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LARGE-TIME BEHAVIOR OF SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH A DYNAMICAL BOUNDARY CONDITION

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Abstract. The main purpose of the paper is to study the large-time behavior of positive solutions of a semilinear elliptic equation with a dynamical boundary condition. We show that small solutions behave asymptotically like suitable multiples of the Poisson kernel.

1. Introduction

This paper is concerned with a semilinear elliptic equation with a dynamical boundary condition,

$$\begin{cases} -\Delta u = u^p, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ u(x,0) = \varphi(x') \geq 0, & x = (x',0) \in \partial\mathbb{R}^N_+, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mathbb{R}^N_+ := \{x = (x', x_N) : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$, $u = u(x,t)$, Δ is the N-dimensional Laplacian (in x), $\partial_t := \partial/\partial t$, $\partial_\nu := -\partial/\partial x_N$, and $p > 1$. In this paper we prove that there is a critical exponent for the existence of positive solutions of problem (1.1). Furthermore, we study the asymptotic behavior of small solutions as $t \to \infty$.

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Throughout this paper we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}^N_+$. We introduce some notation. For any $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, let

$$\mathcal{P}(x',\lambda) := c_N \lambda^{1-N} \Lambda^{-N}(x'/\lambda) \quad \text{with} \quad \Lambda(x') := (1+|x'|^2)^{1/2}, \tag{1.2}$$

where c_N is the constant to be chosen as

$$\int_{\mathbb{R}^{N-1}} \mathcal{P}(x',\lambda)\,dx' = 1, \qquad \lambda > 0. \tag{1.3}$$

Then there holds

$$\mathcal{P}(x',\lambda+\lambda') = \int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y',\lambda)\mathcal{P}(y',\lambda')\,dy' \tag{1.4}$$

for all $x' \in \mathbb{R}^{N-1}$ and $\lambda, \lambda' \in (0,\infty)$. For any $(x', x_N) \in \mathbb{R}^N_+$ and $t > 0$, we put

$$P(x', x_N, t) := \mathcal{P}(x', x_N + t), \tag{1.5}$$

which is the fundamental solution of the Laplace equation in $\mathbb{R}^N_+$ with the homogeneous dynamical boundary condition; that is, P satisfies

$$\begin{cases} -\Delta P = 0, & x \in \mathbb{R}^N_+,\ \ t > 0, \\ \partial_t P + \partial_\nu P = 0, & x \in \partial\mathbb{R}^N_+,\ \ t > 0, \\ P(x,0) = \delta(x), & x \in \partial\mathbb{R}^N_+, \end{cases} \tag{1.6}$$

where $\delta = \delta(\cdot)$ is the Dirac delta function on $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$. Let G be the Green function for the Laplace equation on $\mathbb{R}^N_+$ with the homogeneous Dirichlet boundary condition; that is,

$$G(x,y) := \frac{c_N}{2(N-2)} \left(|x-y|^{-(N-2)} - |x-y_*|^{-(N-2)} \right) \tag{1.7}$$

for $x, y \in \mathbb{R}^N_+$ with $x \neq y$, where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \mathbb{R}^N_+$. For any $1 \le r \le \infty$, we put

$$|\cdot|_r := \|\cdot\|_{L^r(\partial\mathbb{R}^N_+)}, \qquad \|\cdot\|_r := \|\cdot\|_{L^r(\mathbb{R}^N_+)}, \qquad |||\cdot|||_r := \|\cdot\|_{L^r(\mathbb{R}^N_+, x_N\,dx)}.$$

Next we give the definition of a solution of (1.1).

Definition 1.1. *Let φ be a nonnegative measurable function in $\mathbb{R}^{N-1}$.*

(i) *For any nonnegative measurable function u in $\mathbb{R}^N_+ \times (0,\sigma]$ with $\sigma > 0$, we call the function u a solution of* (1.1) *in $\mathbb{R}^N_+ \times [0,\sigma]$ if it satisfies*

$$\begin{aligned} u(x', x_N, t) &= \int_{\mathbb{R}^{N-1}} P(x'-y', x_N, t)\varphi(y')\,dy' + \int_{\mathbb{R}^N_+} G(x,y)u(y,t)^p\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N_+} P(x'-y', x_N + y_N, t-s)u(y,s)^p\,dy\,ds < \infty \end{aligned} \tag{1.8}$$

for almost all $x' \in \mathbb{R}^{N-1}$ *and all* $x_N \in [0,\infty)$ *and* $t \in (0,\sigma]$. *In particular,* u *is said to be a global solution of* (1.1) *if* u *is a solution of* (1.1) *in* $\mathbb{R}^N_+ \times [0,\sigma]$ *for any* $\sigma > 0$.

(ii) *Let* u *be a solution of* (1.1) *in* $\mathbb{R}^N_+ \times (0,\sigma]$ *for some* $\sigma > 0$. *Then* u *is said to be a minimal solution of* (1.1) *in* $\mathbb{R}^N_+ \times (0,\sigma]$ *if, for any solution* v *of* (1.1) *in* $\mathbb{R}^N_+ \times (0,\sigma]$, *there holds*

$$u(x', x_N, t) \le v(x', x_N, t)$$

for almost all $x' \in \mathbb{R}^{N-1}$ *and all* $x_N \ge 0$ *and* $t > 0$. (*Obviously, a minimal solution is uniquely determined.*)

Remark 1.1. (i) Let u be a solution of (1.1) in $\mathbb{R}^N_+ \times (0,\sigma]$ for some $\sigma > 0$. Then, for any $\tau \in (0,\sigma)$, the function $v(x,t) := u(x, t+\tau)$ is a solution of (1.1) in $\mathbb{R}^N_+ \times (0,\sigma-\tau)$ with the initial function

$$\begin{aligned} v(x', 0, 0) = &\int_{\mathbb{R}^{N-1}} P(x'-y', 0, \tau)\varphi(y')\,dy' \\ &+ \int_0^\tau \int_{\mathbb{R}^N_+} P(x'-y', y_N, \tau-s) u(y,s)^p\,dy\,ds. \end{aligned}$$

(ii) Let φ be a nonnegative continuous function in $\mathbb{R}^{N-1}$ and v a classical solution of (1.1) in $\mathbb{R}^N_+ \times (0,\sigma]$ for some $\sigma > 0$. Put

$$g_v(x,t) := \int_{\mathbb{R}^N_+} G(x,y) v(y,t)^p\,dy. \tag{1.9}$$

Then, under a suitable integrability condition on the solution v, the function g_v is a classical solution of

$$-\Delta g = v^p \quad \text{in} \quad \mathbb{R}^N_+ \times (0,\sigma], \qquad g = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ \times (0,\sigma]. \tag{1.10}$$

The function $h_v := v - g_v$ satisfies

$$\begin{cases} \Delta h_v = 0 & \text{in} \quad \mathbb{R}^N_+ \times (0,\sigma], \\ \partial_t h_v + \partial_\nu h_v = -\partial_\nu g_v & \text{on} \quad \partial\mathbb{R}^N_+ \times (0,\sigma], \\ h_v(x,0) = \varphi(x) & \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

Under a suitable integrability condition on v and φ, it follows from (1.6) that

$$\begin{aligned} h_v(x,t) = &\int_{\mathbb{R}^{N-1}} P(x'-y', x_N, t)\varphi(y')\,dy' \\ &- \int_0^t \int_{\mathbb{R}^{N-1}} P(x'-y', x_N, t-s)\partial_\nu g_v(y', 0, s)\,dy'\,ds. \end{aligned} \tag{1.11}$$

On the other hand, since $g_v = 0$ on $\partial\mathbb{R}^N_+$, by (1.6) and (1.9) we have

$$\begin{aligned}
-\int_0^t \int_{\mathbb{R}^{N-1}} & P(x'-y', x_N, t-s)\partial_\nu g_v(y', 0, s)\, dy'\, ds \\
&= -\int_0^t \int_{\mathbb{R}^N_+} P(x'-y', x_N+y_N, t-s)\Delta g_v(y,s)\, dy\, ds \\
&= \int_0^t \int_{\mathbb{R}^N_+} P(x'-y', x_N+y_N, t-s) v(y,s)^p\, dy\, ds. \qquad (1.12)
\end{aligned}$$

By (1.11) and (1.12) we see that v satisfies the integral identity (1.8). This means that a classical solution v of (1.1) is a solution of (1.1) in the sense of Definition 1.1 under a suitable integrability condition on v and φ.

Next we state the main results of this paper. In what follows, we put

$$p_* := \tfrac{N+1}{N-1} = 1 + \tfrac{2}{N-1}.$$

The first theorem is concerned with the nonexistence of global solutions of (1.1). See Remark 3.1 for $N = 2$.

Theorem 1.1. *Assume $N \geq 3$ and $1 < p \leq p_*$. Then problem (1.1) has no nontrivial global solutions.*

In the second theorem we give a sufficient condition for the existence of nontrivial global solutions of (1.1), and prove that the solutions behave asymptotically like suitable multiples of the Poisson kernel.

Theorem 1.2. *Let $N \geq 3$ and $p > p_*$. Assume*

$$\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}). \qquad (1.13)$$

Then the following holds:
(i) *There exists a positive constant $\delta = \delta(N,p)$ such that, if*

$$|\varphi|_1 |\varphi|_\infty^{\frac{N-1}{2}(p-1)-1} < \delta, \qquad (1.14)$$

then problem (1.1) possesses a global minimal solution u satisfying

$$\sup_{t>0}\, (1+t)^{N-1-\frac{N}{q}} \|u(t)\|_q < \infty \quad \textit{for any} \quad q \in (p_*, \infty], \qquad (1.15)$$

$$\sup_{t>0}\, (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u(t)|||_p < \infty; \qquad (1.16)$$

(ii) *Let v be a global solution of* (1.1)*, satisfying* (1.15) *and* (1.16)*. Then v is a classical solution of* (1.1) *in $\mathbb{R}^N_+ \times (0,\infty)$ and there exists a limit*

$$C_* := \lim_{t\to\infty} \int_{\mathbb{R}^{N-1}} v(x',0,t)\,dx' = \int_{\mathbb{R}^{N-1}} \varphi(x')\,dx' + \int_0^\infty \int_{\mathbb{R}^N_+} v(x,t)^p\,dx\,dt \tag{1.17}$$

such that

$$\lim_{t\to\infty} t^{N-1-\frac{N}{q}} \|v(t) - C_* P(t)\|_q = 0 \tag{1.18}$$

for any $q \in (p_*, \infty]$.

For a result on the continuous dependence of minimal solutions on initial data see Proposition 4.2.

A statement similar to Theorem 1.2 also holds for $N = 2$. But, since the Green function G has a different form (see Remark 3.1), we shall treat this case in a forthcoming paper.

Remark 1.2. Let u be a solution of (1.1). For $\mu > 0$, put

$$u_\mu(x', x_N, t) = \mu^{\frac{2}{p-1}} u(\mu x', \mu x_N, \mu t), \qquad \varphi_\mu(x') = \mu^{\frac{2}{p-1}} \varphi(\mu x'). \tag{1.19}$$

Then the function u_μ is also a solution of (1.1) with the initial function φ_μ and

$$|\varphi_\mu|_1 |\varphi_\mu|_\infty^{\frac{N-1}{2}(p-1)-1} = |\varphi|_1 |\varphi|_\infty^{\frac{N-1}{2}(p-1)-1}. \tag{1.20}$$

This means that condition (1.14) is invariant with respect to the similarity transformation (1.19). We remark that $(N-1)(p-1)/2 - 1 = 0$ if $p = p_*$.

By Theorems 1.1 and 1.2 we see that the exponent p_* is critical with respect to the global existence of positive solutions (in the sense of Definition 1.1). Hence, it plays the role of the Fujita exponent for problem (1.1). On the other hand, the exponent p_* is known as the Brézis–Turner exponent, which is critical for the boundedness of very weak nonnegative solutions of the problem

$$-\Delta u = u^p \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$. See [5] and [13, Chapter I, Section 11].

We also mention that the condition $1 < p \le p_*$ is sufficient for the nonexistence of positive classical solutions of the semilinear elliptic equation

$$-\Delta u = u^p \qquad \text{in} \quad \mathbb{R}^N_+ \tag{1.21}$$

(see [3, Theorem 2.1] and [13, Remark 8.5 (iii)]). For $p > p_*$, $(N-3)p < N+1$, there exist positive classical solutions of (1.21) that are singular on the boundary; see [4] and [6].

Nonlinear elliptic systems with dynamical boundary conditions were studied by a different approach in [7] and [8] on bounded domains. The Laplace equation on the half-space with a nonlinear dynamical boundary condition was considered in [1] and [9]. In the present paper, the nonlinear term in the elliptic equation makes the problem more challenging because the integral representation of the solution becomes more involved.

The paper is organized as follows. In Section 2 we give some preliminary inequalities on the Poisson kernel $\mathcal{P} = \mathcal{P}(x', \lambda)$ and the kernel $G = G(x, y)$. In Section 3 we show that problem (1.1) has no nontrivial global solutions in the case $1 < p \le p_*$, and prove Theorem 1.1. In Section 4 we assume that the initial function φ is sufficiently small, and construct a local solution of (1.1) in the case $p > p_*$. In Section 5 we prove that the solution constructed in Section 4 is a global solution of (1.1). Furthermore, we study the large-time behavior of the solution of (1.1), satisfying (1.15) and (1.16), and complete the proof of Theorem 1.2.

2. Preliminaries

In this section we give some preliminary results on the kernel $P = P(x, t)$ and the Green function $G = G(x, y)$.

For any two nonnegative functions f_1 and f_2 defined in a subset D of $[0, \infty)$, we write $f_1(t) \preceq f_2(t)$ for all $t \in D$ if there exists a positive constant C such that $f_1(t) \le C f_2(t)$ for all $t \in D$. In addition, we say $f_1(t) \asymp f_2(t)$ for all $t \in D$ if $f_1(t) \preceq f_2(t)$ and $f_2(t) \preceq f_1(t)$ for all $t \in D$.

We first give some inequalities for the kernel $P = P(x', x_N, t)$. By (1.2)–(1.5), for any $1 \le r \le \infty$, we have

$$|P(t)|_r = |\mathcal{P}(t)|_r \asymp t^{-(N-1)(1-\frac{1}{r})}, \qquad t > 0. \tag{2.1}$$

This yields the following inequalities:

- If $N/(N-1) < q \le \infty$, then

$$\|P(t)\|_q \asymp t^{-(N-1)+\frac{N}{q}}, \qquad t > 0; \tag{2.2}$$

- If $p_* \equiv (N+1)/(N-1) < q \le \infty$, then

$$|||P(t)|||_q \asymp t^{-(N-1)(1-\frac{p_*}{q})}, \qquad t > 0. \tag{2.3}$$

Indeed, in the case $q = \infty$, since $\|P(t)\|_q = |||P(t)|||_q$, inequalities (2.2) and (2.3) follow from (1.2) and (1.5). In the case $N/(N-1) < q < \infty$, by (1.5) and (2.1) we have

$$\|P(t)\|_q = \Big(\int_0^\infty \int_{\mathbb{R}^{N-1}} |\mathcal{P}(x', x_N + t)|^q \, dx' \, dx_N \Big)^{1/q}$$

$$\asymp \Big(\int_0^\infty (x_N + t)^{-(N-1)(q-1)}\, dx_N \Big)^{1/q} \asymp t^{-(N-1)+\frac{N}{q}}$$

for all $t > 0$, and obtain (2.2). Similarly, if $q > p_*$, then we have

$$\begin{aligned} |||P(t)|||_q &= \Big(\int_0^\infty x_N \int_{\mathbb{R}^{N-1}} |\mathcal{P}(x', x_N + t)|^q\, dx'\, dx_N \Big)^{1/q} \\ &\asymp \Big(\Big(\int_0^t + \int_t^\infty \Big) x_N (x_N + t)^{-(N-1)(q-1)}\, dx_N \Big)^{1/q} \\ &\asymp t^{-(N-1)(1-\frac{p_*}{q})} \end{aligned}$$

for all $t > 0$, and obtain (2.3). Thus inequalities (2.2) and (2.3) follow.

For any measurable function φ on $\mathbb{R}^{N-1}$ and $t > 0$, we put

$$[S(t)\varphi](x) := \int_{\mathbb{R}^{N-1}} P(x' - y', x_N, t)\varphi(y')\, dy' \tag{2.4}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$. The function $S(t)\varphi$ has the following properties:

(1) Let t, $t' > 0$. Then

$$[S(t)\varphi](x', x_N) = [S(t + x_N)\varphi](x', 0), \tag{2.5}$$

$$[S(t + t')\varphi](x) = [S(t)(S(t')\varphi)](x), \tag{2.6}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$ if either φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$ or $\varphi \in L^q(\mathbb{R}^{N-1})$ for some $q \in [1, \infty]$. In particular,

$$\sup_{t>0} \|S(t)\varphi\|_\infty \leq |\varphi|_\infty \quad \text{for any } \varphi \in L^\infty(\mathbb{R}^{N-1}), \tag{2.7}$$

$$\lim_{t\to 0} |S(t)\varphi - \varphi|_r = 0 \quad \text{for any } \varphi \in L^r(\mathbb{R}^{N-1}) \text{ with } 1 \leq r < \infty. \tag{2.8}$$

(2) For any $1 \leq q \leq r \leq \infty$,

$$|S(t)\varphi|_r \preceq t^{-(N-1)(\frac{1}{q}-\frac{1}{r})} |\varphi|_q, \qquad t > 0, \tag{2.9}$$

for all $\varphi \in L^q(\mathbb{R}^{N-1})$. In particular, if $q = r$, then

$$|S(t)\varphi|_r \leq |\varphi|_r, \qquad t > 0. \tag{2.10}$$

(3) If $\varphi \in L^1(\mathbb{R}^{N-1})$, then, for any $1 \leq r \leq \infty$,

$$\lim_{t\to\infty} t^{(N-1)(1-\frac{1}{r})} |S(t)\varphi - M\mathcal{P}(t)|_r = 0 \;\text{ with }\; M = \int_{\mathbb{R}^{N-1}} \varphi(x')\, dx'. \tag{2.11}$$

Assertions (1) and (2) follow from (1.4), (1.5), (2.1), and (2.4). For assertion (3), see [9, Lemma 2.2].

Lemma 2.1. *Let $1 \le q < \infty$ and*

$$\tfrac{Nq}{N-1} < r \le \infty. \tag{2.12}$$

Then there exists a constant C_1 such that

$$\|S(t)\varphi\|_r \le C_1 t^{-(N-1)(\frac{1}{q}-\frac{1}{r})+\frac{1}{r}}|\varphi|_q, \qquad t > 0, \tag{2.13}$$

for all $\varphi \in L^q(\mathbb{R}^{N-1})$. Moreover, there exists a constant C_2 such that

$$\|S(t)\varphi\|_r \le C_2(|\varphi|_r + |\varphi|_q), \qquad t \ge 0, \tag{2.14}$$

for all $\varphi \in L^q(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1})$.

Proof. If $\varphi \in L^q(\mathbb{R}^{N-1})$, then, by (2.5), (2.9), and (2.12) we have

$$\begin{aligned}\|S(t)\varphi\|_r^r &= \int_0^\infty |S(t+x_N)\varphi|_r^r\,dx_N \\ &\preceq |\varphi|_q^r \int_0^\infty (t+x_N)^{-(N-1)(\frac{r}{q}-1)}\,dx_N \preceq t^{-(N-1)(\frac{r}{q}-1)+1}|\varphi|_q^r\end{aligned}$$

for all $t > 0$, and we obtain (2.13). If $\varphi \in L^q(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1})$, then, by (2.5), (2.9), and (2.10) we have

$$\begin{aligned}\|S(t)\varphi\|_r^r &= \Big(\int_0^1 + \int_1^\infty\Big)|S(t+x_N)\varphi|_r^r\,dx_N \\ &\preceq |\varphi|_r^r + |\varphi|_q^r \int_1^\infty (t+x_N)^{-(N-1)(\frac{r}{q}-1)}\,dx_N \preceq |\varphi|_r^r + |\varphi|_q^r\end{aligned}$$

for all $t > 0$, and we obtain (2.14). Thus Lemma 2.1 follows. □

Lemma 2.2. *Let $p_* < r \le \infty$. Then there exists a constant C such that*

$$|||S(t)\varphi|||_r \le Ct^{-(N-1)(1-\frac{p_*}{r})}|\varphi|_1, \qquad t > 0, \tag{2.15}$$

for all $\varphi \in L^1(\mathbb{R}^{N-1})$. Furthermore,

$$|||S(t)\varphi|||_r \le |\varphi|_r + |\varphi|_1, \qquad t \ge 0, \tag{2.16}$$

for all $\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^r(\mathbb{R}^{N-1})$.

Proof. Let $\varphi \in L^1(\mathbb{R}^{N-1})$. In the case $r = \infty$, since $|||S(t)\varphi|||_r = \|S(t)\varphi\|_r$, Lemma 2.2 follows from Lemma 2.1. In the case $r < \infty$, if $r > p_*$, then, by (2.5) and (2.9) we have

$$\begin{aligned}|||S(t)\varphi|||_r^r &= \int_0^\infty x_N|S(t+x_N)\varphi|_r^r\,dx_N \\ \preceq |\varphi|_1^r\Big(\int_0^t &+ \int_t^\infty\Big)x_N(t+x_N)^{-(N-1)(r-1)}\,dx_N \preceq t^{-(N-1)(r-p_*)}|\varphi|_1^r\end{aligned}$$

for all $t > 0$, and we obtain (2.15). Similarly, by (2.5), (2.9), and (2.10) we have (2.16) (see also the proof of (2.14)), and the proof of Lemma 2.2 follows. $\square$

Next we put

$$W[f](x') := \int_{\mathbb{R}^N_+} \mathcal{P}(x'-y', y_N) f(y', y_N)\, dy, \qquad x' \in \mathbb{R}^{N-1}, \tag{2.17}$$

for any measurable function f in $\mathbb{R}^N_+$, and prove the following lemma.

Lemma 2.3. *Let $1 \le q < N < r \le \infty$. Then there exists a constant C such that*

$$|W[f]|_\infty \le C(\|f\|_q + \|f\|_r) \tag{2.18}$$

for all $f \in L^q(\mathbb{R}^N_+) \cap L^r(\mathbb{R}^N_+)$.

Proof. By the Hölder inequality, (1.2), and (2.1) we have

$$\begin{aligned} |W[f](x')| &\le \Big(\int_0^1 \int_{\mathbb{R}^{N-1}} |\mathcal{P}(x'-y', y_N)|^{\tilde{r}}\, dy'\, dy_N \Big)^{1/\tilde{r}} \|f\|_r \\ &\quad + \Big(\int_1^\infty \int_{\mathbb{R}^{N-1}} |\mathcal{P}(x'-y', y_N)|^{\tilde{q}}\, dy'\, dy_N \Big)^{1/\tilde{q}} \|f\|_q \\ &\le \Big(\int_0^1 |\mathcal{P}(y_N)|_{\tilde{r}}^{\tilde{r}}\, dy_N \Big)^{1/\tilde{r}} \|f\|_r + \Big(\int_1^\infty |\mathcal{P}(y_N)|_{\tilde{q}}^{\tilde{q}}\, dy_N \Big)^{1/\tilde{q}} \|f\|_q \\ &\le C \Big(\int_0^1 y_N^{-(N-1)(\tilde{r}-1)}\, dy_N \Big)^{1/\tilde{r}} \|f\|_r \\ &\qquad + C \Big(\int_1^\infty y_N^{-(N-1)(\tilde{q}-1)}\, dy_N \Big)^{1/\tilde{q}} \|f\|_q \end{aligned} \tag{2.19}$$

for almost all $x' \in \mathbb{R}^{N-1}$, where C is a constant. Here $\tilde{q}$ and $\tilde{r}$ are the Hölder conjugates of q and r, respectively. It follows from $1 \le q < N < r \le \infty$ that

$$1 \le \tilde{r} < \tfrac{N}{N-1} < \tilde{q} \le \infty.$$

This together with (2.19) implies (2.18), and Lemma 2.3 follows. $\square$

In the rest of this section we give some properties for the integral operator

$$[(-\Delta_D)^{-1} f](x) := \int_{\mathbb{R}^N_+} G(x,y) f(y)\, dy, \qquad x \in \mathbb{R}^N_+. \tag{2.20}$$

We first give a Hardy–Littlewood–Sobolev-type inequality for the operator $(-\Delta_D)^{-1}$.

Lemma 2.4. *Let $1 < q < \infty$. Then there exists a constant C such that*

$$\|(-\Delta_D)^{-1} f\|_{q_*} \le C\|f\|_q^{\frac{N-1}{N+1}} \, |||f|||_1^{\frac{2}{N+1}} \tag{2.21}$$

for all $f \in L^q(\mathbb{R}^N_+) \cap L^1(\mathbb{R}^N_+, x_N\, dx)$, where $q_* = p_* q$.

Proof. Let $1 < q < \infty$ and $f \in L^q(\mathbb{R}^N_+) \cap L^1(\mathbb{R}^N_+, x_N\, dx)$. Put

$$[e^{t\Delta_D} f](x) := \int_{\mathbb{R}^N_+} \Gamma_D(x, y, t) f(y)\, dy, \qquad t > 0.$$

Here Γ_D is the Dirichlet heat kernel on $\mathbb{R}^N_+$; that is,

$$\Gamma_D(x, y, t) := (4\pi t)^{-\frac{N}{2}} \Big[\exp\Big(-\frac{|x-y|^2}{4t} \Big) - \exp\Big(-\frac{|x-y_*|^2}{4t} \Big) \Big]$$

for $(x, y, t) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (0, \infty)$, where $y_* = (y', -y_N)$ for $y = (y', y_N) \in \mathbb{R}^N_+$. Then, for any $1 \le q \le r \le \infty$, we have

$$\|e^{t\Delta_D} f\|_r \preceq t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|f\|_q, \qquad t > 0. \tag{2.22}$$

By the mean-value theorem we have

$$|\Gamma_D(x, y, t)| \preceq t^{-\frac{N+1}{2}} y_N, \quad (x, y, t) \in \mathbb{R}^N_+ \times \mathbb{R}^N_+ \times (0, \infty), \tag{2.23}$$

and we obtain

$$\|e^{t\Delta_D} f\|_\infty \preceq t^{-\frac{N+1}{2}} \, |||f|||_1 \,, \qquad t > 0. \tag{2.24}$$

On the other hand, since

$$G(x, y) = \int_0^\infty \Gamma_D(x, y, t)\, dt, \qquad x, y \in \mathbb{R}^N_+, \quad x \ne y$$

(see (1.7)), it follows from (2.20) and the Fubini theorem that

$$(-\Delta_D)^{-1} f(x) = \int_{\mathbb{R}^N_+} \Big(\int_0^\infty \Gamma_D(x, y, t)\, dt \Big) f(y)\, dy = \int_0^\infty e^{t\Delta_D} f(x)\, dt \tag{2.25}$$

for all $x \in \mathbb{R}^N_+$. Then we apply the same argument as in [11, Section 6.2] with the aid of (2.22) and (2.24), and obtain (2.21). Thus Lemma 2.4 follows. □

Next we give an $L^\infty(\mathbb{R}^N_+)$-estimate of $(-\Delta_D)^{-1} f$.

Lemma 2.5. *Let $1 \le q < N/2 < r \le \infty$. Then there exists a constant C such that*

$$\|(-\Delta_D)^{-1} f\|_\infty \le C(\|f\|_q + \|f\|_r) \tag{2.26}$$

for all $f \in L^q(\mathbb{R}^N_+) \cap L^r(\mathbb{R}^N_+)$.

Proof. Let $1 \le q < N/2 < r \le \infty$ and $f \in L^q(\mathbb{R}^N_+) \cap L^r(\mathbb{R}^N_+)$. Then, by (2.22) and (2.25) we can find constants C_1 and C_2 such that

$$\begin{aligned}(-\Delta_D)^{-1} f(x) &\le \int_0^\infty \|e^{t\Delta_D} f\|_\infty \, dt \\ &\le C_1 \|f\|_r \int_0^1 t^{-\frac{N}{2r}} \, dt + C_1 \|f\|_q \int_1^\infty t^{-\frac{N}{2q}} \, dt \le C_2 \|f\|_r + C_2 \|f\|_q\end{aligned}$$

for all $x \in \mathbb{R}^N_+$. This implies (2.26), and Lemma 2.5 follows. ☐

Next we give an $L^1(\mathbb{R}^N_+, x_N \, dx)$-estimate of $(-\Delta_D)^{-1} f$.

Lemma 2.6. *Let $p_* < q \le \infty$. Then there exists a constant C such that*

$$\big|\big|\big|(-\Delta_D)^{-1} f\big|\big|\big|_q \le C(\|f\|_{L^1(\mathbb{R}^N_+, (1+x_N)\, dx)} + \|f\|_\infty) \tag{2.27}$$

for all $f \in L^\infty(\mathbb{R}^N_+) \cap L^1(\mathbb{R}^N_+, (1+x_N)\, dx)$.

In order to prove Lemma 2.6, we prepare the following lemma.

Lemma 2.7. *Let $p_c := N/(N-2)$. For any $q \in (p_*, p_c)$, there exists a constant C such that*

$$\int_{\mathbb{R}^N_+} x_N G(x,y)^q \, dx \le C(1 + y_N)^q, \qquad y \in \mathbb{R}^N_+. \tag{2.28}$$

Proof. We first prove that

$$G(x,y) \le C \frac{x_N y_N}{|x-y|^{N-2} |x_N + y_N|^2}, \qquad x, y \in \mathbb{R}^N_+, \quad x \ne y, \tag{2.29}$$

for some constant C. In the case $N = 3$, since $|x - y| \le |x - y_*|$ and $|x_N + y_N| \le |x - y_*|$, we have

$$\begin{aligned} G(x,y) &= \frac{c_N}{|x-y|} \frac{|x-y_*|^2 - |x-y||x-y_*|}{|x-y_*|^2} \le \frac{c_N}{|x-y|} \frac{|x-y_*|^2 - |x-y|^2}{|x-y_*|^2} \\ &= \frac{c_N}{|x-y|} \frac{4x_N y_N}{|x-y_*|^2} \le \frac{c_N}{|x-y|} \frac{4x_N y_N}{|x_N + y_*|^2}, \end{aligned}$$

and obtain (2.29). In the case $N \ge 4$, putting

$$h(t) := \big||x-y_*|^2 - 4t x_N y_N\big|^{(N-2)/2},$$

by the mean-value theorem we have for some $\alpha \in (0,1)$

$$\begin{aligned} G(x,y) &= \frac{c_N}{2(N-2)} \frac{h(0) - h(1)}{|x-y|^{N-2} |x-y_*|^{N-2}} \\ &= \frac{c_N}{2(N-2)} \frac{-h'(\alpha)}{|x-y|^{N-2} |x-y_*|^{N-2}} \end{aligned}$$

$$= \frac{c_N}{4}\frac{1}{|x-y|^{N-2}|x-y_*|^{N-2}}\left||x-y_*|^2-4\alpha x_N y_N\right|^{(N-2)/2-1}4x_n y_n$$
$$\le \frac{c_N}{4}\frac{4x_N y_N}{|x-y|^{N-2}|x-y_*|^2} \le \frac{c_N}{4}\frac{4x_N y_N}{|x-y|^{N-2}|x_N+y_N|^2},$$

and we obtain (2.29). Thus inequality (2.29) holds. See also [2, Proposition 1].

Next, for any $y \in \mathbb{R}^N_+$, we divide $\mathbb{R}^N_+$ into the following three sets,

$$D_1(y) := \{x \in \mathbb{R}^N_+ : |x'-y'| \le 1 \quad \text{and} \quad |x_N - y_N| \le 1\},$$
$$D_2(y) := \{x \in \mathbb{R}^N_+ : |x'-y'| \le 1 \quad \text{and} \quad |x_N - y_N| > 1\},$$
$$D_3(y) := \{x \in \mathbb{R}^N_+ : |x'-y'| > 1\},$$

and prove (2.28). Since $\mathbb{R}^N_+ = \bigcup_{k=1}^3 D_k(y)$, it suffices to prove

$$\int_{D_k(y)} x_N G(x,y)^q\,dx \le (1+y_N)^q, \qquad y_N \in \mathbb{R}^N_+, \tag{2.30}$$

for $k = 1,2,3$. Since $x_N \le 1 + y_N$ for $x \in D_1(y)$ and $(N-2)q < N$, by (2.29) we have

$$\int_{D_1(y)} x_N G(x,y)^q\,dx \preceq \int_{D_1(y)} \frac{x_N}{|x-y|^{(N-2)q}}\,dx \preceq 1+y_N \le (1+y_N)^q \tag{2.31}$$

for all $y \in \mathbb{R}^N_+$. Furthermore, since $x_N+y_N \ge |x_N-y_N|$ and $(N-1)q-1 > 1$, by (2.29) we have

$$\begin{aligned}
&\int_{D_2(y)} x_N G(x,y)^q\,dx \\
&\preceq \int_{\{x_N>0\}\cap\{|x_N-y_N|>1\}} \Big(\int_{|x'-y'|\le 1} \frac{x_N^{q+1}y_N^q|x_N+y_N|^{-2q}}{|x_N-y_N|^{(N-2)q}}\,dx'\Big)\,dx_N \\
&\preceq \int_{\{x_N>0\}\cap\{|x_N-y_N|>1\}} \frac{y_N^q}{|x_N-y_N|^{(N-2)q}|x_N+y_N|^{q-1}}\,dx_N \\
&\le y_N^q \int_{\{x_N>0\}\cap\{|x_N-y_N|>1\}} \frac{1}{|x_N-y_N|^{(N-1)q-1}}\,dx_N \preceq y_N^q
\end{aligned} \tag{2.32}$$

for all $y \in \mathbb{R}^N_+$.

On the other hand, since $N-2-(N-2)q < -1$, we have

$$\int_{|x'-y'|>1} \frac{1}{|x-y|^{(N-2)q}}\,dx'$$

$$\preceq \Big(\int_1^{1+|x_N-y_N|} + \int_{1+|x_N-y_N|}^\infty \Big) \frac{r^{N-2}}{(r^2+|x_N-y_N|^2)^{(N-2)q/2}}\,dr$$
$$\preceq \int_1^{1+|x_N-y_N|} \frac{r^{N-2}}{(1+|x_N-y_N|)^{(N-2)q}}\,dr + \int_{1+|x_N-y_N|}^\infty r^{N-2-(N-2)q}\,dr$$
$$\preceq (1+|x_N-y_N|)^{N-1-(N-2)q} \tag{2.33}$$

for all $y \in \mathbb{R}^N_+$. Then, since $x_N + y_N \geq |x_N - y_N|$ and $N-(N-1)q < -1$, by (2.29) and (2.33) we have

$$\int_{D_3(y)} x_N G(x,y)^q\,dx$$
$$\preceq \int_0^\infty \frac{x_N^{q+1}y_N^q}{|x_N+y_N|^{2q}} \Big(\int_{|x'-y'|>1} \frac{1}{|x'-y'|^{(N-2)q}}\,dx' \Big)\,dx_N$$
$$\preceq \Big(\int_{|x_N-y_N|\leq 1} + \int_{|x_N-y_N|>1} \Big) \frac{y_N^q(1+|x_N-y_N|)^{N-1-(N-2)q}}{|x_N+y_N|^{q-1}}\,dx_N$$
$$\preceq \int_{|x_N-y_N|\leq 1} y_N\,dx_N + y_N^q \int_{|x_N-y_N|>1} |x_N-y_N|^{N-(N-1)q}\,dx_N$$
$$\preceq y_N + y_N^q \preceq (1+y_N)^q \tag{2.34}$$

for all $y \in \mathbb{R}^N_+$. Therefore we have inequality (2.30) for $k = 1, 2, 3$, and we obtain (2.28). Thus Lemma 2.7 follows. □

By using Lemma 2.7 we prove Lemma 2.6.

Proof of Lemma 2.6. In the case $q \in (p_*, p_c)$, by (2.20) and (2.28) we obtain

$$\big|\big|\big|(-\Delta_D)^{-1}f\big|\big|\big|_q^q \leq \int_{\mathbb{R}^N_+} x_N \Big(\int_{\mathbb{R}^N_+} G(x,y)|f(y)|\,dy \Big)^q\,dx$$
$$\leq \|f\|^{q-1}_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} \int_{\mathbb{R}^N_+} x_N \Big\{ \int_{\mathbb{R}^N_+} (1+y_N)^{-(q-1)} G(x,y)^q |f(y)|\,dy \Big\}\,dx$$
$$\leq C_1 \|f\|^{q-1}_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} \int_{\mathbb{R}^N_+} (1+y_N)|f(y)|\,dy$$
$$= C_1 \|f\|^q_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} \tag{2.35}$$

for some constant C_1. This implies inequality (2.27). In the case $q \in [p_c, \infty)$, by (2.20), (2.26), and (2.35), for any $\tilde{q} \in (p_*, p_c)$, we have

$$\big|\big|\big|(-\Delta_D)^{-1}f\big|\big|\big|_p^p \leq \int_{\mathbb{R}^N_+} x_N \Big| \int_{\mathbb{R}^N_+} G(x,y)f(y)\,dy \Big|^{\tilde{p}}\,dx \cdot \|(-\Delta_D)^{-1}f\|_\infty^{p-\tilde{p}}$$

$$\le C_2\|f\|^{\tilde{p}}_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)}(\|f\|_1+\|f\|_\infty)^{p-\tilde{p}}$$
$$\le C_2(\|f\|_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)}+\|f\|_\infty)^p$$

for some constant C_2. So we have (2.27), and the proof of Lemma 2.6 is complete. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1, which means that problem (1.1) has no nontrivial global solutions in the case $1 < p \le p_*$. The proof of Theorem 1.1 is based on the arguments in [1, Lemma 2], [14, Theorem 5], and [15, Theorem 1].

We first prove the following lemma.

Lemma 3.1. *Let u be a solution of (1.1) in $\mathbb{R}^N_+ \times (0,T)$ with $0 < T \le \infty$. Then there exists a constant C, independent of φ and T, such that*

$$t^{\frac{2}{p-1}}\|S(t)\varphi\|_\infty \le C \quad \textit{for any } t \in [0,T). \tag{3.1}$$

Proof. By (1.8) and (2.4) we have

$$[S(t)\varphi](x) \le u(x,t) < \infty \tag{3.2}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0,\infty)$ and $t \in (0,T)$. This together with (1.5) and (1.8) implies

$$u(x,t) \ge \int_0^t \int_{\mathbb{R}^N_+} \mathcal{P}(x'-y', x_N+y_N+t-s)\left([S(s)\varphi](y',y_N)\right)^p\,dy\,ds \tag{3.3}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0,\infty)$ and $t \in (0,T)$. Then we apply the Jensen inequality with the aid of (1.3) to (3.3), and by (2.4), (2.5), and (2.6) we obtain

$$\begin{aligned} &u(x,t)\\ &\ge \int_0^t \int_0^\infty \Big(\int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y', x_N+y_N+t-s)[S(s)\varphi](y',y_N)\,dy'\Big)^p dy_N\,ds\\ &= \int_0^t \int_0^\infty \Big(\int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y', x_N+2y_N+t)\varphi(y')\,dy'\Big)^p dy_N\,ds. \end{aligned} \tag{3.4}$$

On the other hand, by (1.2) we have

$$\begin{aligned} \frac{\mathcal{P}(x',\lambda+\lambda')}{\mathcal{P}(x',\lambda)} &= \frac{(\lambda+\lambda')^{1-N}}{\lambda^{1-N}}\Big[1+\frac{|x'|^2}{\lambda^2}\Big]^{N/2}\Big/\Big[1+\frac{|x'|^2}{(\lambda+\lambda')^2}\Big]^{N/2}\\ &\ge \frac{(\lambda+\lambda')^{1-N}}{\lambda^{1-N}} \end{aligned} \tag{3.5}$$

for $x' \in \mathbb{R}^{N-1}$ and $\lambda, \lambda' \in (0, \infty)$. This together with (3.4) yields

$$\begin{aligned} u(x,t) &\geq t \int_0^\infty \frac{(x_N + 2y_N + t)^{-r}}{(x_N+t)^{-r}} \Big(\int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y', x_N+t)\varphi(y')\, dy' \Big)^p \, dy_N \\ &= t\,([S(t)\varphi](x))^p \int_0^\infty \frac{(x_N+t)^r}{(x_N+2y_N+t)^r}\, dy_N \\ &\geq \frac{t(x_N+t)}{2(r-1)} ([S(t)\varphi](x))^p \geq \frac{t^2}{2(r-1)} \big([S(t)\varphi](x', x_N)\big)^p \end{aligned} \tag{3.6}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty)$ and $t \in (0, T)$, where $r := (N-1)p > p$. We repeat the above argument with (3.2) replaced by (3.6), and obtain

$$\begin{aligned} u(x,t) &\geq \int_0^t \int_{\mathbb{R}^N_+} \mathcal{P}(x'-y', x_N+y_N+t-s) \left(\frac{s^2\,([S(s)\varphi](y', y_N))^p}{2(r-1)} \right)^p dy\, ds \\ &\geq \int_0^t \frac{s^{2p}}{2^p(r-1)^p} \\ &\times \int_0^\infty \Big(\int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y', x_N+y_N+t-s)[S(s)\varphi](y', y_N)\, dy' \Big)^{p^2} dy_N\, ds \\ &\geq \int_0^t \frac{s^{2p}}{2^p(r-1)^p} \int_0^\infty \Big(\int_{\mathbb{R}^{N-1}} \mathcal{P}(x'-y', x_N+2y_N+t)\varphi(y')\, dy' \Big)^{p^2} dy_N\, ds. \end{aligned}$$

This together with (3.5) implies

$$\begin{aligned} u(x,t) &\geq \frac{1}{2^p(r-1)^p} \big([S(t)\varphi](x', x_N)\big)^{p^2} \frac{t^{2p+1}}{2p+1} \int_0^\infty \frac{(x_N+t)^{rp}}{(x_N+2y_N+t)^{rp}}\, dy_N \\ &\geq \frac{1}{2^p(r-1)^p} ([S(t)\varphi](x))^{p^2} \frac{t^{2p+1}}{2p+1} \frac{x_N+t}{2(rp-1)} \\ &\geq \frac{1}{2^{p+1}(r-1)^p} ([S(t)\varphi](x))^{p^2} \frac{t^{2p+2}}{2p+1} \frac{1}{r^2-1} \\ &= \Big(\frac{1}{2(r-1)} \Big)^{p+1} \frac{1}{r+1} \frac{t^{2p+2}}{2p+1} ([S(t)\varphi](x))^{p^2} \end{aligned}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty)$ and $t \in (0, T)$. Repeating the above argument, for any $k = 2, 3, \dots$, we have

$$u(x,t) \geq A_k B_k t^{\frac{2}{p-1}(p^k - 1)} ([S(t)\varphi](x))^{p^k} \tag{3.7}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0,\infty)$ and $t \in (0,T)$, where

$$\begin{aligned} A_k &:= \Big(\frac{1}{2(r-1)}\Big)^{\frac{p^k-1}{p-1}} \Big(\frac{1}{1+r}\Big)^{p^{k-2}} \Big(\frac{1}{1+r+r^2}\Big)^{p^{k-3}} \\ &\qquad \cdots \Big(\frac{1}{1+r+\cdots+r^{k-1}}\Big) \\ &= \Big(\frac{1}{2(r-1)}\Big)^{\frac{p^k-1}{p-1}} \prod_{j=1}^{k-1} \Big(\frac{r-1}{r^{j+1}-1}\Big)^{p^{k-j-1}}, \end{aligned}$$

and

$$\begin{aligned} B_k &:= \Big(\frac{1}{2p+1}\Big)^{p^{k-2}} \Big(\frac{1}{2(p+1)p+1}\Big)^{p^{k-3}} \cdots \Big(\frac{1}{2(1+p+\cdots+p^{k-2})p+1}\Big) \\ &\ge \Big(\frac{1}{2(p+1)}\Big)^{p^{k-2}} \Big(\frac{1}{2(p+1)(p+1)}\Big)^{p^{k-3}} \\ &\qquad \cdots \Big(\frac{1}{2(1+p+\cdots+p^{k-2})(p+1)}\Big) \\ &= \Big(\frac{1}{2(p+1)}\Big)^{\frac{p^{k-1}-1}{p-1}} \prod_{j=1}^{k-2} \Big(\frac{p-1}{p^{j+1}-1}\Big)^{p^{k-j-2}}. \end{aligned}$$

Therefore, by (3.7) we have

$$\begin{aligned} u(x,t) &\ge \Big(\frac{1}{2(r-1)}\Big)^{\frac{p^k-1}{p-1}} \prod_{j=1}^{k-1} \Big(\frac{r-1}{r^{j+1}-1}\Big)^{p^{k-j-1}} \\ &\times \Big(\frac{1}{2(p+1)}\Big)^{\frac{p^{k-1}-1}{p-1}} \prod_{j=1}^{k-2} \Big(\frac{p-1}{p^{j+1}-1}\Big)^{p^{k-j-2}} t^{\frac{2}{p-1}(p^k-1)} \Big([S(t)\varphi](x)\Big)^{p^k}, \end{aligned}$$

and we obtain

$$\begin{aligned} t^{\frac{2}{p-1}(1-\frac{1}{p^k})}[S(t)\varphi](x) &\le (2(r-1))^{\frac{1}{p-1}\frac{p^k-1}{p^k}} (2(p+1))^{\frac{1}{p-1}\frac{p^{k-1}-1}{p^k}} u(x,t)^{p^{-k}} \\ &\times \Big(\prod_{j=1}^{k-1} \Big(\frac{r-1}{r^{j+1}-1}\Big)^{p^{k-j-1}}\Big)^{-p^{-k}} \Big(\prod_{j=1}^{k-2} \Big(\frac{p-1}{p^{j+1}-1}\Big)^{p^{k-j-2}}\Big)^{-p^{-k}} \end{aligned} \tag{3.8}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0,\infty)$ and $t \in (0,T)$. On the other hand, we have

$$\log\Big(\prod_{j=1}^{\infty} \Big(\frac{p^{j+1}-1}{p-1}\Big)^{p^{-j-2}}\Big) = \sum_{j=1}^{\infty} p^{-j-2} \log\Big(\frac{p^{j+1}-1}{p-1}\Big)$$

$$\leq \sum_{j=1}^{\infty} p^{-j-2} \log\big((j+1)p^j\big) < \infty, \tag{3.9}$$

$$\log\Big(\prod_{j=1}^{\infty}\Big(\frac{r^{j+1}-1}{r-1}\Big)^{p^{-j-1}}\Big) = \sum_{j=1}^{\infty} p^{-j-1}\log\Big(\frac{r^{j+1}-1}{r-1}\Big)$$

$$\leq \sum_{j=1}^{\infty} p^{-j-1}\log\big((j+1)r^j\big) < \infty. \tag{3.10}$$

Then, by (3.8), (3.9), and (3.10) we can find a constant C, independent of T and the initial function φ, such that $t^{\frac{2}{p-1}}[S(t)\varphi](x) \leq C < \infty$ for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0,\infty)$ and $t \in (0,T)$. This implies (3.1), and Lemma 3.1 follows. □

We prove Theorem 1.1 by using Lemma 3.1.

Proof of Theorem 1.1. The proof is by contradiction. We assume that there exists a nontrivial global solution u of (1.1). For any $T > 0$, we put

$$\varphi_T(x') := \int_{\mathbb{R}^{N-1}} P(x'-y',0,T)\varphi(y')\,dy' + \int_0^T \int_{\mathbb{R}^N_+} P(x'-y',y_N,T-s)u(y,s)^p\,dy\,ds \tag{3.11}$$

for almost all $x' \in \mathbb{R}^{N-1}$. Then, by the Fubini theorem we see that the function $u_T(x,t) := u(x,t+T)$ is a global solution of (1.1) with $u_T(x,0) = \varphi_T(x')$ on $\partial\mathbb{R}^N_+$ (see also Remark 1.1 (i)). On the other hand, since u is a nontrivial global solution, there exists a constant $T > 0$ such that

$$\varphi_T(x') \geq \int_0^T \int_{\mathbb{R}^N_+} P(x'-y',y_N,T-s)u(y,s)^p\,dy\,ds > 0$$

for almost all $x' \in \mathbb{R}^{N-1}$. Therefore we can assume, without loss of generality, that

$$\varphi(x') > 0 \quad \text{for almost all } x' \in \mathbb{R}^{N-1}. \tag{3.12}$$

We consider the case $1 < p < p_*$. By (3.12) we can find a function $\tilde{\varphi} \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $\tilde{\varphi} \not\equiv 0$, such that $0 \leq \tilde{\varphi}(x') \leq \varphi(x')$ for almost all $x \in \mathbb{R}^{N-1}$. Put

$$M := \int_{\mathbb{R}^{N-1}} \tilde{\varphi}(y')\,dy' > 0.$$

Then, by (2.5), (2.6), and (2.11) with $r = \infty$ we have

$$[S(t)\varphi](x) - MP(x,t) = S(x_N)\big[[S(t)\varphi](x') - M\mathcal{P}(x',t)\big] = o(t^{-(N-1)})$$

as $t \to \infty$ uniformly for almost all $x \in \mathbb{R}^N_+$. This together with (2.2) implies

$$\lim_{t\to\infty} t^{1-N}\|S(t)\tilde{\varphi}\|_\infty = \lim_{t\to\infty} t^{1-N}\|MP(t)\|_\infty > 0. \tag{3.13}$$

On the other hand, by Lemma 3.1 we have

$$t^{\frac{2}{p-1}}\|S(t)\tilde{\varphi}\|_\infty \le t^{\frac{2}{p-1}}\|S(t)\varphi\|_\infty \le C, \qquad t > 0, \tag{3.14}$$

for some constant C. Then, since $p < p_*$, inequality (3.14) contradicts (3.13). Thus Theorem 1.1 follows in the case $p < p_*$.

We consider the case $p = p_*$. By an argument similar to that used in the case $p < p_*$ with minor modification we can find a constant C', independent of the initial function φ, such that $|\varphi|_1 \le C'$. This together with (3.11) implies

$$|\varphi_T|_1 = |u(T)|_1 \le C' \quad \text{for any } T > 0. \tag{3.15}$$

On the other hand, by (3.12) we can find a positive constant R such that

$$m := \int_{|y'|<R} \varphi(y')\,dy' > 0.$$

Then, since

$$1 + |x'-y'|^2 \le 2\left(1+|x'|^2\right)\left(1+|y'|^2\right), \qquad x', y' \in \mathbb{R}^{N-1},$$

by (1.2), (1.5), and (3.11) we have

$$\begin{aligned}\varphi_1(x') &\ge c_N \int_{\mathbb{R}^{N-1}} \left(1+|x'-y'|^2\right)^{-N}\varphi(y')\,dy' \\ &\ge 2^{-N}c_N(1+|x'|^2)^{-N}\int_{|y'|<R}(1+|y'|^2)^{-N}\varphi(y')\,dy' \ge c_1 m\mathcal{P}(x',1)\end{aligned}$$

for all $x' \in \mathbb{R}^{N-1}$, where c_1 is a positive constant. This implies that

$$\begin{aligned} u(x,t+1) = u_1(x,t) &\ge [S(t)\varphi_1](x) \\ &\ge c_1 m[S(t)\mathcal{P}(1)](x) = c_1 m P(x,t+1) \end{aligned} \tag{3.16}$$

for almost all $x = (x', x_N) \in \mathbb{R}^N_+$ and all $t > 0$. Then, by (1.7), (3.11), and (3.16) we have

$$\begin{aligned} |\varphi_T|_1 &\ge \int_{\mathbb{R}^{N-1}} \Big(\int_0^T \int_{\mathbb{R}^N_+} P(x'-y', y_N, t-s)u(y,s)^p\,dy\,ds\Big)\,dx' \\ &= \int_0^T \int_{\mathbb{R}^N_+} u(y,s)^p\,dy\,ds \ge (c_1 m)^p \int_1^T \int_{\mathbb{R}^N_+} P(y,s)^p\,dy\,ds \end{aligned} \tag{3.17}$$

for all $T > 1$. Furthermore, since $p = p_*$, there holds

$$\int_{\mathbb{R}^N_+} P(y,s)^p\,dy = c_N^p \|\Lambda^{-Np}\|_{1,\mathbb{R}^{N-1}} s^{-1}. \tag{3.18}$$

Therefore, by (3.17) and (3.18) we obtain $\lim_{T\to\infty} |\varphi_T|_1 = \infty$. This contradicts (3.15). Thus Theorem 1.1 follows in the case $p = p_*$, and the proof of Theorem 1.1 is complete. □

Remark 3.1. In the case $N = 2$, similarly to Definition 1.1, we can define a solution of (1.1) with G replaced by

$$-\frac{1}{2\pi}\log\frac{|x-y|}{|x-y_*|} > 0.$$

Then, applying the same argument as in the proof of Theorem 1.1, we see that the conclusion of Theorem 1.1 holds; that is, problem (1.1) has no nontrivial global solutions in the case $1 < p \le p_*$.

4. Local existence

Let $p > p_*$. In this section we assume that the initial function φ is sufficiently small, and construct a local solution of (1.1).

Let $\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, and put

$$\lambda_\varphi := \max\{|\varphi|_1, |\varphi|_\infty\}. \tag{4.1}$$

Then we have

$$|\varphi|_q \le |\varphi|_\infty^{1-\frac{1}{q}} |\varphi|_1^{\frac{1}{q}} \le \lambda_\varphi \quad \text{for any} \quad q \in [p,\infty]. \tag{4.2}$$

Let $u_1(x,t) \equiv 0$ for $(x,t) \in \mathbb{R}^N_+ \times (0,\infty)$. Then, by induction, for any $n = 2, 3, \dots$, we define

$$u_{n+1}(x,t) := [S(t)\varphi](x) + g_n(x,t) + W_n(x,t) \tag{4.3}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$ and $t > 0$, where

$$\begin{aligned} f_n(x,t) &:= u_n(x,t)^p, & g_n(x,t) &:= \left[(-\Delta_D)^{-1} f_n(t)\right](x),\\ w_n(x,t) &:= W[f_n(t)](x), & W_n(x,t) &:= \int_0^t [S(t-s)w_n(s)](x)\,ds. \end{aligned} \tag{4.4}$$

Then we can prove inductively that

$$0 \le u_{n-1}(x,t) \le u_n(x,t) \tag{4.5}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$ and $t > 0$, and we can define the function

$$u_*(x,t) := \lim_{n\to\infty} u_n(x,t) \in [0,\infty] \tag{4.6}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$ and $t > 0$. In this section we prove the following proposition, which ensures that u_* is a solution of (1.1) in $\mathbb{R}^N_+ \times (0, \sigma]$ for some $\sigma > 0$ if λ_φ is sufficiently small.

Proposition 4.1. *Assume $p > p_*$. Let $\sigma > 0$. Then the following holds:*
(i) *There exists a positive constant ε_1 such that, if $\lambda_\varphi \leq \varepsilon_1$, then the function u_* defined by (4.6) is a minimal solution of (1.1) in $\mathbb{R}^N_+ \times [0, \sigma]$ such that*

$$\sup_{0<t\leq\sigma} \Big(\|u_*(t)\|_p + \|u_*(t)\|_\infty + |||u_*(t)|||_p \Big) < \infty. \tag{4.7}$$

(ii) *Let v be a solution of (1.1) in $\mathbb{R}^N_+ \times [0, \sigma]$ with the initial function $\phi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. If v satisfies (4.7), then*

$$\sup_{0<t\leq\sigma} \|v(t)\|_q < \infty \quad \text{for any } q \in (p_*, \infty], \qquad \sup_{0<t\leq\sigma} |v(t)|_\infty < \infty \tag{4.8}$$

and

$$v \in C([0, \sigma] : L^r(\partial\mathbb{R}^N_+)) \quad \text{for any } r \in [1, \infty). \tag{4.9}$$

Furthermore,

$$\begin{aligned} &\partial_x^\alpha v \in C(\mathbb{R}^N_+ \times (0, \sigma]) \ (|\alpha| \leq 2), \ \ v, \partial_x v \in C(\overline{\mathbb{R}^N_+} \times (0, \sigma]), \\ &\partial_t v \in C(\partial\mathbb{R}^N_+ \times (0, \sigma]) \end{aligned} \tag{4.10}$$

and v satisfies

$$-\Delta v = v^p \quad \text{in} \quad \mathbb{R}^N_+ \times (0, \sigma], \qquad \partial_t v + \partial_\nu v = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ \times (0, \sigma], \tag{4.11}$$

pointwise. In addition, if the initial function ϕ is a continuous function in $\mathbb{R}^{N-1}$, then v is a classical solution of (1.1) in $\mathbb{R}^N_+ \times [0, \sigma]$.

In order to prove Proposition 4.1, we prepare the following lemma.

Lemma 4.1. *Assume $p > p_*$. Assume that*

$$d_n := \sup_{0<t\leq\sigma} [\|u_n(t)\|_p + \|u_n(t)\|_\infty + |||u_n(t)|||_p] < \infty$$

for some $\sigma > 0$ and $n \in \{1, 2, \dots\}$. Then there exists a constant c_1, independent of n, such that

$$\|g_n(t)\|_p + \|g_n(t)\|_\infty + |||g_n(t)|||_p \leq c_1 d_n^p, \tag{4.12}$$

$$\|W_n(t)\|_p + \|W_n(t)\|_\infty + |||W_n(t)|||_p \leq c_1 d_n^p, \tag{4.13}$$

for all $t \in (0, \sigma]$. Furthermore, there exists a constant c_2, independent of n, such that

$$d_{n+1} \leq c_2(\lambda_\varphi + d_n^p). \tag{4.14}$$

Proof. Similarly to (4.2), we have

$$\|u_n(t)\|_q \le \|u_n(t)\|_\infty^{1-\frac{p}{q}} \|u_n(t)\|_p^{\frac{p}{q}} \le d_n \quad \text{for any} \quad q \in [p,\infty], \tag{4.15}$$

and we obtain

$$\begin{aligned} &\|f_n(t)\|_r \le \|u_n(t)\|_{pr}^p \le d_n^p, \qquad |||f_n(t)|||_1 \le |||u_n(t)|||_p^p \le d_n^p, \\ &\|f_n(t)\|_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} \le \|f_n(t)\|_1 + |||f_n(t)|||_1 \le 2d_n^p, \end{aligned} \tag{4.16}$$

for all $t \in (0,\sigma]$ and $1 \le r \le \infty$. Then, by (4.16) and $p > p_*$, applying Lemmata 2.4, 2.5, and 2.6, we can find a constant C_1 such that

$$\begin{aligned} &\|g_n(t)\|_p \le C_1 \|f_n(t)\|_{p/p_*}^{\frac{1}{p_*}} |||f_n(t)|||_1^{\frac{2}{N+1}} \le C_1 d_n^p, \\ &\|g_n(t)\|_\infty \le C_1(\|f_n(t)\|_1 + \|f_n(t)\|_\infty) \le C_1 d_n^p, \\ &|||g_n(t)|||_p \le C_1(\|f_n(t)\|_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} + \|f_n(t)\|_\infty) \le 3C_1 d_n^p \end{aligned} \tag{4.17}$$

for all $t \in (0,\sigma]$. These imply inequality (4.12).

On the other hand, by Lemma 2.3 and (4.4) we can find a constant $C_2 > 1$ such that

$$|w_n(t)|_\infty \le C_2(\|f_n(t)\|_1 + \|f_n(t)\|_\infty) \le C_2 d_n^p \tag{4.18}$$

for all $t \in (0,\sigma]$. Then, since it follows from (1.3) and (4.16) that

$$|w_n(t)|_1 = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^N_+} \mathcal{P}(x'-y', y_N) f_n(y,t)\,dy\,dx' = \int_{\mathbb{R}^N_+} f_n(y,t)\,dy \le d_n^p, \tag{4.19}$$

similarly to (4.15), by (4.18) we have

$$|w_n(t)|_q \le C_2^{1-1/q} d_n^p \le C_2 d_n^p, \qquad 1 \le q \le \infty, \tag{4.20}$$

for all $t \in (0,\sigma]$. Furthermore, by (2.7) and (4.18) we have

$$\|W_n(t)\|_\infty \le \int_0^t \|S(t-s)w_n(s)\|_\infty\,ds \le \int_0^t |w_n(s)|_\infty\,ds \le C_2\sigma d_n^p \tag{4.21}$$

for all $t \in (0,\sigma]$. In addition, since $p > p_* > N/(N-1)$, we apply Lemmata 2.1 and 2.2, and by (4.20) we obtain

$$\begin{aligned} \|W_n(t)\|_p &\le \int_0^t \|S(t-s)w_n(s)\|_p\,ds \\ &\qquad \le C_3 \int_0^t (|w_n(s)|_p + |w_n(s)|_1)\,ds \le 2C_2C_3\sigma d_n^p, \\ |||W_n(t)|||_p &\le \int_0^t |||S(t-s)w_n(s)|||_p\,ds \end{aligned}$$

$$\leq C_3 \int_0^t (|w_n(s)|_p + |w_n(s)|_1)\, ds \leq 2C_2C_3d_n^p,$$

for all $t \in (0, \sigma]$, where C_3 is a constant. These together with (4.21) imply inequality (4.13). By (4.3), (4.12), and (4.13) we have

$$d_{n+1} \leq \sup_{0<t\leq\sigma} [\|S(t)\varphi\|_p + \|S(t)\varphi\|_\infty + |||S(t)\varphi|||_p] + C_5d_n^p.$$

This together with (2.7), (2.14), (2.16), and (4.2) implies (4.14). Therefore, the proof of Lemma 4.1 is complete. □

We prove Proposition 4.1.

Proof of Proposition 4.1. We prove assertion (i). Let $\tau > 0$ and $\sigma > 0$. Let ε_1 be a sufficiently small constant such that

$$(2c_2)^p\varepsilon_1^{p-1} \leq 1, \tag{4.22}$$

where c_2 is the constant given in Lemma 4.1. Assume $\lambda_\varphi \leq \varepsilon_1$. Then, since $u_2(x,t) = [S(t)\varphi](x)$, by (2.7), (2.14), (2.16), and (4.2) we see that there exists a constant C_1 such that

$$\sup_{0<t\leq\sigma} \Big(\|u_2(t)\|_p + \|u_2(t)\|_\infty + |||u_2(t)|||_p\Big) \leq C_1\lambda_\varphi \leq C_1\varepsilon_1. \tag{4.23}$$

Taking a sufficiently small ε_1 if necessary, by Lemma 4.1 we have

$$\sup_{0<t\leq\sigma} \Big(\|u_3(t)\|_p + \|u_3(t)\|_\infty + |||u_3(t)|||_p\Big) \leq c_2(\lambda_\varphi + (C_1\varepsilon_1)^p) \leq 2c_2\varepsilon_1.$$

Applying Lemma 4.1 again, by (4.22) we have

$$\sup_{0<t\leq\sigma} \Big(\|u_4(t)\|_p + \|u_4(t)\|_\infty + |||u_4(t)|||_p\Big) \leq c_2(\lambda_\varphi + (2c_2\varepsilon_1)^p) \leq 2c_2\varepsilon_1.$$

Repeating this argument, we have

$$\sup_{0<t\leq\sigma} \Big(\|u_n(t)\|_p + \|u_n(t)\|_\infty + |||u_n(t)|||_p\Big) \leq 2c_2\varepsilon_1 \tag{4.24}$$

for all $n = 2, 3, \ldots$. Then, by (4.3), (4.4), (4.5), and (4.24) we see that the limit function $u_*(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a minimal solution of (1.1) in $\mathbb{R}^N_+ \times (0, \sigma]$, satisfying

$$\sup_{0<t\leq\sigma} \Big(\|u_*(t)\|_p + \|u_*(t)\|_\infty + |||u_*(t)|||_p\Big) \leq 2c_2\varepsilon_1. \tag{4.25}$$

On the other hand, for any solution $\tilde{u}$ of (1.1) in $\mathbb{R}^N_+ \times (0, \sigma]$, by (1.8) we have $\tilde{u}(x,t) \geq u_2(x,t) = [S(t)\varphi](x)$ for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty]$ and $t \in (0, \sigma]$. Similarly to (4.5), we can prove inductively that $\tilde{u}(x,t) \geq u_n(x,t)$, $n = 2, 3, \ldots$, for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty]$

and $t \in (0,\sigma]$. This together with (4.6) implies that the function u_* is a minimal solution of (1.1) in $\mathbb{R}^N_+ \times (0,\sigma]$. Thus assertion (i) follows.

We prove assertion (ii). Let v be a solution of (1.1) in $\mathbb{R}^N_+ \times (0,\sigma]$ for some $\sigma > 0$, satisfying (4.7). Put

$$\begin{aligned} &f_v(x,t) := v(x,t)^p, \qquad g_v(x,t) := \left[(-\Delta_D)^{-1} f_v(t)\right](x),\\ &w_v(x') := W[f_v(t)](x'). \end{aligned} \tag{4.26}$$

Then, by the same argument as in Lemma 4.1 we have

$$\sup_{0<t\le\sigma} \|f_v(t)\|_r < \infty, \qquad \sup_{0<t\le\sigma} |w_v(t)|_r < \infty, \tag{4.27}$$

for any $r \in [1,\infty]$, and obtain

$$\sup_{0<t\le\sigma} \|g_v(t)\|_\infty < \infty, \quad \int_0^\sigma \|S(t-s)w_v(s)\|_\infty \, ds < \infty. \tag{4.28}$$

On the other hand, for any $q \in (p_*,\infty)$, we apply Lemmata 2.2 and 2.4, and by (4.27) we obtain

$$\begin{aligned} &\|g_v(t)\|_q \le C_2 \|f_v(t)\|_{q/p_*}^{\frac{1}{p_*}} \, |||f_v(t)|||_1^{\frac{2}{N+1}} \le C_3,\\ &\int_0^t \|S(t-s)w_v(s)\|_q \, ds \le C_2 \int_0^t (|w_v(s)|_q + |w_v(s)|_1) \, ds \le C_3 \end{aligned} \tag{4.29}$$

for all $t \in (0,\sigma]$, where C_2 and C_3 are constants. Therefore, by (1.8), (2.10), (4.27), (4.28), and (4.29) we have

$$\sup_{0<t\le\sigma} \|v(t)\|_q < \infty, \qquad \sup_{0<t\le\sigma} |v(t)|_r < \infty, \tag{4.30}$$

for any $q \in (p_*,\infty]$ and $r \in [1,\infty]$. This implies (4.8). Moreover, by (1.8) (see also Remark 1.1 (i)), (2.8), (2.10), (4.27), and (4.30), for any $r \in [1,\infty)$, we have

$$\begin{aligned} &\lim_{\tau\to+0} |v(t+\tau) - v(t)|_r\\ &\le \lim_{\tau\to0} |S(\tau)\varphi_t - \varphi_t|_r + \lim_{\tau\to0} \int_t^{t+\tau} |S(t+\tau-s)w_v(s)|_r \, ds\\ &\le C_4 \lim_{\tau\to0} \int_t^{t+\tau} |w_v(s)|_r \, ds = 0 \end{aligned}$$

for all $0 \le t < \sigma$, where $\varphi_t(x') := v(x',0,t)$ and C_4 is a constant. Similarly we have

$$\lim_{\tau\to-0} |v(t+\tau) - v(t)|_r = 0$$

for all $0 < t \le \sigma$. These imply (4.9). In addition, by [11, Section 6.3.5] and [9, Appendix], for any $\eta \in (0, \sigma)$, we can find constants C_5 and $\alpha \in (0, 1)$ such that

$$|v(x,t) - v(y,s)| \le C_5 \left(|x-y|^\alpha + |t-s|^\alpha\right)$$

for all $(x,t), (y,s) \in \overline{\mathbb{R}^N_+} \times [\eta, \sigma]$. Then, by an argument similar to that in [12, Section 4] we see that $g_v(t) \in C^2(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}}^N_+)$, $t \in (0, \sigma]$, and g_v is a solution of (1.10). Therefore, by the same argument as in (1.12) we see that the function $h_v := v - g_v$ satisfies the integral identity (1.11) for all $x' \in \mathbb{R}^{N-1}$ and $t \in (0, \sigma]$. Finally we apply arguments similar to those in [10, Section 3, Chapter 1] to h_v, and see that (4.10) holds. Moreover we see that v satisfies (4.11) pointwise, and that v is a classical solution of (1.1) if the initial function φ is continuous in $\mathbb{R}^{N-1}$. Thus assertion (ii) follows, and the proof of Proposition 4.1 is complete. □

Proposition 4.2. *Let $\varphi_i \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ $(i = 1, 2)$. For any $T > 0$, there exists a positive constant ϵ such that, if $\max\{\lambda_{\varphi_1}, \lambda_{\varphi_2}\} \le \epsilon$, the following holds:*

(i) *For $i = 1, 2$, there exists a minimal solution v_i of (1.1) in $\mathbb{R}^N_+ \times (0, T]$ with the initial data φ_i. Furthermore, for any $q \in (p_*, \infty]$, there holds*

$$\sup_{0<t\le T} \|v_1(t) - v_2(t)\|_q \le C(|\varphi_1 - \varphi_2|_1 + |\varphi_1 - \varphi_2|_\infty) \tag{4.31}$$

for some constant C;

(ii) *If $\varphi_1(x') \le \varphi_2(x')$ for almost all $x' \in \mathbb{R}^{N-1}$, then*

$$v_1(x', x_N, t) \le v_2(x', x_N, t)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$ and $t > 0$.

Proof. By Proposition 4.1 and the uniqueness of minimal solutions we see that, if λ_{φ_i} is sufficiently small, then there exists a minimal solution v_i of (1.1) in $\mathbb{R}^N_+ \times (0, T]$ with the initial data φ_i. Furthermore, assertion (ii) follows from the construction of u_* in (4.3)–(4.6).

It remains to prove (4.31). Taking a sufficiently small ϵ if necessary, by (4.25) we have

$$\sup_{0<t\le T} (\|v_i(t)\|_p + \|v_i(t)\|_\infty + |||v_i(t)|||_p) \le C_1\epsilon, \qquad i = 1, 2, \tag{4.32}$$

for some constant C_1. Put

$$\tilde{v} := v_1 - v_2, \quad F(x,t) := v_1(x,t)^p - v_2(x,t)^p,$$

$$w(x,t) := \begin{cases} \dfrac{v_1(x,t)^p - v_2(x,t)^p}{v_1(x,t) - v_2(x,t)} & \text{if} \quad v_1(x,t) \neq v_2(x,t), \\ pv_1(x,t)^{p-1} & \text{if} \quad v_1(x,t) = v_2(x,t). \end{cases}$$

Then, by (4.32) we have

$$\begin{aligned} &\|F(t)\|_1 = \|w(t)\tilde{v}(t)\|_1 \leq \|w(t)\|_{p/(p-1)}\|\tilde{v}(t)\|_p \leq C_2\epsilon^{p-1}\|\tilde{v}(t)\|_p, \\ &\|F(t)\|_\infty = \|w(t)\tilde{v}(t)\|_\infty \leq \|w(t)\|_\infty\|\tilde{v}(t)\|_\infty \leq C_2\epsilon^{p-1}\|\tilde{v}(t)\|_\infty, \\ &|||F(t)|||_1 = |||w(t)\tilde{v}(t)|||_1 \leq |||w(t)|||_{p/(p-1)} |||\tilde{v}(t)|||_p \leq C_2\epsilon^{p-1} |||\tilde{v}(t)|||_p \,, \end{aligned}$$

for all $t \in (0,T]$, where C_2 is a constant. Since $\tilde{v}$ satisfies

$$\tilde{v}(x,t) = [S(t)(\varphi_1 - \varphi_2)](x) + [(-\Delta_D)^{-1}F(t)](x) + \int_0^t [S(t-s)W[F(s)]](x)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$ and $t \in (0,T]$, by arguments similar to those in the proof of Proposition 4.1 we have

$$\begin{aligned} \sup_{0<t\leq T}[\|\tilde{v}(t)\|_p + \|\tilde{v}(t)\|_\infty + |||\tilde{v}(t)|||_p] &\leq C_3(|\varphi_1 - \varphi|_1 + |\varphi_1 - \varphi|_\infty) \\ &+ C_3\epsilon^{p-1} \sup_{0<t\leq T}[\|\tilde{v}(t)\|_p + \|\tilde{v}(t)\|_\infty + |||\tilde{v}(t)|||_p] \end{aligned}$$

for some constant C_3. Then, taking a sufficiently small ϵ if necessary, we have

$$\sup_{0<t\leq T}[\|\tilde{v}(t)\|_p + \|\tilde{v}(t)\|_\infty + |||\tilde{v}(t)|||_p] \leq 2C_3(|\varphi_1 - \varphi|_1 + |\varphi_1 - \varphi|_\infty).$$

This implies (4.31) in the case $q \in [p,\infty]$. Furthermore, applying the same argument as in the proof of (4.8), we have (4.31) in the case $q \in (p_*,\infty)$. Thus we obtain (4.31), and the proof of Proposition 4.2 is complete. □

5. Proof of Theorem 1.2

In this section we prove that the solution u_* constructed in Proposition 4.1 (i) is a global solution of (1.1) if λ_φ is sufficiently small, and complete the proof of Theorem 1.1. We use the same notation as in Section 4.

We first prove the following lemma.

Lemma 5.1. *Assume $p > p_*$. Assume that*

$$\begin{aligned} D_n := \sup_{0<t<\infty} \Big[&(1+t)^{N-1-\frac{N}{p}}\|u_n(t)\|_p + (1+t)^{N-1}\|u_n(t)\|_\infty \\ &+ (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u_n(t)|||_p\Big] < \infty \qquad (5.1) \end{aligned}$$

for some $n \in \{1, 2, \dots\}$*. Then there exists a constant* c_3*, independent of* n*, such that*

$$(1+t)^{N-1-\frac{N}{p}}\|g_n(t)\|_p + (1+t)^{N-1}\|g_n(t)\|_\infty + (1+t)^{(N-1)(1-\frac{p_*}{p})}\,|||g_n(t)|||_p \le c_3 D_n^p, \tag{5.2}$$

$$(1+t)^{N-1-\frac{N}{p}}\|W_n(t)\|_p + (1+t)^{N-1}\|W_n(t)\|_\infty + (1+t)^{(N-1)(1-\frac{p_*}{p})}\,|||W_n(t)|||_p \le c_3 D_n^p, \tag{5.3}$$

for all $t > 0$*. Furthermore, there exists a constant* c_4*, independent of* n*, such that*

$$D_{n+1} \le c_4(\lambda_\varphi + D_n^p). \tag{5.4}$$

Proof. Similarly to (4.15), by (5.1) we have

$$\|u_n(t)\|_q \le \|u_n(t)\|_\infty^{1-\frac{p}{q}}\|u_n(t)\|_p^{\frac{p}{q}} \le D_n(1+t)^{-(N-1)+\frac{N}{q}}, \qquad t > 0,$$

for $q \in [p, \infty]$. This together with (5.1) and $p > p_*$ implies

$$\begin{aligned} &\|f_n(t)\|_r \le \|u_n(t)\|_{pr}^p \le D_n^p(1+t)^{-(N-1)p+\frac{N}{r}},\\ &|||f_n(t)|||_1 \le |||u_n(t)|||_p^p \le D_n^p(1+t)^{-(N-1)p(1-\frac{p_*}{p})}\\ &\|f_n(t)\|_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} \le D_n^p(1+t)^{-(N-1)p(1-\frac{p_*}{p})}, \end{aligned} \tag{5.5}$$

for all $t > 0$. Let $q_1 \in [1, N/2)$ be such that $q_1 > \frac{N}{(N-1)(p-p_*)+2}$. Then, since $p > p_*$, similarly to (4.17), we can find a constant C_1 such that

$$\begin{aligned} \|g_n(t)\|_p &\le C_1\|f_n(t)\|_{p/p_*}^{\frac{1}{p_*}}\,|||f_n(t)|||_1^{\frac{2}{N+1}}\\ &\le C_1 D_n^p(1+t)^{-\frac{N+1}{p_*}(p-p_*)-N+1+\frac{N}{p}} \le C_1 D_n^p(1+t)^{-N+1+\frac{N}{p}},\\ \|g_n(t)\|_\infty &\le C_1(\|f_n(t)\|_\infty + \|f_n(t)\|_{q_1})\\ &\le C_1 D_n^p(1+t)^{-(N-1)p+\frac{N}{q_1}} \le C_1 D_n^p(1+t)^{-N+1},\\ |||g_n(t)|||_p &\le C_1(\|f_n(t)\|_{L^1(\mathbb{R}^N_+,(1+x_N)\,dx)} + \|f_n(t)\|_\infty)\\ &\le 2C_1 D_n^p(1+t)^{-(N-1)p(1-\frac{p_*}{p})} \le 2C_1 D_n^p(1+t)^{-(N-1)(1-\frac{p_*}{p})} \end{aligned} \tag{5.6}$$

for all $t > 0$. These imply inequality (5.2).

Let $q_2 \in [1, N)$ be such that

$$q_2 > \frac{N}{(N-1)(p-p_*)+1}. \tag{5.7}$$

By Lemma 2.3 and (5.5), for any $q \in [1, N)$, we can find a constant C_2 such that

$$|w_n(t)|_\infty \le C_2(\|f_n\|_q + \|f_n\|_\infty) \le C_2 D_n^p(1+t)^{-(N-1)p+\frac{N}{q}} \tag{5.8}$$

for all $t > 0$. Similarly to (4.19), by (5.5), we have

$$|w_n(t)|_1 = \|f_n(t)\|_1 \le D_n^p (1+t)^{-(N-1)p+N} \tag{5.9}$$

for all $t > 0$. Then, since $p > p_*$, by Lemma 2.1, (2.7), (5.7), (5.8), and (5.9) we have

$$\begin{aligned}
&\|W_n(t)\|_\infty \le \int_0^t \|S(t-s)w_n(s)\|_\infty \, ds \\
&\le C_2 \int_0^{t/2} (t-s)^{-(N-1)} |w_n(s)|_1 \, ds + C_2 \int_{t/2}^t |w_n(s)|_\infty \, ds \\
&\preceq D_n^p t^{-(N-1)} \int_0^\infty (1+s)^{-(N-1)p+N} \, ds + D_n^p \int_{t/2}^t (1+s)^{-(N-1)p+\frac{N}{q_2}} \, ds \\
&\preceq D_n^p (t^{-(N-1)} + t^{-(N-1)p+\frac{N}{q_2}+1}) \le C_3 D_n^p t^{-(N-1)} \qquad (5.10)
\end{aligned}$$

for all $t \ge 1$, where C_3 is a constant.

Let $\gamma = (N+1)/N$, and put

$$\begin{aligned}
\eta(\xi) &:= \gamma(N-1)\xi(\xi - p_*) + \gamma\xi - N(\xi - \gamma) \\
&= \gamma(N-1)\xi^2 + (-\gamma(N-1)p_* + \gamma - N)\xi + N\gamma
\end{aligned}$$

for $\xi \in [p_*, \infty)$. Then we have

$$\begin{aligned}
\eta(p_*) &= \gamma p_* - N(p_* - \gamma) = \frac{N+1}{N(N-1)} = \frac{p_*}{N}, \\
\eta'(p_*) &= \gamma(N-1)p_* + \gamma - N > 0, \\
\Big(\frac{\eta(\xi)}{\xi}\Big)' &= \frac{\gamma(N-1)\xi^2 - N\gamma}{\xi^2} > 0, \qquad \xi \in [p_*, \infty),
\end{aligned}$$

and obtain $\eta(\xi) > 0$, $\frac{\eta(\xi)}{\xi} > \frac{\eta(p_*)}{p_*} = \frac{1}{N}$ for $\xi \in (p_*, \infty)$. So we can find a constant $q_3 \in [1, N)$ such that

$$q_3 > \frac{p}{\eta(p)}. \tag{5.11}$$

Since $p > p_* = N\gamma/(N-1)$, by Lemma 2.1 we have

$$\begin{aligned}
&\|W_n(t)\|_p \le \int_0^t \|S(t-s)w_n(s)\|_p \, ds \\
&\preceq \int_0^{t/2} (t-s)^{-(N-1)(1-\frac{1}{p})+\frac{1}{p}} |w_n(s)|_1 \, ds + \int_{t/2}^t (|w_n(s)|_p + |w_n(s)|_\gamma) \, ds
\end{aligned}$$

for all $t > 0$. This together with (5.7), (5.8), (5.9), and (5.11) implies

$$\begin{aligned}
\|W_n(t)\|_p &\preceq D_n^p t^{-(N-1)+\frac{N}{p}} \int_0^\infty (1+s)^{-(N-1)p+N} ds \\
&\quad + D_n^p \int_{t/2}^t \left\{ s^{-(N-1)p+\frac{N}{q_2}(1-\frac{1}{p})+\frac{N}{p}} ds + s^{-(N-1)p+\frac{N}{q_3}(1-\frac{1}{\gamma})+\frac{N}{\gamma}} \right\} ds \\
&\preceq D_n^p \left(t^{-(N-1)+\frac{N}{p}} + t^{-(N-1)p+\frac{N}{q_2}(1-\frac{1}{p})+\frac{N}{p}+1} + t^{-(N-1)p+\frac{N}{q_3}(1-\frac{1}{\gamma})+\frac{N}{\gamma}+1} \right) \\
&\le C_4 D_n^p t^{-(N-1)+\frac{N}{p}} \qquad (5.12)
\end{aligned}$$

for all $t \ge 1$, where C_4 is a constant. Here we remark

$$-(N-1)p + \frac{N}{q_2}(1-\frac{1}{p}) + \frac{N}{p} + 1 < -(N-1) + \frac{N}{p}$$

and

$$\begin{aligned}
&-(N-1)p + \frac{N}{q_3}(1-\frac{1}{\gamma}) + \frac{N}{\gamma} + 1 \\
&< -(N-1)p + \frac{N\eta(p)}{p}(1-\frac{1}{\gamma}) + \frac{N}{\gamma} + 1 = -(N-1) + \frac{N}{p}.
\end{aligned}$$

Moreover, by Lemma 2.2, (5.8), and (5.9) we have

$$\begin{aligned}
|||W_n(t)|||_p &\le \int_0^t |||S(t-s)w_n(s)|||_p \, ds \\
&\preceq \int_0^{t/2} (t-s)^{-(N-1)(1-\frac{p_*}{p})} |w_n(s)|_1 \, ds + \int_{t/2}^t (|w_n(s)|_p + |w_n(s)|_1) \, ds \\
&\preceq D_n^p t^{-(N-1)(1-\frac{p_*}{p})} \int_0^\infty (1+s)^{-(N-1)p+N} \, ds + D_n^p \int_{t/2}^t s^{-(N-1)p+N} \, ds \\
&\preceq D_n^p (t^{-(N-1)(1-\frac{p_*}{p})} + t^{-(N-1)(p-p_*)}) \le C_5 D_n^p t^{-(N-1)(1-\frac{p_*}{p})} \qquad (5.13)
\end{aligned}$$

for all $t \ge 1$, where C_5 is a constant. On the other hand, by Lemma 4.1 we can find a constant C_6 such that

$$\|W_n(t)\|_p + \|W_n(t)\|_\infty + |||W_n(t)|||_p \le C_6 D_n^p, \qquad (5.14)$$

for all $0 < t \le 1$. Therefore, by (5.10), (5.12), (5.13), and (5.14) we have inequality (5.3). Furthermore, by (4.3), (5.2), and (5.3) we have

$$\begin{aligned}
D_{n+1} \le \sup_{0<t<\infty} \Big[&(1+t)^{N-1-\frac{N}{p}} \|S(t)\varphi\|_p + (1+t)^{N-1} \|S(t)\varphi\|_\infty \\
&+ (1+t)^{(N-1)(1-\frac{p_*}{p})} |||S(t)\varphi|||_p \Big] + C_7 D_n^p
\end{aligned}$$

for some constant C_7, and we obtain (5.4). Thus Lemma 5.1 follows. $\square$

Lemma 5.2. *Assume $p > p_*$ and (4.1). Let u_* be the function defined by (4.6). Then there exists a constant ε_2 such that, if $\lambda_\varphi \le \varepsilon_2$, then the function u_* is a global minimal solution of (1.1), satisfying (1.15) and (1.16).*

Proof. Let ε_2 be a sufficiently small constant such that

$$(2c_4)^p \varepsilon_2^{p-1} \le 1, \tag{5.15}$$

where c_4 is the constant given in Lemma 5.1. Assume $\lambda_\varphi \le \varepsilon_2$. Then, since $u_2(x,t) = [S(t)\varphi](x)$, by Lemmata 2.1 and 2.2 we see that there exists a constant C_1 such that

$$\sup_{0<t<\infty} \Big[(1+t)^{N-1-\frac{N}{p}} \|u_2(t)\|_p + (1+t)^{N-1} \|u_2(t)\|_\infty. \\ + (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u_2(t)|||_p \Big] \le C_1 \lambda_\varphi \le C_1 \varepsilon_2.$$

Taking a sufficiently small ε_2 if necessary, by Lemma 5.1 we have

$$\sup_{0<t<\infty} \Big[(1+t)^{N-1-\frac{N}{p}} \|u_3(t)\|_p + (1+t)^{N-1} \|u_3(t)\|_\infty \\ + (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u_3(t)|||_p \Big] \le c_4(\lambda_\varphi + (C_1\varepsilon_2)^p) \le 2c_4 \varepsilon_2.$$

Similarly to the proof of Proposition 4.1 (i), repeating the above argument, we have

$$\sup_{0<t<\infty} \Big[(1+t)^{N-1-\frac{N}{p}} \|u_n(t)\|_p + (1+t)^{N-1} \|u_n(t)\|_\infty \\ + (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u_n(t)|||_p \Big] \le 2c_4 \varepsilon_2$$

for all $n = 2, 3, \ldots$, and see that the limit function $u_*(x,t) = \lim_{n\to\infty} u_n(x,t)$ is a global minimal solution of (1.1), satisfying

$$\sup_{0<t<\infty} \Big[(1+t)^{N-1-\frac{N}{p}} \|u_*(t)\|_p + (1+t)^{N-1} \|u_*(t)\|_\infty \\ + (1+t)^{(N-1)(1-\frac{p_*}{p})} |||u_*(t)|||_p \Big] \le 2c_2 \varepsilon_2. \tag{5.16}$$

Then, by the same arguments as in Proposition 4.1 (ii) and Lemma 5.1 with (5.16), for any $q \in (p_*, \infty)$, we have

$$\sup_{0<t<\infty} (1+t)^{N-1-\frac{N}{q}} \|u_*(t)\|_q < \infty.$$

This together with (5.16) implies (1.15) and (1.16), and Lemma 5.2 follows. $\square$

Now we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We prove assertion (i). Let $\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, and assume

$$|\varphi|_1 |\varphi|_\infty^{\frac{N-1}{2}(p-1)-1} < \delta := \varepsilon_2^{(N-1)(p-1)/2},$$

where ε_2 is the constant given in Lemma 5.2. Then there exists a constant $\mu > 0$ such that $|\varphi_\mu|_1 = |\varphi_\mu|_\infty \le \varepsilon_2$, where $\varphi_\mu(x') := \mu^{\frac{2}{p-1}}\varphi(\mu x')$ (see Remark 1.2). Then, by Lemma 5.2 we can find a global minimal solution u_μ of (1.1) with the initial function φ_μ, satisfying (1.15) and (1.16). This implies that the function $u(x', x_N, t) := \mu^{-\frac{2}{p-1}} u_\mu(\mu^{-1}x', \mu^{-1}x_N, \mu^{-1}t)$ is a global minimal solution of (1.1) with the initial function φ, satisfying (1.15) and (1.16). Thus assertion (i) follows.

We prove assertion (ii). Let v be a global solution of (1.1), satisfying (1.15) and (1.16). By Proposition 4.1 (ii) we see that v satisfies (1.1) in $\mathbb{R}^N_+ \times (0,\infty)$ in the classical sense. Put

$$\begin{aligned} c(t) &:= \int_{\mathbb{R}^{N-1}} v(x', 0, t)\, dx' \\ &= \int_{\mathbb{R}^{N-1}} \varphi(x')\, dx' + \int_0^t \int_{\mathbb{R}^N_+} v(x,s)^p\, dx\, ds, \quad t > 0. \end{aligned} \tag{5.17}$$

Then, by (1.15) and (5.17) we have

$$c(t_2) - c(t_1) = \int_{t_1}^{t_2} \int_{\mathbb{R}^N_+} v(x,s)^p\, dx\, ds \preceq \int_{t_1}^{t_2} (1+s)^{-(N-1)p+N} ds \tag{5.18}$$

for all $t_2 \ge t_1 \ge 0$. This implies that $c(t)$ converges to some constant C_* as $t \to \infty$ and (1.17) holds. Furthermore, by (5.18) we have

$$|c(t) - C_*| = O(t^{-(N-1)(p-p_*)}) \qquad \text{as} \quad t \to \infty. \tag{5.19}$$

Let $T \ge 1$ and put

$$\begin{aligned} R_T(x,t) &:= v(x,t) - c(T)P(x', x_N, t), \\ R_T^1(x,t) &:= [S(t-T)R_T(\cdot, 0, T)](x', x_N), \\ R_T^2(x,t) &:= \int_T^t S(t-s) w_v(x', x_N)\, ds \end{aligned}$$

(see also (4.26)). Then, by (1.8) and (2.6) we have

$$R_T(x,t) = R_T^1(x,t) + R_T^2(x,t) + g_v(x,t). \tag{5.20}$$

Since it follows from (1.3) and (5.17) that

$$\int_{\mathbb{R}^{N-1}} R_T^1(x',0,t)\,dx' = \int_{\mathbb{R}^{N-1}} R_T(x',0,T)\,dx' = 0,$$

by (2.11) and (2.13) we have

$$\lim_{t\to\infty} t^{N-1+\frac{N}{q}} \|R_T^1(t)\|_q \le C_1 \lim_{t\to\infty} |S((t-T)/2)R_T(T)|_1 = 0 \tag{5.21}$$

for any $q \in (p_*, \infty]$, where C_1 is a constant. Since v satisfies (1.15) and (1.16), by the same argument as in (4.29) and (5.6), for any $q \in (p_*, \infty)$, we have

$$t^{N-1-\frac{N}{q}} \|g_v(t)\|_q \preceq t^{N-1-\frac{N}{q}} \|f_v(t)\|_{q/p_*}^{\frac{1}{p_*}} |||f_v(t)|||_1^{\frac{2}{N+1}} \preceq t^{-\frac{N+1}{p_*}(p-p_*)} \tag{5.22}$$

for all sufficiently large t. Furthermore, we obtain

$$t^{N-1}\|g_v(t)\|_\infty \preceq t^{N-1}\|f_v(t)\|_{q_1} \preceq t^{-(N-1)(p-p_*)-2+\frac{N}{q_1}} \tag{5.23}$$

for all sufficiently large t. On the other hand, applying arguments similar to those in (5.10) and (5.12) with the aid of (1.15), for any $q \in (p_*, \infty]$, we have

$$\limsup_{t\to\infty} t^{(N-1)-\frac{N}{q}} \|R_T^2(t)\|_q \le C_2 T^{-(N-1)(p-p_*)}, \quad T \ge 1,$$

for some constant C_2. This together with (5.20), (5.21), (5.22), and (5.23) yields

$$\limsup_{t\to\infty} t^{N-1-\frac{N}{q}} \|R_T(t)\|_q \le C_2 T^{-(N-1)(p-p_*)}, \quad T \ge 1.$$

Therefore, for any $q \in (p_*, \infty]$, by (2.2) we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} t^{N-1-\frac{N}{q}} \|v(t) - C_* P(t)\|_q \\
&\le \limsup_{t\to\infty} t^{(N-1)-\frac{N}{q}} \|c(T)P(t) - C_*P(t)\|_1 + \limsup_{t\to\infty} t^{(N-1)-\frac{N}{q}} \|R_T(t)\|_q \\
&\le |c(T) - C_*| + C_2 T^{-(N-1)(p-p_*)}
\end{aligned}$$

for all $T \ge 1$. This together with (5.19) yields

$$\lim_{t\to\infty} t^{N-1-\frac{N}{q}} \|v(t) - C_* P(t)\|_q = 0$$

for any $q \in (p_*, \infty]$. Consequently we have (1.18), and assertion (ii) holds. Thus Theorem 1.2 follows. □

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