1/q}\|f\|_{L^q} \quad \text{for all } 1 \le q \le p \le \infty,$$

it suffices to show

$$\|\Gamma(t) \star f\|_{L^p} \le Ct^{-1/p+1/q}\|f\|_{L^q}$$

if $1 \le q < p \le \infty$ or $1 < q \le p < \infty$. To this end we first write

$$\Gamma(t) \star f = \Xi(-\Delta_{\mathbb{R}^2})^{-1}\big(G(t) \star f\big)$$

and observe that the symbol $p(\xi)$ of the operator $\Xi(-\Delta_{\mathbb{R}^2})^{-1}$ is given by

$$p(\xi) = 2\frac{-\xi_1^2 + i|\xi_1|\xi_2}{|\xi|^2}. \tag{3.16}$$

Thus, $\Xi(-\Delta_{\mathbb{R}^2})^{-1}$ is a singular integral operator in $\mathbb{R}^2$ (see [6, Theorem 8.14]), so we have for $1 \le q < p < \infty$ or $1 < q \le p < \infty$,

$$\|\Gamma(t) \star f\|_{L^p(\mathbb{R}^2)} \le C\|G(t) \star f\|_{L^p(\mathbb{R}^2)} \le Ct^{-\frac{1}{q}+\frac{1}{p}}\|f\|_{L^q}.$$

Let $1 \le q < p = \infty$. Then by the Gagliardo–Nirenberg inequality we have for $\max\{q,2\} < \tilde{q} < \infty$ and $\sigma = 1 - 2/\tilde{q}$,

$$\begin{aligned}\|\Gamma(t) \star f\|_{L^\infty(\mathbb{R}^2)} &\le C\|\nabla\Gamma(t) \star f\|^{1-\sigma}_{L^{\tilde{q}}(\mathbb{R}^2)}\|\Gamma(t) \star f\|^{\sigma}_{L^{\tilde{q}}(\mathbb{R}^2)}\\ &\le C\|\nabla G(t) \star f\|^{1-\sigma}_{L^{\tilde{q}}(\mathbb{R}^2)}\|G(t) \star f\|^{\sigma}_{L^{\tilde{q}}(\mathbb{R}^2)} \le Ct^{-\frac{1}{q}}\|f\|_{L^q}.\end{aligned}$$

(ii) Again it suffices to show $\|\Gamma(t) \star g\mathcal{H}^1_{\{x_2=0\}}\| \le Ct^{-(1+1/q-2/p)/2}\|g\|_{L^q_{x_1}}$. To prove this we use the pointwise estimate (5.8) and observe that

$$|\Gamma(t) \star (g\mathcal{H}^1_{\{x_2=0\}})(x)| \tag{3.17}$$

$$\leq Ct^{-1}\int_{\mathbb{R}}\big(1+\frac{|(x_1-y_1)/\sqrt{t}|^2}{\log(e+|(x_1-y_1)/\sqrt{t}|^2)}+\frac{x_2^2}{t}\big)^{-1}|g(y_1)|\,dy_1.$$

Then the Young inequality implies that

$$\begin{aligned}&\|\Gamma(t)\star(g\mathcal{H}^1_{\{x_2=0\}})(\cdot,x_2)\|_{L^p_{x_1}}\\ &\leq Ct^{-\frac12(\frac1q-\frac1p+1)}\big(1+\frac{|x_2|}{\sqrt{t}}\big)^{-1+\frac1p-\frac1q}\big(\log(e+\frac{x_2^2}{t})\big)^{1+\frac1p-\frac1q}\ \|g\|_{L^q_{x_1}}.\end{aligned}$$

Hence, we get (3.13) since $p>1$ and $p\geq q$.

(iii) It suffices to consider $\|\nabla^k\Gamma(t)\star f\|_{L^p}$, but the desired estimate follows from the pointwise estimate (5.8) and the Young inequality. The details are omitted here.

(iv) By the definition of e^{tB} and integration by parts we have

$$\begin{aligned}&e^{tB}(f-g\mathcal{H}^1_{\{x_2=0\}})(x)\\ &=\int_{\mathbb{R}^2_+}\nabla^\perp_y\big(G(t,x-y)+G(t,x-y^*)+\Gamma(t,x-y^*)\big)\cdot\nabla^\perp(-\Delta_D)^{-1}f(y)\,dy.\end{aligned}$$

Then it is easy to get (3.15) from (5.8) and the Young inequality. This completes the proof.

If $b\in L^p(\mathbb{R}^2_+)$ for some $p\in[1,2)$, then it is not difficult to see that the trace of $\partial_2(-\Delta_D)^{-1}b$ on the boundary makes sense, and in particular, it belongs to $L^p(\mathbb{R})+L^\infty(\mathbb{R})$. Moreover, we have

Proposition 3.5. *If $b\in L^1(\mathbb{R}^2_+)$ and $\{\partial_2(-\Delta_D)^{-1}b\}|_{\partial\mathbb{R}^2_+}=0$, then*

$$\int_{\mathbb{R}^2_+}b(x)\,dx=0.$$

Proof. Since

$$\big(\partial_2(-\Delta_D)^{-1}b\big)|_{\partial\mathbb{R}^2_+}(x_1)=\pi^{-1}\int_{\mathbb{R}^2_+}\frac{y_2}{|x_1-y_1|^2+y_2^2}b(y)\,dy,$$

we have from the Fubini theorem

$$\begin{aligned}0&=\int_{\mathbb{R}}\big(\partial_2(-\Delta_D)^{-1}b\big)|_{\partial\mathbb{R}^2_+}(x_1)\,dx_1\\ &=\frac1\pi\int_{\mathbb{R}^2_+}y_2b(y)\int_{\mathbb{R}}\frac{1}{|x_1-y_1|^2+y_2^2}\,dx_1\,dy=\int_{\mathbb{R}^2_+}b(y)\,dy.\end{aligned}$$

This completes the proof.

Proposition 3.5 shows that if $b \in L^1(\mathbb{R}^2_+)$ is the vorticity field of a velocity field satisfying the no-slip boundary condition, then b must have zero integral mean over $\mathbb{R}^2_+$. In other words, there are no nontrivial vorticity fields which are nonnegative and decay fast enough at spatial infinity. This is contrastive if compared with the case $\Omega = \mathbb{R}^2$, where there are nontrivial and nonnegative vorticity fields which decay rapidly at $|x| \to \infty$.

We conclude this section by stating the local solvability of (3.11).

Theorem 3.6. *Assume that $b \in L^p(\mathbb{R}^2_+)$ for some $p \in (1,2)$. Then there is $T > 0$ such that (3.11) has a unique solution $\omega \in C([0,T); L^p(\mathbb{R}^2_+))$ satisfying*

$$\sup_{0<t<T} t^{1/p-1/4} \|\omega(t)\|_{L^4} < \infty.$$

If b satisfies the compatibility condition $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$ in addition, then the solution $\omega(t)$ converges to b as $t \to 0$ in $L^p(\mathbb{R}^2_+)$.

Proof. The solution is constructed in the space

$$X_T = \{f \in C([0,T); L^p(\mathbb{R}^2_+)) : \sup_{0<t<T} t^{1/p-1/4}\|f(t)\|_{L^4} < \infty\}$$

with the norm

$$\|f\|_{X_T} = \sup_{0<t<T} \|f(t)\|_{L^p} + \sup_{0<t<T} t^{1/p-1/4}\|f(t)\|_{L^4}$$

by the contraction-mapping theorem, thanks to Lemma 3.4 and well-known estimates such as

$$\|\nabla^\perp(-\Delta_D)^{-1}\nabla f\|_{L^q} \le C\|f\|_{L^q}$$

for $1 < q < \infty$ and

$$\|\nabla^\perp(-\Delta_D)^{-1} f\|_{L^\infty} \le C\|f\|^{\sigma}_{L^4}\|f\|^{1-\sigma}_{L^p}$$

with $\sigma = (4-2p)/(4-p)$ for $1 < p < 2$. To show the convergence to the initial data, it suffices to write

$$e^{\nu t B} b = e^{\nu t \Delta_N} b + \big(\Gamma(\nu t) - \Gamma(0)\big) \star b + \Gamma(0) \star b$$

and note that the last term vanishes by Proposition 3.2. Then by the density argument and the estimate $\sup_{t\ge 0} \|\Gamma(t) \star b\|_{L^p} \le C\|b\|_{L^p}$ one can check that $\|(\Gamma(\nu t) - \Gamma(0)\big) \star b\|_{L^p}$ goes to zero as $t \to 0$. It is easy to see that $e^{\nu t\Delta_N} b$ converges to b in L^p as $t \to 0$. This completes the proof.

Remark 3.7. Even for $b \in L^1(\mathbb{R}^2_+)$ we can construct a local unique solution ω to (3.11) such that $t^{1-1/r}\omega(t) \in L^\infty(0,T; L^r(\mathbb{R}^2_+))$ and

$$\lim_{t\to 0} t^{1-1/r}\|\omega(t)\|_{L^r} = 0$$

with $r = 4/3, 4$. Furthermore, under the smallness assumption of $\|b\|_{L^1}$ it is also possible to show that the solution exists globally in time. However, the author does not know if the solution ω belongs to $L^\infty(0, T; L^1(\mathbb{R}^2_+))$ in general, due to the lack of the L^1-L^1 estimate of e^{tB}.

Remark 3.8. By the bootstrap argument using Lemma 3.4 the solution ω in Theorem 3.6 is shown to be smooth in positive time. We note that, in order to ensure that $u = \nabla^\perp(-\Delta_D)^{-1}\omega$ solves (NS), we need the compatibility condition $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$ for the initial data.

Remark 3.9. When $b \in L^p(\mathbb{R}^2_+)$ for some $p \in (1, 2)$ the related velocity a belongs to $L^q_\sigma(\mathbb{R}^2_+)$ with $1/q = 1/p - 1/2$ by the Hardy–Littlewood–Sobolev inequality. Since $q > 2$ the solvability of (NS) is already well known in this case from L^q theory of the Navier–Stokes equations in the half space; e.g., see [34, 39].

4. Application to analysis of vorticity at zero viscosity limit

The inviscid-limit behavior of solutions to the Navier–Stokes equations is a classical theme in fluid mechanics. However, if the no-slip boundary conditions are imposed on velocity fields, only partial results are known even in the two-dimensional case; so far we need either the analyticity of initial data or special symmetry of the domain and the solutions (see the examples in [38, Section 2]). More precisely, if the initial data is analytic it is proved in [2, 29, 30] that the inviscid limit is described by the Euler equations and the Prandtl equations. When Ω is a disk and the solution possesses a radial symmetry, the inviscid limit is already well studied in various functional settings [26, 4, 22, 23, 19]; see also [38, 27] and references therein for weakly coupled nonlinear case. On the other hand, [16] gave necessary and sufficient conditions for the convergence of weak solutions of (NS) to that of the Euler equations in the energy class. The analysis in this direction has been developed by [36, 38, 5, 17, 18, 19].

Making use of (3.9), in this section we study the behavior of vorticity fields at the zero viscosity limit when $\Omega = \mathbb{R}^2_+$ and establish the asymptotic expansion near the initial time. The main result is stated as follows.

Theorem 4.1. *Assume that $b = \mathrm{Rot}\, a$ with $a \in L^q_\sigma(\mathbb{R}^2_+) \cap (W^{1,q}_0(\mathbb{R}^2_+))^2$ for some $1 < q < \infty$ and $b \in W^{l,4/3}(\mathbb{R}^2_+)$ for sufficiently large l. Let ω be the solution to (V)–(BC) with $\Omega = \mathbb{R}^2_+$. Then there are $c_0, C > 0$ such that the following estimates hold for sufficiently small $\nu > 0$:*

$$\|u(t)\|_{L^\infty} \leq C \text{ for } 0 < t \leq c_0\nu^{\frac{1}{3}}, \tag{4.1}$$

$$\|\omega(t) - \omega_E(t) - \omega_{BL}(t)\|_{L^p} \le C\nu^{-\frac{1}{2}(\frac{1}{3}-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \tag{4.2}$$

for $0 < t \le c_0\nu^{\frac{1}{3}}$, $\frac{4}{3} \le p \le \infty$*. Here* c_0 *is independent of* ν*, and* C *is independent of* ν *and* $t \in [0, c_0\nu^{1/3}]$*. The function* ω_E *is the vorticity field of the solution to the Euler equation with the initial velocity* a*. The function* ω_{BL} *is given by*

$$\omega_{BL}(t,x) = 2\int_0^t (4\pi\nu s)^{-\frac{1}{2}} \exp(-\frac{x_2^2}{4\nu s})\, ds \cdot \{\partial_2(-\Delta_D)^{-1}\big(a\cdot\nabla b\big)\}(x_1, 0). \tag{4.3}$$

By (4.3) the function ω_{BL} is nontrivial if and only if

$$\partial_2(-\Delta_D)^{-1}\big(a\cdot\nabla b\big) \not\equiv 0 \text{ on } \partial\mathbb{R}^2_+. \tag{4.4}$$

In particular, when (4.4) holds ω_{BL} satisfies

$$\|\omega_{BL}(t)\|_{L^p} \le C'\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \text{ for } t > 0,\ 1 \le p \le \infty, \tag{4.5}$$

$$\|\omega_{BL}(t)\|_{L^p(\{0\le x_2 \le (\nu t)^{\frac{1}{2}}\})} \ge c_1\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \text{ for } t > 0,\ 1 \le p \le \infty. \tag{4.6}$$

Here the positive constants c_1 and C' are independent of ν and t. Hence we immediately get

Corollary 4.2. *Under the assumptions of Theorem* 4.1*, if* (4.4) *holds in addition, then the high creation of vorticity near the boundary in* L^p *occurs in the following sense:*

$$\|\omega(c_0\nu^{\frac{1}{3}})\|_{L^p(\{0\le x_2\le c_0^{\frac{1}{2}}\nu^{\frac{2}{3}}\})} \ge c_2\nu^{-\frac{1}{3}(1-\frac{2}{p})} \ \to\ \infty\ (\nu \to 0) \text{ if } 2 < p \le \infty. \tag{4.7}$$

Here c_0 *is the constant in Theorem* 4.1*, and* $c_2 > 0$ *is independent of* ν *and* $t \in [0, c_0\nu^{1/3}]$*.*

Remark 4.3. In Theorem 4.1 the assumption on b in $L^{4/3}(\mathbb{R}^2_+)$ is just for technical reasons, and we can also handle with b in $W^{l,p}(\mathbb{R}^2_+)$ if $p \in [1,2)$ and l is large enough.

The proof of Theorem 4.1 requires lengthy calculations, and we divide it into several steps. Let $J(f)$ be the velocity field recovered from f via the Biot–Savart law; i.e.,

$$J(f) = (J_1(f), J_2(f)) = \nabla^\perp(-\Delta_D)^{-1}f,\ \nabla^\perp = (\partial_2, -\partial_1). \tag{4.8}$$

Note that $J(f)$ satisfies $\nabla \cdot J(f) = 0$ in $\mathbb{R}^2_+$ and $J_2(f) = 0$ on $\partial\mathbb{R}^2_+$. The function ω_E satisfies the equation

$$\begin{cases} \partial_t \omega_E + u_E \cdot \nabla \omega_E = 0, & t > 0,\ x \in \mathbb{R}^2_+, \\ u_E = J(\omega_E), & t > 0,\ x \in \mathbb{R}^2_+, \\ \omega_E|_{t=0} = b, & x \in \mathbb{R}^2_+. \end{cases} \tag{4.9}$$

We note that (4.9) is equivalent to the Euler equations with the boundary condition $u_{E,2} = 0$ on $\partial\mathbb{R}^2_+$. Hence, under the assumption $b \in W^{l,4/3}(\mathbb{R}^2_+)$ with $l \gg 1$ the existence and the uniqueness of solutions to (4.9) follow from the methods developed in the literature [3, 15, 38, 41]. In particular, we can show that $\omega_E \in C^1([0,T); W^{l',4/3}(\mathbb{R}^2_+))$ with $l' \gg 1$ for all $T > 0$, and this fact will be freely used in the rest of the paper. Next we consider the second and third expansions of ω which are directly related with ω_E:

$$\begin{cases} \partial_t w_{E,1} - \nu \Delta w_{E,1} = 0, & t > 0,\ x \in \mathbb{R}^2_+, \\ \nu(\partial_2 w_{E,1} + (-\partial_1^2)^{\frac{1}{2}} w_{E,1}) = -J_1(u_E \cdot \nabla \omega_E), & t > 0,\ x \in \partial\mathbb{R}^2_+, \\ w_{E,1}|_{t=0} = 0, & x \in \mathbb{R}^2_+, \end{cases} \tag{4.10}$$

$$\begin{cases} \partial_t w_{E,2} - \nu \Delta w_{E,2} = \nu \Delta \omega_E, & t > 0,\ x \in \mathbb{R}^2_+, \\ \nu(\partial_2 w_{E,2} + (-\partial_1^2)^{\frac{1}{2}} w_{E,2}) = -\nu J_1(\Delta \omega_E), & t > 0,\ x \in \partial\mathbb{R}^2_+, \\ w_{E,2}|_{t=0} = 0, & x \in \mathbb{R}^2_+. \end{cases} \tag{4.11}$$

The function $w_{E,1}$ is responsible for the creation of vorticity near the boundary. Set

$$w_E = w_{E,1} + w_{E,2},\ F = J(\omega_E + w_E) \cdot \nabla w_E + J(w_E) \cdot \nabla \omega_E. \tag{4.12}$$

Then $W = \omega - \omega_E - w_E$ satisfies $W|_{t=0} = 0$ and

$$\begin{cases} \partial_t W - \nu \Delta W = -L(\omega_E + w_E) W - N(W,W) - F,\ t > 0,\ x \in \mathbb{R}^2_+, \\ \nu(\partial_2 W + (-\partial_1^2)^{\frac{1}{2}} W) = -J_1(L(\omega_E + w_E) W + N(W,W) + F), \\ \qquad t > 0,\ x \in \partial\mathbb{R}^2_+. \end{cases} \tag{4.13}$$

Here

$$L(f)W = J(f) \cdot \nabla W + J(W) \cdot \nabla f,\ N(f,g) = J(f) \cdot \nabla g. \tag{4.14}$$

By the above definitions we can check that each of $J(\omega_E + w_{E,1})$, $J(w_{E,2})$, and $J(W)$, satisfies the no-slip boundary condition (see the proof of Theorem 2.3), and this property will be essentially used in the proof of Theorem 4.1. We note that the above decomposition of ω should be effective only near the initial time $0 < t \le \nu^\beta$ for some $\beta > 0$. For a longer time period we need to take into account the vorticity counterpart of the Prandtl equations, where the verification of such an expansion is widely open except for the analytic initial data. Finally we set for $\delta > 0$,

$$\Omega_\delta = \{x \in \mathbb{R}^2_+ : 0 \le x_2 \le \delta^{\frac{1}{2}}\},\ \Omega^c_\delta = \mathbb{R}^2_+ \backslash \Omega_\delta = \{x \in \mathbb{R}^2_+ : x_2 \ge \delta^{\frac{1}{2}}\}. \quad (4.15)$$

In the sequel we will focus on the a priori estimates of ω (especially, of W) based on the above decompositions. The basic strategy is as follows: we will use the integral equations (3.6) or (3.9) for the estimates of $w_{E,1}$ and $w_{E,2}$, and also of W near the boundary. The estimates of W away from the boundary will be obtained by the energy argument. Theorem 4.1 then follows from these a priori estimates.

4.1. **Preliminary estimates.** In this section we prepare several linear estimates which will be used to study w_E and W.

4.1.1. *Estimate for layer potential.* We first establish the estimate for layer potentials. Set

$$\begin{aligned} G_i(t,x) &= \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp(-\frac{|x_i|^2}{4t}), && (4.16)\\ K_{0,\nu}(g)(t,x) &= 2\int_0^t G_2(\nu(t-s),x)\,ds\ g(0,x_1), && (4.17)\\ K_{1,\nu}(g)(t) &= 2\int_0^t G(\nu(t-s)) \star g(s)\mathcal{H}^1_{\{x_2=0\}}\,ds. && (4.18)\end{aligned}$$

Lemma 4.4. *Let $1 \le p \le \infty$, $k \in \mathbb{N}\cup\{0\}$, and $l = 0,1,2$. Then we have*

$$\|\partial_1^k K_{0,\nu}(g)(t)\|_{L^p(\Omega_{\nu t})} \ge c\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \|\partial_1^k g(0)\|_{L^p_{x_1}}, \quad (4.19)$$

$$\begin{aligned}\|\partial_1^k\partial_2^l K_{1,\nu}(g)(t)\|_{L^p} &\le C\nu^{-\frac{1}{2}(1-\frac{1}{p}+l)} t^{\frac{1}{2}(1+\frac{1}{p}-l)} \quad (4.20)\\ &\times \big(\|\partial_1^k g\|_{L^\infty_t L^p_{x_1}} + lt\|\partial_t\partial_1^k g\|_{L^\infty_t L^p_{x_1}} + l\nu t\|\partial_1^{2+k} g\|_{L^\infty_t L^p_{x_1}}\big),\end{aligned}$$

$$\|\partial_1^k K_{1,\nu}(g)(t)\|_{L^1_{x_2}L^\infty_{x_1}} \le Ct\|\partial_1^k g\|_{L^\infty_t L^\infty_{x_1}}, \quad (4.21)$$

$$\begin{aligned}\|\partial_1^k\big(K_{1,\nu} - K_{0,\nu}\big)(g)(t)\|_{L^p} &\le C\nu^{-\frac{1}{2}(1-\frac{1}{p})} t^{1+\frac{1}{2p}} \quad (4.22)\\ &\big(t^{\frac{1}{2}}\|\partial_t\partial_1^k g\|_{L^\infty_t L^p_{x_1}} + \nu^{\frac{1}{2}}\|\partial_1^{1+k} g\|_{L^\infty_t L^p_{x_1}}\big).\end{aligned}$$

Proof. We may assume that $k = 0$. Since

$$|K_0(g)(t,x)| = 2|g(0,x_1)| \int_0^t G_2(\nu(t-s),x)\,ds$$

$$\geq 2|g(0,x_1)|\int_0^{\frac{t}{2}} G_2(\nu(t-s),x)\,ds \geq \frac{t^{\frac{1}{2}}|g(0,x_1)|}{2(\pi\nu)^{\frac{1}{2}}}e^{-\frac{x_2^2}{2\nu t}},$$

we have

$$\|K_0(g)(t)\|_{L^p(\Omega_{\nu t})} \geq c\nu^{-1/2+1/(2p)}t^{1/2+1/(2p)}\|\partial_1^k g(0)\|_{L^p_{x_1}}.$$

This proves (4.19). When $l=0$, (4.20) is a direct consequence of the Young inequality. When $l=1$ we rewrite $\partial_2 K_{1,\nu}(g)$ as

$$\partial_2 K_{1,\nu}(g)(t,x) = 2\int_0^t \int_{\mathbb{R}} \partial_2 G((\nu(t-s),x_1-y_1,x_2)g(s,y_1)\,dy_1\,ds \tag{4.23}$$

$$= -\nu^{-1}g(t,x_1) + 2\nu^{-1}\int_{\mathbb{R}}\int_0^{x_2} G(\nu t,x_1-y_1,y_2)\,dy_2\,g(t,y_1)\,dy_1$$
$$+ 2\int_0^t\int_{\mathbb{R}}\int_0^{x_2} G((\nu(t-s),x_1-y_1,y_2)\,dy_2\left(\nu^{-1}\partial_s g(s,y_1)-\partial_1^2 g(s,y_1)\right)dy_1\,ds. \tag{4.24}$$

Thus, the Young inequality implies

$$\|\partial_2 K_{1,\nu}(g)(t)\|_{L^\infty_{x_2}L^p_{x_1}} \leq C\nu^{-1}\big(\|g\|_{L^\infty_t L^p_{x_1}} + t\|\partial_t g\|_{L^\infty_t L^p_{x_1}} + \nu t\|\partial_1^2 g\|_{L^\infty_t L^p_{x_1}}\big),$$

which leads to

$$\|\partial_2 K_{1,\nu}(g)(t)\|_{L^p(\{x_2\leq R\})}$$
$$\leq CR^{\frac{1}{p}}\nu^{-1}\big(\|g\|_{L^\infty_t L^p_{x_1}} + t\|\partial_t g\|_{L^\infty_t L^p_{x_1}} + \nu t\|\partial_1^2 g\|_{L^\infty_t L^p_{x_1}}\big).$$

On the other hand, we have from (4.23) and $|\partial_2 G(t,x)| \leq Cx_2^{-1}|G(2t,x)|$,

$$\|\partial_2 K_{1,\nu}(g)(t)\|_{L^p(\{x_2\geq R\})} \leq CR^{-1}\nu^{-\frac{1}{2}(1-\frac{1}{p})}t^{\frac{1}{2}(1+\frac{1}{p})}\|g\|_{L^\infty_t L^p_{x_1}}.$$

Taking $R=(\nu t)^{1/2}$ we get (4.20) with $l=1$. The case $l=2$ for (4.20) is obtained by the equality

$$\partial_2^2 K_{1,\nu}(g)(t,x) = 2\nu^{-1}\int_{\mathbb{R}} G(\nu t,x_1-y_1,x_2)g(t,y_1)\,dy_1 \tag{4.25}$$
$$+ 2\int_0^t\int_{\mathbb{R}} G((\nu(t-s),x_1-y_1,x_2)\big(\nu^{-1}\partial_s g(s,y_1)-\partial_1^2 g(s,y_1)\big)\,dy_1\,ds,$$

which is derived from (4.24). The details are omitted here. Estimate (4.21) is easily checked from the definition of $K_{1,\nu}(g)$. To prove (4.22) we observe that

$$K_{1,\nu}(g)(t,x) - K_{0,\nu}(g)(t,x)$$

$$
\begin{aligned}
&= 2\int_0^t \int_{\mathbb{R}} G(\nu(t-s), x_1-y_1, x_2)\big(g(s,y_1)-g(0,y_1)\big)\,dy_1\,ds \\
&+ 2\int_0^t \int_{\mathbb{R}} G(\nu(t-s), x_1-y_1, x_2)\big(g(0,y_1)-g(0,x_1)\big)\,dy_1\,ds \\
&= I_1(t,x)+I_2(t,x).
\end{aligned}
$$

Then it is easy to see $\|I_1(t)\|_{L^p} \le C\nu^{-(1-1/p)/2}t^{(3+1/p)/2}\|\partial_t g\|_{L^\infty_t L^p_{x_1}}$. As for I_2, we have

$$
\begin{aligned}
\|I_2(t)\|_{L^p} &\le C\int_0^t \|x_1 G_1(\nu(t-s))\|_{L^\infty_{x_2}L^1_{x_1}} \|G_2(\nu(t-s))\|_{L^p_{x_2}L^\infty_{x_1}}\,ds\,\|\partial_1 g\|_{L^\infty_t L^p_{x_1}} \\
&\le C\nu^{\frac{1}{2p}} t^{1+\frac{1}{2p}} \|\partial_1 g\|_{L^\infty_t L^p_{x_1}}.
\end{aligned}
$$

This completes the proof.

4.1.2. *Estimate for* $\big(\Gamma(t)-\Gamma(0)\big)\star$.

Proposition 4.5. *Let* $1\le p\le\infty$, $k\in\mathbb{N}\cup\{0\}$, $l=0,1$, *and* $m>0$. *Then*

$$
\begin{aligned}
&\|\partial_1^k\partial_2^l\big(\Gamma(t)-\Gamma(0)\big)\star f\|_{L^p} \\
&\le Ct^{\frac12}\big(\|\partial_1^{1+k+l} f\|_{L^p} + \|\partial_1^{k+l}(-\partial_1^2)^{\frac12} f\|_{L^p}\big) + Cl\|\partial_1^k(-\partial_1^2)^{\frac12} f\|_{L^p}, \qquad (4.26)\\
&\|\partial_1^k\big(\Gamma(t)-\Gamma(0)\big)\star(g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \\
&\qquad + t^{-\frac{m-l}{2}}\|x_2^m\partial_1^k\partial_2^l\big(\Gamma(t)-\Gamma(0)\big)\star(g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \\
&\le Ct^{\frac{1}{2p}}\Big(\|\partial_1^k(-\partial_1^2)^{\frac12} g\|_{L^p_{x_1}} + t^{\frac12}\|\partial_1^{2+k} g\|_{L^p_{x_1}} + t\|\partial_1^{2+k}(-\partial_1^2)^{\frac12} g\|_{L^p_{x_1}}\Big). \qquad (4.27)
\end{aligned}
$$

Proof. We may assume that $k=0$. In view of (5.5) and (5.6) we have

$$
(\Gamma(t)-\Gamma(0))\star f = -2\int_0^t \partial_1 G(\tau)\star\partial_1 f\,d\tau - 2\int_0^t \partial_2 G(\tau)\star(-\partial_1^2)^{\frac12} f\,d\tau. \qquad (4.28)
$$

Hence, (4.26) with $l=0$ follows from the Young inequality. When $l=1$ by the equality

$$
\partial_2^2 G(\tau) = \partial_\tau G(\tau) - \partial_1^2 G(\tau)
$$

we observe that in $\mathbb{R}^2_+$,

$$
\begin{aligned}
&\partial_2(\Gamma(t)-\Gamma(0))\star f \\
&= -2\int_0^t \partial_2 G(\tau)\star\partial_1^2 f\,d\tau - 2G(t)\star(-\partial_1^2)^{\frac12} f - 2\int_0^t \partial_1 G(\tau)\star\partial_1(-\partial_1^2)^{\frac12} f\,d\tau.
\end{aligned}
$$

Hence it is easy to get (4.26) also for $l = 1$ by the Young inequality. As for (4.27), we have the equality (4.28) with f replaced by $g\mathcal{H}^1_{\{x_2=0\}}$, and thus,

$$(\Gamma(t) - \Gamma(0)) \star (g\mathcal{H}^1_{\{x_2=0\}}) = -K_{1,1}(\partial_1^2 g)(t) - \partial_2 K_{1,1}((-\partial_1^2)^{1/2} g)(t).$$

Here $K_{1,1}$ is defined by (4.18), and we have used the fact that g is time-independent in this case. Then by using (4.20) and by noting that $\partial_t g = 0$, we conclude that $\|\big(\Gamma(t) - \Gamma(0)\big) \star (g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p}$ is bounded by the right-hand side of (4.27). As for $\|x_2^m \partial_2^l \big(\Gamma(t) - \Gamma(0)\big) \star (g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p}$, from the inequality

$$|x_2^m \partial_2^\gamma G(\tau, x)| \le C\tau^{(m-\gamma)/2} G(2\tau, x)$$

it is not difficult to see that $t^{-(m-l)/2} \|x_2^m \partial_2^l \big(\Gamma(t) - \Gamma(0)\big) \star (g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p}$ is estimated just the same as $\|\big(\Gamma(t) - \Gamma(0)\big) \star (g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p}$. We omit the details here. This completes the proof.

4.1.3. *Estimate for velocity field.*

Proposition 4.6. *Let $J(f)$ be the vector field defined by* (4.8). *Then it follows that*

$$\|J(f)\|_{L^p} \le C\|f\|_{L^{\frac{4}{3}}}^{\frac{1}{2}+\frac{2}{p}} \|f\|_{L^4}^{\frac{1}{2}-\frac{2}{p}}, \ 4 \le p \le \infty, \tag{4.29}$$

$$\|J(f)\|_{L^\infty(\{x_2 \ge 1\})} \le C\|f\|_{L^1} + C_m \|x_2^m f\|_{L^{\frac{4}{3}}}^{\frac{1}{2}} \|x_2^m f\|_{L^4}^{\frac{1}{2}}, \ m \ge 0, \tag{4.30}$$

$$\|J_1(f)\|_{L^\infty_{x_2} L^p_{x_1}} \le C\|f\|_{L^1_{x_2} L^p_{x_1}}, \ 1 \le p \le \infty, \tag{4.31}$$

$$\|x_2^{-1} J_2(f)\|_{L^\infty_{x_2} L^p_{x_1}} \le C\|\partial_1 f\|_{L^1_{x_2} L^p_{x_1}}, \ 1 \le p \le \infty, \tag{4.32}$$

$$\|\nabla J(f)\|_{L^p} + \|J(\nabla f)\|_{L^p} \le C\|f\|_{L^p}, \ 1 < p < \infty. \tag{4.33}$$

Proof. We first note that (4.33) follows from the Calderón–Zygmund inequality. As for (4.29), the case $p = 4$ is derived from the Hardy–Littlewood–Sobolev inequality, and the case $p = \infty$ follows from the Gagliardo–Nirenberg inequality

$$\|J(f)\|_{L^\infty(\mathbb{R}^2)} \le C\|\nabla J(f)\|_{L^4}^{1/2} \|J(f)\|_{L^4}^{1/2}$$

and by applying (4.33) and (4.29) with $p = 4$. The case $4 < p < \infty$ is then obtained by interpolation. Estimate (4.30) is derived from the inequality

$$|J(f)(x)| \le C\int_{y_2 \le \frac{1}{2}} |f(y)|\, dy + C_m \int_{y_2 \ge \frac{1}{2}} \frac{1}{|x-y|} y_2^m |f(y)|\, dy \text{ for } x_2 \ge 1,$$

and then by applying the same argument as in the proof of (4.29) with $p = \infty$ to the second term of the right-hand side of the above inequality. To prove (4.31) we note that

$$|J_1(f)(x)| \le C \int_{\mathbb{R}^2_+} \Big(\frac{|x_2 - y_2|}{|x-y|^2} + \frac{|x_2+y_2|}{|x-y^*|^2}\Big)|f(y)|\, dy;$$

then the desired estimate follows from the Young inequality:

$$\begin{aligned}\|J_1(f)(\cdot, x_2)\|_{L^p_{x_1}} &\le C\int_{\mathbb{R}^2_+} \Big(\frac{|x_2-y_2|}{y_1^2 + (x_2-y_2)^2} + \frac{|x_2+y_2|}{y_1^2+(x_2+y_2)^2}\Big)\|f(\cdot, y_2)\|_{L^p_{x_1}} dy \\ &\le C\|f\|_{L^1_{x_2}L^p_{x_2}}.\end{aligned}$$

Finally, we have

$$J_2(f)(x) = \int_0^{x_2} \partial_2 J_2(f)(x_1, y_2)\, dy_2 = -\int_0^{x_2} J_1(\partial_1 f)(x_1, y_2)\, dy_2.$$

Thus, (4.32) holds by (4.31). This completes the proof.

4.2. **Estimate for $w_{E,1}$.** Set $g = -J_1(u_E \cdot \nabla\omega_E)\,|_{x_2=0}$. Then Theorem 3.1 implies

$$\begin{aligned} w_{E,1}(t) &= -\int_0^t e^{\nu(t-s)\Delta_N}(g(s)\mathcal{H}^1_{\{x_2=0\}})\, ds \\ &\quad - \int_0^t \big(\Gamma(\nu(t-s)) - \Gamma(0)\big) \star (g(s)\mathcal{H}^1_{\{x_2=0\}})\, ds =: w_{E,1,1}(t) + w_{E,1,2}(t). \end{aligned} \tag{4.34}$$

We also recall that ω_{BL} is defined by (4.3).

Proposition 4.7. *Let $1 \le p \le \infty$, $0 \le k \le 4$, $l = 0, 1$, and $m > 0$. Let $0 < t \le 1$. Then we have*

$$\|\partial_1^k \omega_{BL}(t)\|_{L^p} \ge c_1 \nu^{-\frac12(1-\frac1p)} t^{\frac12(1+\frac1p)} \|\partial_1^k g(0)\|_{L^p_{x_1}}, \tag{4.35}$$

$$\|\partial_1^k \omega_{BL}(t)\|_{L^p} \le C \nu^{-\frac12(1-\frac1p)} t^{\frac12(1+\frac1p)}, \tag{4.36}$$

$$\|\partial_1^k w_{E,1,1}(t) - \partial_1^k \omega_{BL}(t)\|_{L^p} \le C\nu^{-\frac12(1-\frac1p)} t^{1+\frac{1}{2p}} (t^{\frac12} + \nu^{\frac12}), \tag{4.37}$$

$$\|\partial_1^k w_{E,1,1}(t)\|_{L^p} + (\nu t)^{-\frac{m-l}{2}} \|x_2^m \partial_1^k \partial_2^l w_{E,1,1}(t)\|_{L^p} \le C\nu^{-\frac12(1-\frac1p)} t^{\frac12(1+\frac1p)}, \tag{4.38}$$

$$\|\partial_1^k w_{E,1,2}(t)\|_{L^p} + (\nu t)^{-\frac{m-l}{2}} \|x_2^m \partial_1^k \partial_2^l w_{E,1,2}(t)\|_{L^p} \le C\nu^{\frac{1}{2p}} t^{1+\frac{1}{2p}}, \tag{4.39}$$

$$\|\partial_1^k w_{E,1}(t)\|_{L^1_{x_2}L^\infty_{x_1}} \le Ct. \tag{4.40}$$

Proof. Since $w_{E,1,1}(t) = -K_{1,\nu}(g)(t)$ and $\omega_{BL}(t) = -K_{0,\nu}(g)(t)$ by the definitions, (4.35)–(4.37) and (4.39) follow from Lemma 4.4 and Proposition 4.5. Estimate (4.38) is obtained by using the representation of the kernel of $e^{t\Delta_N}$ and the Young inequality; see Lemma 4.4 and the proof of (4.27). The details are omitted here since their calculations are straightforward. As for (4.40), we already have $\|\partial_1^k w_{E,1,1}(t)\|_{L^1_{x_2}L^\infty_{x_1}} \le Ct$ by Lemma 4.4, so it suffices to consider $w_{E,1,2}$. By the Sobolev inequality and (4.39) we have

$$\|w_{E,1,2}(t)\|_{L^1_{x_2}L^\infty_{x_1}} \le C\|\partial_1 w_{E,1,2}(t)\|_{L^1} \le C\nu^{1/2}t^{3/2}.$$

We note that (4.39) holds also for $k = 5$ if b is smooth enough; hence, (4.40) holds for $k = 4$ by the same argument. This completes the proof.

4.3. **Estimate for $w_{E,2}$.** By Theorem 3.1 the function $w_{E,2}$ is written as

$$\begin{aligned} w_{E,2}(t) &= \nu \int_0^t \big(e^{\nu(t-s)B} - \Gamma(0)\star\big)\Delta\omega_E\, ds \qquad (4.41)\\ &\quad - \nu\int_0^t \big(e^{\nu(t-s)B} - \Gamma(0)\star\big)\big(J_1(\Delta\omega_E)\mathcal{H}^1_{\{x_2=0\}}\big)ds =: w_{E,2,1}(t) + w_{E,2,2}(t). \end{aligned}$$

Proposition 4.8. *Let $4/3 \le p \le \infty$, $0 \le k \le 4$, $l = 0, 1$, and $m > 0$. Then we have*

$$\|\partial_1^k\partial_2^l w_{E,2,1}(t)\|_{L^p} \le C\nu t, \qquad (4.42)$$

$$\|\partial_1^k w_{E,2,2}(t)\|_{L^p} + (\nu t)^{-\frac{m-l}{2}}\|x_2^m\partial_1^k\partial_2^l w_{E,2,2}(t)\|_{L^p} \le C(\nu t)^{\frac{1}{2}(1+\frac{1}{p})}, \qquad (4.43)$$

$$\|\partial_1^k w_{E,2,2}(t)\|_{L^1_{x_2}L^\infty_{x_1}} \le C\nu t. \qquad (4.44)$$

Proof. The proof of (4.43) and (4.44) is the same as in (4.38)–(4.40). Indeed, in this case it suffices to take g as $-\nu J_1(\Delta\omega_E)\,|_{x_2=0}$. So we omit the details. To estimate $w_{E,2,1}$ we decompose it as

$$\begin{aligned} w_{E,2,1}(t) &= \nu\int_0^t e^{\nu(t-s)\Delta_N}\Delta\omega_E\, ds + \nu\int_0^t \big(\Gamma(\nu(t-s)) - \Gamma(0)\big)\star\Delta\omega_E\, ds\\ &=: w_{E,2,1,1}(t) + w_{E,2,1,2}(t). \end{aligned}$$

From $\partial_2 e^{t\Delta_N} f = e^{t\Delta_D}\partial_2 f$ and the Young inequality we have

$$\|\partial_1^k\partial_2^l w_{E,2,1,1}(t)\|_{L^p} \le C\nu t.$$

By using (4.26) the function $w_{E,2,1,2}$ is estimated as

$$\|\partial_1^k\partial_2^l w_{E,2,1,2}(t)\|_{L^p} \le C(\nu t)^{(3-l)/2}.$$

This completes the proof.

4.4. **Estimate for F.** Let F be the function defined by (4.12), which is decomposed as $F = \sum_{i=1}^{3} F_i$ with

$$F_1 = J(\omega_E + w_E) \cdot \nabla w_{E,1},\ F_2 = J(\omega_E + w_E) \cdot \nabla w_{E,2},\ F_3 = J(w_E) \cdot \nabla \omega_E. \tag{4.45}$$

Proposition 4.9. *Let $1 \le p \le \infty$, $4/3 \le q \le \infty$, $k = 0, 1$, and $m \ge 0$. Let $0 < t \le 1$. Then*

$$\|x_2^m \partial_1^k F_1(t)\|_{L^p} \le C\nu^{-1}(\nu^{-\frac12} t^{\frac12} + 1)(\nu t)^{\frac12(2+\frac1p+m)}, \tag{4.46}$$

$$\|\partial_1^k F_2(t)\|_{L^p} \le C\nu t + C\nu^{\frac12(1+\frac1p)} t^{\frac12(3+\frac1p)}, \tag{4.47}$$

$$\|\partial_1^k F_3(t)\|_{L^p} \le C\nu^{\frac{1}{8p}} t^{1+\frac{1}{8p}} + C(\nu t)^{\frac34}, \tag{4.48}$$

$$\|\partial_1^k F_3(t)\|_{L^q} \le C\nu^{\frac{1}{2q}} t^{1+\frac{1}{2q}} + C(\nu t)^{\frac34}. \tag{4.49}$$

In particular, we have

$$\|\partial_1^k F(t)\|_{L^p} \le C\nu^{-2}(\nu t)^{\frac12(3+\frac1p)} + C\nu^{-1}(\nu t)^{1+\frac{1}{8p}} + C(\nu t)^{\frac34},\ 1 \le p < \tfrac43, \tag{4.50}$$

$$\|\partial_1^k F(t)\|_{L^p} \le C\nu^{-2}(\nu t)^{\frac12(3+\frac1p)} + C\nu^{-1}(\nu t)^{1+\frac{1}{2p}} + C(\nu t)^{\frac34},\ \tfrac43 \le p \le \infty. \tag{4.51}$$

Remark 4.10. Although it is possible to derive slightly better estimates for F_2 and F_3, (4.47) and (4.48) are enough for the proof of Theorem 4.1. If $b \in W^{1,1}(\mathbb{R}^2_+)$ in addition, then (4.49) and (4.51) hold also for $1 \le q\ (p) \le 4/3$.

Proof. It suffices to consider the case $k = 0$. Since $J(\omega_E + w_E)$ satisfies the no-slip boundary condition, we have

$$\begin{aligned} J_1(\omega_E + w_E)(x) = &- \int_0^{x_2} (\omega_E + w_E)(x_1, y_2)\, dy_2 \\ &- \int_0^{x_2}\int_0^{y_2} J_1(\partial_1^2(\omega_E + w_E))(x_1, z_2)\, dz_2\, dy_2, \end{aligned}$$

and

$$\begin{aligned} J_2(\omega_E + w_E)(x) = &\int_0^{x_2}\int_0^{y_2} \partial_1(\omega_E + w_E)(x_1, z_2)\, dz_2\, dy_2 \\ &- \int_0^{x_2}\int_0^{y_2} J_2(\partial_1^2(\omega_E + w_E))(x_1, z_2)\, dz_2\, dy_2. \end{aligned}$$

The estimates (4.29)–(4.32) with $m \gg 1$ and Propositions 4.7–4.8 imply

$$\|J(\partial_1^2(\omega_E + w_E))\|_{L^\infty} \le C \quad \text{for } 0 < t \le 1.$$

Thus, we have from (4.40), (4.42), and (4.44),

$$|J_1(\omega_E+w_E)(x)| \le C(x_2+x_2^2+t),\ |J_2(\omega_E+w_E)(x)| \le Cx_2(x_2+t). \quad (4.52)$$

This yields

$$|F_1(t,x)| \le C(x_2+x_2^2+t)|\partial_1 w_{E,1}(t,x)| + Cx_2(x_2+t)|\partial_2 w_{E,1}(t,x)|,$$

and hence, (4.46) holds by (4.38) and (4.39). Next we consider (4.47). We decompose $w_{E,2}$ as in (4.41), and then (4.42) yields

$$\|J(\omega_E+w_E)\cdot\nabla w_{E,2,1}\|_{L^p} \le C\|J(\omega_E+w_E)\|_{L^\infty}\|\nabla w_{E,2,1}\|_{L^p} \le C\nu t.$$

On the other hand, the term $J(\omega_E+w_E)\cdot\nabla w_{E,2,2}$ is estimated as in the proof of (4.46) by using (4.43), and we get

$$\|J(\omega_E+w_E)\cdot\nabla w_{E,2,2}\|_{L^p} \le C\big(t+(\nu t)^{\frac12}\big)(\nu t)^{\frac12(1+\frac1p)}.$$

This shows (4.47). As for (4.48), we write $F_3 = F_{3,1}+F_{3,2}$ with $F_{3,i} = J(w_{E,i})\cdot\nabla\omega_E$. In order to estimate $F_{3,1}$ we observe that if $x_2 \ge (\nu t)^{1/4}$, then

$$\begin{aligned} |J(w_{E,1})(t,x)| &\le C\int_{\mathbb{R}^2_+} \frac{y_2}{|x-y||x-y^*|}|w_{E,1}(t,y)|\,dy \\ &\le C(\nu t)^{-\frac14}\int_{\mathbb{R}^2_+}\frac{y_2}{|x-y|}|w_{E,1}(t,y)|\,dy. \qquad (4.53)\end{aligned}$$

Here, in the first inequality of the above estimate we have used

$$|x_1-y_1|\Big|\frac{1}{|x-y|^2}-\frac{1}{|x-y^*|^2}\Big| + \Big|\frac{x_2-y_2}{|x-y|^2}-\frac{x_2+y_2}{|x-y^*|^2}\Big| \le \frac{Cy_2}{|x-y||x-y^*|}.$$

Hence by the Hölder inequality and the estimates for the Riesz potential we have

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^1(\{x_2\ge(\nu t)^{\frac14}\})} \le C(\nu t)^{-\frac14}\|x_2 w_{E,1}\|_{L^{\frac43}} \le C\nu^{\frac18}t^{\frac98},$$

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^\infty(\{x_2\ge(\nu t)^{\frac14}\})} \le C(\nu t)^{-\frac14}\|x_2 w_{E,1}\|^{\frac12}_{L^{\frac43}}\|x_2 w_{E,1}\|^{\frac12}_{L^4} \le Ct,$$

where we have used (4.38)–(4.39). Then by the interpolation we have

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^p(\{x_2\ge(\nu t)^{1/4}\})} \le C\nu^{1/(8p)}t^{1+1/(8p)}.$$

On the other hand, we have from Proposition 4.6,

$$\begin{aligned}&\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^p(\{x_2\le(\nu t)^{\frac14}\})} \\ &\le C(\nu t)^{\frac{1}{4p}}\|J_1(w_{E,1})\partial_1\omega_E\|_{L^\infty_{x_2}L^p_{x_1}} + C(\nu t)^{1+\frac{1}{4p}}\Big\|\frac{J_2(w_{E,1})}{x_2}\partial_2\omega_E\Big\|_{L^\infty_{x_2}L^p_{x_1}}\end{aligned}$$

$$\le C(\nu t)^{\frac{1}{4p}}\|w_{E,1}\|_{L^1_{x_2}L^p_{x_1}} + C(\nu t)^{1+\frac{1}{4p}}\|\partial_1 w_{E,1}\|_{L^1_{x_2}L^p_{x_1}} \le C\nu^{\frac{1}{4p}}t^{1+\frac{1}{4p}}.$$

In the last line we have used

$$\|f\|_{L^1_{x_2}L^p_{x_1}} \le C\|f\|_{L^1}^{1/p}\|\partial_1 f\|_{L^1}^{1-1/p}$$

and the estimates (4.38)–(4.39). Next we have from (4.42) and (4.43) that

$$\|J(w_{E,2})\cdot\nabla\omega_E\|_{L^1} \le C\|w_{E,2}\|_{L^{4/3}} \le C(\nu t)^{7/8},$$

$$\|J(w_{E,2})\cdot\nabla\omega_E\|_{L^\infty} \le C\|w_{E,2}\|_{L^{4/3}}^{1/2}\|w_{E,2}\|_{L^4}^{1/2} \le C(\nu t)^{3/4}.$$

This yields

$$\|J(w_{E,2})\cdot\nabla\omega_E\|_{L^p} \le C(\nu t)^{3/4},$$

and (4.48) is proved. For (4.49) it suffices to consider the case $q < \infty$. Instead of (4.53), we use

$$|J(w_{E,1})(t,x)| \le C\int_{\mathbb{R}^2_+}\frac{y_2}{|x-y||x-y^*|}|w_{E,1}(t,y)|\,dy$$

$$\le C\int_0^\infty \frac{y_2\|w_{E,1}(t,\cdot,y_2)\|_{L^\infty_{x_1}}}{|x_2-y_2|^{\frac{1}{2}}|x_2+y_2|^{\frac{1}{2}}}\,dy_2.$$

Then, if $x_2 \ge (\nu t)^{1/2}$ and $0<\alpha<1$ we have

$$\|J(w_{E,1})(t,\cdot,x_2)\|_{L^\infty_{x_1}} \le C(\nu t)^{-\frac{\alpha}{4}}\int_0^\infty \frac{1}{|x_2-y_2|^{1-\frac{\alpha}{2}}}\|y_2 w_{E,1}(t,\cdot,y_2)\|_{L^\infty_{x_1}}\,dy_2.$$

Set

$$h(t,x_2) = \|J(w_{E,1})(t,\cdot,x_2)\|_{L^\infty_{x_1}}\|\nabla\omega_E(t)\|_{L^\infty_{x_2}L^q_{x_1}}.$$

Then the Hardy–Littlewood–Sobolev inequality leads to

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^q(\{x_2\ge(\nu t)^{1/2}\})} \le C\|h\|_{L^q_{x_2}(\{x_2\ge(\nu t)^{\frac{1}{2}}\})}$$

$$\le C(\nu t)^{-\frac{\alpha}{4}}\|y_2 w_{E,1}(t)\|_{L^r_{x_2}L^\infty_{x_1}}$$

$$\le C(\nu t)^{-\frac{\alpha}{4}}\|y_2\partial_1 w_{E,1}(t)\|_{L^1}^{\frac{1}{r}}\|y_2 w_{E,1}(t)\|_{L^\infty}^{1-\frac{1}{r}}$$

where $1/q = 1/r - \alpha/2$. Hence (4.38)–(4.39) imply that

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^q(\{x_2\ge(\nu t)^{1/2}\})} \le C\nu^{-1}(\nu t)^{1+1/(2q)}.$$

The same argument as in the proof of (4.48) yields

$$\|J(w_{E,1})\cdot\nabla\omega_E\|_{L^q(\{x_2\le(\nu t)^{1/2}\})} \le C\nu^{-1}(\nu t)^{1+1/(2q)},$$

and thus (4.49) holds if $4/3 \le q < \infty$. This completes the proof.

4.5. **Estimate for W.** By (4.13) and Corollary 3.3 the function W is expressed as

$$W(t) = -\sum_{i=0}^{3}\int_0^t e^{\nu(t-s)B}\big(f_i(s) - g_i(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds =: -\sum_{i=0}^{3} W_i(t). \quad (4.54)$$

Here $g_i(t) = J_1(f_i(t))(x_1,0)$ and

$$f_0 = F,\ f_1 = J(\omega_E + w_E)\cdot\nabla W,\ f_2 = J(W)\cdot\nabla(\omega_E + w_E),\ f_3 = N(W,W). \quad (4.55)$$

In this section we will prove

Proposition 4.11. *Assume that $\nu > 0$ is sufficiently small. Then there are $c_0, C > 0$ such that*

$$\|W(t)\|_{L^p} \le C\nu^{\frac{2}{3p}} t \text{ for } 0 < t \le c_0\nu^{\frac13},\ \frac43 \le p \le 4, \quad (4.56)$$

$$\|W(t)\|_{L^p} \le C\nu^{-\frac16+\frac{4}{3p}} t \text{ for } 0 < t \le c_0\nu^{\frac13},\ 4 < p \le \infty. \quad (4.57)$$

Here c_0 is independent of ν, and C is independent of ν and $t \in [0, c_0\nu^{1/3}]$.

4.5.1. *Estimate away from the boundary.* In the region away from the boundary the energy argument is useful to estimate W. The divergence-free condition of u plays an essential role. We recall that Ω_δ and Ω_δ^c are defined by (4.15).

Proposition 4.12. *Let $4 \le p < \infty$. Assume that $\delta = \nu^{2/3}$ and $0 < t \le \nu^{1/3}$. Then*

$$\begin{aligned}\|W(t)\|_{L^{\frac43}(\Omega_\delta^c)} &\le C\int_0^t \|W\|_{L^{\frac43}}\,ds + C\nu^{-\frac13}\int_0^t \|W\|_{L^{\frac43}}^{\frac32}\|W\|_{L^4}^{\frac12}\,ds \\ &\quad + C\nu^{\frac34}t^{\frac74} + C\nu^{\frac38}t^{\frac{19}{8}}, \end{aligned} \quad (4.58)$$

$$\begin{aligned}\|W(t)\|_{L^p(\Omega_\delta^c)} &\le Cp\nu^{-\frac13}\int_0^t \|W\|_{L^{\frac43}}^{\frac12+\frac2p}\|W\|_{L^4}^{\frac12-\frac2p}\,ds + Cp^2\nu^{\frac13}\int_0^t \|W\|_{L^p}\,ds \\ &\quad + C\nu^{-\frac13}\int_0^t \|W\|_{L^p}\|W\|_{L^{\frac43}}^{\frac12}\|W\|_{L^4}^{\frac12}\,ds + Cp\nu^{\frac34}t^{\frac74} + Cp\nu^{\frac{1}{2p}}t^{2+\frac{1}{2p}}. \end{aligned} \quad (4.59)$$

Here the constant C is taken independently also of p.

Proof. Let $\epsilon > 0$, and let $\chi_\epsilon(x_2)$ be a cutoff function such that $\chi_\epsilon = 1$ if $x_2 \ge 2\epsilon$ and $\chi = 0$ if $x_2 \le \epsilon$, and $|\partial_2^j\chi_\epsilon| \le C\epsilon^{-j}$. Note that we can take χ_ϵ of the form $(\tilde\chi_\epsilon)^2$ for a suitable $\tilde\chi_\epsilon$. Set $w_\epsilon = W\chi_\epsilon$. Then w_ϵ satisfies

$$\partial_t w_\epsilon - \nu\Delta w_\epsilon + u\cdot\nabla w_\epsilon = -\chi_\epsilon J(W)\cdot\nabla(\omega_E + w_E) - F\chi_\epsilon$$

$$+ W u_2 \partial_2 \chi_\epsilon - \nu(\partial_2\chi_\epsilon \partial_2 W + W\partial_2^2 \chi_\epsilon), \qquad (4.60)$$

which is now considered as the equation in $\mathbb{R}^2$. For $\eta > 0$ we set $\Psi_\eta(z) = (z^2+\eta^2)^{\frac12} - \eta$, which satisfies

$$0 \le \Psi_\eta(z) \le |z|,\ |\Psi_\eta'(z)| \le 1,\ \Psi_\eta''(z) > 0,\ |z|^q \Psi_\eta''(z) \le \eta^{q-1} \text{ for } q \ge 1. \qquad (4.61)$$

Let $1 < p < \infty$. Then by integration by parts we have

$$\begin{aligned}
\frac{d}{dt}\|\Psi_\eta(w_\epsilon)\|_{L^p}^p &= -\nu p \int \Psi_\eta''(w_\epsilon)|\nabla w_\epsilon|^2 \Psi_\eta^{p-1}(w_\epsilon) \\
&- \nu p(p-1)\int |\Psi_\eta'(w_\epsilon)|^2 |\nabla w_\epsilon|^2 \Psi_\eta^{p-2}(w_\epsilon) \\
&- p\int \Psi_\eta'(w_\epsilon)\Psi_\eta^{p-1}(w_\epsilon)\chi_\epsilon J(W)\cdot\nabla(\omega_E + w_E) - p\int \Psi_\eta'(w_\epsilon)\Psi_\eta^{p-1}(w_\epsilon) F\chi_\epsilon \\
&+ 2p\nu \int W\partial_2\chi_\epsilon \partial_2 w_\epsilon \big\{\Psi_\eta''(w_\epsilon)\Psi_\eta^{p-1}(w_\epsilon) + (p-1)|\Psi_\eta'(w_\epsilon)|^2 \Psi_\eta^{p-2}(w_\epsilon)\big\} \\
&+ p\nu\int W\partial_2^2\chi_\epsilon \Psi_\eta'(w_\epsilon)\Psi_\eta^{p-1}(w_\epsilon) + p\int \Psi_\eta'(w_\epsilon)\Psi_\eta^{p-1}(w_\epsilon) W u_2 \partial_2\chi_\epsilon \\
&\le -\frac{\nu p}{2}\int \Psi_\eta''(w_\epsilon)|\nabla w_\epsilon|^2\Psi_\eta^{p-1}(w_\epsilon) - \frac{\nu p(p-1)}{2}\int |\Psi_\eta'(w_\epsilon)|^2|\nabla w_\epsilon|^2 \Psi_\eta^{p-2}(w_\epsilon) \\
&+ p\int |w_\epsilon|^{p-1}\chi_\epsilon |J(W)\cdot\nabla(\omega_E + w_E)| + p\int |w_\epsilon|^{p-1}|F|\chi_\epsilon \\
&+ Cp\nu\int W^2|\partial_2\chi_\epsilon|^2 \Psi_\eta''(w_\epsilon)|w_\epsilon|^{p-1} + Cp^2\nu\epsilon^{-2}\int |W||w_\epsilon|^{p-1} \\
&+ Cp\epsilon^{-1}\int_{\epsilon\le x_2\le 2\epsilon} |W||w_\epsilon|^{p-1}|u_2|.
\end{aligned}$$

From (4.61) and $\chi_\epsilon = (\tilde\chi_\epsilon)^2$ it is easy to check that

$$\int W^2|\partial_2\chi_\epsilon|^2\Psi_\eta''(w_\epsilon)|w_\epsilon|^{p-1} \to 0 \quad \text{as } \eta \to 0$$

by the Lebesgue convergence theorem. Hence by integrating over 0 to t and letting $\eta \to 0$ we get

$$\begin{aligned}
\|w_\epsilon(t)\|_{L^p}^p &\le p\int_0^t \|w_\epsilon\|_{L^p}^{\frac{p}{p'}} \big(\|\chi_\epsilon J(W)\cdot\nabla(\omega_E + w_E)\|_{L^p} + \|\chi_\epsilon F\|_{L^p}\big)\, ds \\
&+ Cp^2\nu\epsilon^{-2}\int_0^t \|w_\epsilon\|_{L^p}^{\frac{p}{p'}}\|W\|_{L^p}\, ds + Cp\epsilon^{-1}\int_0^t \|w_\epsilon\|_{L^p}^{\frac{p}{p'}} \|Wu_2\|_{L^p(\{\epsilon\le x_2\le 2\epsilon\})}\, ds.
\end{aligned}$$

Let $\epsilon = p\delta^{1/2} \geq \delta^{1/2}$. Then this inequality implies

$$\sup_{0\leq s\leq t} \|W(s)\|_{L^p(\Omega_\delta^c)} \leq p\int_0^t \big(\|J(W)\cdot\nabla(\omega_E+w_E)\|_{L^p(\Omega_\delta^c)} + \|F\|_{L^p(\Omega_\delta^c)}\big)\,ds + C\int_0^t \big(\nu\delta^{-1}\|W\|_{L^p} + \delta^{-\frac12}\|Wu_2\|_{L^p(\{x_2\leq 2p\delta^{\frac12}\})}\big)\,ds.$$

From (4.29) and (4.52) we have

$$\begin{aligned}\|Wu_2\|_{L^p(\{x_2\leq 2p\delta^{\frac12}\})} &\leq C\|W\|_{L^p}\big(\|J_2(W)\|_{L^\infty} + \|J_2(\omega_E+w_E)\|_{L^\infty(\{x_2\leq 2p\delta^{\frac12}\})}\big)\\ &\leq C\|W\|_{L^p}\big(\|W\|_{L^{\frac43}}^{\frac12}\|W\|_{L^4}^{\frac12} + Cp\delta^{\frac12}(p\delta^{\frac12}+s)\big).\end{aligned}$$

Let us consider $\|J(W)\cdot\nabla(\omega_E+w_E)\|_{L^p(\Omega_\delta^c)}$. For $p = 4/3$ we have from Propositions 4.6, 4.7, and 4.8,

$$\begin{aligned}&\|J(W)\cdot\nabla(\omega_E+w_E)\|_{L^{\frac43}(\Omega_\delta^c)}\\ &\quad\leq \|J(W)\|_{L^4}\big(\|\nabla\omega_E\|_{L^2} + \|\nabla w_{E,2,1}\|_{L^2} + \delta^{-1}\|x_2^2\nabla(w_{E,1}+w_{E,2,2})\|_{L^2}\big)\\ &\quad\leq C\|W\|_{L^{\frac43}}\big(1+\nu s+\delta^{-1}\nu^{\frac14}s^{\frac54}\big).\end{aligned}$$

Similarly, for $p\in[4,\infty)$ we have from (4.29) and Propositions 4.7 and 4.8,

$$\begin{aligned}&\|J(W)\cdot\nabla(\omega_E+w_E)\|_{L^p(\Omega_\delta^c)}\\ &\quad\leq \|J(W)\|_{L^p}\big(\|\nabla\omega_E\|_{L^\infty} + \|\nabla w_{E,2,1}\|_{L^\infty} + \delta^{-1}\|x_2^2\nabla(w_{E,1}+w_{E,2,2})\|_{L^\infty}\big)\\ &\quad\leq C\|W\|_{L^{\frac43}}^{\frac12+\frac2p}\|W\|_{L^4}^{\frac12-\frac2p}\big(1+\nu s+\delta^{-1}s\big).\end{aligned}$$

As for F, we have from Proposition 4.9,

$$\begin{aligned}\|F\|_{L^p(\Omega_\delta^c)} &\leq \|F_1\|_{L^p(\Omega_\delta^c)} + \|F_2\|_{L^p} + \|F_3\|_{L^p}\\ &\leq \delta^{-\frac m2}\|x_2^mF_1\|_{L^p} + C(\nu t)^{\frac34} + C\nu^{\frac{1}{2p}}t^{1+\frac{1}{2p}}\\ &\leq C\delta^{-\frac m2}\nu^{-2}(\nu t)^{\frac12(3+m+\frac1p)} + C(\nu t)^{\frac34} + C\nu^{\frac{1}{2p}}t^{1+\frac{1}{2p}}\end{aligned}$$

for $4/3\leq p\leq\infty$. Let us take $m>0$ large enough. Then, collecting these above, we have for $0<t\leq 1$,

$$\|W(t)\|_{L^{\frac43}(\Omega_\delta^c)} \leq C\big(1+\nu^{-\frac{5}{12}}t^{\frac54}\big)\int_0^t \|W\|_{L^{\frac43}}\,ds + C\nu^{-\frac13}\int_0^t \|W\|_{L^{\frac43}}^{\frac32}\|W\|_{L^4}^{\frac12}\,ds + C\nu^{\frac34}t^{\frac74} + C\nu^{\frac38}t^{\frac{19}{8}},$$

and for $p \in [4, \infty)$,

$$
\begin{aligned}
&\|W(t)\|_{L^p(\Omega_\delta^c)} \\
&\le Cp\big(1+\nu^{-\frac{2}{3}}t\big)\int_0^t \|W\|_{L^{\frac{4}{3}}}^{\frac{1}{2}+\frac{2}{p}}\|W\|_{L^4}^{\frac{1}{2}-\frac{2}{p}}\,ds + C\big(p^2\nu^{\frac{1}{3}}+pt\big)\int_0^t \|W\|_{L^p}\,ds \\
&\quad + C\nu^{-\frac{1}{3}}\int_0^t \|W\|_{L^p}\|W\|_{L^{\frac{4}{3}}}^{\frac{1}{2}}\|W\|_{L^4}^{\frac{1}{2}}\,ds + Cp\nu^{\frac{3}{4}}t^{\frac{7}{4}} + Cp\nu^{\frac{1}{2p}}t^{2+\frac{1}{2p}}.
\end{aligned}
$$

Hence Proposition 4.12 follows from $0 < t \le \nu^{1/3}$. This completes the proof.

4.5.2. *Estimate for W_0.*

Proposition 4.13. *Let $1 < p < \infty$ and $0 < \kappa < 1$. Assume that $0 < t \le \nu^{1/3}$. Then*

$$
\|W_0(t)\|_{L^p} \le C\nu^{-2+\frac{2}{3p}}(\nu t)^{\frac{7}{4}}, \tag{4.62}
$$

$$
\|W_0(t)\|_{L^\infty} \le C_\kappa \nu^{-2-\kappa}(\nu t)^{\frac{7}{4}}. \tag{4.63}
$$

Proof. If $1 < p < \infty$, then we have from (3.12) and (3.13),

$$
\|W_0(t)\|_{L^p} \le C\int_0^t \|F(s)\|_{L^p}\,ds + C\int_0^t \big(\nu(t-s)\big)^{-\frac{1}{2}(1-\frac{1}{p})}\|g_0(s)\|_{L^p_{x_1}}\,ds. \tag{4.64}
$$

By (4.31) we have

$$
\|g_0(s)\|_{L^p_{x_1}} \le C\|F(s)\|_{L^1_{x_2}L^p_{x_1}} \le C\|F(s)\|_{L^1}^{1/p}\|\partial_1 F(s)\|_{L^1}^{1-1/p}.
$$

Thus, (4.50) yields

$$
\begin{aligned}
\|W_0(t)\|_{L^p} &\le C\nu^{-3}(\nu t)^{\frac{1}{2}(5+\frac{1}{p})} + C\nu^{-2}(\nu t)^{2+\frac{1}{8p}} + C\nu^{-1}(\nu t)^{\frac{7}{4}} \\
&\quad + C\nu^{-1}(\nu t)^{\frac{1}{2}(1+\frac{1}{p})}\big(\nu^{-3}(\nu t)^3 + \nu^{-2}(\nu t)^{\frac{17}{8}} + \nu^{-1}(\nu t)^{\frac{7}{4}}\big) \\
&\le C\nu^{-2+\frac{2}{3p}}(\nu t)^{\frac{7}{4}}
\end{aligned}
$$

if $0 < t \le \nu^{1/3}$. This gives (4.62). If $p = \infty$, then we have, instead of (4.64),

$$
\|W_0(t)\|_{L^\infty} \le C_q\int_0^t \big((\nu(t-s)\big)^{-\frac{1}{q}}\|F(s)\|_{L^q}\,ds + C\int_0^t \big(\nu(t-s)\big)^{-\frac{1}{2}}\|g_0(s)\|_{L^\infty_{x_1}}\,ds
$$

for all $2 < q < \infty$. This implies

$$
\|W_0(t)\|_{L^\infty} \le C\nu^{-2-2/(3q)}(\nu t)^{7/4}
$$

if $0 < t \le \nu^{1/3}$, which completes the proof.

4.5.3. *Estimate for* W_1.

Proposition 4.14. *Let* $4 \le p \le \infty$. *Assume that* $0 < t \le \nu^{1/3}$. *Then*

$$\|W_1(t)\|_{L^{\frac{4}{3}}} \le C\nu^{-\frac{1}{6}} \int_0^t (t-s)^{-\frac{1}{2}} \|W\|_{L^{\frac{4}{3}}}\, ds + C\nu^{-\frac{1}{3}} \int_0^t \|W\|_{L^{\frac{4}{3}}}\, ds$$
$$+ C\nu^{-\frac{2}{3}} \int_0^t \|W\|_{L^{\frac{4}{3}}}^{\frac{3}{2}} \|W\|_{L^4}^{\frac{1}{2}}\, ds + C\nu^{\frac{1}{4}} t^{\frac{9}{4}} + C\nu^{-\frac{1}{8}} t^{\frac{23}{8}}, \tag{4.65}$$

$$\|W_1(t)\|_{L^p} \le C\nu^{-\frac{5}{12}+\frac{1}{p}} \int_0^t (t-s)^{-\frac{3}{4}+\frac{1}{p}} \|W\|_{L^4}\, ds \tag{4.66}$$
$$+ C\nu^{-1+\frac{4}{3p}} \int_0^t \|W\|_{L^{\frac{4}{3}}}\, ds + C\nu^{-\frac{1}{3}+\frac{4}{3p}} \int_0^t \|W\|_{L^4}\, ds$$
$$+ C\nu^{-1+\frac{4}{3p}} \int_0^t \|W\|_{L^{\frac{4}{3}}}^{\frac{1}{2}} \|W\|_{L^4}^{\frac{3}{2}} ds + C\nu^{-2}(\nu t)^{2+\frac{1}{p}} + C\nu^{-3}(\nu t)^{\frac{19}{8}+\frac{1}{p}}.$$

Proof. We give the proof only for (4.66) since (4.65) is obtained in the same manner. From (3.15) and (4.33) we have

$$\|W_1(t)\|_{L^p} \le C \int_0^t \big(\nu(t-s)\big)^{-\frac{3}{4}+\frac{1}{p}} \|\nabla^\perp(-\Delta_D)^{-1}\big(J(\omega_E + w_E)\cdot \nabla W\big)\|_{L^4}\, ds$$
$$\le C \int_0^t \big(\nu(t-s)\big)^{-\frac{3}{4}+\frac{1}{p}} \|J(\omega_E + w_E) W\|_{L^4}\, ds.$$

Set $\delta = \nu^{2/3}$. Then by (4.52) we have for $0 < s \le t \le \nu^{1/3}$,

$$\|J(\omega_E + w_E)W\|_{L^4(\Omega_\delta)} \le C(\delta^{1/2} + s)\|W\|_{L^4} \le C\nu^{1/3}\|W\|_{L^4}.$$

On the other hand, from (4.29) for $J(\omega_E)$, (4.31) for $J_1(w_E)$, and (4.30)–(4.32) with $m \gg 1$ for $J_2(w_E)$, we have

$$\|J(\omega_E + w_E)W\|_{L^4(\Omega_\delta^c)}$$
$$\le C(\|J(\omega_E)\|_{L^\infty} + \|J_1(w_E)\|_{L^\infty} + \|J_2(w_E)\|_{L^\infty})\|W\|_{L^4(\Omega_\delta^c)} \le C\|W\|_{L^4(\Omega_\delta^c)}.$$

Here we have also used Propositions 4.7, 4.8, and $0 < t \le 1$. Then (4.66) follows from (4.59). This completes the proof.

4.5.4. *Estimate for* W_2. Set $f_{2,1}(t) = J(W)\cdot\nabla\omega_E$, $f_{2,2}(t) = J(W)\cdot \nabla w_E$, $g_{2,j}(t, x_1) = J_1(f_{2,j}(t))(x_1, 0)$, and

$$W_{2,j}(t) = \int_0^t e^{\nu(t-s)B}\big(f_{2,j}(s) - g_{2,j}(s)\mathcal{H}^1_{\{x_2=0\}}\big)\, ds. \tag{4.67}$$

Proposition 4.15. *Let* $4 \le p \le \infty$. *Assume that* $0 < t \le \nu^{1/3}$. *Then*

$$\|W_{2,1}(t)\|_{L^{\frac43}} \le C\nu^{-\frac18}\int_0^t (t-s)^{-\frac18}\|W\|_{L^{\frac43}}\, ds, \tag{4.68}$$

$$\|W_{2,1}(t)\|_{L^p} \le C\nu^{-\frac12(\frac54-\frac2p)}\int_0^t (t-s)^{-\frac12(\frac54-\frac2p)}\|W\|_{L^{\frac43}}\, ds, \tag{4.69}$$

$$\|W_{2,2}(t)\|_{L^{\frac43}} \le C\nu^{-\frac16}\int_0^t (t-s)^{-\frac12}\|W\|_{L^{\frac43}}\, ds, \tag{4.70}$$

$$\|W_{2,2}(t)\|_{L^p} \le C\nu^{-\frac{5}{12}+\frac1p}\int_0^t (t-s)^{-\frac34+\frac1p}\big(\|W\|_{L^4} + \nu^{\frac13}\|W\|_{L^{\frac43}}\big)\, ds. \tag{4.71}$$

Proof. We give the proof only for (4.69) and (4.71). The other estimates are proved by similar arguments. By the definition of $W_{2,1}$, (3.12) and (3.13) yield

$$\|W_{2,1}(t)\|_{L^p} \le C\int_0^t \Big(\big(\nu(t-s)\big)^{-\frac14+\frac1p}\|f_{2,1}\|_{L^4} + \big(\nu(t-s)\big)^{-\frac12(\frac54-\frac2p)}\|g_{2,1}\|_{L^4_{x_1}}\Big)\, ds.$$

Then (4.69) follows from

$$\|f_{2,1}(s)\|_{L^4} \le C\|J(W)\|_{L^4} \le C\|W\|_{L^{4/3}}$$

and

$$\|g_{2,1}(s)\|_{L^4_{x_1}} \le C\|J(W)\cdot\nabla\omega_E\|_{L^1_{x_2}L^4_{x_1}} \le C\|J(W)\|_{L^4}\|\omega_E\|_{L^{4/3}_{x_2}L^\infty_{x_1}} \le C\|W\|_{L^{4/3}}.$$

Here we have used Proposition 4.6. Next we consider $W_{2,2}$. By (3.15) we have

$$\begin{aligned}\|W_{2,2}(t)\|_{L^p} &\le C\int_0^t \big(\nu(t-s)\big)^{-\frac34+\frac1p}\|\nabla^\perp(-\Delta_D)^{-1}\big(J(W)\cdot\nabla w_E\big)\|_{L^4}\, ds\\ &\le C\int_0^t \big(\nu(t-s)\big)^{-\frac34+\frac1p}\big(\|J(W)w_{E,1}\|_{L^4} + \|J(W)w_{E,2}\|_{L^4}\big)\, ds.\end{aligned}$$

Since $J(W)$ satisfies the no-slip boundary condition we have from (4.38)–(4.39),

$$\begin{aligned}\|J(W)w_{E,1}\|_{L^4} &= \Big\|\int_0^{x_2}\partial_2 J(W)\, dy_2\, w_{E,1}\Big\|_{L^4}\\ &\le C\|\partial_2 J(W)\|_{L^4}\|x_2^{\frac34}w_{E,1}\|_{L^4_{x_2}L^\infty_{x_1}} \le Cs\|W\|_{L^4}.\end{aligned}$$

By using Proposition 4.8 the term $J(W)w_{E,2}$ is estimated as

$$\|J(W)w_{E,2}\|_{L^4} \le \|J(W)\|_{L^4}\|w_{E,2}\|_{L^\infty} \le C(\nu t)^{\frac12}\|W\|_{L^{\frac43}}.$$

This shows (4.71) since $0 < t \le \nu^{1/3}$. The proof of Proposition 4.15 is completed.

4.5.5. *Estimate for* W_3.

Proposition 4.16. *Let* $4 \le p \le \infty$. *Then*

$$\|W_3(t)\|_{L^{\frac{4}{3}}} \le C\int_0^t (\nu(t-s))^{-\frac{1}{2}} \|W(s)\|_{L^{\frac{4}{3}}}^{\frac{3}{2}} \|W(s)\|_{L^4}^{\frac{1}{2}}\, ds, \tag{4.72}$$

$$\|W_3(t)\|_{L^p} \le C\int_0^t (\nu(t-s))^{-\frac{3}{4}+\frac{1}{p}} \|W(s)\|_{L^{\frac{4}{3}}}^{\frac{1}{2}} \|W(s)\|_{L^4}^{\frac{3}{2}}\, ds. \tag{4.73}$$

Proof. By (3.15) we have

$$\|W_3(t)\|_{L^{\frac{4}{3}}} \le C\int_0^t \big(\nu(t-s)\big)^{-\frac{1}{2}} \|\nabla^\perp(-\Delta_D)^{-1}\big(J(W)\cdot\nabla W\big)\|_{L^{\frac{4}{3}}}\, ds$$

$$\|W_3(t)\|_{L^p} \le C\int_0^t \big(\nu(t-s)\big)^{-\frac{3}{4}+\frac{1}{p}} \|\nabla^\perp(-\Delta_D)^{-1}\big(J(W)\cdot\nabla W\big)\|_{L^4}\, ds.$$

Then (4.72) and (4.73) are obtained by

$$\begin{aligned}\|\nabla^\perp(-\Delta_D)^{-1}\big(J(W)\cdot\nabla W\big)\|_{L^{\frac{4}{3}}} &\le C\|J(W)W\|_{L^{\frac{4}{3}}}\\ &\le C\|J(W)\|_{L^\infty}\|W\|_{L^{\frac{4}{3}}} \le C\|W\|_{L^{\frac{4}{3}}}^{\frac{3}{2}}\|W\|_{L^4}^{\frac{1}{2}}\end{aligned}$$

and

$$\|\nabla^\perp(-\Delta_D)^{-1}\big(J(W)\cdot\nabla W\big)\|_{L^4} \le C\|W\|_{L^{4/3}}^{1/2}\|W\|_{L^4}^{3/2}.$$

Here we have used Proposition 4.6. This completes the proof.

4.5.6. *Proof of Proposition* 4.11. Let $0 < t \le c_0\nu^{1/3}$, where $0 < c_0 < 1$ will be taken small enough. Set $\|W\|_{Z_p} = \sup_{0<t\le c_0\nu^{1/3}} \|W(t)\|_{L^p}$. Collecting the estimates in Propositions 4.13–4.16, we get for $0 < t \le c_0\nu^{1/3}$,

$$\|W(t)\|_{L^{\frac{4}{3}}} \le \sum_{i=0}^{3} \|W_i(t)\|_{L^{\frac{4}{3}}} \le Cc_0^{\frac{1}{2}}\|W\|_{Z_{\frac{4}{3}}} + C\nu^{-\frac{1}{3}}\|W\|_{Z_{\frac{4}{3}}}^{\frac{3}{2}}\|W\|_{Z_4}^{\frac{1}{2}} + C\nu^{\frac{5}{6}},$$

and

$$\begin{aligned}\|W(t)\|_{L^4} &\le \sum_{i=0}^{3}\|W_i(t)\|_{L^4}\\ &\le Cc_0^{\frac{1}{2}}\|W\|_{Z_4} + Cc_0\nu^{-\frac{1}{3}}\|W\|_{Z_{\frac{4}{3}}} + C\nu^{-\frac{1}{3}}\|W\|_{Z_{\frac{4}{3}}}^{\frac{1}{2}}\|W\|_{Z_4}^{\frac{3}{2}} + C\nu^{\frac{1}{2}}.\end{aligned}$$

Then it is easy to see that $\|W\|_{Z_{4/3}} \le C\nu^{\frac{5}{6}}$ and $\|W\|_{Z_4} \le C\nu^{\frac{1}{2}}$ for some $C > 0$ if c_0 and ν are sufficiently small. Note that c_0 and C are taken independent of ν if $0 < \nu \ll 1$. Then Propositions 4.13–4.16, $\|W\|_{Z_{4/3}} \le C\nu^{\frac{5}{6}}$, and $\|W\|_{Z_4} \le C\nu^{\frac{1}{2}}$ yield

$$\|W_i(t)\|_{L^{\frac{4}{3}}} \le C\nu^{\frac{2}{3}} t^{\frac{1}{2}}, \ \|W_i(t)\|_{L^4} \le C\nu^{\frac{1}{3}} t^{\frac{1}{2}}, \ i = 0, 1, 2, 3,$$

which implies

$$\|W(t)\|_{L^{4/3}} \le C\nu^{3/2} t^{1/2} \quad \text{and} \quad \|W(t)\|_{L^4} \le C\nu^{1/3} t^{1/2}.$$

Repeating this argument again, we get

$$\|W_i(t)\|_{L^{\frac{4}{3}}} \le C\nu^{\frac{1}{2}} t, \ \|W_i(t)\|_{L^4} \le C\nu^{\frac{1}{6}} t, \ i = 0, 1, 2, 3.$$

Thus, we get

$$\|W(t)\|_{L^{4/3}} \le C\nu^{1/2} t \quad \text{and} \quad \|W(t)\|_{L^4} \le C\nu^{1/6} t.$$

By interpolation we have $\|W(t)\|_{L^p} \le C\nu^{2/(3p)} t$ if $4/3 \le p \le 4$. Then Propositions 4.13–4.16 yield $\|W(t)\|_{L^p} \le C\nu^{-1/6+4/(3p)} t$ for all $4 < p \le \infty$. This completes the proof of Proposition 4.11.

4.6. **Proof of Theorem 4.1.** By Propositions 4.6, 4.7, 4.8, and 4.11, one can check that $u = J(\omega) = J(\omega_E) + J(w_{E,1}) + J(w_{E,2}) + J(W)$ is uniformly bounded in $(L^\infty(\mathbb{R}^2))^2$ with respect to $0 < \nu \ll 1$ and $0 < t \le c_0\nu^{1/3}$. Let ω_{BL} be the function defined by (4.3). Then from (4.37), (4.42), (4.43), and (4.57), we have for $0 < t \le c_0\nu^{1/3}$,

$$\begin{aligned}
&\|\omega(t) - \omega_E(t) - \omega_{BL}(t)\|_{L^p} \le \|w_{E,1,1}(t) - \omega_{BL}(t)\|_{L^p} \\
&\qquad + \|w_{E,1,2}(t)\|_{L^p} + \|w_{E,2,1}(t)\|_{L^p} + \|w_{E,2,2}(t)\|_{L^p} + \|W(t)\|_{L^p} \\
&\le C\nu^{-\frac{1}{6}+\frac{1}{2p}} t^{\frac{1}{2}(1+\frac{1}{p})} + C\nu^{\frac{2}{3p}} t + C\nu t + C(\nu t)^{\frac{1}{2}(1+\frac{1}{p})} + C\nu^{-\frac{1}{6}+\frac{4}{3p}} t \\
&\le C\nu^{-\frac{1}{6}+\frac{1}{2p}} t^{\frac{1}{2}(1+\frac{1}{p})}.
\end{aligned}$$

This proves (4.2). The other statements in the theorem follow from the definition of ω_{BL} and Proposition 4.7. This completes the proof of Theorem 4.1.

5. Appendix

5.1. **Proof of solution formula.** For simplicity we consider the case $b = 0$ in (LV)–(LBC). It is easy to recover the case $b \ne 0$ from this case. Let $\tilde{\omega}$ be the Fourier–Laplace transform of ω defined by

$$\tilde{\omega}(s,\xi_1,x_2)=\frac{1}{(2\pi)^{\frac12}}\int_0^\infty\int_{\mathbb{R}}\omega(t,x_1,x_2)e^{-ix_1\xi_1-ts}\,dx_1\,dt. \tag{5.1}$$

Then (LV)–(LBC) is converted to

$$\partial_2^2\tilde{\omega}-(\frac{s}{\nu}+\xi_1^2)\tilde{\omega} = -\frac{\tilde{f}}{\nu}, \qquad x_2>0, \tag{5.2}$$

$$\partial_2\tilde{\omega}+|\xi_1|\tilde{\omega} = \frac{\tilde{g}}{\nu}, \quad x_2=0. \tag{5.3}$$

Set $\alpha = s/\nu+\xi_1^2$. Solving this ODE under a decay condition at spatial infinity, we get

$$\begin{aligned}\tilde{\omega}(s,\xi_1,x_2)&=\big(\frac{1}{s}(\frac{\xi_1^2}{\sqrt{\alpha}}+|\xi_1|)+\frac{1}{2\nu\sqrt{\alpha}}\big)\int_0^\infty e^{-\sqrt{\alpha}(x_2+y_2)}\tilde{f}(s,\xi_1,y_2)\,dy_2\\&+\frac{1}{2\nu\sqrt{\alpha}}\Big(\int_0^{x_2}e^{-\sqrt{\alpha}(x_2-y_2)}\tilde{f}(s,\xi_1,y_2)\,dy_2+\int_{x_2}^\infty e^{-\sqrt{\alpha}(y_2-x_2)}\tilde{f}(s,\xi_1,y_2)\,dy_2\Big)\\&-\big(\frac{1}{s}(\frac{\xi_1^2}{\sqrt{\alpha}}+|\xi_1|)+\frac{1}{\nu\sqrt{\alpha}}\big)e^{-\sqrt{\alpha}x_2}\tilde{g}(s,\xi_1).\end{aligned} \tag{5.4}$$

Inverting the Fourier–Laplace transform, we have

$$\begin{aligned}\omega(t,x_1,x_2)&=\int_0^t\int_{\mathbb{R}^2_+}\big(G(\nu(t-s),x-y)+G(\nu(t-s),x-y^*)\big)f(s,y)\,dy\,ds\\&-2\int_0^t\int_{\mathbb{R}}G(\nu(t-s),x_1-y_1,x_2)g(s,y_1)\,dy_1\,ds\\&-2\nu\int_0^t\int_0^s\int_{\mathbb{R}^2_+}\big(\partial_1^2+(-\partial_1^2)^{\frac12}\partial_2\big)G(\nu(s-\tau),x-y^*)f(\tau,y)\,dy\,d\tau\,ds\\&+2\nu\int_0^t\int_0^s\int_{\mathbb{R}}\big(\partial_1^2+(-\partial_1^2)^{\frac12}\partial_2\big)G(\nu(s-\tau),x_1-y_1,x_2)g(\tau,y_1)\,dy_1\,d\tau\,ds\\&=\int_0^t e^{\nu(t-s)\Delta_N}\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds\\&\quad-\nu\int_0^t\int_0^s\big(\Xi G(\nu(s-\tau))\big)\star\big(f(\tau)-g(\tau)\mathcal{H}^1_{\{x_2=0\}}\big)\,d\tau\,ds.\end{aligned} \tag{5.5}$$

From the equality $G(t) = -\partial_t(-\Delta_{\mathbb{R}^2})^{-1}G(t)$, the second term of the right-hand side of (5.5) is written in $\mathbb{R}^2_+$ as

$$\begin{aligned}
&-\nu\int_0^t\int_0^s \big(\Xi G(\nu(s-\tau))\big)\star\big(f(\tau)-g(\tau)\mathcal{H}^1_{\{x_2=0\}}\big)\,d\tau\,ds\\
&=\int_0^t\int_0^s \partial_s\big(\Xi(-\Delta_{\mathbb{R}^2})^{-1}G(\nu(s-\tau))\big)\star\big(f(\tau)-g(\tau)\mathcal{H}^1_{\{x_2=0\}}\big)\,d\tau\,ds\\
&=\int_0^t \Gamma(\nu(t-s))\star\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds\\
&\quad-\int_0^t \Gamma(0)\star\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,ds. \qquad (5.6)
\end{aligned}$$

This completes the proof.

5.2. Pointwise estimate of $\Gamma(t,x)$.

Proposition 5.1. *Let $k,l\in\mathbb{N}\cup\{0\}$. Then there is C such that*

$$|\partial_1^k\partial_2^l\Gamma(1,x)|\le C\Big(1+\frac{|x_1|^{2+k}}{\{\log(e+x_1^2)\}^{\delta_{0l}}}+|x_2|^{2+k+l}\Big)^{-1}. \qquad (5.7)$$

Here δ_{0l} is Kronecker's delta. In particular, it follows that

$$|\partial_1^k\partial_2^l\Gamma(t,x)|\le Ct^{-\frac{k+l+2}{2}}\Big(1+\frac{|x_1/\sqrt{t}|^{2+k}}{\{\log(e+|x_1/\sqrt{t}|^2)\}^{\delta_{0l}}}+|x_2/\sqrt{t}|^{2+k+l}\Big)^{-1}. \qquad (5.8)$$

Proof. Let $0<R<1$, and let χ_R be a cutoff function on $\mathbb{R}$ such that $\chi_R(r)=1$ if $|r|\le R$ and $\chi_R(r)=0$ if $|r|\ge 2R$, and $|\partial_r^k\chi_R(r)|\le CR^{-k}$. Set $\chi_R^c=1-\chi_R$. With the definition of $p(\xi)$ in (3.16) we observe that

$$\begin{aligned}
\partial_1^k\partial_2^l\Gamma(1,x) &= \frac{i^{k+l}}{2\pi}\int_{\mathbb{R}^2}\xi_1^k\xi_2^l p(\xi)e^{-|\xi|^2+ix\cdot\xi}\,d\xi\\
&= \frac{i^{k+l}}{2\pi}\int_{\mathbb{R}^2}\big(\chi_R(\xi_1)\chi_R(\xi_2)+\chi_R(\xi_1)\chi_R^c(\xi_2)+\chi_R^c(\xi_1)\chi_R(\xi_2)+\chi_R^c(\xi_1)\chi_R^c(\xi_2)\big)\\
&\quad\times\xi_1^k\xi_2^l p(\xi)e^{-|\xi|^2+ix\cdot\xi}\,d\xi=\sum_{j=1}^4 I_j(x). \qquad (5.9)
\end{aligned}$$

Then we have $|I_1(x)|\le CR^{k+l+2}$ and

$$|I_2(x)|\le CR^k\int_{|\xi_1|\le 2R,\ |\xi_2|\ge R}\frac{|\xi_1||\xi_2|^l(|\xi_1|+|\xi_2|)}{|\xi|^2}e^{-|\xi|^2}\,d\xi$$

$$\le CR^{k+2}\int_R^\infty |\xi_2|^{l-1}e^{-\xi_2^2}\,d\xi_2 \le C(1+|\log R|^{\delta_{0l}})R^{k+2}.$$

For I_3 we use the equality

$$x_1^m I_3(x) = (-1)^m i^{k+l-m}\int_{\mathbb{R}^2}\chi_R(\xi_2)\xi_2^l e^{ix\cdot\xi}\partial_1^m\big(\chi_R^c(\xi_1)\xi_1^k p(\xi)e^{-|\xi|^2}\big)\,d\xi.$$

If $m \ge k+2$ we have

$$|\partial_1^m\big(\chi_R^c(\xi_1)\xi_1^k p(\xi)e^{-|\xi|^2}\big)| \le C\big(1+\frac{|\xi_1|^{k-m+1}}{|\xi|}\big)e^{-\frac{\xi_1^2}{2}-\xi_2^2}\chi^c_{\frac{R}{2}}(\xi_1). \tag{5.10}$$

Hence $|x_1^m I_3(x)| \le CR^{l+1}(1+R^{k-m+1}) \le CR^{k+l+2-m}$. The term I_4 is estimated similarly. Indeed, we have from (5.10),

$$|x_1^m I_4(x)| \le C\int_{|\xi_1|\ge R,\ |\xi_2|\ge R}|\xi_2|^l\big(1+\frac{|\xi_1|^{k-m+1}}{|\xi|}\big)e^{-\frac{\xi_1^2}{2}-\xi_2^2}\,d\xi$$
$$\le C\int_R^\infty|\xi_2|^{l-1}e^{-\xi_2^2}\,d\xi_2\int_R^\infty|\xi_1|^{k-m+1}e^{-\frac{\xi_1^2}{2}}\,d\xi_1 \le C(1+|\log R|^{\delta_{0l}})R^{k-m+2}.$$

Then by taking $R=|x_1|^{-1}$ for $|x_1|>2$ we get

$$|\partial_1^k\partial_2^l\Gamma(1,x)| \le C|x_1|^{-k-2}\{\log(e+|x_1|)\}^{\delta_{0l}},$$

which implies

$$|\partial_1^k\partial_2^l\Gamma(1,x)| \le C\big(1+\frac{|x_1|^{k+2}}{\{\log(e+|x_1|)\}^{\delta_{0l}}}\big)^{-1} \tag{5.11}$$

for all $x\in\mathbb{R}^2$. To show the spatial decay in the x_2 direction, instead of (5.9) we write

$$\partial_1^k\partial_2^l\Gamma(1,x) = \frac{i^{k+l}}{2\pi}\int_{\mathbb{R}^2}\big(\chi_R(|\xi|)+\chi_R^c(|\xi|)\big)\xi_1^k\xi_2^l p(\xi)e^{-|\xi|^2+ix\cdot\xi}\,d\xi$$
$$= \sum_{i=1}^2 II_i(x). \tag{5.12}$$

It is clear that $|II_1(x)| \le CR^{k+l+2}$, while we have

$$x_2^m II_2(x) = (-1)^m i^{k+l-m}\int_{\mathbb{R}^2}\xi_1^k e^{ix\cdot\xi}\partial_2^m\big(\chi_R^c(|\xi|)\xi_2^l p(\xi)e^{-|\xi|^2}\big)\,d\xi.$$

From

$$|\xi_1^k\partial_2^m\big(\chi_R^c(|\xi|)\xi_2^l p(\xi)e^{-|\xi|^2}\big)| \le C|\xi|^{k+l-m}e^{-\frac{|\xi|^2}{2}}\chi^c_{\frac{R}{2}}(|\xi|), \tag{5.13}$$

for $m \gg 1$, we get $|x_2^m II_2(x)| \le CR^{k+l-m+2}$ for $m \gg 1$. By taking $R = |x_2|^{-1}$ for $|x_2| > 2$ we see $|\partial_1^k \partial_2^l \Gamma(1,x)| \le C|x_2|^{-k-l-2}$ for $|x_2| > 2$, which implies

$$|\partial_1^k \partial_2^l \Gamma(1,x)| \le C\big(1 + |x_2|^{k+l+2}\big)^{-1} \tag{5.14}$$

for all $x \in \mathbb{R}^2$. Then (5.11) and (5.14) yield (5.7). Estimate (5.8) is then obtained by the relation $\Gamma(t,x) = t^{-1}\Gamma(1, x/\sqrt{t})$. This completes the proof.

5.3. **Semigroup property of $\{e^{tB}\}_{t\ge 0}$.** Here we discuss the property of $\{e^{tB}\}_{t\ge 0}$ or the one-parameter family $\{S(t)\}_{t\ge 0}$ defined by

$$S(t)f = e^{tB}f - \Gamma(0) \star f \text{ for } t > 0, \text{ and } S(0)f = f. \tag{5.15}$$

Since $\Gamma(0)\star = \Xi E \star$ is bounded in $L^q(\mathbb{R}^2_+)$ for any $1 < q < \infty$ by the Calderón–Zygmund inequality, $S(t)$ is a bounded linear operator in $L^q(\mathbb{R}^2_+)$ for $1 < q < \infty$.

Proposition 5.2. *Let $q \in (1,\infty)$. Then $\{S(t)\}_{t\ge 0}$ defined by* (5.15) *is a C_0 analytic semigroup in $L^q(\mathbb{R}^2_+)$.*

Remark 5.3. It is well known that the Stokes operator $A = \mathbf{P}\Delta$, $D(A) = L^q_\sigma(\mathbb{R}^n_+) \cap W_0^{1,q}(\mathbb{R}^n_+)^n \cap W^{2,q}(\mathbb{R}^n_+)^n$, generates a C_0 analytic semigroup in $L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$, $n \ge 2$; see, e.g., [34, 25, 37]. Here $\mathbf{P} : L^q(\mathbb{R}^n_+)^n \to L^q_\sigma(\mathbb{R}^n_+)$ is the Helmholtz projection.

Proof of Proposition 5.2. By the representation of $S(t)$ it is straightforward to see the continuity $\lim_{h\downarrow 0} S(h)f = f$ in $L^q(\mathbb{R}^2_+)$ and the estimate $\|\dot{S}(t)f\|_{L^q} \le Ct^{-1}\|f\|_{L^q}$, where $\dot{S}(t) = dS(t)/dt = de^{tB}/dt$. Thus it remains to prove the semigroup property of $\{S(t)\}_{t\ge 0}$; cf. [24, Proposition 2.1.9]. By the density argument it suffices to show $S(t+s)f = S(t)S(s)f$ for all $f \in C_0^\infty(\mathbb{R}^2_+)$. For $f \in C_0^\infty(\mathbb{R}^2_+)$, direct calculations yield

$$\|\nabla^k e^{tB} f\|_{L^p} \le C\|f\|_{W^{2,p}}, \quad \|\nabla^k \Gamma(0) \star f\|_{L^p} \le C\|f\|_{W^{2,p}},$$

and $\|\dot{S}(t)f\|_{L^p} \le C\|f\|_{W^{2,p}}$ for all $1 < p < \infty$ and $|k| \le 2$. We note that $\Gamma(0) \star f$ satisfies the equation

$$\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\Gamma(0) \star f = 0 \text{ on } \overline{\mathbb{R}^2_+}. \tag{5.16}$$

In particular, $\Gamma(0) \star f$ is harmonic in $\mathbb{R}^2_+$ by the relation

$$\Delta = \big(\partial_2 - (-\partial_1^2)^{1/2}\big)\big(\partial_2 + (-\partial_1^2)^{1/2}\big).$$

Hence, again from direct calculations, we observe that $\omega(t) = S(t)f$ satisfies the equation

$$\begin{aligned} &\partial_t\omega - \Delta\omega = 0 \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2_+, \\ &\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\omega = 0 \ (t,x) \in \mathbb{R}_+ \times \partial\mathbb{R}^2_+, \ \lim_{t\to 0}\omega(t) = f. \end{aligned} \tag{5.17}$$

Then integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2_{L^2(\mathbb{R}^2_+)} = -\|\nabla\omega(t)\|^2_{L^2(\mathbb{R}^2_+)} + \|(-\partial_1^2)^{\frac{1}{4}}\omega(t)\|^2_{L^2(\partial\mathbb{R}^2_+)}. \tag{5.18}$$

Since $\|(-\partial_1^2)^{\frac{1}{4}}\omega(t)\|_{L^2(\partial\mathbb{R}^2_+)} \le \|\nabla\omega(t)\|_{L^2(\mathbb{R}^2_+)}$ in general, we have

$$d\|\omega(t)\|^2_{L^2}/\,dt \le 0;$$

i.e., $\|\omega(t)\|_{L^2} \le \|f\|_{L^2}$ for $t \ge 0$. Now, since both $S(t+s)f$ and $S(t)S(s)f$ are solutions to (5.17) with f replaced by $S(s)f$, the energy calculation as above, which is satisfied even for $w(t) = S(t+s)f - S(t)S(s)f$, implies $w(t) = 0$ for $t \ge 0$. The proof is complete.

Due to the term $\Gamma(0)\star$ the semigroup $\{S(t)\}_{t\ge 0}$ does not possess spatial smoothing effects near the boundary. To recover the spatial smoothing effects and to specify the domain of the generator we need to restrict $\{S(t)\}_{t\ge 0}$ to a suitable invariant subspace of $L^q(\mathbb{R}^2_+)$. To state the result we denote by $\dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$ the completion with respect to the homogeneous norm $\|\nabla u\|_{L^q}$ of the space of all smooth, divergence-free vector fields with compact support in $\mathbb{R}^2_+$, and set

$$X_q = \{\operatorname{Rot} u \in L^q(\mathbb{R}^2_+) : u \in \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)\}. \tag{5.19}$$

It is not difficult to see that X_q, $1 < q < \infty$, is a closed subspace of $L^q(\mathbb{R}^2_+)$. Indeed, let $\{f_n\} \subset X_q$ satisfy $f_n \to f$ as $n \to \infty$ in $L^q(\mathbb{R}^2_+)$. Then there is $u_n \in \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$ such that $f_n = \operatorname{Rot} u_n$. Take $u_{n,n} \in C^\infty_{0,\sigma}(\mathbb{R}^2_+)$ such that $\|\nabla u_n - \nabla u_{n,n}\|_{L^q} \le 2^{-n}$, and set $f_{n,n} = \operatorname{Rot} u_{n,n} \in C^\infty_0(\mathbb{R}^2_+)$. Since $u_{n,n}$ vanishes on the boundary we have the Biot–Savart law $u_{n,n} = \nabla^\perp(-\Delta_D)^{-1} f_{n,n}$. By the Calderón–Zygmund inequality and the triangle inequality we have

$$\|\nabla u_{n,n} - \nabla u_{m.m}\|_{L^q} \le C\|f_{n,n} - f_{m,m}\|_{L^q} \le C(2^{-n} + 2^{-m} + \|f_n - f_m\|_{L^q}).$$

Since $u_{n,n}$ vanishes on the boundary, this estimate implies that there is $u \in \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$ such that $u_{n,n} \to u$ in $L^q_{loc}(\overline{\mathbb{R}^2_+})$ and $\nabla u_{n,n} \to \nabla u$ in $L^q(\mathbb{R}^2_+)$. Since $\operatorname{Rot} u_{n,n} = f_{n,n} \to f$, we have the desired relation $f = \operatorname{Rot} u \in X_q$. Hence X_q is closed. The main result here is now stated as follows.

Proposition 5.4. *Let $q \in (1,\infty)$. Then X_q is invariant under the action of $\{S(t)\}_{t\geq 0}$ and $S(t) = e^{tB}$ in X_q. Moreover, the generator B_q of $\{S(t)\}_{t\geq 0}$ $(= \{e^{tB}\}_{t\geq 0})$ in X_q is given by*

$$D(B_q) = \{f \in X_q \cap W^{2,q}(\mathbb{R}^2_+) : \big(\partial_2 + (-\partial_1^2)^{\frac12}\big) f = 0 \text{ on } \partial\mathbb{R}^2_+\},$$
$$B_q f = \Delta f, \;\; f \in D(B_q),$$

and it follows that

$$\|\nabla^2 f\|_{L^q} \leq C\|B_q f\|_{L^q} \quad \text{for all } f \in D(B_q).$$

Proof. Let $f \in X_q$. Then $f = \mathrm{Rot}\, u$ for some $u \in \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$. Let $\{u_n\} \subset C^\infty_{0,\sigma}(\mathbb{R}^2_+)$ be such that $\nabla u_n \to \nabla u$ in $L^q(\mathbb{R}^2_+)$ as $n \to \infty$. Set $f_n = \mathrm{Rot}\; u_n \in C^\infty_0(\mathbb{R}^2_+)$. Then by Proposition 3.2 we have $\Gamma(0) \star f_n = 0$ in $\mathbb{R}^2_+$, for the Biot–Savart law $u_n = \nabla^\perp(-\Delta_D)^{-1} f_n$ holds and u_n vanishes on the boundary. Since $\Gamma(0)\star$ is bounded in $L^q(\mathbb{R}^2_+)$ we have $\Gamma(0) \star f = 0$ in $L^q(\mathbb{R}^2_+)$; that is, $S(t)f = e^{tB} f$ for $f \in X_q$. To prove the invariant property, for $f \in X_q$ let $f_n = \mathrm{Rot}\; u_n$ be the function defined as above, and set

$$\omega_n(t) = S(t) f_n = e^{tB} f_n, \;\; v_n(t) = (v_{n,1}(t), v_{n,2}(t)) = \nabla^\perp(-\Delta_D)^{-1}\omega_n(t).$$

Note that $\omega_n(t) \in W^{2,p}(\mathbb{R}^2_+)$ for all $1 < p < \infty$. Since ω_n satisfies (5.17) with f replaced by f_n in the classical sense, we can verify that

$$\partial_t v_{n,1}(t)|_{\partial\mathbb{R}^2_+} = -\partial_2\omega_n(t)|_{\partial\mathbb{R}^2_+} - (-\partial_1^2)^{1/2}\omega_n(t)|_{\partial\mathbb{R}^2_+} = 0$$

as in the proof of Theorem 2.3. Hence, $v_{n,1}(t)|_{\partial\mathbb{R}^2_+} = v_{n,1}(0)|_{\partial\mathbb{R}^2_+} = 0$ holds, which implies that $v_n(t) \in W^{1,p}_{0,\sigma}(\mathbb{R}^2_+) \cap \dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$ for $2 < p < \infty$; see [9] for the characterization of $\dot W^{1,q}_{0,\sigma}(\mathbb{R}^2_+)$. Then the relation $\omega_n(t) = \mathrm{Rot}\, v_n(t)$ shows $\omega_n(t) \in X_q$. Since

$$\|S(t)f - \omega_n(t)\|_{L^q} \leq C\|f - f_n\|_{L^q} \to 0 \quad \text{as } n \to \infty,$$

the closedness of X_q implies $S(t)f \in X_q$, as desired. Next we see from Lemma 3.4 that the growth bound of $\{e^{tB_q}\}_{t\geq 0}$ is less than or equal to 0. Thus any positive λ belongs to the resolvent set of B_q, and we have

$$(\lambda - B_q)^{-1} = \int_0^\infty e^{-\lambda t} e^{tB_q}\, dt.$$

But by the definition of e^{tB} we have

$$(\lambda - B_q)^{-1} f = \int_0^\infty e^{-\lambda t} e^{t\Delta_N} f\, dt + \int_0^\infty e^{-\lambda t}\Gamma(t) \star f\, dt.$$

It is classical that the estimate

$$\|\nabla^2 \int_0^\infty e^{-\lambda t} e^{t\Delta_N} f\, dt\|_{L^q} \le C\|f\|_{L^q}$$

holds with C independent of $\lambda > 0$. As for the second term, note that we can write

$$\int_0^\infty e^{-\lambda t}\Gamma(t) \star f\, dt = \Xi E * \big(\int_0^\infty e^{-\lambda t} G(t) \star f\, dt\big).$$

Thus, we have

$$\|\nabla^2 \int_0^\infty e^{-\lambda t}\Gamma(t)\star f\, dt\|_{L^q(\mathbb{R}^2_+)} \le C\|\nabla^2 \int_0^\infty e^{-\lambda t} G(t)\star f\, dt\|_{L^q(\mathbb{R}^2)} \le C\|f\|_{L^q(\mathbb{R}^2_+)}$$

with C independent of $\lambda > 0$. Hence for $f \in D(B_q)$ we arrive at

$$\|\nabla^2 f\|_{L^q} \le C\|(\lambda - B_q)f\|_{L^q},$$

and by letting $\lambda \to 0$ the estimate

$$\|\nabla^2 f\|_{L^q} \le C\|B_q f\|_{L^q}$$

follows. The fact that the function $f \in X_q \cap W^{2,q}(\mathbb{R}^2_+)$ satisfying

$$\partial_2 f + (-\partial_1^2)^{1/2} f = 0 \quad \text{on } \partial\mathbb{R}^2_+$$

belongs to $D(B_q)$ is directly checked from the integral representation of e^{tB}. Conversely, if $f \in D(B_q)$, then $f \in W^{2,q}(\mathbb{R}^2_+)$, as we have already seen. The function $\omega(t) = e^{tB_q} f = e^{tB} f$ solves (5.17), and in particular,

$$\partial_2 \omega(t) + (-\partial_1^2)^{1/2}\omega(t) = 0 \quad \text{on } \partial\mathbb{R}^2_+$$

for each $t > 0$. The trace theorem and the above coercive estimate implies that

$$\begin{aligned}
&\|\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\big(f - \omega(t)\big)\|_{W^{1-\frac{1}{q},q}(\partial\mathbb{R}^2_+)} \\
&\le C\|f - \omega(t)\|_{W^{2,q}(\mathbb{R}^2_+)} \le C(\|f - \omega(t)\|_{L^q} + \|B_q(f - \omega(t))\|_{L^q}) \\
&\le C(\|e^{tB_q} f - f\|_{L^q} + \|e^{tB_q} B_q f - B_q f\|_{L^q}) \to 0, \quad t \to 0,
\end{aligned}$$

which proves

$$\partial_2 f + (-\partial_1^2)^{1/2} f = 0 \quad \text{on } \partial\mathbb{R}^2_+$$

in the sense of trace. The fact $B_q f = \Delta f$ for $f \in D(B_q)$ is proved in the same manner. The proof is complete.

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