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WELL-POSEDNESS AND AVERAGING OF NLS WITH TIME-PERIODIC DISPERSION MANAGEMENT

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Abstract. We consider the Cauchy problem for dispersion-managed nonlinear Schrödinger equations, where the dispersion map is assumed to be periodic and piecewise constant in time. We establish local and global well-posedness results and the possibility of finite time blow-up. In addition, we shall study the scaling limit of fast dispersion management and establish convergence to an effective model with averaged dispersion.

1. Introduction

In this work, we study the Cauchy problem for the following class of *dispersion-managed nonlinear Schrödinger equations* (NLS):

$$i\partial_t u + \gamma(t)\Delta u + |u|^{p-1}u = 0 \qquad u(t_0, x) = \varphi(x) \tag{1.1}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, a given $t_0 \in \mathbb{R}_+$, and $p > 1$ to be specified below. In addition, we assume that the *dispersion map* γ is 1-*periodic*; i.e., $\gamma(t + 1) =$

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$\gamma(t)$ and is piecewise constant:

$$\gamma(t) = \begin{cases} \gamma_+ & 0 < t \leqslant t_+ \,, \\ -\gamma_- & t_+ < t \leqslant 1 \,, \end{cases} \tag{1.2}$$

where $\gamma_\pm > 0$ are some positive constants and $t_+ \in (0,1)$.

Our main motivation stems from models in nonlinear fiber optics. Indeed, the case of a cubic NLS, i.e., $p = 3$, in $d = 1$ spatial dimension naturally arises as an envelope equation for electromagnetic wave propagation in optical fibers exhibiting a (weak) Kerr nonlinearity. In this context, the variable $t \in \mathbb{R}_+$ actually corresponds to the distance along the fiber, and $x \in \mathbb{R}$ denotes the (retarded) time. The coefficient $\gamma(t)$ consequently models a periodically varying dispersion along the fiber; cf. [2, 32]. Dispersion-managed NLS in $d = 2$ spatial dimensions are also physically relevant; see, e.g., [1]. The technique of dispersion management was invented to balance the effects of nonlinearity and dispersion in such a way that stable nonlinear pulses (solitary waves) are supported over long distances; cf. [22, 25, 26, 34]. Due to the enormous practical implications, there is a huge literature concerned with the qualitative properties of (1.1). Most often, however, the results are based on non-rigorous asymptotics and/or numerical simulations. A notable exception is the regime of so-called strong dispersion management where several rigorous results are available for the corresponding asymptotic model; see the discussion below.

In contrast to that, we shall work directly on the dispersion-managed NLS and in the following prove several rigorous results concerning the well-posedness of (1.1) and its asymptotic behavior in the case of rapidly varying dispersion. Having in mind the physics background of fiber optics, we shall focus on the well-posedness theory for (large) data $\varphi \in L^2(\mathbb{R}^d)$ or $\varphi \in H^1(\mathbb{R}^d)$. By multiplying (1.1) with $\overline{u}$, integrating with respect to $x \in \mathbb{R}^d$, and taking the imaginary part of the resulting identity, we (formally) obtain *mass conservation*:

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Note that in the case *without* dispersion management, i.e., $\gamma(t) = \gamma \in \mathbb{R}$, equation (1.1) also conserves the *energy*

$$E(t) = \frac{\gamma}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1} \, dx = E(0). \tag{1.3}$$

Thus, in the *defocusing case* $\gamma < 0$ and when $p < \frac{d+2}{d-2}$, one immediately infers a uniform bound on the H^1 norm of u, prohibiting the appearance of *finite-time blow-up*. In view of (1.2), we can think of (1.1) as switching in between focusing and defocusing behavior (with respective dispersion coefficient $\gamma_+ > 0$ and $-\gamma_- < 0$), and we consequently expect the appearance of finite-time blow-up. That this is indeed the case will be proved in Section 3.2.

Remark 1.1. Note that this switching between focusing and defocusing behavior makes our problem very different from the NLS-type models studied in [16, 23]. In [16] the author considers an equation of the form (1.1) with time-dependent dispersion $\gamma(t) \geqslant 0$, possibly vanishing (with finite order) on a discrete set of points, a typical example being $\gamma(t) = |t - t_0|^\lambda$ for some $\lambda > 0$. In [23] an NLS-type model with time-dependent coefficients but with an additional constant-coefficient third-order spatial derivative is studied. The behavior of this model is similar to the KdV equation.

Even if finite-time blow-up in general can not be ruled out, one may still wonder if sufficiently *fast* switching between the focus and defocusing step can at least delay the appearance of blow-up. In order to gain more insight we shall therefore study the scaling limit corresponding to *fast dispersion management*; i.e., we shall consider

$$i\partial_t u_\varepsilon + \gamma\left(\tfrac{t}{\varepsilon}\right)\Delta u_\varepsilon + |u_\varepsilon|^{p-1}u_\varepsilon = 0, \qquad u_\varepsilon(t_0, x) = \varphi(x), \tag{1.4}$$

where $0 < \varepsilon \ll 1$ denotes a small parameter. This regime has been studied using formal asymptotics and numerical simulations in, e.g., [6, 7, 35]. As $\varepsilon \to 0_+$ we expect the behavior of the solution u^ε to be close (in some sense to be made precise; see Section 4) to the solution of the *averaged NLS*

$$i\partial_t u_0 + \langle\gamma\rangle\Delta u_0 + |u_0|^{p-1}u_0 = 0, \qquad u_0(t_0, x) = \varphi(x), \tag{1.5}$$

where we denote by

$$\langle\gamma\rangle := \int_0^1 \gamma(\tau)\, d\tau, \tag{1.6}$$

the average dispersion coefficient (which can be either positive or negative). In the case of mean zero dispersion $\langle\gamma\rangle = 0$ this scaling limit provides a possible explanation for the stabilizing effects of dispersion management; see Corollary 4.6.

The situation above should, however, be distinguished from the case of *strong dispersion management;* cf. [25, 17] for some physical motivation. One thereby considers a dispersion map of the form $\frac{1}{\varepsilon}\gamma\left(\frac{t}{\varepsilon}\right)$, which, as $\varepsilon \to 0_+$, leads to an effective description by a non-local equation, originally introduced in [17]. This model has been rigorously studied by several authors; see, e.g., [14, 20, 30, 31, 34]. In particular, it provides a mathematical basis for the definition of *dispersion-managed solitons* [14].

Remark 1.2. A similar situation is analyzed in [5, 13], where the authors consider NLS with fast *random* dispersion management. More precisely, they consider dispersion maps of the form $\frac{1}{\varepsilon}\gamma\left(\frac{t}{\varepsilon^2}\right)$, where γ is is a (smooth) stationary random process, and prove convergence of the equation towards an NLS with white noise dispersion. Finally, we want to mention that in [11, 12] the somewhat dual problem of NLS with (rapidly) *time-oscillating nonlinearity* has been analyzed (see also [9, 28] for a related studies on the KdV equation).

This paper is organized as follows: In Section 2, we shall collect some basic properties of the linear equation to be used in the following. Section 3 establishes global well-posedness in the L^2 subcritical regime. In addition, we set up a local well-posedness result in H^1 theory and prove the existence of finite-time blow-up. Section 4 is concerned with the fast dispersion limit. Finally, we shall collect some concluding remarks on possible generalizations and closely related problems in Section 5.

2. Basic properties of the linear equation

2.1. **The linear propagator.** Before studying the nonlinear Cauchy problem we shall collect some basic facts about the associated linear equation

$$i\partial_t u_{\text{lin}} + \gamma(t)\Delta u_{\text{lin}} = 0 \qquad u_{\text{lin}}(t_0, x) = \varphi(x). \tag{2.1}$$

In the following, we shall denote by

$$\Gamma(t,s) := \int_s^t \gamma(\tau)\, d\tau$$

the *cumulative dispersion* on the time interval $[s,t] \subset \mathbb{R}$ and the associated propagator $U(t,s)$ by

$$U(t,s)f(x) := e^{i\Gamma(t,s)\Delta} f(x) = \int_{\mathbb{R}^d} e^{-i\Gamma(t,s)|\xi|^2} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{2.2}$$

where $\widehat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R}^d)$. We directly infer the following result.

Lemma 2.1. *Let $t_0 \in \mathbb{R}$ be fixed. Then the mapping $t \mapsto U(t,t_0)$ defines a family of strongly continuous unitary operators on $L^2(\mathbb{R}^d)$, such that for all $t \in \mathbb{R}_+$ it holds that $u_{\text{lin}}(t,x) = U(t,t_0)\varphi(x)$. Explicitly, we find*

$$u_{\text{lin}}(t,x) = \frac{e^{-i\pi d/4}}{|4\pi\Gamma(t,t_0)|^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/(4\Gamma(t,t_0))} \varphi(y)\, dy, \tag{2.3}$$

provided $\Gamma(t,t_0) \neq 0$.

Some remarks are in order: First, one should note that for all $s, t \in \mathbb{R}$, $U(t, t_0)U(t_0, s) = U(t, s)$, and we also have $U(t, s)^* = U(t, s)^{-1} = U(s, t)$. However, $U(t, s) \neq U(t - s, 0)$, since

$$\int_s^t \gamma(\tau)\, d\tau = \int_0^{t-s} \gamma(\tau + s)\, d\tau \neq \int_0^{t-s} \gamma(\tau)\, d\tau,$$

unless $\gamma(t) = \text{const.}$ *for all* $t \in \mathbb{R}$. In other words, $U(t, s)$ is not a group. Second, we note that for $\varphi \in L^2(\mathbb{R}^d)$, it holds that $\partial_t u \in BV(\mathbb{R}; H^{-2}(\mathbb{R}^d))$, in view of (1.1) and the fact that $\gamma(t)$ is piecewise constant (including jump discontinuities). In particular, $\partial_t u$ is *not* continuous in time, in contrast to the case with constant dispersion.

Next, we recall the decomposition $\gamma(t) = \langle\gamma\rangle + \gamma_0(t)$, where $\langle\gamma\rangle \in \mathbb{R}$ denotes the average defined in (1.6), and $\gamma_0(t)$ is 1-periodic with mean zero. Using this, we can write

$$\Gamma(t, s) = \langle\gamma\rangle(t - s) + \int_s^t \gamma_0(\tau)\, d\tau. \tag{2.4}$$

We consequently infer that if $\langle\gamma\rangle = 0$, then $\Gamma(t, t_0) = 0$ for all $t \in \mathbb{R}$ such that $(t - t_0) \in \mathbb{N}$. On the other hand, if $\langle\gamma\rangle \neq 0$, then $\Gamma(t, t_0)$ vanishes at most finitely many times, since

$$-\left(\gamma_- + \langle\gamma\rangle\right) \leqslant \int_{t_0}^t \gamma_0(\tau)\, d\tau \leqslant \gamma_+ - \langle\gamma\rangle, \tag{2.5}$$

and thus $|\Gamma(t, t_0)| \gtrsim |t|$, provided $t \geqslant t_0$ is sufficiently large. Also note that if $t_0 \in \mathbb{N}$, i.e., if we start our time-evolution at the beginning of a focusing step, then

$$\vartheta(t) := \int_{t_0}^t \gamma_0(\tau)\, d\tau \geqslant 0,$$

for all times $t \in \mathbb{R}$, whereas if we start at $t = t_0$ at the beginning of a defocusing step, then $\vartheta(t) \leqslant 0$ for all times.

Remark 2.2. In the particular situation of mean-zero dispersion, i.e., $\langle\gamma\rangle = 0$, the solution u_{lin} is found to be 1-periodic in time. Indeed, in view of (2.2), we infer that the Fourier transformed solution is given by

$$\widehat{u}_{\text{lin}}(t, \xi) = \exp\Big(i|\xi|^2 \int_{t_0}^t \gamma_0(\tau)\, d\tau\Big)\widehat{\varphi}(\xi),$$

which satisfies $\widehat{u}_{\text{lin}}(n, x) = \widehat{\varphi}(x)$, for all $n \in \mathbb{N}$.

2.2. **Dispersive properties.** From what is said above it is clear that $U(t,s)$ in general does *not* allow one to infer uniform (in time) dispersive estimates (analogous to the usual Schrödinger group), since for arbitrary $s, t \in \mathbb{R}$

$$|\Gamma(t,s)| = \Big| \int_s^t \gamma(\tau)\, d\tau \Big| \not\gtrsim |t-s|,$$

due to fact that $\gamma(t)$ changes sign.

Remark 2.3. Clearly, this would not be the case if $|\gamma(t)| > 0$ and 1-periodic, in which case the behavior of (1.1), for all times $t \in \mathbb{R}$, would be either focusing or defocusing (but with time-dependent dispersion coefficient); see, e.g., [16].

The fact that $\gamma(t)$ is assumed to be piecewise constant allows us to obtain the following result.

Lemma 2.4. *Let $t_+ \in (0,1)$ be given and $t, s \in (n, n+t_+]$, or $t, s \in (n + t_+, n+1]$, for $n \in \mathbb{N}$. Then, for $t \neq s$, it holds that*

$$\|U(t,s)f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t-s|^{-d/2} \|f\|_{L^1(\mathbb{R}^d)}.$$

The point is that both t and s have to be in the *same* time interval corresponding to either a focusing or defocusing step.

Proof. The proof follows directly from the representation formula (2.3) and the fact that if $t, s \in (n, n+t_+]$, or $t, s \in (n+t_+, n+1]$ we have

$$\Gamma(t,s) = \int_s^t \gamma(\tau)\, d\tau = \pm \gamma_\pm (t-s),$$

in view of (1.2). □

Clearly, we have that

$$\bigcup_{n\in\mathbb{Z}} \big((n, n+t_+] \cup (n+t_+, n+1] \big) = \mathbb{R}, \tag{2.6}$$

and thus we can split the time axis $\mathbb{R}$ into a union of subintervals on which $U(t,s)$ allows for the usual dispersive behavior of the Schrödinger group. As a consequence, we know that the usual Strichartz estimates hold on each such time interval [19, 21]. To this end, let us recall that a pair (q,r) is *admissible* if $2 \leqslant r \leqslant \frac{2d}{d-2}$ (respectively $2 \leqslant r \leqslant \infty$ if $d=1$, $2 \leqslant r < \infty$ if $d=2$) and [10]: $\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right)$.

Lemma 2.5. *Let $n \in \mathbb{N}$ and $t_+ \in (0,1)$ be given. Then, for each admissible (q,r), (q_1,r_1), and(q_2,r_2), and each time interval $I_n \subset (n, n+t_+]$, or $I_n \subset (n+t_+, n+1]$, we have the following:*

1. *There exists $C_r = C(r, I_n)$, such that for $t_0 \in I_n$ and for any $f \in L^2(\mathbb{R}^d)$ it holds that*

$$\|U(\cdot, t_0)f\|_{L^q(I_n;L^r(\mathbb{R}^d))} \leqslant C_r \|f\|_{L^2(\mathbb{R}^d)}.$$

2. *There exists $C_{r_1,r_2} = C(r_1, r_2, I)$, such that for any $F \in L^{q_2'}(I_n; L^{r_2'})$ it holds that*

$$\Big\| \int_{I_n \cap \{s \leqslant t\}} U(t,s)F(s)\, ds \Big\|_{L^{q_1}(I_n;L^{r_1}(\mathbb{R}^d))} \leqslant C_{r_1,r_2} \|F\|_{L^{q_2'}(I_n;L^{r_2'}(\mathbb{R}^d))}.$$

Again, we can only conclude the existence of a Strichartz estimate locally on each time interval corresponding to either a focusing or defocusing step.

Remark 2.6. One may want to compare this to the case with *smooth* periodic dispersion management. For example, let $\gamma(t) = \cos(t)$ and $t_0 = 0$. In this case we find

$$u_{\text{lin}}(t,x) = \frac{e^{-i\pi d/4}}{|4\pi \sin(t)|^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/(4\sin(t))} \varphi(y)\, dy,$$

for which the corresponding dispersive estimate is again valid only for small times; i.e.,

$$\|U(t,0)f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2} \|f\|_{L^1(\mathbb{R}^d)}, \quad \text{for } |t| < \frac{\pi}{2}.$$

This situation is very similar to the case of a Schrödinger equation with quadratic confinement [8]. The corresponding Schrödinger group is formally given by $S(t,0)\varphi(x) = e^{it(\Delta+|x|^2)}\varphi(x)$, $t \in \mathbb{R}_+$, the kernel of which can be explicitly computed via Mehler's formula. Due to the existence of eigenfunctions, this time evolution enjoys dispersive properties only for sufficiently small $0 < |t| < \delta$; see [8] for more details.

3. Well-posedness results

3.1. **The L^2 subcritical case.** As we have seen, the propagator $U(t,s)$ does not allow for uniform dispersive estimates. Nevertheless, one can prove global well-posedness of (1.1) in the L^2 subcritical case.

Theorem 3.1. *Consider* (1.1) *with $\varphi \in L^2(\mathbb{R}^d)$ and $1 < p < 1 + \frac{4}{d}$. Then, for any $t_0 \in \mathbb{R}$, there exists a unique solution $u \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ satisfying $\|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}$. In addition, for each admissible pair (q,r) and for any compact time interval $I \subset \mathbb{R}$, $\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \leqslant C\big(|I|, \|u_0\|_{L^2(\mathbb{R}^d)}\big)$.*

Proof. Using Duhamel's formula the solution of (1.1) can be written as

$$u(t,x) = U(t,t_0)\varphi(x) + i\int_{t_0}^{t} U(t,s)|u(s,x)|^{p-1}u(s,x)\,ds. \tag{3.1}$$

Next, we use the decomposition (2.6) to split the time interval $[t_0,t] \in \mathbb{R}$ into countably many subintervals $[t_0,t] = \bigcup_{n\in\mathbb{Z}} \big([t_0,t]\cap I_n^1\big) \cup \big([t_0,t]\cap I_n^2\big)\big)$, with $I_n^1 = (n, n+t_+]$ and $I_n^2 = (n+t_+, n+1]$. In each of the intervals I_n^1 and I_n^2 we are able to apply the Strichartz estimates stated in Lemma 2.5. Indeed, for $1 < p < 1 + \frac{4}{d}$, the pair $(q,r) = \big(\frac{4p}{d(p-1)}, 2p\big)$ is admissible, and hence we have

$$\begin{aligned}\|u\|_{L_t^{\frac{4p}{d(p-1)}} L_x^{2p}(I\times\mathbb{R}^d)} &\lesssim \|u_0\|_{L^2} + \||u|^{p-1}u\|_{L_t^1 L_x^2(I\times\mathbb{R}^d)} \\ &\lesssim \|u_0\|_{L^2} + |I|^{\alpha}\|u\|^p_{L_t^{\frac{4p}{d(p-1)}} L_x^{2p}(I\times\mathbb{R}^d)},\end{aligned}$$

where $\alpha = \frac{4+d-dp}{4} > 0$. Hence, if $I \subset I_n^{1,2}$ is sufficiently small, a standard continuity argument implies that there exists a solution u to (1.1) in $I\times\mathbb{R}^d$ such that

$$\|u\|_{L_t^{\frac{4p}{d(p-1)}} L_x^{2p}(I\times\mathbb{R}^d)} \leqslant C(\|u_0\|_{L^2(\mathbb{R}^d)}).$$

The right-hand side only depends on the $L^2(\mathbb{R}^d)$-norm of the solution, which is uniformly bounded for all times (and in fact equal to $\|\varphi\|_{L^2}$). Continuity therefore implies that on each subinterval the solution exists for all times t in one of the subintervals I_n^1 and I_n^2, and, in addition, we have

$$\|u\|_{L_t^q L_x^r(I_n^{1,2}\times\mathbb{R}^d)} \leqslant C(\|u_0\|_{L^2}, |I_n^{1,2}|).$$

By considering the union of subintervals I_n we consequently infer the existence of a solution $u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^n))$.

In order to obtain the asserted continuity in time, we consider two different times $t_2 \neq t_1 \in \mathbb{R}_+$ for which we need to show that

$$\|u(t_1,\cdot) - u(t_2,\cdot)\|_{L^2(\mathbb{R}^n)} \to 0, \quad \text{as} \quad |t_1 - t_2| \to 0.$$

In view of Duhamel's formula (3.1) this requires

$$\|U(t_1,t_0)\varphi - U(t_2,t_0)\varphi\|_{L^2(\mathbb{R}^n)} \stackrel{t_1\to t_2}{\longrightarrow} 0,$$

which is nothing but the strong continuity of $U(t,t_0)$ stated in Lemma 2.1. For the second term on the right-hand side of (3.1) we write

$$\int_{t_0}^{t_1} U(t_1,s)|u(s,x)|^{p-1}u(s,x)\,ds - \int_{t_0}^{t_2} U(t_2,s)|u(s,x)|^{p-1}u(s,x)\,ds = i_1 + i_2,$$

where

$$i_1 = \big(U(t_1,t_0) - U(t_2,t_0)\big)\int_{t_0}^{t_2} U(t_0,s)|u(s,x)|^{p-1}u(s,x)\,ds$$

and

$$i_2 = \int_{t_2}^{t_1} U(t,s)|u(s,x)|^{p-1}u(s,x)\,ds.$$

By using the strong continuity of $U(t,s)$ for i_1, and Strichartz estimates for i_2 (after splitting the time interval $[t_2,t_1]$ into unions of I_n^1 and I_n^2), we see that both are $o(1)$ as $|t_1-t_2|\to 0$. Thus, the solution is continuous in time with values in $L^2(\mathbb{R}^d)$. □

3.2. **Local well-posedness in H^1 and finite-time blow-up.** In order to allow for $p = 1+\frac{4}{d}$, which, in particular, includes the physically relevant situation of a cubic nonlinearity in $d=2$ spatial dimensions, we require $\varphi\in H^1(\mathbb{R}^d)$. In the following, we shall always denote by $(a)_+$ the positive part of $a\in\mathbb{R}$.

Lemma 3.2. *Consider* (1.1) *with* $\varphi\in H^1(\mathbb{R}^d)$ *and* $1<p<1+\frac{4}{(d-2)_+}$. *Then there exists a* $T^* = T^*(\|\varphi\|_{H^1}) > t_0$ *and a unique solution* $u\in C([t_0,T],H^1(\mathbb{R}^d))$, *for all* $t_0<T<T^*$. *The solution is maximal in the sense that if* $T^*<+\infty$, *then*

$$\lim_{t\to T^*}\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} = +\infty.$$

Note that in dimensions $d=1,2$ we can allow for any exponent $1<p<+\infty$.

Proof. Differentiating (1.1) with respect to x we see that ∇u solves the following integral equation:

$$\nabla u(t,x) = U(t,t_0)\nabla\varphi(x) + i\int_{t_0}^{t} U(t,s)G(u(s,x))\,ds,$$

where $G(u) = \frac{p+1}{2}|u|^{p-1}\nabla u + \frac{p-1}{2}|u|^{p-3}u^2\nabla\overline{u}$. The latter satisfies $G(u)\lesssim |u|^{p-1}|\nabla u|$. By decomposing as before $[t_0,t]$ into unions of I_n^1 and I_n^2 and applying Strichartz estimates together with Sobolev's imbedding on each of these subintervals, we obtain the existence of a local-in-time solution $u(t,\cdot)\in H^1(\mathbb{R}^d)$. The existence of $T^*>0$ time thereby depends on $\|\varphi\|_{H^1}$. □

Clearly, if at the initial time $t_0\in\mathbb{R}$ it holds that $\gamma(t_0) = -\gamma_-$; i.e., if we start with a defocusing step, then we know that the solution exists at least up to $T^*\geqslant t_+$. This follows from the fact that $\gamma(t)$ is constant (and negative) for all $t \in [t_0, t_+]$, and thus the energy (1.3) is conserved on this time interval, which in turn provides a uniform bound on the H^1 norm of u.

Remark 3.3. In the case of constant dispersion and if $p < 1 + \frac{4}{d}$, one can infer $T^* = +\infty$, i.e., a global H^1 solution, using Strichartz estimates and the conservation of energy. Since conservation of energy does not hold in our case one has to argue differently. Namely, if $\varphi \in H^1(\mathbb{R}^d)$ and if $p < 1 + \frac{4}{d}$, the global L^2 solution constructed in Theorem 3.1 admits propagation of regularity, and thus we indeed obtain that $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$. The proof of this result is given in [10, Theorem 5.2.1].

In the following we denote by $Q \in H^1(\mathbb{R}^d)$ the unique radially symmetric solution of the following elliptic equation:

$$\Delta Q - Q + Q^{1+\frac{4}{d}} = 0. \tag{3.2}$$

The existence of such ground states Q has been proved in [3, 4].

Theorem 3.4. *Let $p = 1 + \frac{4}{d}$ and $\varphi \in H^1(\mathbb{R}^d)$ with*

$$\|\varphi\|_{L^2} < \gamma_+^{-d/4} \|Q\|_{L^2}.$$

Then $T^* = +\infty$. Moreover, this criterion is sharp.

Up to the inclusion of the scaling factor $\gamma_+^{-d/4}$, this criterion is analogous to the case without dispersion management; cf. [24]. We see that the stronger the dispersion in the focusing step, the larger the mass of the initial data can be in order to infer global existence.

Proof. With the local-in-time existence result for H^1 solution at hand, we need to show that $\|\nabla u(t, \cdot)\|_{L^2(\mathbb{R}^d)} < +\infty$ for any finite time $t \in \mathbb{R}$. To this end, let us assume for the moment that at $t = t_0$ we start with a defocusing step; i.e., $\gamma(t_0) = -\gamma_- < 0$. Denoting by $\lceil t_0 \rceil \in \mathbb{N}$ the ceiling of $t_0 \in \mathbb{R}$, we consequently have that for all $t \in [t_0, \lceil t_0 \rceil]$ the following energy is conserved:

$$E_-(t) := \frac{\gamma_-}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx + \frac{d}{2d + 4} \int_{\mathbb{R}^d} |u(t, x)|^{2+\frac{4}{d}} \, dx = E_-(t_0),$$

where we have set $p = 1 + 4/d$. Next, let us recall the Gagliardo–Nirenberg-type inequality (see [33])

$$\|f\|_{L^{2+4/n}}^{2+4/d} \leqslant \left(1 + \tfrac{2}{d}\right) \|Q\|_{L^2}^{-4/d} \|f\|_{L^2}^{4/d} \|\nabla f\|_{L^2}^2, \tag{3.3}$$

where Q is the solution of (3.2). Using that $u(t_0) \in H^1(\mathbb{R}^d)$, by assumption, together with the fact that for all times $t \in \mathbb{R}$, $\|u(t)\|_{L^2} = \|\varphi\|_{L^2} < +\infty$, the inequality (3.3) implies that $E_-(t) < +\infty$ for all $t \in [t_0, \lceil t_0 \rceil]$. We consequently obtain that $u(t) \in H^1(\mathbb{R}^d)$ exists up to $t = \lceil t_0 \rceil \in \mathbb{N}$ and that

$$\|\nabla u(\lceil t_0 \rceil, \cdot)\|_{L^2}^2 \lesssim E_-(t_0), \quad \|u(\lceil t_0 \rceil, \cdot)\|_{L^{2+4/d}}^{2+4/d} \lesssim E_-(t_0). \tag{3.4}$$

On the time interval $[\lceil t_0 \rceil, \lceil t_0 \rceil + t_+]$ we are in the focusing regime. The associated energy is

$$E_+(t) := \frac{\gamma_+}{2} \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \, dx - \frac{d}{2d+4} \int_{\mathbb{R}^d} |u(t,x)|^{2+\frac{4}{d}} \, dx = E_+(\lceil t_0 \rceil) < +\infty,$$

where the last inequality follows from (3.4). Using again (3.3) we infer that for all $t \in [\lceil t_0 \rceil, \lceil t_0 \rceil + t_+]$ it holds that

$$E_+(\lceil t_0 \rceil) = E_+(t) \geqslant \frac{1}{2}\Big(\gamma_+ - \frac{\|\varphi\|_{L^2}^{4/d}}{\|Q\|_{L^2}^{4/d}}\Big)\|\nabla u(t,\cdot)\|_{L^2}^2.$$

This consequently implies that if $\|\varphi\|_{L^2} < \gamma_+^{-d/4}\|Q\|_{L^2}$ we have

$$\|\nabla u(t)\|_{L^2}^2 \leqslant 2\Big(\gamma_+ - \frac{\|\varphi\|_{L^2}^{4/d}}{\|Q\|_{L^2}^{4/d}}\Big)^{-1} E_+(\lceil t_0 \rceil) < +\infty,$$

for all $t \in [\lceil t_0 \rceil, \lceil t_0 \rceil + t_+]$. Thus, we infer the existence of $u(t) \in H^1(\mathbb{R}^d)$ up to the time $t = \lceil t_0 \rceil + t_+$, after which we are again in the defocusing regime and the same argument as above applies. Clearly, this also shows that we could have started at time $t = t_0$ with a focusing step, instead of a defocusing. In both cases we obtain that $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} < +\infty$ for any finite time $t \in \mathbb{R}$.

Next, let us show that this result is indeed sharp; i.e., we need to find an initial data $\varphi \in H^1(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2} = \gamma_+^{-d/4}\|Q\|_{L^2}$ and the corresponding maximal time of existence for u is $T^* < +\infty$. By the pseudo-conformal symmetry of the mass-critical NLS with $p = 1 + \frac{4}{d}$, it is well-known that (see, e.g., [10, 24])

$$v_a(t,x) = (1-at)^{-\frac{d}{2}} Q\Big(\frac{\sqrt{\gamma_+}\, x}{1-at}\Big) e^{ia\frac{|x|^2}{4(1-at)}} e^{i\frac{t}{1-at}} \tag{3.5}$$

is a solution to the mass-critical focusing NLS

$$i\partial_t v_a + \gamma_+ \Delta v_a + |v_a|^{\frac{4}{d}} v_a = 0,$$

with initial datum

$$v_a(0,x) = Q(\sqrt{\gamma_+}\, x) e^{ia\frac{|x|^2}{4}},$$

satisfying $\|v_a(0,\cdot)\|_{L^2} = \gamma_+^{-d/4}\|Q\|_{L^2}$. In addition, we see from (3.5) that v_a blows up at time $t = \frac{1}{a}$. Now if t_0 is such that $\gamma(t_0) = \gamma_+$, i.e., we start with a focusing step, then all we need to do is choose $a = a_*$ such that $\frac{1}{a_*} < t_+ - \{t_0\}$, where $[0,1) \ni \{t_0\} = t_0 - \lfloor t_0 \rfloor$ is the fractional part of t_0. Setting $u(t_0,x) = v_{a_*}(0,x)$ ensures that the solution to (1.1) will blow up before the time $\lfloor t_0 \rfloor + t_+$, i.e., before switching to the defocusing regime. If on the other hand we start with a defocusing step $\gamma(t_0) = -\gamma_-$, then we choose $u(t_0,x) = w(t_0,x)$, where w solves the time-reversed defocusing NLS

$$i\partial_t w + \gamma_- \Delta w - |w|^{\frac{4}{d}} w = 0,$$

with initial data $w(\lceil t_0 \rceil, x) = v_{a_*}(0,x)$. □

In particular, the proof shows that a defocusing step in general can not prevent the appearance of finite-time blow-up. Note, however, that the construction of blow-up solutions given above is based on the pseudo-conformal symmetry of the mass-critical NLS, and the associated blow-up scenario is known to be *unstable*; see, e.g., [24] for further discussions.

Remark 3.5. Also note that in the proof Theorem 3.4 we have used the fact that $\gamma(t)$ is piecewise constant several times. Indeed, the situation where $\gamma(t)$ is smooth and 1-periodic would be considerably more complicated.

4. AVERAGING FOR FAST DISPERSION MANAGEMENT

4.1. **Preliminaries.** In this section we consider the ε-scaled NLS (1.4). Clearly, we have that for every *fixed* $\varepsilon > 0$ all the results of the foregoing section remain valid. In the following, though, we shall consider the scaling limit as $\varepsilon \to 0_+$, i.e., the regime of rapidly varying dispersion.

As a first step, we shall show that the expected averaging result is true for the linear equation. To this end, we denote by

$$U_\varepsilon(t,s) = e^{i\Gamma_\varepsilon(t,s)\Delta}, \quad \Gamma_\varepsilon(t,s) = \varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \gamma(\tau)\, d\tau,$$

the propagator associated to the fast-dispersion map $\gamma\left(\frac{t}{\varepsilon}\right)$. In addition,

$$U_0(t,s) = e^{i\langle\gamma\rangle(t-s)\Delta} = e^{it\langle\gamma\rangle\Delta} \circ e^{-is\langle\gamma\rangle\Delta}$$

is the propagator associated to the linear Schrödinger equation with averaged dispersion $\langle\gamma\rangle \in \mathbb{R}$ defined in (1.6). Note that in fact $U_0(t,s) = U_0(t-s,0)$ and is thus a group.

Lemma 4.1. *For any $s, t \in \mathbb{R}$ and $f \in H^\sigma(\mathbb{R}^d)$ with $\sigma \in \mathbb{N}$ we have*

$$\lim_{\varepsilon \to 0_+} \sup_{t,s\in\mathbb{R}} \|U_\varepsilon(t,s)f - U_0(t,s)f\|_{H^\sigma(\mathbb{R}^d)} = 0.$$

Proof. Using (2.4) we can decompose Γ_ε as

$$\Gamma_\varepsilon(t,s) = \langle\gamma\rangle(t-s) + \varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \gamma_0(\tau)\, d\tau \equiv \langle\gamma\rangle(t-s) + \varepsilon\vartheta_\varepsilon(t,s),$$

where we note that $\vartheta_\varepsilon(t,s) \in L^\infty(\mathbb{R}^d)$ uniformly, in view of (2.5). Fourier transformation and Plancherel's identity allow us to write

$$\sup_{t,s\in\mathbb{R}} \|U_\varepsilon(t,s)f - U_0(t,s)f\|^2_{H^\sigma(\mathbb{R}^d)} =$$
$$= \sup_{t,s\in\mathbb{R}} \int_{\mathbb{R}^d} (1+|\xi|^2)^\sigma \left| e^{i\langle\gamma\rangle(t-s)|\xi|^2} \left(e^{i\varepsilon\vartheta_\varepsilon(t,s)} - 1\right)\right|^2 |\widehat{f}(\xi)|^2\, d\xi.$$

The assertion then follows by the Lebesgue dominated convergence theorem and the fact that $\lim_{\varepsilon\to 0_+}\left(e^{i\varepsilon\vartheta_\varepsilon(t,s)} - 1\right) = 0$, pointwise for all $t, s \in \mathbb{R}$. □

To prove the desired averaging result as $\varepsilon \to 0_+$, we will require sufficiently smooth solutions $u_\varepsilon(t,\cdot) \in H^\sigma(\mathbb{R}^d)$ with $\sigma > \frac{d}{2}$. In the following we shall concentrate on the physically relevant cases of $d \leqslant 3$ spatial dimensions. The generalization to higher-order dimensions will be indicated below. For $d \leqslant 3$ it is sufficient to consider solutions in $H^2(\mathbb{R}^d)$, whose existence is guaranteed by the following lemma.

Lemma 4.2. *Let $\varphi \in H^2(\mathbb{R}^d)$ and $2 \leqslant p < \infty$. Then, for any $\varepsilon > 0$, there exists a time $T^* = T^*(\|\varphi\|_{H^2};\varepsilon) > t_0$ and a unique solution $u_\varepsilon \in C([t_0,T], H^2(\mathbb{R}^d))$, for all $t_0 < T < T^*$. The solution is maximal in the sense that if $T^* < +\infty$ then*

$$\lim_{t\to T_*} \|u_\varepsilon(t,\cdot)\|_{H^2(\mathbb{R}^d)} = +\infty.$$

Furthermore,

$$\tau := \inf_{0<\varepsilon\leqslant 1} T^*(\|\varphi\|_{H^2};\varepsilon) > 0.$$

Proof. The proof follows from arguments analogous to those given in the proof of Theorem 1.4 in Chapter 6 of [29] (see also [10, Section 4.8]). In particular, this establishes the fact that $\tau > 0$, bearing in mind that

$$\sup_{0<\varepsilon\leqslant 1} \|U_\varepsilon(t,s)\| = 1.$$ □

Standard arguments also imply that for $\varphi \in H^2(\mathbb{R}^d)$ there exists a maximal time of existence $T_* = T_*(\|\varphi\|_{H^2}) > t_0$ and a unique solution $u_0 \in C([t_0, T], H^2(\mathbb{R}^d))$, for all $t_0 < T < T_*$, satisfying the averaged equation (1.5); cf. [10, 24] for more details. Depending on the power of the nonlinearity p and on the sign of $\langle\gamma\rangle \in \mathbb{R}$ the maximal time of existence for the averaged equation might be infinite or not. In particular, for H^1-subcritical nonlinearities in the defocusing case $\langle\gamma\rangle < 0$ we have $T_* = +\infty$; cf. the results given in Section 5.3 of [10].

4.2. **Averaging of NLS with rapidly varying dispersion.** The main result of this section is as follows:

Theorem 4.3. *Let $d \leqslant 3$, $\varphi \in H^2(\mathbb{R}^d)$, and $2 \leqslant p < \infty$. Denote by $u_0 \in C([t_0, T_*), H^2(\mathbb{R}^d))$ the maximal solution to the averaged equation* (1.5). *In addition, let $u_\varepsilon \in C([t_0, T^*), H^2(\mathbb{R}^d))$ be the maximal solution of* (1.4) *for given $\varepsilon > 0$. Then, we have that $u_\varepsilon \overset{\varepsilon\to 0_+}{\longrightarrow} u_0$, in $L^\infty([t_0, T], H^2(\mathbb{R}^d))$, for all $t_0 < T < T_*$.*

In particular, we know that as $\varepsilon \to 0_+$ the solution u_ε of the original equation (1.1) can not blow up before $T_* > 0$, the maximal time of existence of a smooth solution to the averaged equation.

Proof. In the following, we denote the nonlinearity by $f(u) = |u|^{p-1}u$, for simplicity. In view of Duhamel's formula (3.1) we need to estimate the difference

$$\begin{aligned} u_\varepsilon(t,x) - u_0(t,x) &= (U_\varepsilon(t,t_0) - U_0(t,t_0))\varphi(x) \\ &+ i \int_{t_0}^t U_\varepsilon(t,s) f(u_\varepsilon(s,x)) - U_0(t,s) f(u_0(s,x))\, ds. \end{aligned}$$

For the first term on the right-hand side we can directly use Lemma 4.1, whereas the second term can be rewritten as

$$\int_{t_0}^t U_\varepsilon(t,s) f(u_\varepsilon(s,x)) - U_0(t,s) f(u_0(s,x)\, ds = \mathcal{I}_1^\varepsilon + \mathcal{I}_2^\varepsilon,$$

where

$$\mathcal{I}_1^\varepsilon = \int_{t_0}^t U_\varepsilon(t,s)\left(f(u_\varepsilon(s,x)) - f(u_0(s,x))\right) ds,$$

and

$$\mathcal{I}_2^\varepsilon = \int_{t_0}^t \left(U_\varepsilon(t,s) - U_0(t,s)\right) f(u_0(s,x))\, ds.$$

We first start by deriving an estimate in $L^2(\mathbb{R}^d)$; i.e., $\sigma = 0$. Using Minkowski's inequality,

$$\|\mathcal{I}_2^\varepsilon\|_{L_t^\infty L_x^2} \leqslant |T - t_0| \| \left(U_\varepsilon(t,s) - U_0(t,s)\right) f(u_0(s))\|_{L_{t,s}^\infty L_x^2}.$$

As long as $u_0(t,\cdot) \in H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $d \leqslant 3$ we have

$$\|f(u_0)\|_{L^2} \leqslant \|u_0\|_{L^\infty}^{p-1} \|u_0\|_{L^2} < +\infty,$$

and thus Lemma 4.1 implies that for $\varepsilon = \varepsilon(T)$ sufficiently small there exists a $\delta(\varepsilon) > 0$ such that

$$\|\mathcal{I}_2^\varepsilon\|_{L_t^\infty L_x^2} \lesssim \delta(\varepsilon) \stackrel{\varepsilon \to 0_+}{\longrightarrow} 0.$$

Next, we note that for all $u, v \in \mathbb{C}$ it holds that

$$|f(u) - f(v)| \leqslant C(|u|^{p-1} + |v|^{p-1})|u - v|.$$

With this we can estimate $\mathcal{I}_1^\varepsilon$, using again Minkowski's inequality and the fact that U_ε is unitary on L^2, via

$$\|\mathcal{I}_2^\varepsilon\|_{L_t^\infty L_x^2} \lesssim \|f(u_\varepsilon) - f(u_0)\|_{L_t^1 L_x^2} \lesssim (\|u_\varepsilon\|_{L_{t,x}^\infty}^{p-1} + \|u_0\|_{L_{t,x}^\infty}^{p-1}) \|u_\varepsilon - u_0\|_{L_t^1 L_x^2}.$$

As long as $u_\varepsilon, u_0 \in H^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ this implies

$$\|u_\varepsilon - u_0\|_{L_t^\infty L_x^2} \leqslant \delta(\varepsilon) + C\|u_\varepsilon - u_0\|_{L_t^1 L_x^2}. \tag{4.1}$$

Next, we recall the following result proved in [11] (see also [9, 28]):

Lemma 4.4 ([11, Lemma A.1])**.** *Let $T > 0$ and $1 \leqslant p < q \leqslant \infty$ and $A, B \geqslant 0$ be some constants. Assume that $f \in L^q(0,T)$ satisfies, for all $t \in (0,T)$,*

$$\|f\|_{L^q(0,t)} \leqslant A + B\|f\|_{L^p(0,t)}.$$

Then there exists a $K = K(B,p,q,T)$ such that $\|f\|_{L^q(0,t)} \leqslant AK$.

Using this lemma with $q = \infty$ and $p = 1$, we infer from (4.1) that

$$\|u_\varepsilon - u_0\|_{L_t^\infty L_x^2} \lesssim \delta(\varepsilon) \stackrel{\varepsilon \to 0_+}{\longrightarrow} 0.$$

To obtain the analogous estimate for H^σ with $\sigma = 1, 2$ we differentiate the equation with respect to $x \in \mathbb{R}^d$, using the fact that the nonlinearity is sufficiently smooth. Then, similar arguments as before imply that

$$\|u_\varepsilon - u_0\|_{L_t^\infty H_x^2} \stackrel{\varepsilon \to 0_+}{\longrightarrow} 0. \tag{4.2}$$

The result then follows from a continuity argument similar to the one given in [11]: Fix $0 < T < T_*$ and set $M = \sup_{0 \leqslant t \leqslant T} \|u_0\|_{H^2}$. It follows from Lemma 4.2 that for $\|\varphi\|_{H^2} \leqslant M$ there exists a $\tau > t_0$ such that for any $\varepsilon > 0$, u_ε exists on $[t_0, \tau]$ and

$$\sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon\|_{L^\infty((t_0,\tau),H^2)} \lesssim \|\varphi\|_{H^2}. \tag{4.3}$$

Next, let $t_0 < \ell \leqslant T$ be such that u_ε exists on $[t_0, \ell]$ for ε sufficiently small and

$$\limsup_{\varepsilon\to 0_+} \|u_\varepsilon\|_{L^\infty((t_0,\ell),H^2)} < +\infty.$$

Note that $\tau = \ell$ is always a possible choice. Then we deduce from (4.2) above that

$$\|u_\varepsilon(\ell,\cdot) - u_0(\ell,\cdot)\|_{H^2} \stackrel{\varepsilon\to 0_+}{\longrightarrow} 0,$$

from which we consequently infer that

$$\|u_\varepsilon(\ell,\cdot)\|_{H^2} \leqslant M,$$

for $\varepsilon \ll 1$ sufficiently small. Applying Lemma 4.2 to the NLS (1.4) translated by ℓ we deduce that for ε sufficiently small, u_ε exists on $[t_0, \tau + \ell]$ and that

$$\sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon\|_{L^\infty((t_0,\tau+\ell),H^2)} \lesssim \|\varphi\|_{H^2}.$$

In other words, the estimate (4.3) holds with ℓ replaced by $\ell + \tau$, provided that $\ell + \tau \leqslant T < T_*$. Iterating this argument we infer that

$$\sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon\|_{L^\infty((t_0,T),H^2)} \lesssim \|\varphi\|_{H^2},$$

and the assertion is proved. □

As a first corollary we can state the following result valid in arbitrary spatial dimensions $d \in \mathbb{N}$, provided the nonlinearity is sufficiently smooth.

Corollary 4.5. *Let $d \in \mathbb{N}$ and $\varphi \in H^\sigma(\mathbb{R}^d)$ with $\mathbb{N} \ni \sigma > \frac{d}{2}$, and assume that the nonlinearity satisfies $p = 1 + 2\alpha$ with $\alpha \in \mathbb{N}$. Then $u_\varepsilon \stackrel{\varepsilon\to 0_+}{\longrightarrow} u_0$, in $L^\infty([t_0, T], H^\sigma(\mathbb{R}^d))$, for all $t_0 < T < T_*$, the maximal time of existence of solutions $u_0(t,\cdot) \in H^\sigma(\mathbb{R}^d)$, satisfying the averaged equation.*

Proof. In higher dimensions, existence of smooth solutions $u_0, u_\varepsilon(t,\cdot) \in H^\sigma(\mathbb{R}^d)$ with $\sigma > \frac{d}{2}$ can be proved as in [10, Section 4.10], provided $p = 1+2\alpha$ with $\alpha \in \mathbb{N}$. The averaging result then follows by the same arguments as given in the proof of Theorem 4.3 above. □

More interestingly, in the case of zero average dispersion $\langle\gamma\rangle = 0$ the averaged equation (1.4) simplifies to an ordinary differential equation

$$i\partial_t u_0 + |u_0|^{p-1} u_0 = 0, \qquad u_0(t_0, x) = \varphi(x), \tag{4.4}$$

which has been formally derived in [7] (including higher-order corrections in ε). Equation (4.4) can be solved explicitly, resulting in the following corollary (stated for $d \leqslant 3$, for simplicity).

Corollary 4.6. *Let $d \leqslant 3$ and $\langle\gamma\rangle = 0$. Under the same assumptions as in Theorem 4.3 we have, for any compact time interval $I \subset \mathbb{R}$, that $u_\varepsilon \overset{\varepsilon\to 0_+}{\longrightarrow} u_0$, in $L^\infty(I; H^2(\mathbb{R}^d))$, where $u_0(t,x) = \varphi(x) e^{i(t-t_0)|\varphi(x)|^{p-1}}$.*

Proof. In the case $\langle\gamma\rangle = 0$, Theorem 4.3 implies that u_ε converges for $\varepsilon \to 0_+$ to the solution of (4.4). Multiplying the latter by $\overline{u}_0$ and taking the real part, we find $\partial_t |u_0|^2 = 0$, $u_0(t_0, x) = \varphi(x)$, and thus $|u_0(t,x)| = |\varphi(x)|$ for all $t \in \mathbb{R}$. Writing $u_0 = |u_0| e^{i\theta}$ and, analogously, $\varphi = |\varphi| e^{i\theta_0}$, we find the following equation for the phase:

$$\partial_t \theta = |\varphi(x)|^{p-1}\theta, \quad \theta(t_0, x) = \theta_0(x).$$

Integration with respect to t then yields the result. □

The particular form of u_0 found in the case of zero average dispersion corresponds to a solution which *does not disperse*. Indeed, the spatial density (corresponding to the energy density of an electromagnetic pulse) is seen to be time *independent;* i.e., $|u_0(t,x)|^2 = |\varphi(x)|^2$, for all $t \in \mathbb{R}$. However, the solution u_0 oscillates with an x-dependent frequency $\omega(x) = |\varphi(x)|^{p-1}$. From the physics point of view, Corollary 4.6 provides a possible justification for the stabilizing effect of dispersion management in optical fibers with mean-zero dispersion. Note, however, that this effect should be distinguished from the ones established in [34].

Remark 4.7. The case for rapidly varying mean-zero dispersion $\langle\gamma\rangle = 0$ is radically different from the corresponding situation found for NLS with time-periodic nonlinearity management [11, 12]. In the latter case, the effective model obtained after averaging is given by a *linear free* Schrödinger equation, whose solution is purely dispersive, in contrast to dispersion-managed NLS.

5. Concluding remarks

5.1. **The case $|\gamma(t)| > 0$.** In this case, the behavior of (1.1) is purely focusing or defocusing. In particular, we have that, for any given $t_0 \in \mathbb{R}$, the mapping $t \mapsto \Gamma(t, t_0) \in \mathbb{R}$ is strictly monotone and we can define a new unknown $v(t, x) = u(\Gamma(t, t_0), x)$, which solves

$$i\partial_t v + \Delta v + \kappa(t)|v|^{p-1}v = 0, \qquad v(0, x) = \varphi(x), \tag{5.1}$$

with coefficient $\kappa(t) = \frac{1}{\gamma(t)}$. Equation (5.1) is a NLS with time-dependent *nonlinearity management*, similar to the models studied in [11, 16]. In particular, if $\gamma(t) < 0$ for all $t \in \mathbb{R}$, this equation is defocusing, and one can prove global-in-time existence of solutions in $H^1(\mathbb{R}^d)$ along the lines of [16].

Of course, the (physically and mathematically) most interesting case of nonlinearity management is the one with sign-changing time-dependent coefficient $\kappa(t)$. Such a situation, however, is no longer equivalent to the one with dispersion management.

5.2. **Possible generalizations.** Let us mention that it is straightforward to generalize all of our results to the following NLS-type equation with linear dissipation:

$$i\partial_t u + \gamma(t)\Delta u + |u|^{p-1}u + i\sigma u = 0, \quad \sigma > 0.$$

This equation models wave propagation in dispersion-managed fibers including the effects of absorption (or damping) by the fiber. In the case without dispersion management the influence of the damping term on the possibility of finite-time blow-up is well studied; cf. [15, 27].

Another possibility would be to consider models with only partial dispersion management, i.e., where the dispersion management only appears in one space direction. An example of this sort can be found in [1], where the authors study the following equation (in $d = 2$):

$$i\partial_t u + \big(\gamma(t)\partial_{xx} + \partial_{yy}\big)u + |u|^2 u = 0.$$

Note that with our choice of $\gamma(t)$ this equation periodically switches in between the usual "elliptic" NLS and a *non-elliptic*, or *hyperbolic*, NLS (in the terminology of [18] and [32], respectively). To this end, we recall that the Cauchy problem for the non-elliptic NLS

$$i\partial_t u - \partial_{xx} u + \partial_{yy} u + |u|^2 u = 0$$

is locally well-posed for initial data in $H^1(\mathbb{R}^2)$; see [18]. It is conjectured that this local solution is actually global. One of the main differences between elliptic and non-elliptic NLS is the absence of localized solitary waves for the latter.

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