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EXISTENCE THEOREMS FOR QUASILINEAR ELLIPTIC EIGENVALUE PROBLEMS IN UNBOUNDED DOMAINS

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Abstract. The paper focuses on the existence of nontrivial solutions of a nonlinear eigenvalue problem depending on a real parameter λ under Robin boundary conditions in unbounded domains, with (possibly noncompact) smooth boundary. The problem involves a weighted p-Laplacian operator and subcritical nonlinearities, and even in the case $p = 2$ the main existence results are new. Denoting by λ_1 the first eigenvalue of the underlying Robin eigenvalue problem, we prove the existence of (weak) solutions, with different methods, according to the case $\lambda \geq \lambda_1$ or $\lambda < \lambda_1$. In the first part of the paper we show the existence of a nontrivial solution for all $\lambda \in \mathbb{R}$ for the problem under *Ambrosetti–Rabinowitz*-type conditions on the nonlinearities involved in the model. In detail, we apply the mountain-pass theorem of *Ambrosetti* and *Rabinowitz* if $\lambda < \lambda_1$, while we use mini-max methods and linking structures over cones, as in *Degiovanni* [10] and in *Degiovanni* and *Lancelotti* [11], if $\lambda \geq \lambda_1$. In the latter part of the paper we do not require any longer the *Ambrosetti–Rabinowitz* condition at ∞, but the so-called *Szulkin–Weth* conditions, and we obtain the same result for all $\lambda \in \mathbb{R}$. More precisely, using the Nehari-manifold method for C^1 functionals developed by *Szulkin* and *Weth* in [38], we prove existence of ground states, multiple solutions, and least-energy sign-changing solutions, whenever $\lambda < \lambda_1$. On the other hand, in the case $\lambda \geq \lambda_1$, we establish the existence of solutions again by linking methods.

1. Introduction

In this paper we are concerned with problems arising in the study of physical phenomena related to the equilibrium of anisotropic continuous media

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which possibly are somewhere *perfect* insulators; cf. *Dautray* and *Lions* [9] and also [37]. More precisely, let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with (possibly noncompact) smooth boundary $\partial\Omega$ and take $1 < p < N$. Consider the problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \nu + b(x)|u|^{p-2}u = h(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a real parameter and ν denotes the unit outward normal on $\partial\Omega$. The main result of the paper consists in proving the existence of a nontrivial weak solution u of (1.1) for all $\lambda \in \mathbb{R}$, requiring on the nonlinearities g and h either some *Ambrosetti–Rabinowitz* assumptions at ∞ or the so-called *Szulkin–Weth* conditions.

The Robin boundary condition in (1.1) arises naturally in heat conduction problems as well as in physical geodesy; cf. [22]. Problem (1.1) may also be viewed as a prototype in the study of activator-inhibitor systems modeling biological pattern formation, in addition to a steady-state problem for a chemotactic aggregation model. For a more exhaustive physical discussion on the topic we refer to [19].

Problems of type (1.1) were studied recently in [19]–[21] and [25]–[27]. In these papers the authors use mountain-pass-type theorems, the fibering method due to *Pohozaev* and *Ricceri*'s critical-point-type theorems in order to prove the existence and multiplicity of solutions. Most of these papers deal with the case in which Ω is a bounded domain.

The loss of compactness of the Sobolev imbeddings on unbounded domains renders variational techniques more delicate. Some of the papers treating problems on unbounded domains use special function spaces where the compactness is preserved, such as spaces of radially symmetric functions. We point out that even if Ω is unbounded, standard compact imbeddings still remain true, e.g., if Ω is *thin at infinity*, in the sense that

$$\lim_{R\to\infty} \sup \left\{ \operatorname{meas}(\Omega \cap B(x,1)) : x \in \mathbb{R}^N,\ |x| = R \right\} = 0,$$

where meas denotes the Lebesgue measure and $B(x,1)$ is the unit ball centered at x. Such arguments cannot be applied to the general unbounded domains Ω we consider in this paper. Indeed, since Ω is not necessarily *"thin"* and it may look like $\mathbb{R}^N$ at infinity (e.g., when Ω is an exterior domain), the analysis of the compactness failure shows that a Palais–Smale sequence, briefly (PS) sequence, of the associated energy functional (see

Bahri and *Lions* [5]) differs from its weak limit by *"waves"* that go to infinity. However, the definition of an appropriate solution space E, defined in Section 2, combined with the main assumptions ensures that E is compactly embedded into the weighted Lebesgue spaces involved.

In the first part of this paper, using the method developed by *Degiovanni* and *Lancelotti* in [11], *Degiovanni* in [10], and the *Ambrosetti* and *Rabinowitz* mountain-pass theorem, we prove that (1.1) has a nontrivial solution for all $\lambda \in \mathbb{R}$, when g and h satisfy an *Ambrosetti–Rabinowitz*-type condition. In the latter part we complete the picture, proving the existence of nontrivial solutions of (1.1) for all $\lambda \in \mathbb{R}$, without requiring the famous *Ambrosetti* and *Rabinowitz* condition on the nonlinearities g and h, but rather the so-called *Szulkin–Weth* conditions.

In [11] *Degiovanni* and *Lancelotti* prove the existence of nontrivial solutions for the equation given in (1.1) for any $\lambda \in \mathbb{R}$, when $a \equiv 1$ and $b = h \equiv 0$, that is, under homogeneous Dirichlet boundary conditions on bounded domains Ω. They use a mini-max approach and construct linking structures over cones for an associated eigenvalue problem. In the case $p \neq 2$, even when all the weights involved in (1.1) are just trivial constants, $a = f \equiv 1$ and $b = h \equiv 0$, and Ω is a bounded domain of $\mathbb{R}^N$, the spectrum of $-\Delta_p$ with homogeneous Dirichlet boundary conditions is not at all clear. We refer to [11] and the references therein for a complete discussion on the complexity of this question, as well as on the difficulty in covering the case $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of the natural underlying eigenvalue problem.

For the above reasons, if $\lambda \geq \lambda_1$, we prove existence of nontrivial solutions of (1.1) by constructing a suitable non-decreasing sequence $(\lambda_k)_k$ diverging to ∞ as $k \to \infty$, via a mini-max argument based on a $\mathbb{Z}_2$-cohomological index of the most canonical Finsler manifold $\mathcal{M}$ associated to the eigenvalue problem related to (1.1); see (3.1) below.

The case $\lambda < \lambda_1$ is either simply proved via the mountain-pass theorem of *Ambrosetti* and *Rabinowitz* [3] (see also [27] for similar problems), or using the Nehari manifold method for C^1 functionals developed by *Szulkin* and *Weth* in [38].

The paper is organized as follows. In Section 2 we present notation and some auxiliary results, some of them crucial to constructing the main sequence $(\lambda_k)_k$ used in the main proofs of Theorems 4.3 and 7.4. In Section 3 a simple minimization argument shows the existence of a positive first eigenvalue λ_1 of the Robin nonlinear boundary eigenvalue problem corresponding to (1.1). The main properties of λ_1 are briefly presented in Propositions 3.1 and 3.3. In Section 4 we prove the existence Theorem 4.3

under the *Ambrosetti–Rabinowitz* condition on g and h. In Section 5, we describe the abstract Nehari method and prove results of independent interest. From Section 6 on we assume the *Szulkin–Weth* conditions on g and h and prove several prelimary results of the main functionals associated to (1.1) in Section 6. In Section 7 we first show when $\lambda < \lambda_1$ in Theorem 7.1 that problem (1.1) has a ground-state solution, that is, a nontrivial solution of minimal energy; that (1.1) admits a least-energy sign-changing solution; and that (1.1) has infinitely many pairs of solutions if furthermore g and h are assumed to be odd in u. In the final part of Section 7, when $\lambda \geq \lambda_1$, we prove in Theorem 7.4 that (1.1) admits a nontrivial solution. The last part of the paper contains an Appendix in which we show that $E = (E, \|\cdot\|)$ is uniformly convex, where $\|\cdot\|$ is an appropriate norm in the solution space E.

2. Preliminaries and auxiliary results

In this section we collect a series of notations and preliminaries which are used throughout the paper. By w we denote a weight on Ω, that is, a measurable function with $w > 0$ a.e. in Ω, so that $L^\sigma(\Omega, w)$, $\sigma \geq 1$, is the weighted Lebesgue space equipped with the norm

$$\|u\|_{\sigma,w} = \Big(\int_\Omega w(x)|u(x)|^\sigma \, dx \Big)^{1/\sigma}.$$

Similarly, if $\tilde{w}$ is a weight on $\partial\Omega$, that is, $\tilde{w}$ is measurable and a.e. positive in $\partial\Omega$ with respect to the $N-1$-dimensional measure on $\partial\Omega$, then $L^\sigma(\partial\Omega, \tilde{w})$, $\sigma \geq 1$, denotes the weighted Lebesgue space equipped with the norm

$$\|u\|_{\sigma,\tilde{w},\partial\Omega} = \Big(\int_{\partial\Omega} \tilde{w}(x)|u(x)|^\sigma dS \Big)^{1/\sigma}.$$

The next lemma, stated here for the weighted space $L^\sigma(A, w)$, where A is a measurable subset of $\mathbb{R}^d$, $d \geq 1$, w is a weight on A, is well-known in the usual Lebesgue spaces (see, for instance, Theorem 4.9 of [7]). The proof is left to the reader, since it is standard; see also [34].

Lemma 2.1. *If $(u_n)_n$ and u are in $L^\alpha(A, w)$, with $\alpha \in [1, \infty)$, and $u_n \to u$ in $L^\alpha(A, w)$ as $n \to \infty$, then there exist a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and a function $h \in L^\alpha(A, w)$ such that a.e. in Ω*

(*i*) $u_{n_k} \to u$ *as* $k \to \infty$; (*ii*) $|u_{n_k}(x)| \leq h(x)$ *for all* $k \in \mathbb{N}$.

The case (*i*) of the result below is proved for the standard Lebesgue spaces in Theorem 2.3 of [13], while for (*ii*) we refer to Lemma 4.2. of [11].

Lemma 2.2. *Let $\alpha, \beta \in [1, \infty)$, $\varphi : A \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and let $N_\varphi(u) = \varphi(x, u)$ be the corresponding Nemytskii operator.*

(i) If $|\varphi(x, s)| \le \varphi_0(x) + c|s|^{\alpha/\beta}$ for a.a. $x \in A$ and all $s \in \mathbb{R}$, where $\varphi_0 \in L^\beta(A, w)$ and $c > 0$ is a constant, then N_φ is continuous and bounded from $L^\alpha(A, w)$ into $L^\beta(A, w)$.

(ii) Assume that for every $\varepsilon > 0$ there exists a nonnegative function $a_\varepsilon \in L^\beta(A, w)$ such that $|\varphi(x, s)| \le a_\varepsilon(x) + \varepsilon|s|^{\alpha/\beta}$ for a.a. $x \in A$ and all $s \in \mathbb{R}$. If $(u_n)_n$ is a bounded sequence in $L^\alpha(A, w)$ and $u_n \to u$ a.e. in A, then $N_\varphi(u_n) \to N_\varphi(u)$ in $L^\beta(A, w)$.

Proof. (i) From $\|N_\varphi(u)\|_{\beta,w} \le 2^{\beta-1} \max\{1, c\}\big(\|\varphi_0\|^\beta_{\beta,w} + \|u\|^\alpha_{\alpha,w}\big)$, it follows at once that N_φ is well defined and bounded from $L^\alpha(A, w)$ into $L^\beta(A, w)$.

Now, let $(u_n)_n \subset L^\alpha(A, w)$ and $u \in L^\alpha(A, w)$ be such that $u_n \to u$ in $L^\alpha(A, w)$. Fix a subsequence $(u_{n_k})_k$ of $(u_n)_n$. By Lemma 2.1 there exists a subsequence, say, $(v_{n_k})_k$ of $(u_{n_k})_k$ such that $v_{n_k} \to u$ and $|v_{n_k}| \le h$ a.e. in A, for some appropriate $h \in L^\alpha(A, w)$. Hence $N_\varphi(v_{n_k}) \to N_\varphi(u)$ a.e. in A and $|N_\varphi(v_{n_k})| \le \varphi_0 + ch^{\alpha/\beta} \in L^\beta(A, w)$. Therefore, by the Lebesgue dominated convergence theorem we get that $N_\varphi(v_{n_k}) \to N_\varphi(u)$ in $L^\beta(A, w)$. Thus, the entire sequence $(N_\varphi(u_n))_n$ converges to $N_\varphi(u)$ in $L^\beta(A, w)$ as $n \to \infty$. This proves the continuity of N_φ.

(ii) Clearly, for a.a. $x \in A$ and all $n \in \mathbb{N}$ we have

$$|\varphi(x, u_n(x)) - \varphi(x, u(x))| \le 4^{\beta-1}\{2a_\varepsilon(x)^\beta + \varepsilon^\beta|u_n(x)|^\alpha + \varepsilon^\beta|u(x)|^\alpha\}.$$

By the Fatou lemma

$$\begin{aligned} 2 \cdot 4^{\beta-1} \Big\{\|a_\varepsilon\|^\beta_{\beta,w} + \varepsilon^\beta\|u\|^\alpha_{\alpha,w}\Big\} &\le 2 \cdot 4^{\beta-1}\|a_\varepsilon\|^\beta_{\beta,w} \\ &+ 4^{\beta-1}\varepsilon^\beta\Big\{\|u\|^\alpha_{\alpha,w} + \sup_n \|u_n\|^\alpha_{\alpha,w}\Big\} - \limsup_n \|\varphi(x, u_n) - \varphi(x, u)\|^\beta_{\beta,w}. \end{aligned}$$

Therefore,

$$0 \le \limsup_n \|\varphi(x, u_n) - \varphi(x, u)\|^\beta_{\beta,w} \le 4^{\beta-1}\varepsilon^\beta\Big\{\sup_n \|u_n\|^\alpha_{\alpha,w} - \|u\|^\alpha_{\alpha,w}\Big\}.$$

By the arbitrariness of $\varepsilon > 0$, we get the assertion. □

Let $C^\infty_\delta(\Omega)$ be the space of functions of class $C^\infty_0(\mathbb{R}^N)$ restricted on Ω and let E be the completion of $C^\infty_\delta(\Omega)$ with respect to the norm

$$\|u\|_E = \Big(\int_\Omega \Big(|\nabla u(x)|^p + \frac{|u(x)|^p}{(1 + |x|)^p}\Big)dx\Big)^{1/p}.$$

Of course, since Ω is a smooth domain, by the celebrated density Theorem 3.18 of *Adams* [1] it is apparent that $E = W^{1,p}(\Omega)$ and $\|\cdot\|_E$ is an equivalent norm on $W^{1,p}(\Omega)$, whenever Ω is bounded. On the other hand, if Ω is unbounded, then $E \subset W^{1,p}_{\mathrm{loc}}(\Omega)$. We are now able to state the main result of the paper, which is proved in Section 5.

Throughout the paper, without further mentioning it, we assume that $0 < a_0 \le a \in C^1(\Omega) \cap L^\infty(\Omega)$, $b : \partial\Omega \to \mathbb{R}$ *is a continuous function, with*

$$c_b(1+|x|)^{1-p} \le b(x) \le C_b(1+|x|)^{1-p},$$

for some constants $0 < c_b \le C_b$; moreover, f *is a measurable weight defined on* Ω *and satisfying*

$$0 < f(x) \le C_f w_1(x), \quad w_1(x) = (1+|x|)^{-\alpha_1}, \quad p < \alpha_1 < N, \tag{2.1}$$

for a.a. $x \in \Omega$.

Since the embedding $E \hookrightarrow L^p(\partial\Omega, b)$ is continuous by Theorem 1 of [31], for all $u \in E$ the quantity

$$\|u\| = \left(\int_\Omega a(x)|\nabla u(x)|^p\,dx + \int_{\partial\Omega} b(x)|u(x)|^p\,dS \right)^{1/p}$$

is well defined and $\|\cdot\|$ is an equivalent norm in E, as shown in [31, Lemma 2]. From now on we endow E with the norm $\|\cdot\|$. Proposition A.2 of the Appendix shows that $(E, \|\cdot\|)$ is a uniformly convex Banach space. From now on $E^\star = (E^\star, \|\cdot\|_{E^\star})$ denotes the dual space of $E = (E, \|\cdot\|)$.

Proposition 2.3. *The map* $\mathcal{F} : E \to E^\star$, *defined by* $\mathcal{F}(u) = f|u|^{p-2}u$, *is weak-to-strong sequentially continuous; i.e.,* $u_n \rightharpoonup u$ *in* E *implies* $\|\mathcal{F}(u_n) - \mathcal{F}(u)\|_{E^\star} \to 0$ *as* $n \to \infty$.

Let w *be a weight on* Ω *such that the embedding* $E \hookrightarrow L^p(\Omega, w)$ *is compact. Then, also* $\mathcal{I} : E \to L^1(\Omega)$, *defined by* $\mathcal{I}(u) = w|u|^p$, *is weak-to-strong sequentially continuous; that is,* $u_n \rightharpoonup u$ *in* E *implies* $\|\mathcal{I}(u_n) - \mathcal{I}(u)\|_1 \to 0$ *as* $n \to \infty$.

Similarly, if $\tilde{w}$ *be a weight on* $\partial\Omega$ *such that the embedding* $E \hookrightarrow L^p(\partial\Omega, \tilde{w})$ *is compact, then* $\tilde{\mathcal{I}} : E \to L^1(\partial\Omega)$, $\tilde{\mathcal{I}}(u) = \tilde{w}|u|^p$, *is weak-to-strong sequentially continuous; that is,* $u_n \rightharpoonup u$ *in* E *implies* $\|\tilde{\mathcal{I}}(u_n) - \tilde{\mathcal{I}}(u)\|_{1,\partial\Omega} \to 0$ *as* $n \to \infty$.

Proof. Let $(u_n)_n \subset E$ be such that $u_n \rightharpoonup u$ in E. Hence $u_n \to u$ in $L^p(\Omega, f)$, since the embedding $E \hookrightarrow L^p(\Omega, f)$ is compact, because $\alpha_1 \in (p, N)$. Thus, in particular, there exists $C = C(p, N, \Omega) > 0$ such that

$$\|v\|_{p,f} \le C\|v\| \quad \text{for all } v \in E. \tag{2.2}$$

Clearly, $\|u_n\|_{p,f} \to \|u\|_{p,f}$, that is $\|v_n\|_{p',f} \to \|v\|_{p',f}$, where $v_n = |u_n|^{p-2}u_n$ and $v = |u|^{p-2}u$. We claim that $v_n \to v$ in $L^{p'}(\Omega, f)$. Indeed, fix any subsequence $(v_{n_k})_k$ of $(v_n)_n$. The related subsequence $(u_{n_k})_k$ of $(u_n)_n$ converges in $L^p(\Omega, f)$ and admits a subsequence, say $(u_{n_{k_j}})_j$, converging to u a.e. in Ω by Lemma 2.1. Hence, the corresponding subsequence $(v_{n_{k_j}})_j$ of $(v_{n_k})_k$ converges to v a.e. in Ω. Therefore, since $1 < p' < \infty$, by the Clarkson and Mil'man theorems it follows that $v_{n_{k_j}} \rightharpoonup v$ in $L^{p'}(\Omega, f)$, since the sequence $(\|v_n\|_{p',f})_n$ is bounded, and so by Radon's theorem we get that $v_{n_{k_j}} \to v$ in $L^{p'}(\Omega, f)$, since $\|v_n\|_{p',f} \to \|v\|_{p',f}$. This shows the claim, since the subsequence $(v_{n_k})_k$ of $(v_n)_n$ is arbitrary.

By Hölder's inequality we have for all $\phi \in E$, with $\|\phi\| = 1$,

$$|\langle \mathcal{F}(u_n) - \mathcal{F}(u), \phi\rangle| \le \int_\Omega f(x)|v_n - v|\cdot|\phi|\,dx \le \|v_n - v\|_{p',f}\|\phi\|_{p,f} \le C\|v_n - v\|_{p',f},$$

where $C > 0$ is given in (2.2). In other words, $\|\mathcal{F}(u_n) - \mathcal{F}(u)\|_{E^\star} \to 0$ as $n \to \infty$.

Since $E \hookrightarrow L^p(\Omega, w)$ and $E \hookrightarrow L^p(\partial\Omega, \tilde{w})$ are compact by assumption, the second and the third parts of the statement are trivial. Indeed, by the elementary inequality $||s|^p - |t|^p| \le C_p(|s-t|^p + |y|^{p-1}|s-t|)$ for all $s, t \in \mathbb{R}$, where C_p is an appropriate constant depending only on $p > 1$, we find

$$\begin{aligned}\int_\Omega |\mathcal{I}(u_n) - \mathcal{I}(u)|\,dx &= \int_\Omega w(x)||u_n|^p - |u|^p|\,dx \\ &\le C_p \int_\Omega w(x)\left(|u_n - u|^p + |u|^{p-1}|u_n - u|\right)dx \\ &\le C_p\left(\|u_n - u\|_{p,w}^p + \|u\|_{p,w}^{p-1}\|u_n - u\|_{p,w}\right) \to 0,\end{aligned}$$

since $u_n \rightharpoonup u$ in E and so $u_n \to u$ in $L^p(\Omega, w)$ as $n \to \infty$.

Similarly, we get the claim for $\tilde{\mathcal{I}}$. □

In the case $\lambda \ge \lambda_1$ we are going to solve problem (1.1) with mini-max and linking arguments based on a $\mathbb{Z}_2$-cohomological index. For this purpose let us present some crucial auxiliary results.

Let $X = (X, \|\cdot\|_X)$ be a real normed space and $X^\star = (X^\star, \|\cdot\|_{X^\star})$ its dual. We denote by $\mathcal{S}(X)$ the set of all center-symmetric subsets of X not containing the origin of X.

For $A \in \mathcal{S}(X)$, put $\overline{A} = A/\mathbb{Z}_2$. Let $\varphi : \overline{A} \to \mathbb{R}P^\infty$ be the *classifying map* and $\varphi^\star : H^\star(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega] \to H^\star(\overline{A})$ the *induced homomorphism of the cohomology rings*. The *cohomological index of* A, denoted by $i(A)$, is defined by $\sup\{k \geq 1 : \varphi^\star(\omega^{k-1}) \neq 0\}$. Let $A, B \in \mathcal{S}(X)$, and let us list some properties which will be used in the sequel.

(i_1) Monotonicity*: if* $A \subset B$, *then* $i(A) \leq i(B)$.
(i_2) Invariance*: if* $\varphi : A \to B$ *is an odd homeomorphism, then* $i(A) = i(B)$.
(i_3) Continuity*: if* C *is a closed symmetric subset of* A, *then there exists a closed symmetric neighborhood* $\mathcal{N}$ *of* C *in* A, *such that* $i(\mathcal{N}) = i(C)$*; hence, the interior of* $\mathcal{N}$ *is also a neighborhood of* C *in* A *and* $i(\operatorname{int} C) = i(C)$.
(i_4) Neighborhood of the origin*: if* U *is a bounded closed symmetric neighborhood of the origin in* X, *then* $i(\partial U) = \dim X$.

Definition 2.4. Let $\mathcal{I} : X \to \mathbb{R}$ be a functional of class $C^1(X)$. A sequence $(u_n)_n \subset X$ is said to be a *Palais–Smale sequence for* $\mathcal{I}$, (PS) *sequence* for short, if $(\mathcal{I}(u_n))_n$ is bounded and $\|\mathcal{I}'(u_n)\|_{X^\star} \to 0$ as $n \to \infty$. The functional $\mathcal{I}$ satisfies the *Palais–Smale condition*, (PS) *condition*, if any (PS) sequence admits a convergent subsequence.

Similarly, $\mathcal{I}$ satisfies the *Palais–Smale condition at level* c, $(PS)_c$ *condition* for short, if any (PS) sequence, with $\mathcal{I}(u_n) \to c$ as $n \to \infty$, admits a convergent subsequence.

Hence, equivalently, $\mathcal{I}$ satisfies the (PS) condition if and only if $(PS)_c$ holds for $\mathcal{I}$ at every level $c \in \mathbb{R}$. We recall a well-known result which is crucial in the proof that problem (3.1) has infinitely many distinct eigenvalues which tend to infinity.

Proposition 2.5 (Proposition 3.14.7 of [30])**.** *Let* $\mathcal{M}$ *be a* C^1 *Finsler complete manifold with free* $\mathbb{Z}_2$*-action, and let* $\mathcal{I} \in C^1(\mathcal{M})$ *be even (i.e.,* $\mathbb{Z}_2$*-invariant). Set* $\mathcal{F}_k = \{M \subset \mathcal{M} : M$ *is symmetric and* $i(M) \geq k\}$ *and* $\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \mathcal{I}(u)$. *Then the following statements are true:*

(i) *If* $-\infty < \lambda_k = \cdots = \lambda_{k+m-1} = c < \infty$ *and* $\mathcal{I}$ *satisfies* $(PS)_c$, *then* $i(K^c) \geq m$, *where* $K^c = \{u \in \mathcal{M} : \mathcal{I}(u) = c,\ \mathcal{I}'(u) = 0\}$.
(ii) *If* $-\infty < \lambda_k \leq \cdots \leq \lambda_{k+m-1} < \infty$ *and* $\mathcal{I}$ *satisfies* $(PS)_c$ *for* c *in* $\{\lambda_k, \ldots, \lambda_{k+m-1}\}$, *then all* $\lambda_k, \ldots, \lambda_{k+m-1}$ *are critical values for* $\mathcal{I}$ *and* $\mathcal{I}$ *has* m *distinct pairs of associated critical points.*
(iii) *If* $-\infty < \lambda_k < \infty$ *for all sufficiently large* k *and* $\mathcal{I}$ *satisfies* (PS), *then* $\lambda_k \nearrow \infty$ *as* $k \to \infty$.

Let us first introduce a special definition useful in stating the main result contained in [11]; see also [10].

Definition 2.6. *Let Q, R, and S be three subsets of X, with $R \subset Q$, and let k be a nonnegative number. We say that (Q, R) links S cohomologically in dimension k over $\mathbb{Z}_2$, if $R \cap S = \emptyset$ and the restriction homomorphism $H^k(X, X \setminus S; \mathbb{Z}_2) \to H^k(Q, R; \mathbb{Z}_2)$ is not identically zero.*

In the next statement we summarize Corollary 2.9, Theorem 2.8, and Proposition 2.4 of *Degiovanni* and *Lancelotti* [11], and Theorem 6.10 of *Degiovanni* in [10]. See also Theorem 3.2 and Remark 3.3. of *Degiovanni* in [10]. This result is crucial in the proof of Theorem 4.3.

Theorem 2.7. *Let $(X, \|\cdot\|_X)$ be a Banach space and C_- and C_+ be two symmetric cones in X such that C_+ is closed in X, $C_- \cap C_+ = \{0\}$, and*

$$i(C_- \setminus \{0\}) = i(X \setminus C_+) = k < \infty.$$

Let r_-, $r_+ > 0$ and let $e \in X$ with $-e \notin C_-$. Define the following sets:

$$D_- = \{u \in C_- : \|u\|_X \le r_-\}, \quad S_+ = \{u \in C_+ : \|u\|_X = r_+\},$$
$$Q = \{u + te : u \in C_-,\ t \ge 0,\ \|u + te\|_X \le r_-\},$$
$$H = \{u + te : u \in C_-,\ t \ge 0,\ \|u + te\|_X = r_-\}.$$

Then $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $k+1$ over $\mathbb{Z}_2$.

Let $F \in C^1(X)$ satisfy the (PS) condition. If furthermore

$$\sup_{x \in D_- \cup H} F(x) < \inf_{x \in S_+} F(x) \quad \text{and} \quad \sup_{x \in Q} F(x) < \infty,$$

then there exists a critical point u of F at some level c, with

$$\inf_{x \in S^+} F(x) \le c \le \sup_{x \in Q} F(x).$$

3. The underlying Robin boundary eigenvalue problem

Consider the Robin boundary eigenvalue problem associated to (1.1),

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \nu + b(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and put

$$\lambda_1 = \inf_{u \in \mathcal{M}} \mathcal{I}(u), \qquad \mathcal{M} = \{u \in E : \|u\|_{p,f} = 1\}, \tag{3.2}$$

where

$$\mathcal{I}(u) = \|u\|^p \quad \text{and} \quad \|u\|_{p,f} = \Big(\int_\Omega f(x)|u(x)|^p \, dx \Big)^{1/p}.$$

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* for (3.1) and that $u \in E \setminus \{0\}$ is a corresponding *eigenfunction*, if

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx + \int_{\partial\Omega} b(x)|u|^{p-2}u\phi \, dS = \lambda \int_\Omega f(x)|u|^{p-2}u\phi \, dx \tag{3.3}$$

for all $\phi \in E$. We call (λ, u) an *eigenpair* for (3.1).

For a wide study on nonlinear eigenvalue problems for the p-Laplace operator via variational methods in bounded domains, under several different boundary conditions of great interest, we refer to [23] and the references therein.

Proposition 3.1. *The following statements hold.*

(i) *The infimum λ_1 in* (3.2) *is attained; that is, there exists $\varphi_1 \in \mathcal{M}$ which realizes the minimum in* (3.2) *and represents an eigenfunction for* (3.1) *corresponding to the eigenvalue λ_1. Moreover, $\lambda_1 > 0$.*

(ii) *The set E_1 of eigenfunctions corresponding to λ_1 is a vector space of dimension* 1*. The elements of E_1 are either positive or negative and of class $C^{1,\delta}_{\text{loc}}(\Omega)$.*

(iii) *Let v be an eigenfunction of* (3.1) *corresponding to an eigenvalue $\lambda \neq \lambda_1$. Then v is a nodal solution of* (3.1)*; that is, v changes sign in Ω.*

Proof. (i) First, $\mathcal{I}(u) = \|u\|^p$ and $\mathcal{J}(u) = \|u\|^p_{p,f}$ are continuously Fréchet differentiable and convex in E. Clearly $\mathcal{I}'(0) = \mathcal{J}'(0) = 0$ and $\mathcal{J}'(u) = 0$ implies $u = 0$. In particular, $\mathcal{I}$ and $\mathcal{J}$ are weak lower semicontinuous on E. Actually, $\mathcal{J}$ is weak sequentially continuous on E. Indeed, if $(u_n)_n \subset E$ and $u_n \rightharpoonup u$ in E, then $u_n \to u$ in $L^p(\Omega, f)$, being the natural embedding between E and $L^p(\Omega, f)$ compact. Hence $\mathcal{J}(u_n) = \|u_n\|^p_{p,f} \to \mathcal{J}(u) = \|u\|^p_{p,f}$, as claimed. Of course, $\mathcal{I}$ is coercive in E, so that an easy contradiction argument shows that $\mathcal{I}$ is coercive in $E \cap \{u \in E : \mathcal{J}(u) = \|u\|^p_{p,f} \le 1\}$. In conclusion, all the assumptions of Theorem 6.3.2 of [6] are fulfilled, and so λ_1 is attained in $\mathcal{M}$.

Finally, $\mathcal{J}(u) = \|u\|^p_{p,f} \le B\|u\|^p = B\mathcal{I}(u)$ for all $u \in E$, with $B = C^p$ and C given in (2.2). Hence, $\lambda_1 \ge 1/B > 0$ by (3.2).

(ii) Let φ_1 be an eigenfunction corresponding to λ_1. Since $|\varphi_1|$ is also a minimizer for $\mathcal{I}$ in $\mathcal{M}$, we may assume that $\varphi_1 \ge 0$. Now $\varphi_1 \in W^{1,p}_{\text{loc}}(\Omega)$ since $\varphi_1 \in E$, so that Theorem 2.2 of *Pucci* and *Servadei* [35] can be applied with $A(x, u, \xi) = -a(x)|\xi|^{p-2}\xi$ and $B(x, u, \xi) = \lambda f(x)|u|^{p-2}u$, since the main condition (2.17) of [35] is satisfied because $f \in L^\infty(\Omega)$. Hence $\varphi_1 \in L^\infty_{\rm loc}(\Omega)$. From Corollary of Theorem 2 of *DiBenedetto* [14] it follows that $\varphi_1 \in C^{1,\delta}_{\rm loc}(\Omega)$, and clearly φ_1 is a non-negative weak solution of the differential inequality $\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) \le 0$ in Ω. Thus $\varphi_1 > 0$ in Ω by the strong maximum principle given in Theorem 5.4.1 of *Pucci* and *Serrin* [32] and the Remarks 3 and 4 on page 117 therein, since clearly $\varphi_1 \in C^1(\Omega)$; cf. also [33].

It remains to show that E_1 is a vector space of dimension one. We in some sense follow the main arguments of [2, Theorem 1.1], but for completeness we report here all the principal steps of the proof.

Let u and v be two eigenfunctions associated to λ_1. Without loss of generality, we assume that both u and v are positive in Ω. By the above argument, u and v are of class $C^1(\Omega)$ and so the functional

$$L(u,v) = |\nabla u|^p + (p-1)\left(\frac{u}{v}\right)^p |\nabla v|^p - p\left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2}\nabla v \nabla u$$

is well defined and non-negative in Ω as a consequence of Theorem 1.1 of [2]. Since v could be zero on $\partial\Omega$, let us define $v_n = v + 1/n$. Then, by (3.3) with $\phi = (u/v_n)^{p-1}u \in E$ it follows that

$$\begin{aligned} 0 &\le \int_\Omega L(u,v_n)\,dx = \int_\Omega \Big[|\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{v_n^{p-1}}\Big)\nabla v\Big]dx \\ &\le \frac{1}{a_0}\Big\{\int_\Omega a(x)|\nabla u|^p\,dx + \int_{\partial\Omega} b(x)v^{p-1}\frac{u^p}{v_n^{p-1}}\,dS \\ &\qquad\qquad - \lambda_1\int_\Omega f(x)v^{p-1}\frac{u^p}{v_n^{p-1}}\,dx\Big\} \\ &\le \frac{1}{a_0}\Big\{\|u\|^p - \lambda_1\int_\Omega f(x)v^{p-1}\frac{u^p}{v_n^{p-1}}\,dx\Big\} \to \frac{1}{a_0}\Big\{\|u\|^p - \lambda_1\|u\|^p_{p,f}\Big\} = 0 \end{aligned} \tag{3.4}$$

by the Lebesgue dominated convergence theorem and the fact that (λ_1, u) is an eigenpair. Clearly, $L(u,v_n) \to L(u,v)$ as $n \to \infty$ in Ω, since $v > 0$ in Ω. Therefore by the Fatou lemma and (3.4) we have $0 \le \int_\Omega L(u,v)\,dx \le 0$; that is, $L(u,v) = 0$ a.e. in Ω. Hence by Theorem 1.1 of [2] there exists $k \in \mathbb{R}$ such that $u = kv$ in Ω. Of course $k > 0$, and this shows that E_1 has dimension 1.

(iii) Suppose for the sake of contradiction that v does not change sign in Ω, and let u be an eigenfunction of (3.1) corresponding to λ_1. Without loss of generality we assume that both u and v are positive in Ω. Using the first argument of step (ii) we get that u and v are of class $C^1(\Omega)$ and so, proceeding as above, we find again (3.4). Hence $L(u,v) = 0$ a.e. in Ω; that is, there exists $k > 0$ such that $u = kv$ in Ω. In particular, $v \in E_1$. This is impossible, because $\lambda \neq \lambda_1$. Therefore, v is a nodal solution of (3.1). □

The next step is to prove that the spectrum of the Robin problem (3.1) is closed. For similar results, when Ω is bounded and $E = W^{1,p}(\Omega)$, we refer to Lemma 2.3 of [23]. To this aim, let us first establish a preliminary result.

Lemma 3.2. *The C^1 functional $\mathcal{I}$ has the property that for all $u, v \in E$*

$$\langle \mathcal{I}'(u) - \mathcal{I}'(v), u - v\rangle \geq p\left(\|u\|^{p-1} - \|v\|^{p-1}\right)\left(\|u\| - \|v\|\right) \geq 0.$$

Proof. Straightforward computations yield

$$\begin{aligned}
&\langle \mathcal{I}'(u) - \mathcal{I}'(v), u - v\rangle \\
&= p\int_\Omega a(x)\left\{|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u\cdot\nabla v - |\nabla v|^{p-2}\nabla v\cdot\nabla u\right\}dx \\
&\qquad + p\int_{\partial\Omega} b(x)\left\{|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu\right\}dS \\
&\geq p\int_\Omega a(x)\left\{|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-1}|\nabla v| - |\nabla v|^{p-1}|\nabla u|\right\}dx \\
&\qquad + p\int_{\partial\Omega} b(x)\left\{|u|^p + |v|^p - |u|^{p-1}|v| - |v|^{p-1}|u|\right\}dS.
\end{aligned}$$

By Hölder's inequality we obtain

$$\int_\Omega a(x)|\nabla u|^{p-1}|\nabla v|\,dx \leq \left(\int_\Omega a(x)|\nabla u|^p\,dx\right)^{1/p'}\left(\int_\Omega a(x)|\nabla v|^p\,dx\right)^{1/p}$$

and

$$\int_{\partial\Omega} b(x)|u|^{p-1}|v|\,dS \leq \left(\int_{\partial\Omega} b(x)|u|^p\,dS\right)^{1/p'}\left(\int_{\partial\Omega} b(x)|v|^p\,dS\right)^{1/p}.$$

Applying the inequality

$$a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha} \leq (a+b)^\alpha (c+d)^{1-\alpha},$$

which holds for any $\alpha \in (0,1)$ and for any $a, b, c, d > 0$, with $\alpha = 1/p'$,

$$\begin{aligned}
a &= \int_\Omega a(x)|\nabla u|^p\,dx, \quad b = \int_{\partial\Omega} b(x)|u|^p\,dS, \\
c &= \int_\Omega a(x)|\nabla v|^p\,dx, \quad d = \int_{\partial\Omega} b(x)|v|^p\,dS,
\end{aligned}$$

we conclude that

$$\int_\Omega a(x)|\nabla u|^{p-1}|\nabla v|\,dx + \int_{\partial\Omega} b(x)|u|^{p-1}|v|\,dS \leq \|u\|^{p-1}\|v\|.$$

Similarly, we have

$$\int_\Omega a(x)|\nabla v|^{p-1}|\nabla u|\,dx + \int_{\partial\Omega} b(x)|v|^{p-1}|u|\,dS \le \|v\|^{p-1}\|u\|.$$

This concludes the proof. □

Proposition 3.3. *The following statements hold.*

(*i*) *The set of the eigenvalues of* (3.1) *is closed in* E.

(*ii*) λ_1 *is isolated; i.e., there exists* $\delta > 0$ *such that the open interval* $(0, \lambda_1 + \delta)$ *does not contain any eigenvalue of* (3.1) *other than* λ_1.

Proof. We somehow follow the main arguments of [23, Theorems 5.9, 5.13] and [20, Proposition 3.1], based on the techniques developed by *Lindqvist* in [24]. For completeness we report here all the principal steps of the proof.

(*i*) Let $n \mapsto (\mu_n, u_n)$ be a sequence of eigenpairs of (3.1), with $\mu_n \to \lambda \ge 0$ as $n \to \infty$. Without loss of generality we assume that $\|u_n\| = 1$ for all n and that $(u_n)_n$ converges weakly to some $u \in E$. By (3.3) with $\phi = u_n - u$, we obtain

$$\langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle = p\mu_n\langle \mathcal{F}(u_n), u_n - u\rangle - \langle \mathcal{I}'(u), u_n - u\rangle \to 0$$

as $n \to \infty$ by Proposition 2.3. Hence $\|u_n\| \to \|u\|$ as $n \to \infty$ by Lemma 3.2. In conclusion, $u_n \to u$ in E by the Clarkson uniform convexity Proposition A.2 of E. Therefore, $\langle \mathcal{I}'(u_n), \phi\rangle \to \langle \mathcal{I}'(u), \phi\rangle$ and $\langle \mathcal{F}(u_n), \phi\rangle \to \langle \mathcal{F}(u), \phi\rangle$ for all $\phi \in E$, by the fact that $\mathcal{I} \in C^1(E)$ and by Proposition 2.3. Since for all n and all $\phi \in E$

$$\langle \mathcal{I}'(u_n), \phi\rangle = p\mu_n\langle \mathcal{F}(u_n), \phi\rangle$$

by (3.3), passing to the limit as $n \to \infty$, we get the claim at once.

(*ii*) Assume for the sake of contradiction that λ_1 is not isolated. Then by (*i*) there exists a sequence of eigenpairs $n \mapsto (\mu_n, u_n)$ such that $u_n \to u$, $\mu_n \to \lambda_1$ as $n \to \infty$ and (λ_1, u) is an eigenpair. Without loss of generality we assume that $\|u_n\| = \|u\| = 1$ and that $u > 0$ in Ω by Proposition 3.1-(*ii*). Hence $u = \varphi_1$ and there exists $\delta > 0$ such that $\mu_n \in (\lambda_1, \lambda_1 + \delta)$ for all n.

Put $\mathcal{U}_n^- = \{x \in \overline{\Omega} : u_n(x) < 0\}$ for all n. Clearly $\text{meas}(\mathcal{U}_n^-) > 0$ by Proposition 3.1-(*iii*). Since (μ_n, u_n) is an eigenpair for (3.1), by (3.3), with $\phi = u_n^- = \min\{u_n, 0\} \in E$, it follows that

$$\|u_n^-\|^p = \int_{\mathcal{U}_n^-} a(x)|\nabla u_n^-|^p\,dx + \int_{\partial\Omega\cap\mathcal{U}_n^-} b(x)|u_n^-|^p\,dS = \mu_n\int_{\mathcal{U}_n^-} f(x)|u_n^-|^p\,dx,$$

which gives by (2.1) and the fact that $\mu_n \in (\lambda_1, \lambda_1 + \delta)$,

$$\|u_n^-\|^p \le c\|u_n^-\|^p_{L^p(\mathcal{U}_n^-, w_1)},$$

where $c = C_f(\lambda_1 + \delta) > 0$. Since $\mathrm{meas}\,(\mathcal{U}_n^-) > 0$, then $\|u_n^-\| > 0$. Thus, taking $\varepsilon = 1/2c$ and $R_n > 0$ so large that $\|u_n^-\|^p_{L^p(\mathcal{U}_n^-,w_1)} - \varepsilon\|u_n^-\|^p < \|u_n^-\|^p_{L^p(\mathcal{U}_n^-\cap B_{R_n},w_1)}$, where B_{R_n} is the ball of center 0 and radius R_n of $\mathbb{R}^N$. Now, combining the above inequalities, we get

$$\begin{aligned}\frac{1}{2}\|u_n^-\|^p &< c\int_{\mathcal{U}_n^-\cap B_{R_n}} w_1(x)|u_n^-|^p\,dx \le c\|u_n^-\|^p_{p^*,w_1}\Big(\int_{\mathcal{U}_n^-\cap B_{R_n}}(1+|x|)^{-\alpha_1}\,dx\Big)^{p/N}\\ &\le d\|u_n^-\|^p\left[\mathrm{meas}(U_n^-\cap B_{R_n})\right]^{p/N},\end{aligned}$$

where $d = c_{w_1}c > 0$ and $c_{w_1} > 0$ is such that $\|v\|^p_{p^*,w_1} \le c_{w_1}\|v\|^p$ for all $v \in E$. In other words, recalling that $\|u_n^-\| > 0$, we have that $\inf_n \mathrm{meas}(\mathcal{U}_n^-\cap B_{R_n}) \ge \sigma > 0$, where $\sigma = (2d)^{-N/p}$. In particular, there exists $R > 0$ such that $\inf_n \mathrm{meas}(B_R\cap\mathcal{U}_n^-\cap B_{R_n}) \ge \sigma/2$. Since $\|u_n - \varphi_1\| \to 0$ as $n\to\infty$, then $\|u_n - \varphi_1\|_{p^*,w_1} \to 0$ as $n \to \infty$. Thus, by Lemma 2.1 and Egorov's theorem, up to a subsequence, $u_n \to \varphi_1$ quasi-uniformly in $\Omega\cap B_R$. This is impossible, since $\varphi_1 > 0$ by (ii).

Indeed, since φ_1 is continuous and positive in Ω, there exists $\varepsilon > 0$ such that

$$\mathrm{meas}(\Omega_\varepsilon) > \mathrm{meas}(\Omega\cap B_R) - \sigma/4,$$

where $\Omega_\varepsilon = \{x \in \Omega\cap B_R : \varphi_1(x) > \varepsilon\}$. Moreover, since $u_n \to \varphi_1$ quasi-uniformly in $\Omega\cap B_R$, there exist $A_\varepsilon \subset \Omega_\varepsilon$ and n_ε such that $\mathrm{meas}(A_\varepsilon) > \mathrm{meas}(\Omega_\varepsilon) - \sigma/4$ and $|u_n(x) - \varphi_1(x)| \le \varepsilon/2$ for all $x \in A_\varepsilon$ and all $n \ge n_\varepsilon$. Thus, $u_{n_\varepsilon} \ge \varphi_1 - \varepsilon/2 > \varepsilon/2$ in $A_\varepsilon \subset U^+_{n_\varepsilon}\cap B_R$, where $U^+_{n_\varepsilon} = \{x \in \Omega : u_{n_\varepsilon}(x) > 0\}$. Hence, since $B_R\cap U^-_{n_\varepsilon}\cap B_{n_\varepsilon} \subset (\Omega\cap B_R\cap B_{n_\varepsilon})\setminus(U^+_{n_\varepsilon}\cap B_R) \subset (\Omega\cap B_R)\setminus A_\varepsilon$, we get

$$\begin{aligned}\mathrm{meas}(B_R\cap\mathcal{U}^-_{n_\varepsilon}\cap B_{R_{n_\varepsilon}}) &\le \mathrm{meas}(\Omega\cap B_R) - \mathrm{meas}(A_\varepsilon)\\ &< \mathrm{meas}(\Omega\cap B_R) - \mathrm{meas}(\Omega_\varepsilon) + \sigma/4\\ &< \mathrm{meas}(\Omega\cap B_R) - \mathrm{meas}(\Omega\cap B_R) + \sigma/2 = \sigma/2.\end{aligned}$$

This contradiction shows that λ_1 is isolated. □

The set $\mathcal{M}$ defined in (3.2) is a closed $\mathbb{Z}_2$-invariant Finsler manifold of E of class C^1. Of course $\mathcal{M} \ne \emptyset$, the space E being compactly embedded in $L^p(\Omega, f)$. Furthermore, $\mathcal{I}(u) = \|u\|^p$ is even and of class $C^1(E)$ and so of class $C^1(\mathcal{M})$. Let $\mathcal{F}_k$ and λ_k be respectively the corresponding sets and numbers defined as in Proposition 2.5. Now, by Proposition 3.1 of [11] we get that

$$i(A) = \sup\{i(K) : K \subset A \textit{ is compact and symmetric}\} \tag{3.5}$$

for all open symmetric subsets A of $\mathcal{M}$. Hence

$$\begin{gathered}\lambda_k = \inf_{K\in\mathcal{G}_k}\max_{u\in K}\mathfrak{I}(u),\\ \mathcal{G}_k=\{K\subset\mathcal{M}: K \text{ is compact and symmetric, with } i(K)\ge k\}.\end{gathered}\tag{3.6}$$

To prove the validity of (3.6) we first show that $\mathcal{G}_k\neq\emptyset$ for all k. Fix $k\in\mathbb{N}$. Since $\operatorname{meas}(\{x\in\Omega: f(x)>0\})>0$, there exist k open balls $B_1,\dots,B_k$ such that $B_i\cap B_j=\emptyset$ if $i\neq j$ and

$$\operatorname{meas}(\{x\in\Omega:\ f(x)>0\}\cap B_i)>0\quad\text{for all } i=1,\dots,k.$$

Choose $u_i\in C_0^\infty(\mathbb{R}^N)$, with $\operatorname{supp}u_i\subset\Omega\cap B_i$ and $\mathcal{J}(u_i)>0$ for every $i=1,\dots,k$. Normalize u_i so that $\mathcal{J}(u_i)=1$, and let E_k be the span of $\{u_i\}_{i=1}^k$. For every $u\in E_k$, we have $u=\sum_{i=1}^k a_iu_i$ and

$$\mathcal{J}(u)=\int_{\cup_{i=1}^k\operatorname{supp}u_i} f(x)\Big|\sum_{i=1}^k a_iu_i\Big|^p\,dx=\sum_{i=1}^k|a_i|^p\int_{\operatorname{supp}u_i}f(x)|u_i|^p\,dx=\sum_{i=1}^k|a_i|^p.$$

Hence, $u\mapsto\mathcal{J}(u)^{1/p}$ defines a norm on E_k which is equivalent to the norm $\|\cdot\|\big|_{E_k}$, E_k being finite dimensional. Therefore, $K=\{u\in E_k:\mathcal{J}(u)=1\}$ is compact and symmetric, and $i(K)=k$ by (i_4). In conclusion, $K\in\mathcal{G}_k\neq\emptyset$, as required.

Now, call the right-hand side of (3.6) μ_k. Of course, $\mu_k\ge\lambda_k$. If $\mu_k>\lambda_k$, there would exist $M\in\mathcal{F}_k$ such that $M\subset\mathfrak{I}_k=\{u\in\mathcal{M}:\mathfrak{I}(u)<\mu_k\}$ and $i(M)\ge k$. Hence, there exists a compact and symmetric subset K of $\mathfrak{I}_k$, with $i(K)\ge k$ by (3.5). Therefore, we get the required contradiction $\mu_k<\mu_k$, and so (3.6) holds.

In particular, $\lambda_1>0$ coincides with the number defined in (3.2). Indeed, if $\tilde\lambda_1$ denotes the number in (3.6) when $k=1$, we get immediately $\tilde\lambda_1=\inf_{K\in\mathcal{G}_1}\mathfrak{I}(u_K)\le\lambda_1$, since $\mathcal{M}\in\mathcal{G}_1$. On the other hand, $\mathfrak{I}(u_K)\ge\mathfrak{I}(\varphi_1)=\lambda_1$ for all $K\in\mathcal{G}_1$, since $K\subset\mathcal{M}$ and φ_1 is the first eigenfunction in Proposition 3.1-(i). Therefore, $\tilde\lambda_1\ge\lambda_1$ and in turn $\tilde\lambda_1=\lambda_1$, as claimed.

Since $\mathcal{M}=\mathcal{J}^{-1}(1)$ and $\mathcal{J}(u)=\|u\|_{p,f}^p$, $u\in\mathcal{M}$ is a critical point of $\mathfrak{I}\big|_{\mathcal{M}}$ if and only if there is some $\mu\in\mathbb{R}$ such that $\mathfrak{I}'(u)=\mu\mathcal{J}'(u)$ and $\mathcal{J}'(u)=p\mathcal{F}(u)$. Clearly, $\mathfrak{I}'(u)\neq0$ whenever $u\in\mathcal{M}$, so that $\mu\neq0$. Actually $\mathfrak{I}'(u)=0$ if and only if $u=0\in E$. Similarly, every $\lambda_k\in\mathbb{R}^+$ for all k.

Now, if we show that $\mathfrak{I}\big|_{\mathcal{M}}$ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}^+$, we are in a position to apply the crucial Proposition 2.5.

Following the main ideas of [11] we prove:

Proposition 3.4. $\mathcal{I}\big|_{\mathcal{M}}$ *satisfies the* $(PS)_c$ *condition for all* c,

$$\lambda_1 = \min_{u\in\mathcal{M}} \mathcal{I}(u), \quad \lambda_k \in \mathbb{R}^+, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

For all k, *with* $\lambda_k < \lambda_{k+1}$, *we have*

$$i\left(\{u \in E \setminus \{0\} : \mathcal{I}(u) \le \lambda_k \mathcal{J}(u)\}\right) = i\left(\{u \in E : \mathcal{I}(u) < \lambda_{k+1}\mathcal{J}(u)\}\right) = k.$$

Proof. Let $c \in \mathbb{R}$ and $(u_n)_n \subset \mathcal{M}$ be a $(PS)_c$ sequence of $\mathcal{I}\big|_{\mathcal{M}}$. Hence $\mathcal{I}(u_n) \to c$ and $\mathcal{I}'(u_n) - p\mu_n\mathcal{F}(u_n) \to 0$ in $E^\star$ for a suitable sequence $(\mu_n)_n$ in $\mathbb{R}$. Clearly, $(u_n)_n$ is bounded in E, and so passing to a subsequence $u_n \rightharpoonup u$ in E, the space E being reflexive. Hence, $\mathcal{I}(u_n) \to \mathcal{I}(u)$ in $L^1(\Omega)$ by Proposition 2.3 with $w = f$, so that $\|u\|_{p,f} = 1$, and in turn $u \neq 0$ and $\mathcal{I}(u) > 0$. Now,

$$p\left(\mathcal{I}(u_n) - \mu_n\right) = \langle \mathcal{I}'(u_n), u_n\rangle - p\mu_n\langle \mathcal{F}(u_n), u_n\rangle \to 0.$$

Hence, $\mu_n \to c$ and $0 < \mathcal{I}(u) \le c$ by the weak lower semicontinuity of the norm in E. Therefore, $c > 0$.

Taking $\phi = u_n - u \in E$ in (3.3), we obtain

$$\begin{aligned}\langle \mathcal{I}'(u_n), u_n - u\rangle &- p\mu_n\langle \mathcal{F}(u_n), u_n - u\rangle \\ &= \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle - p\mu_n\langle \mathcal{F}(u_n) - \mathcal{F}(u), u_n - u\rangle \\ &\quad + \langle \mathcal{I}'(u), u_n - u\rangle - p\mu_n\langle \mathcal{F}(u), u_n - u\rangle\end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, $0 \le \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle \to 0$ as $n \to \infty$ by Proposition 2.3. Clearly, $\langle \mathcal{I}'(u), u_n - u\rangle \to 0$ as $n \to \infty$, and $u_n \rightharpoonup u$ in E. Hence

$$\langle \mathcal{I}'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Moreover, $\mathcal{I}(u) \le \liminf_n \mathcal{I}(u_n)$ due to the weak lower semicontinuity of $\mathcal{I}$ on E. On other hand, by convexity,

$$\mathcal{I}(u) + \langle \mathcal{I}'(u_n), u_n - u\rangle \ge \mathcal{I}(u_n),$$

so that $\mathcal{I}(u) \ge \limsup_n \mathcal{I}(u_n)$. In other words, $\mathcal{I}(u) = \lim_n \mathcal{I}(u_n)$. Thanks to the Clarkson uniform convexity Proposition A.2 of E, it follows that $\|u_n - u\| \to 0$ as $n \to \infty$, as required. In other words, $\mathcal{I}\big|_{\mathcal{M}}$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}^+$. An application of Proposition 2.5 shows that $\lambda_k \to \infty$ as $k \to \infty$.

Assume now that $k \ge 1$ and $\lambda_k < \lambda_{k+1}$. Put

$$C = \{u \in \mathcal{M} : \mathcal{I}(u) \le \lambda_k\} \quad \text{and} \quad U = \{u \in \mathcal{M} : \mathcal{I}(u) < \lambda_{k+1}\}.$$

From now on the proof can follow word by word the proof of Theorem 3.2 of [11] by virtue of Proposition 2.3. $\square$

4. Existence in the Ambrosetti–Rabinowitz case

Throughout the section we assume that $g : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ *and* $h : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ *are Carathéodory functions,* $G(x,s) = \int_0^s g(x,t)\,dt$*,* $H(x,s) = \int_0^s h(x,t)dt$*,* g_0*,* g_1*,* γ_0*, and* γ_1 *are measurable functions on* Ω*, while* h_0*,* h_1*,* τ_0*, and* τ_1 *are measurable functions on* $\partial\Omega$*, such that*

(g_1) $|g(x,s)| \le g_0(x)|s|^{p-1} + g_1(x)|s|^{r-1}$ *for a.a.* $x \in \Omega$ *and all* $s \in \mathbb{R}$*, where* $p < r < p^* = pN/(N-p)$*, with*

$$0 < g_0(x) \le C_g w_2(x) \quad \textit{and} \quad 0 \le g_1(x) \le C_g g_0(x) \quad \textit{for a.a. } x \in \Omega,$$

$$g_0/w_2 \in L^{r/(r-p)}(\Omega, w_2), \qquad g_0 \in L^{\tilde r/(\tilde r-p)}(\Omega, w_2),$$

where $w_2(x) = (1+|x|)^{-\alpha_2}$*,* $N - r(N-p)/p < \alpha_2 < N$*, and* $\tilde r$ *is an appropriate exponent, with* $r < \tilde r < \min\{pr, p^*\}$*;*

(g_2) $\lim\limits_{s\to 0} \dfrac{G(x,s)}{g_0(x)|s|^p} = 0$ *and* $\lim\limits_{|s|\to\infty} \dfrac{G(x,s)}{g_0(x)|s|^p} = \infty$*, both uniformly in* $\Omega \setminus \mathcal{N}$*, with* $\mathrm{meas}(\mathcal{N}) = 0$*;*

(g_3) $\gamma_0 \in L^1(\Omega)$ *and* $0 < \gamma_1(x) \le C_\gamma w_1(x)$ *a.e. in* Ω*, and there exists* $\mu > p$ *such that for a.a.* $x \in \Omega$ *and all* $s \in \mathbb{R}$

$$\mu G(x,s) \le s g(x,s) + \gamma_0(x) + \gamma_1(x)|s|^p;$$

(g_4) $G(x,s) \ge 0$ *for a.a.* $x \in \Omega$ *and all* $s \in \mathbb{R}$*;*

(h_1) $|h(x,s)| \le h_0(x)|s|^{p-1} + h_1(x)|s|^{q-1}$ *for a.a.* $x \in \partial\Omega$ *and all* $s \in \mathbb{R}$*, where* $p < q < p_* = p(N-1)/(N-p)$*, with*

$$0 < h_0(x) \le C_h w_3(x) \quad \textit{and} \quad 0 \le h_1(x) \le C_h h_0(x) \quad \textit{for a.a. } x \in \partial\Omega,$$

$$h_0/w_3 \in L^{q/(q-p)}(\partial\Omega, w_3), \qquad h_0 \in L^{\tilde q/(\tilde q-p)}(\partial\Omega, w_3),$$

where $w_3(x) = (1+|x|)^{-\alpha_3}$*,* $N - 1 - q(N-p)/p < \alpha_3 < N$*, and* $\tilde q$ *is an appropriate exponent, with* $q < \tilde q < \min\{qr, p_*\}$*;*

(h_2) $\lim\limits_{s\to 0} \dfrac{H(x,s)}{h_0(x)|s|^p} = 0$ *and* $\lim\limits_{|s|\to\infty} \dfrac{H(x,s)}{h_0(x)|s|^p} = \infty$*, both uniformly in* $\partial\Omega\setminus\mathcal{N}$*, with* $\mathrm{meas}_{N-1}(\mathcal{N}) = 0$*;*

(h_3) $w_4(x) = (1+|x|)^{-\alpha_4}$*,* $p - 1 < \alpha_4 < N$*,* $\tau_0 \in L^1(\partial\Omega)$*,* $0 < \tau_1(x) \le C_\tau w_4(x)$ *a.e. in* $\partial\Omega$*, and there exists* $\tilde\mu > p$ *such that*

$$\tilde\mu H(x,s) \le s h(x,s) + \tau_0(x) + \tau_1(x)|s|^p \ \textit{for a.a. } x \in \partial\Omega \textit{ and all } s \in \mathbb{R};$$

(h_4) $H(x,s) \ge 0$ *for a.a.* $x \in \partial\Omega$ *and all* $s \in \mathbb{R}$*.*

Note that if $g_0 \in L^{r'}(\Omega, w_2^{1/(1-r)})$, then $g_0/w_2 \in L^{r/(r-p)}(\Omega, w_2)$ and $g_0 \in L^{\tilde r/(\tilde r-p)}(\Omega, w_2)$. Also $h_0 \in L^{q'}(\partial\Omega, w_3^{1/(1-q)})$ yields $h_0/w_3 \in L^{q/(q-p)}(\partial\Omega, w_3)$

and $h_0 \in L^{\tilde{q}/(\tilde{q}-p)}(\partial\Omega, w_3)$. Therefore, conditions (g_1) and (h_1) are weaker than the corresponding assumption A1 in [31]; see also [27].

By (g_1) and (h_1) we have $g(x,0) = 0$ for a.a. $x \in \overline{\Omega}$ and $h(x,0) = 0$ for a.a. $x \in \partial\Omega$, so that problem (1.1) always admits the trivial solution $u \equiv 0$.

Let $\Phi : E \to \mathbb{R}$ be the Euler–Lagrange functional associated to (1.1) defined for all $u \in E$ by

$$\Phi(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|^p_{p,f} - \int_\Omega G(x,u)\,dx - \int_{\partial\Omega} H(x,u)\,dS. \tag{4.1}$$

The embeddings given in Section 2 assure that Φ is well-defined and of class $C^1(E)$.

A *weak solution* of (1.1) is a function $u \in E$ such that for all $\phi \in E$

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\phi\,dx + \int_{\partial\Omega} b(x)|u|^{p-2}u\phi\,dx$$
$$= \lambda\int_\Omega f(x)|u|^{p-2}u\phi\,dx + \int_\Omega g(x,u)\phi\,dx + \int_{\partial\Omega} h(x,u)\phi\,dS.$$

Lemma 4.1. *(i)* $\|u\|^{-p}\Big(\int_\Omega G(x,u)\,dx + \int_{\partial\Omega} H(x,u)\,dS\Big) \to 0$ *as* $\|u\| \to 0$.

(ii) Let w and $\tilde{w}$ be two weights on Ω and on $\partial\Omega$, respectively, such that the embeddings $E \hookrightarrow L^p(\Omega, w)$ and $E \hookrightarrow L^p(\partial\Omega, \tilde{w})$ are compact. Let $(u_n)_n$ be a sequence in E such that $\|u_n\| \le c\,(\|u_n\|_{p,w} + \|u_n\|_{p,\tilde{w},\partial\Omega})$ for all n and some constant $c > 0$ independent of n. If $\|u_n\| \to \infty$, then

$$\|u_n\|^{-p}\Big(\int_\Omega G(x,u_n)\,dx + \int_{\partial\Omega} H(x,u_n)\,dS\Big) \to \infty \quad \text{as } n \to \infty.$$

Proof. *(i)* By (g_1) and (g_2) clearly $G_0(x,s)$, defined by

$$G_0(x,s) = G(x,s)/w_2(x)|s|^p$$

if $s \ne 0$ and $G_0(x,0) = 0$ if $s = 0$, is a Carathéodory function in $\Omega \times \mathbb{R}$. Moreover, using also (g_4), for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$ we have by (g_1)

$$0 \le G_0(x,s) \le C_0\left(\tilde{g}_0(x) + |s|^{r-p}\right),$$

where $C_0 = \max\{1/p,\, C_g^2/r\}$ and $\tilde{g}_0 = g_0/w_2$. Therefore, applying the Hölder inequality, we obtain

$$0 \le \int_\Omega G(x,u)\,dx \le \Big(\int_\Omega w_2(x)G_0(x,u)^{r/(r-p)}\,dx\Big)^{(r-p)/r}\|u\|^p_{r,w_2},$$

with

$$\int_\Omega w_2(x)G_0(x,u)^{r/(r-p)}\,dx \to 0 \quad \text{as } \|u\| \to 0.$$

Indeed, by Lemma 2.2-(i), with $A = \Omega$, $d = N$, $w = w_2$, $\alpha = r$, and $\beta = r/(r-p)$, the Nemytskii operator $N_{G_0}(u) = G_0(x,u)$, $N_{G_0}(0) = 0$, maps continuously $L^r(\Omega, w_2)$ into $L^{r/(r-p)}(\Omega, w_2)$.

Therefore,

$$\int_\Omega G(x,u)\,dx = o(\|u\|^p) \quad \text{as } \|u\| \to 0,$$

since $E \hookrightarrow L^r(\Omega, w_2)$ even compact by Theorem 1 of [31]. Similarly, by (h_1) and (h_2), clearly, $H_0(x,s)$, defined by $H_0(x,s) = H(x,s)/w_3(x)|s|^p$ if $s \neq 0$ and $H_0(x,0) = 0$ if $s = 0$, is a Carathéodory function in $\partial\Omega \times \mathbb{R}$. Moreover, using also (h_4), for a.a. $x \in \partial\Omega$ and all $s \in \mathbb{R}$, we obtain by (h_1)

$$0 \le H_0(x,s) \le \tilde{C}_0 \left(\tilde{h}_0(x) + |s|^{q-p} \right),$$

where $\tilde{C}_0 = \max\{1/p,\, C_h^2/q\}$ and $\tilde{h}_0 = h_0/w_3$. Therefore, applying the Hölder inequality, we obtain

$$0 \le \int_{\partial\Omega} H(x,u)\,dS \le \Big(\int_{\partial\Omega} w_3(x) H_0(x,u)^{q/(q-p)}\,dS \Big)^{(q-p)/q} \|u\|^p_{q,w_3,\partial\Omega},$$

with

$$\int_{\partial\Omega} w_3(x) H_0(x,u)^{q/(q-p)}\,dS \to 0 \quad \text{as } \|u\| \to 0.$$

Indeed, by Lemma 2.2-(i), with $A = \partial\Omega$, $d = N-1$, $w = w_3$, $\alpha = q$, and $\beta = q/(q-p)$, the Nemytskii operator $N_{H_0}(u) = H_0(x,u)$, $N_{H_0}(0) = 0$, maps continuously $L^q(\partial\Omega, w_3)$ into $L^{q/(q-p)}(\partial\Omega, w_3)$, using the argument shown above. Therefore,

$$\int_{\partial\Omega} H(x,u)\,dS = o(\|u\|^p) \quad \text{as } \|u\| \to 0,$$

since $E \hookrightarrow L^q(\partial\Omega, w_3)$ even compact by Theorem 1 of [31].

In conclusion, the assertion follows at once.

(ii) Let $(u_n)_n$ be a sequence in E as in the statement, so that $v_n = u_n/\|u_n\|$ is in the unit sphere of E for all n sufficiently large. Therefore, up to a subsequence, still denoted for simplicity by $(v_n)_n$, there is $v \in E$ such that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^p(\Omega, w)$, $v_n \to v$ in $L^p(\partial\Omega, \tilde{w})$, $v_n \to v$ a.e. in Ω, and $v_n \to v$ a.e. in $\partial\Omega$. From the second and third parts of Proposition 2.3 we get

$$1 \le c \lim_n \left(\|v_n\|_{p,w} + \|v_n\|_{p,\tilde{w},\partial\Omega} \right) = c \left(\|v\|_{p,w} + \|v\|_{p,\tilde{w},\partial\Omega} \right).$$

Hence either $|v| > 0$ in a subset A of Ω, with $\text{meas}(A) > 0$, or $|v| > 0$ in a subset Γ of $\partial\Omega$, with $\text{meas}_{N-1}(\Gamma) > 0$, since $c > 0$ and $w > 0$ a.e. in Ω and $\tilde{w} > 0$ a.e. in $\partial\Omega$.

Case $|v| > 0$ *in* A, $\text{meas}(A) > 0$. For n sufficiently large

$$\frac{G(x, u_n(x))}{\|u_n\|^p} = \frac{G(x, \|u_n\| v_n(x))}{g_0(x)\|u_n\|^p |v_n(x)|^p}\, g_0(x)|v_n(x)|^p \quad \text{a.e. in } A.$$

Hence $\lim_n \|u_n\|^{-p} G(x, u_n(x)) = \infty$ a.e. in A by (g_2). By (g_4) and the Fatou lemma we get

$$\|u_n\|^{-p} \int_\Omega G(x, u_n)\, dx \to \infty \quad \text{as } n \to \infty.$$

Case $|v| > 0$ *in* Γ, $\text{meas}_{N-1}(\Gamma) > 0$. Similarly, for n sufficiently large

$$\frac{H(x, u_n(x))}{\|u_n\|^p} = \frac{H(x, \|u_n\| v_n(x))}{h_0(x)\|u_n\|^p |v_n(x)|^p}\, h_0(x)|v_n(x)|^p \quad \text{a.e. in } \Gamma.$$

Thus $\lim_n \|u_n\|^{-p} H(x, u_n(x)) = \infty$ a.e. in Γ by (h_2). By (h_4) and the Fatou lemma we have

$$\|u_n\|^{-p} \int_{\partial\Omega} H(x, u_n)\, dS \to \infty \quad \text{as } n \to \infty.$$

From (g_4) and (h_4) we get the assertion at once. $\square$

Lemma 4.2. (i) *If* $(u_n)_n$ *is bounded in* $L^{\tilde{r}}(\Omega, w_2)$ *and converges to some* $u \in L^{\tilde{r}}(\Omega, w_2)$ *a.e. in* Ω, *then* $N_g(u_n) \to N_g(u)$ *in* $L^{\tilde{r}'}(\Omega, w_2)$, *and so in* $E^\star$.

(ii) *If* $(u_n)_n$ *is bounded in* $L^{\tilde{q}}(\partial\Omega, w_3)$ *and converges to some* $u \in L^{\tilde{q}}(\partial\Omega, w_3)$ *a.e. in* $\partial\Omega$, *then* $N_h(u_n) \to N_h(u)$ *in* $L^{\tilde{q}'}(\partial\Omega, w_3)$, *and so in* $E^\star$.

(iii) *For all* λ *and* c *in* $\mathbb{R}$ *the functional* Φ *satisfies the* $(PS)_c$ *condition.*

Proof. (i) From (g_1) we clearly get $|s|^{1-\tilde{r}} g(x, s) \to 0$ as $|s| \to \infty$ uniformly in $\Omega \setminus \mathcal{N}$ for some $\mathcal{N}$, with $\text{meas}(\mathcal{N}) = 0$. Hence, for all $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|g(x, s)| \le C_\varepsilon g_0(x)|s|^{p-1} + \varepsilon |s|^{\tilde{r}-1} \quad \text{a.e. in } \Omega \text{ and for all } s \in \mathbb{R}.$$

Applying Young's inequality, we get

$$|g(x, s)| \le \frac{\tilde{r} - p}{\tilde{r} - 1}\left(\frac{C_\varepsilon g_0(x)}{\varepsilon}\right)^{(\tilde{r}-1)/(\tilde{r}-p)} + \left(\frac{p-1}{\tilde{r}-1}\, \varepsilon^{(\tilde{r}-1)/(p-1)} + \varepsilon\right)|s|^{\tilde{r}-1}.$$

Hence, by Lemma 2.2-(ii), with $A = \Omega$, $d = N$, $w = w_2$, $\alpha = \tilde{r}$, and $\beta = \tilde{r}'$, we get the assertion at once, since $g_0 \in L^{\tilde{r}/(\tilde{r}-p)}(\Omega, w_2)$ by (g_1). Naturally the strong convergence in $L^{\tilde{r}'}(\Omega, w_2)$ implies convergence in $E^\star$, since $E \hookrightarrow L^{\tilde{r}}(\Omega, w_2)$ compact.

(ii) Similarly, $|s|^{1-\tilde{q}}h(x,s)\to 0$ as $|s|\to\infty$ uniformly in $\Omega\setminus\mathcal{N}$ for some $\mathcal{N}$, with $\operatorname{meas}(\mathcal{N})=0$, by (h_1). Thus, for all $\varepsilon>0$ there is $K_\varepsilon>0$ such that

$$|h(x,s)|\le K_\varepsilon h_0(x)|s|^{p-1}+\varepsilon|s|^{\tilde{q}-1}\quad \text{a.e. in } \partial\Omega \text{ and for all } s\in\mathbb{R}.$$

Proceeding as above and applying Young's inequality we obtain

$$|h(x,s)|\le \frac{\tilde{q}-p}{\tilde{q}-1}\Big(\frac{K_\varepsilon h_0(x)}{\varepsilon}\Big)^{(\tilde{q}-1)/(\tilde{q}-p)}+\Big(\frac{p-1}{\tilde{q}-1}\,\varepsilon^{(\tilde{q}-1)/(p-1)}+\varepsilon\Big)|s|^{\tilde{q}-1}.$$

Hence, by Lemma 2.2-*(ii)*, with $A=\partial\Omega$, $d=N-1$, $w=w_3$, $\alpha=\tilde{q}$, and $\beta=\tilde{q}'$, we reach the conclusion, since $h_0\in L^{\tilde{q}/(\tilde{q}-p)}(\partial\Omega,w_3)$ by (h_1). Naturally, the strong convergence in $L^{\tilde{q}'}(\partial\Omega,w_3)$ implies convergence in $E^\star$, since $E\hookrightarrow L^{\tilde{q}}(\partial\Omega,w_3)$ compact.

(iii) Let $(u_n)_n\subset E$ be a $(PS)_c$ sequence for Φ. We claim that $(u_n)_n$ is bounded in E. Assume for the sake of contradiction that $\|u_n\|\to\infty$ as $n\to\infty$ up to a subsequence.

Case $\mu\le\tilde{\mu}$. By (g_3), (h_3), and (h_4), we have

$$\begin{aligned}\mu\Phi(u_n)-\langle\Phi'(u_n),u_n\rangle &\ge \Big(\frac{\mu}{p}-1\Big)\Big(\|u_n\|^p-\lambda\|u_n\|^p_{p,f}\Big)\\ &\quad-\int_\Omega[\mu G(x,u_n)-u_n g(x,u_n)]\,dx-\int_{\partial\Omega}[\tilde{\mu}H(x,u_n)-u_n h(x,u_n)]\,dS\\ &\ge 2\kappa\|u_n\|^p-\int_\Omega[\gamma_0(x)+\{2|\lambda|\kappa f(x)+\gamma_1(x)\}|u_n|^p]\,dx\\ &\qquad-\int_{\partial\Omega}[\tau_0(x)+\tau_1(x)|u_n|^p]\,dS,\end{aligned}$$

where $2\kappa=\mu/p-1>0$. Now, $\|u_n\|\to\infty$ as $n\to\infty$, so that for all n sufficiently large

$$\mu\Phi(u_n)-\langle\Phi'(u_n),u_n\rangle+\int_\Omega\gamma_0(x)\,dx+\int_{\partial\Omega}\tau_0(x)\,dS\le\kappa\|u_n\|^p.$$

Combining together the last two inequalities, we get

$$\kappa\|u_n\|^p\le\|u_n\|^p_{p,w}+\|u_n\|^p_{p,\tilde{w},\partial\Omega},\quad w=2|\lambda|\kappa f+\gamma_1,\quad \tilde{w}=\tau_1, \tag{4.2}$$

for all n large enough.

Case $\mu>\tilde{\mu}$. Similarly, by (g_3), (h_3), and (g_4), we get

$$\begin{aligned}\tilde{\mu}\Phi(u_n)-\langle\Phi'(u_n),u_n\rangle &\ge \Big(\frac{\tilde{\mu}}{p}-1\Big)\Big(\|u_n\|^p-\lambda\|u_n\|^p_{p,f}\Big)\\ &\quad-\int_\Omega[\mu G(x,u_n)-u_n g(x,u_n)]\,dx-\int_{\partial\Omega}[\tilde{\mu}H(x,u_n)-u_n h(x,u_n)]\,dS\end{aligned}$$

$$\geq 2\kappa\|u_n\|^p - \int_\Omega [\gamma_0(x) + \{2|\lambda|\kappa f(x) + \gamma_1(x)\}|u_n|^p]\,dx - \int_{\partial\Omega} [\tau_0(x) + \tau_1(x)|u_n|^p]\,dS,$$

where now $2\kappa = \tilde{\mu}/p - 1 > 0$. Since $\|u_n\| \to \infty$ as $n \to \infty$, for all n sufficiently large

$$\tilde{\mu}\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle + \int_\Omega \gamma_0(x)\,dx + \int_{\partial\Omega} \tau_0(x)\,dS \leq \kappa\|u_n\|^p.$$

Combining together the last two inequalities, we again obtain (4.2).

In both cases the embeddings $E \hookrightarrow L^p(\Omega, w)$ and $E \hookrightarrow L^p(\partial\Omega, \tilde{w})$ are compact by (2.1), (g_3), and (h_3). Hence, applying Lemma 4.1-(ii), with $w = 2|\lambda|\kappa f + \gamma_1$, $\tilde{w} = \tau_1$, and $c = 1/\kappa > 0$, we get

$$\|u_n\|^{-p}\Big(\int_\Omega G(x, u_n)\,dx + \int_{\partial\Omega} H(x, u_n)\,dS\Big) \to \infty \quad \text{as } n \to \infty.$$

In conclusion, since $\|u_n\|_{p,f} \leq C\|u_n\|$ for all n by (2.2), we have

$$0 = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^p} \leq \frac{1}{p} + \lim_{n\to\infty}\left(|\lambda|\,\frac{\|u_n\|^p_{p,f}}{\|u_n\|^p} - \frac{\int_\Omega G(x,u_n)\,dx + \int_{\partial\Omega} H(x,u_n)\,dS}{\|u_n\|^p}\right) = -\infty,$$

which is the required contradiction.

Therefore, $(u_n)_n$ is bounded in E, and so there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^{\tilde{r}}(\Omega, w_2)$, $u_n \to u$ in $L^{\tilde{q}}(\partial\Omega, w_3)$, $u_n \to u$ a.e. in Ω, and $u_n \to u$ a.e. in $\partial\Omega$. Now,

$$\langle \Phi'(u_n), u_n - u\rangle = \frac{1}{p}\langle \mathcal{I}'(u_n), u_n - u\rangle - \lambda\langle \mathcal{F}(u_n), u_n - u\rangle - \int_\Omega g(x, u_n)(u_n - u)\,dx - \int_{\partial\Omega} h(x, u_n)(u_n - u)\,dS \to 0,$$

since $(u_n)_n$ is a $(PS)_c$ sequence. Hence, by virtue of (i) and (ii) of this lemma and of Proposition 2.3, we have shown that

$$\langle \mathcal{I}'(u_n), u_n - u\rangle \to 0 \quad \text{as } n \to \infty;$$

that is, (3.7) holds. This implies that actually $u_n \to u$ in E as $n \to \infty$, using the same argument given in the proof of Proposition 3.4. The lemma is now proved. □

Theorem 4.3. *Under assumptions* (g_1)–(g_4) *and* (h_1)–(h_4) *problem* (1.1) *admits at least a nontrivial solution* $u \in E$ *for every* $\lambda \in \mathbb{R}$.

Proof. Consider first the case $\lambda \geq \lambda_1$. By Propositions 3.1, 3.3, and 3.4 we have $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \cdots \to \infty$, so that there exists $k \geq 1$ such that $\lambda_k \leq \lambda < \lambda_{k+1}$. Define the two symmetric closed cones

$$C_- = \{u \in E : \mathcal{I}(u) \leq \lambda_k \mathcal{J}(u)\}, \qquad C_+ = \{u \in E : \mathcal{I}(u) \geq \lambda_{k+1} \mathcal{J}(u)\}.$$

Clearly, $C_- \cap C_+ = \{0\}$ and $i(C_- \setminus \{0\}) = i(E \setminus C_+) = k$ by Proposition 3.4. Furthermore, for all $u \in C_+$

$$\Phi(u) \geq \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big)\|u\|^p - \int_\Omega G(x,u)\,dx - \int_{\partial\Omega} H(x,u)\,dS. \tag{4.3}$$

Thus, by Lemma 4.1-(i) there exists a number $r_+ > 0$ such that $\Phi(u) \geq \alpha$ for all $u \in C_+$, with $\|u\| = r_+$, where $\alpha = r_+^p(1 - \lambda/\lambda_{k+1})/2p > 0$. On the other hand, $\Phi(u) \leq (\lambda_k - \lambda)\|u\|_{p,f}^p/p \leq 0$ for all $u \in C_-$ by (g_4) and (h_4).

Since $E \hookrightarrow L^p(\Omega, f)$ and $E \hookrightarrow L^p(\partial\Omega, w_4)$ are compact by Theorem 1 of [31], the cone C_- is also closed in the real normed space $E = (E, |||\cdot|||)$, where $|||\cdot||| = \|\cdot\|_{p,f} + \|\cdot\|_{p,w_4,\partial\Omega}$. Taking $e \in E \setminus C_-$ and $t > 0$, we easily see as in [11] that

$$\|u + te\| \leq t \max\{\lambda_k^{1/p}, \|e\|/\,|||e|||\}(|||u/t||| + |||e|||).$$

Moreover, by Proposition 2.12. of [11] there exists $\beta \geq 1$ such that for all $u \in C_-$ we have $|||u/t||| + |||e||| \leq \beta\,|||u/t + e|||$. In conclusion,

$$\|u + te\| \leq \kappa\,|||u + te||| \tag{4.4}$$

for all $u \in C_-$ and $t \geq 0$, where $\kappa = \beta \max\{\lambda_k^{1/p}, \|e\|/\,|||e|||\} > 0$.

Now, along any sequence $(u_n)_n \subset C_- + \mathbb{R}^+ e \subset E$ such that $\|u_n\| \to \infty$ Lemma 4.1-(ii) can be applied, since $\|u_n\| \leq \kappa\,(\|u_n\|_{p,f} + \|u_n\|_{p,w_4,\partial\Omega})$ for all n. Therefore, an easy contradiction argument shows the existence of some $r_- > r_+$ such that $\Phi(u) \leq 0$ for all $u \in C_- + \mathbb{R}^+ e$, with $\|u\| \geq r_-$.

The geometrical construction of Theorem 2.7 is completed, so that the corresponding sets satisfy the assertion. In particular, $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $k+1$ over $\mathbb{Z}_2$ and

$$\sup_{u \in D_- \cup H} \Phi(u) \leq 0 < \alpha \leq \inf_{u \in S_+} \Phi(u) \quad \text{and} \quad \sup_{u \in Q} \Phi(u) \leq r_-^p/p < \infty. \tag{4.5}$$

Finally, $\Phi \in C^1(E)$ satisfies the $(PS)_c$ condition by Lemma 4.2-(iii) for all λ and c in $\mathbb{R}$. Therefore, problem (1.1) admits a weak nontrivial solution $u \in E$, with $\Phi(u) \geq \alpha$ by virtue of Theorem 2.7.

On the other hand, if $\lambda < \lambda_1$, the geometrical structure of the mountain-pass theorem of *Ambrosetti* and *Rabinowitz* in [3] is valid. Indeed, $\Phi(0) = 0$ and for all $u \in E$

$$\Phi(u) \geq \frac{1}{p}\Big(1 - \frac{\lambda}{\lambda_1}\Big)\|u\|^p - \int_\Omega G(x, u(x))\,dx - \int_{\partial\Omega} H(x, u)\,dS,$$

so that by virtue of Lemma 4.1-(i) there exists $r > 0$ sufficiently small so that $\Phi(u) \geq \alpha_r$ for all $u \in E$, with $\|u\| = r$, where $\alpha_r = r^p(1-\lambda/\lambda_1)/2p > 0$. Fix a finite-dimensional subspace W of E and $w \in W$, with $\|w\| = 1$. Put $n \mapsto u_n = nw \in W$. Then $\|u_n\| \leq c_W\,(\|u_n\|_{p,f} + \|u_n\|_{p,w_4,\partial\Omega})$ for all n, where $c_W > 0$ is independent of n. This can be done, W being a finite-dimensional subspace of E. Since $\|u_n\| \to \infty$ as $n \to \infty$, by Lemma 4.1-(ii) and by (2.2)

$$\Phi(u_n) \leq \|u_n\|^p \Big(\frac{\max\{1, C^p\lambda^-\}}{p} - \frac{\int_\Omega G(x, u_n(x))\,dx + \int_{\partial\Omega} H(x, u)\,dS}{\|u_n\|^p}\Big) \to -\infty$$

as $n \to \infty$. Hence, fix n so large that $v = nw \in W$ is such that $\|v\| > r$ and $\Phi(v) < 0$. Finally, define

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \Phi(u), \quad \text{where } \Gamma = \{\gamma \in C([0, 1]) : \gamma(0) = 0,\ \gamma(1) = v\}.$$

Then $c \geq \alpha_r$ and c is a critical value for Φ in E by virtue of Theorem 2.1 of [3]. In other words, problem (1.1) admits a weak nontrivial solution $u \in E$, with $\Phi(u) \geq \alpha_r$. □

5. The Nehari manifold method

In the second part of the paper we plan to face (1.1), when $\lambda < \lambda_1$, with the Nehari manifold method, developed by *Szulkin* and *Weth* in [38]. In this section we describe it in an abstract setting.

Let E be a real uniformly convex Banach space and S_E be its unit sphere. Let $\Phi \in C^1(E)$, and suppose that $u \neq 0$ is a critical point of Φ; i.e., $\Phi'(u) = 0$. Then necessarily u is contained in the set $\mathcal{N} = \{u \in E\setminus\{0\} : \langle\Phi'(u), u\rangle = 0\}$. Therefore, $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of Φ. The set $\mathcal{N}$ is called a *Nehari manifold*, even if in general it may not be a manifold. Set $c = \inf_{u\in\mathcal{N}} \Phi(u)$. Clearly, if c is attained at some $u_0 \in \mathcal{N}$, then u_0 is a nontrivial critical point of Φ in E. Since u_0 is a solution of the equation $\Phi'(u) = 0$, with *"minimal energy"* Φ, we call u_0 a *ground state* of Φ.

Before proving the main theorems of the section, let us start with the structural assumptions (i)–(vi) below and some preliminary useful results.

Let I, I_0, $\mathcal{I}_0$, and $\mathcal{J}_0$ be functionals of class $C^1(E)$, satisfying

(i) $\|I'(u)\|_{E^\star} = o(\|u\|^{p-1})$ *as* $\|u\| \to 0$;

(ii) $s \mapsto s^{1-p}\langle I'(su), u\rangle$ *is strictly increasing for* $u \neq 0$ *and* $s > 0$;

(iii) $s^{-p}I(su) \to \infty$ *as* $s \to \infty$, *uniformly for* u *on weakly compact subsets of* $E \setminus \{0\}$;

(iv) I' *is completely continuous;*

(v) I_0 *is positively homogeneous of degree* $p > 1$ *and satisfies*

$$c_0\|u\|^p \leq I_0(u) \leq C_0\|u\|^p \tag{5.1}$$

for some $0 < c_0 \leq C_0$ *and for all* $u \in E$;

(vi) $I_0 = \mathcal{I}_0 + \mathcal{J}_0$, *where* $\mathcal{J}_0'$ *is completely continuous, while* $\mathcal{I}_0$ *is weakly lower semicontinuous and* $\mathcal{I}_0'$ *satisfies*

$$\langle \mathcal{I}_0'(u) - \mathcal{I}_0'(v), u - v\rangle \geq c_1(\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) \tag{5.2}$$

for some $c_1 > 0$ *and for all* $u, v \in E$.

Put $\Phi = I_0 - I$. Hence, $\Phi(0) = 0$, since $I_0(0) = 0$ by (v) and $I(0) = 0$ by (i).

The next lemma is proved following some of the arguments given in [39, Lemma 4.1].

Lemma 5.1. *The following properties hold.*

(a) *For all* $v \in E \setminus \{0\}$ *there exists a unique* $s_v > 0$ *such that* $s_v v \in \mathcal{N}$ *and*

$$\Phi(s_v v) = \max_{s>0} \Phi(sv) > 0;$$

(b) *the function* $\eta : E \setminus \{0\} \to \mathbb{R}^+$, $\eta(v) = s_v$, *where* s_v *is uniquely defined by* (a), *is continuous;*

(c) *the map* $m : S_E \to \mathcal{N}$, $m(z) = s_z z$, *where* s_z *is uniquely defined by* (a), *is a homeomorphism;*

(d) *there exists* $\delta > 0$ *such that* $s_z \geq \delta$ *for all* $z \in S_E$ *and* $\inf_{u\in\mathcal{N}} \|u\| = \inf_{z\in S_E} \|s_z z\| > 0$;

(e) $c = \inf_{u\in\mathcal{N}} \Phi(u) = \inf_{v\in E\setminus\{0\}} \max_{s>0} \Phi(sv) = \inf_{z\in S_E} \max_{s>0} \Phi(sz)$;

(f) *if* c *is attained, then* $c > 0$.

Proof. (a) Fix $v \in E \setminus \{0\}$. For all $s > 0$ define $\psi(s) = \Phi(sv)$. Clearly, $\psi \in C^1(\mathbb{R}^+)$ and $\psi'(s) = \langle \Phi'(sv), v\rangle$, so that $\psi'(s) = 0$ if and only if $sv \in \mathcal{N}$. The positive homogeneity of I_0, assumed in (v), gives

$$\psi'(s) = \langle I_0'(sv), v\rangle - \langle I'(sv), v\rangle = ps^{p-1}I_0(v) - \langle I'(sv), v\rangle.$$

Hence, $\psi'(s) = 0$ if and only if

$$pI_0(v) = s^{1-p}\langle I'(sv), v\rangle, \tag{5.3}$$

since $I_0(v) > 0$ by (5.1). Now, by (*ii*), (*iii*), and the L'Hôpital rule, the function $s \mapsto s^{1-p}\langle I'(sv), v\rangle$ is strictly increasing in $\mathbb{R}^+$ and maps $\mathbb{R}^+$ onto $\mathbb{R}^+$. Hence, there exists a unique value $s_v > 0$ satisfying (5.3).

Moreover, for all $s > 0$

$$\psi(s) = s^p\Big(I_0(v) - \frac{I(sv)}{s^p}\Big), \tag{5.4}$$

and so $\psi(s) \sim s^p I_0(v)$ as $s \to 0^+$, since $I(sv)/\|sv\|^p \to 0$ by (*i*). Thus, $\psi(s) > 0$ for $s > 0$ close to zero, since $I_0(v) > 0$ by (5.1). On the other hand, $\psi(s) \to -\infty$ as $s \to \infty$ by (*iii*). Therefore, the unique critical point $s_v \in \mathbb{R}^+$ of ψ is a strict maximum point and satisfies (5.3) and so (*a*).

(*b*) Take now $(v_n)_n \subset E$, with $v_n \to v \in E \setminus \{0\}$ as $n \to \infty$. The corresponding sequence $(s_n)_n = (\eta(v_n))_n$ is bounded in $\mathbb{R}^+$. Otherwise, there would exist a subsequence, still denoted $(s_n)_n$, converging to ∞ as $n \to \infty$. Clearly, $(v_n)_n$ belongs to a weakly compact set $\mathcal{W} \subset E \setminus \{0\}$, since it is bounded and E is a reflexive Banach space. Hence, $\psi(s_n) \to -\infty$ as $n \to \infty$, by (5.4) and (*iii*). On the other hand, $\psi(s_n) \geq 0$ for all n and in turn $\liminf_{n\to\infty} \psi(s_n) \geq 0$, which is an obvious contradiction. Therefore, $(s_n)_n$ is bounded and so admits a subsequence $(s_{n_k})_k$ converging to some $s \in \mathbb{R}_0^+$ as $k \to \infty$. The regularity of ψ forces that $0 = \psi'(s_{n_k}) \to \psi'(s) = 0$. The first part of the lemma implies that $s = s_v$, and so $s_v > 0$. This also shows that the entire sequence $(s_n)_n$ converges to s_v as $n \to \infty$. Hence, $\eta(v_n) = s_n \to s_v = \eta(v)$ as $n \to \infty$; that is, η is continuous.

(*c*) By the previous argument the map $m : z \mapsto s_z z = \eta(z)z = u$ is continuous from S_E onto $\mathcal{N}$, with continuous inverse $m^{-1} : u \mapsto u/\|u\| = z$.

(*d*) We claim that there exists $\delta > 0$ such that $s_z \geq \delta$ for all $z \in S_E$. Indeed, by (5.1) it follows that $\inf_{z\in S_E} I_0(z) \geq c_0 > 0$ and so by (5.3), for all $z \in S_E$, we get

$$s_z^{1-p}\langle I'(s_z z), z\rangle = pI_0(z) \geq pc_0 > 0. \tag{5.5}$$

On the other hand, (*i*) implies that $s^{1-p}\langle I'(sz), z\rangle \to 0$ as $s \to 0^+$, and this fact, combined with (5.5), proves the claim. In particular, by virtue of the homeomorphism m between S_E and $\mathcal{N}$, for every $u \in \mathcal{N}$ we have that $u = \|u\|u/\|u\| = s_z z$, with $z \in S_E$ and s_z uniquely determined by (*a*). Therefore, $\inf_{u\in\mathcal{N}} \|u\| = \inf_{z\in S_E} \|s_z z\| \geq \delta > 0$; that is, (*d*) holds.

(e) Put $c_1 = \inf_{v \in E \setminus \{0\}} \max_{s>0} \Phi(sv)$ and $c_2 = \inf_{z \in S_E} \max_{s>0} \Phi(sz)$. Fix $v \in E \setminus \{0\}$. There exists a unique $s_v \in \mathbb{R}^+$ satisfying (a). Since $s_v v \in \mathcal{N}$, clearly $\Phi(s_v v) \geq c$, that is, $\max_{s>0} \Phi(sv) \geq c$. Hence, $c_1 \geq c$. Clearly, $c_1 \leq c_2$, because $S_E \subset E \setminus \{0\}$. While the homeomorphism m between S_E and $\mathcal{N}$ gives $c_2 = c$. In conclusion, $c \leq c_1 \leq c_2 = c$, as desired.

(f) This follows immediately from properties (a) and (e). □

The lemma below is a slight modification of Proposition 14 of [38]; cf. Remark 15 of [38].

Lemma 5.2. *The following properties hold.*

(a) *If $(u_n)_n \subset \mathcal{N}$ and $\sup_n \Phi(u_n) < \infty$, then there exist $u \neq 0$ and a number $s_u > 0$ such that, up to a subsequence,*

$$s_u u \in \mathcal{N}, \quad u_n \rightharpoonup u \textit{ in } E, \quad \textit{and} \quad \Phi(s_u u) \leq \liminf_{n \to \infty} \Phi(u_n);$$

(b) *$\Phi\big|_{\mathcal{N}}$ is coercive; that is, $\Phi(v_n) \to \infty$ as $n \to \infty$ along any sequence $(v_n)_n \subset \mathcal{N}$, with $\|v_n\| \to \infty$;*

(c) *Φ satisfies the Palais–Smale condition on $\mathcal{N}$.*

Proof. (a). Let $(u_n)_n \subset \mathcal{N}$ be such that $d = \sup_n \Phi(u_n) < \infty$. We first claim that $(u_n)_n$ is bounded in E. Assume for the sake of contradiction that, up to a subsequence, $0 < \|u_n\| \to \infty$ as $n \to \infty$ and $v_n \rightharpoonup v$ in E, $v_n = u_n/\|u_n\|$, the space E being reflexive. We claim that $v \neq 0$. Suppose for the sake of contradiction that $v = 0$. By (iv) and Corollary 41.9 of [40] the functional I is weakly continuous in E, the space E being reflexive. Hence, $I(sv_n) \to I(0) = 0$ as $n \to \infty$ by (v). Since $v_n \in S_E$ and $\|u_n\| v_n = u_n \in \mathcal{N}$, we get that $s_{v_n} = \|u_n\|$ by Lemma 5.1-(a), s_{v_n} being uniquely determined. Thus, for all $s > 0$ we have

$$d \geq \Phi(u_n) = \Phi(s_{v_n} v_n) \geq \Phi(sv_n) \geq c_0 s^p - I(sv_n) \to c_0 s^p$$

as $n \to \infty$ by (v). This yields a contradiction choosing $s > (d/c_0)^{1/p}$. Hence $v \neq 0$, as claimed.

It is easy to see from the proof of Lemma 5.1-(a) that $\Phi(s_v v) > 0$ for all $v \in E \setminus \{0\}$ and so by Lemma 5.1-(e)

$$c = \inf_{u \in \mathcal{N}} \Phi(u) = \inf_{v \in E \setminus \{0\}} \max_{s>0} \Phi(sv) \geq 0.$$

In particular, $\Phi(u_n) \geq c \geq 0$ for all n. Therefore, for n sufficiently large,

$$0 \leq \frac{\Phi(u_n)}{\|u_n\|^p} \leq C_0 - \frac{I(\|u_n\| v_n)}{\|u_n\|^p} \to -\infty$$

as $n \to \infty$ by (iii) and (v), with $\mathcal{W} = \{v_n : n \in \mathbb{N}\} \cup \{v\} \subset E \setminus \{0\}$. This is again a contradiction.

Therefore, $(u_n)_n$ is bounded and there exists $u \in E$ such that $u_n \rightharpoonup u$ in E, up to a subsequence, E being reflexive. Assume for the sake of contradiction that $u = 0$. As before, by Lemma 5.1-(a) and the fact that $u_n \in \mathcal{N}$, we get for all $s > 0$

$$d \geq \Phi(u_n) \geq \Phi(su_n) \geq d_0 s^p - I(su_n) \to d_0 s^p > 0,$$

where $d_0 = c_0 \inf_{u\in\mathcal{N}} \|u\|^p > 0$ by (v) and Lemma 5.1-(d). Choosing s large enough, we find again a contradiction. Hence, $u \neq 0$, and we denote by s_u the unique number determined by Lemma 5.1-(a). Now, $I(s_u u_n) \to I(s_u u)$ and $\mathcal{J}_0(s_u u_n) \to \mathcal{J}_0(s_u u)$, by (iv) and (vi). Thus,

$$\begin{aligned}\Phi(s_u u) &= \mathcal{I}_0(s_u u) + \mathcal{J}_0(s_u u) - I(s_u u)\\ &\leq \liminf_{n\to\infty} \mathcal{I}_0(s_u u_n) + \lim_{n\to\infty} \mathcal{J}_0(s_u u_n) - \lim_{n\to\infty} I(s_u u_n)\\ &= \liminf_{n\to\infty} \Phi(s_u u_n) \leq \liminf_{n\to\infty} \Phi(u_n).\end{aligned}$$

(b) Let $(v_n)_n \subset \mathcal{N}$ be such that $\|v_n\| \to \infty$ as $n \to \infty$. Suppose for the sake of contradiction that $(\Phi(v_n))_n$ is bounded, up to a subsequence, since $\Phi(v_n) \geq c$. Then, by (a) of this lemma, there exists $v \in E \setminus \{0\}$ such that, up to a further subsequence, $v_n \rightharpoonup v$ as $n \to \infty$. In particular, $(\|v_n\|)_n$ is bounded. This is impossible.

(c) Let $(u_n)_n \subset \mathcal{N}$ be such that $(\Phi(u_n))_n$ is bounded and $\Phi'(u_n) \to 0$ as $n \to \infty$. By (a) of this lemma, there exists $u \in E \setminus \{0\}$ such that $u_n \rightharpoonup u$ in E as $n \to \infty$. Consequently, $I'(u_n) \to I'(u)$ and $\mathcal{J}_0'(u_n) \to \mathcal{J}_0'(u)$ in $E^\star$ by (iv) and (vi), and in turn

$$\begin{aligned}\langle \mathcal{I}_0'(u_n) - \mathcal{I}_0'(u), u_n - u\rangle &= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle\\ &\quad - \langle \mathcal{J}_0'(u_n) - \mathcal{J}_0'(u), u_n - u\rangle + \langle I'(u_n) - I'(u), u_n - u\rangle = o(1)\end{aligned}$$

as $n \to \infty$. Hence, by the main inequality in (vi), it follows that $\|u_n\| \to \|u\|$ as $n \to \infty$, and so $u_n \to u$ in E, the Banach space E being uniformly convex. □

Since we are going to argue on the function $\Psi : S_E \to \mathbb{R}$, defined by $\Psi(z) = \Phi(u)$, where $u = s_z z$ is uniquely determined in Lemma 5.1-(a) and (c), let us recall the notion of *weak slope*, taken from Definition 2.1 of [12], together with some useful facts.

Let (M, d) be a metric space, $\Psi : M \to \mathbb{R}$ a continuous function, and $z \in M$. We denote by $|d\Psi|(z)$ the supremum of the real numbers σ in $\mathbb{R}_0^+$

for which there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(z) \times [0, \delta] \to M$ such that for every $v \in B_\delta(z)$ and for every $t \in [0, \delta]$ it results that

$$d(\mathcal{H}(v,t), v) \leq t, \qquad \Psi(\mathcal{H}(v,t)) \leq \Psi(v) - \sigma t. \tag{5.6}$$

The extended real number $|d\Psi|(z)$ is called the *weak slope* of Ψ at z.

Proposition 5.3. *Let $M = (M, d)$ be a complete metric space. If the function $\Psi : M \to \mathbb{R}$ is bounded below, then for all $\varepsilon > 0$ there exists $z_\varepsilon \in M$, with the property that*

$$\Psi(z_\varepsilon) \leq \inf_{z \in M} \Psi(z) + \varepsilon^2 \quad \textit{and} \quad |d\Psi|(z_\varepsilon) \leq \varepsilon.$$

Proof. By Ekeland's variational principle, for all $\varepsilon > 0$ there exists $z_\varepsilon \in M$ such that $\Psi(z_\varepsilon) \leq \inf_{z \in M} \Psi(z) + \varepsilon^2$ and

$$\Psi(z) > \Psi(z_\varepsilon) - \varepsilon d(z, z_\varepsilon) \text{ for all } z \in M, \text{ with } z \neq z_\varepsilon. \tag{5.7}$$

We claim that $|d\Psi|(z_\varepsilon) \leq \varepsilon$. Otherwise there would exist $\delta > 0$, $\sigma > \varepsilon$, and a continuous function $\mathcal{H} : B_\delta(z_\varepsilon) \times [0, \delta] \to M$ such that (5.6) holds for every $v \in B_\delta(z_\varepsilon)$ and $t \in [0, \delta]$. Putting $z = \mathcal{H}(z_\varepsilon, \delta)$ in the second inequality of (5.6), we get

$$\Psi(z) \leq \Psi(z_\varepsilon) - \sigma\delta \leq \Psi(z_\varepsilon) - \varepsilon d(z, z_\varepsilon),$$

which contradicts (5.7), since $z \neq z_\varepsilon$ by (5.6). □

Let M denote a C^1 Finsler manifold. For the definition and the main properties of M we refer to [29, Section 2 and the theorem on page 201] and [28, Corollary at page 120]. Let $\Psi \in C^1(M)$. Then $\Psi'(z)$ is an element of $T_z(M)^\star$. We say that $z \in M$ is a *critical point* for Ψ if $\Psi'(z) = 0$ and, in this case, $c = \Psi(z)$ is called a *critical value* for Ψ. Moreover,

$$\|\Psi'(z)\|_z^\star = \sup_{\substack{v \in T_z(M) \\ \|v\|_z = 1}} \langle \Psi'(z), v \rangle.$$

Proposition 5.4 (Corollary 2.12 of [12])**.** *Let M be a C^1 Finsler manifold and $\Psi : M \to \mathbb{R}$ be a C^1 function. Then for every $z \in M$, we have*

$$|d\Psi|(z) = \|\Psi'(z)\|_z^\star.$$

The functional Ψ is said to *satisfy the $(PS)_c$ condition, $(PS)_c$ for short, if for every sequence $(z_n)_n \subset M$ such that $\Psi(z_n) \to c$ and $\|\Psi'(z_n)\|_{z_n}^\star \to 0$ as $n \to \infty$, there exists a subsequence of $(z_n)_n$ converging in M.*

Using Propositions 5.3 and 5.4, we now extend to complete Finsler manifolds Proposition 5.1 of [17], due to *Ekeland* for complete Riemannian manifolds.

Proposition 5.5. *Let M be a complete C^1 Finsler manifold. If $\Psi \in C^1(M)$ is bounded below, then for all $\varepsilon > 0$ there exists $z_\varepsilon \in M$ such that*

$$\Psi(z_\varepsilon) \leq \inf_{z\in M} \Psi(z) + \varepsilon^2 \quad \textit{and} \quad \|\Psi'(z_\varepsilon)\|^\star_{z_\varepsilon} \leq \varepsilon.$$

In particular, if Ψ satisfies the $(PS)_c$ condition at $c = \inf_{z\in M} \Psi(z)$, then c is a critical value.

Now we are in a position to present a slight modification of Theorem 13 of [38].

Theorem 5.6. *Assume that E is a real infinite-dimensional uniformly convex Banach space and that S_E is a complete C^1 Finsler manifold. Under the main assumptions (i)–(vi) of the section, equation $\Phi'(u) = 0$ admits a ground-state solution u; that is, $\Phi(u) = c$. Moreover, if Φ is even, then the equation $\Phi'(u) = 0$ has infinitely many pairs of solutions.*

Proof. Define the function $\Psi : S_E \to \mathbb{R}$ by $\Psi(z) = \Phi(u)$, where $u = s_z z \in \mathcal{N}$ is given by Lemma 5.1-(a) and (c). Clearly, $\Psi'(z) = s_z\Phi'(u)$ in $T_z(S_E)$ for each $z \in S_E$ by Lemma 5.1-(c). Hence, $\|\Psi'(z)\|^\star_z \leq s_z\|\Phi'(u)\|_{E^\star}$, where $s_z = \|u\| \geq \delta > 0$ by Lemma 5.2-(e). Furthermore, as shown in the proof of Corollary 10-(b) of [38], there exists a constant $C \geq 1$, independent of $z \in S_E$, such that

$$\|\Psi'(z)\|^\star_z \leq \|u\| \cdot \|\Phi'(u)\|_{E^\star} \leq C\|\Psi'(z)\|^\star_z$$

for all $z = m^{-1}(u) \in S_E$. Now, if $(z_n)_n \subset S_E$ is a (PS) sequence for Ψ, then $n \mapsto u_n = m(z_n) = s_{z_n} z_n \in \mathcal{N}$ is a (PS) sequence for Φ by Lemma 5.1-(d). Therefore, by Lemma 5.2-(c) there exists a subsequence, say $(u_n)_n$, such that $u_n \to u \in \mathcal{N}$ in E, Φ being of class $C^1(E)$ and $\mathcal{N}$ closed. Hence, the corresponding subsequence $z_n \to m^{-1}(u)$, as claimed.

The value $c = \inf_{z\in S_E} \Psi(z)$ is critical by Ekeland's variational principle given in Proposition 5.5; that is, there exists a minimizer $z_c \in S_E$, with $\Psi(z_c) = c$ and $\Psi'(z_c) = 0$. Therefore, $u_c = m(z_c) \in \mathcal{N}$ is a ground-state solution for the equation $\Phi'(u) = 0$.

Assume finally that Φ is even, so that also Ψ is even. Moreover, Ψ is bounded from below, since $\inf_{z\in S_E} \Psi(z) = c > 0$ by Lemma 5.1-(e) and (f). We claim that Ψ has infinitely many pairs of critical points. Indeed, put for each $k = 1, 2, \ldots$

$$\mathcal{E}_k = \{C \subset S_E : C \text{ is compact and symmetric, } i(C) \geq k\},$$

where i is the cohomological index for $\mathbb{Z}_2$-action introduced by *Fadell* and *Rabinowitz;* see [18], [30], and also Section 2. By Proposition 2.5-(ii) and (iii), the values

$$c_k = \inf_{C\in\mathcal{E}_k} \sup_{z\in C} \Psi(z) \to \infty \quad \text{as } k\to\infty,$$

and c_k are also critical, since (PS) holds, $i(S_E) = \dim E = \infty$, and $\mathcal{E}_k \neq \emptyset$ for all k. Hence, $\Psi'(z) = 0$ has infinitely many pairs of solutions, as claimed. Thus, $\Phi'(u) = 0$ has infinitely many pairs of solutions, and the proof is completed. □

6. Preliminary results for the Szulkin–Weth case

We now turn back to the main problem (1.1) and assume throughout the section that *the Carathéodory functions* $g : \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ *and* $h : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ *satisfy conditions* (g_1) *and* (h_1) *of Section* 4, *while conditions* (g_2)–(g_4) *and* (h_2)–(h_4) *are replaced by*

- $(g_2)'$ $\displaystyle\lim_{s\to 0}\frac{g(x,s)}{g_0(x)|s|^{p-1}} = 0$ *and* $\displaystyle\lim_{|s|\to\infty}\frac{G(x,s)}{g_0(x)|s|^{p}} = \infty$, *uniformly in* $\Omega\setminus\mathcal{N}$ *for some* $\mathcal{N}$, *with* $\operatorname{meas}(\mathcal{N}) = 0$;
- $(g_3)'$ *the function* $s\mapsto \dfrac{g(x,s)}{g_0(x)|s|^{p-1}}$ *is increasing in* $\mathbb{R}^-$ *and in* $\mathbb{R}^+$ *for a.a.* $x\in\Omega$;
- $(h_2)'$ $\displaystyle\lim_{s\to 0}\frac{h(x,s)}{h_0(x)|s|^{p-1}} = 0$ *and* $\displaystyle\lim_{|s|\to\infty}\frac{H(x,s)}{h_0(x)|s|^{p}} = \infty$, *uniformly in* $\partial\Omega\setminus\mathfrak{N}$ *for some* $\mathfrak{N}$, *with* $\operatorname{meas}_{N-1}(\mathfrak{N}) = 0$;
- $(h_3)'$ *the function* $s\mapsto \dfrac{h(x,s)}{h_0(x)|s|^{p-1}}$ *is increasing in* $\mathbb{R}^-$ *and in* $\mathbb{R}^+$ *for a.a.* $x\in\partial\Omega$.

Problem (1.1) always admits the trivial solution $u\equiv 0$. Furthermore, $(g_3)'$ and $(h_3)'$ yield that $g(x,s)s\geq 0$ for a.a. $x\in\Omega$ and $h(x,s)s\geq 0$ for a.a. $x\in\partial\Omega$ and for all $s\in\mathbb{R}$. Hence (g_4) and (h_4) of Section 4 are automatic.

By the de l'Hôpital rule, $(g_2)'$ implies (g_2), and so by (g_1)

$$\lim_{s\to 0}\frac{G(x,s)}{w_2(x)|s|^p} = 0, \text{ uniformly in } \Omega\setminus\mathcal{O},$$

with $\operatorname{meas}(\mathcal{O}) = 0$. Similarly, $(h_2)'$ gives (h_2), and in turn by (h_1)

$$\lim_{s\to 0}\frac{H(x,s)}{w_3(x)|s|^p} = 0,$$

uniformly in $\partial\Omega\setminus\mathcal{O}$ for some $\mathcal{O}$, with $\operatorname{meas}_{N-1}(\mathcal{O}) = 0$.

In order to prove the main results in the framework of this section, we put

$$\Phi(u) = I_0(u) - I(u),\ I_0(u) = \mathcal{I}_0(u) + \mathcal{J}_0(u) = \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|_{p,f}^p,$$
$$I(u) = I_1(u) + I_2(u), \tag{6.1}$$
$$I_1(u) = \int_\Omega G(x,u(x))\,dx,\ I_2(u) = \int_{\partial\Omega} H(x,u(x))\,dS.$$

Obviously, I_0, $\mathcal{I}_0$, $\mathcal{J}_0$, I_1, I_2, *and* I *are of class* $C^1(E)$ and

$$\langle I_0'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\,dS$$
$$- \lambda\int_\Omega f(x)|u|^{p-2}uv\,dx, \tag{6.2}$$
$$\langle I'(u), v\rangle = \langle I_1'(u), v\rangle + \langle I_2'(u), v\rangle,$$
$$\langle I_1'(u), v\rangle = \int_\Omega g(x,u(x))v(x)\,dx, \qquad \langle I_2'(u), v\rangle = \int_{\partial\Omega} h(x,u(x))v(x)\,dS$$

for every $u, v \in E$. Thus, the solutions of (1.1) coincide with the critical points of Φ.

Moreover, I_0 *is positively homogeneous of degree* p*, and for all* $\lambda \in \mathbb{R}$

$$I_0(u) \le \frac{1}{p}\Big(1 - \frac{\lambda^-}{\lambda_1}\Big)\|u\|^p, \tag{6.3}$$

while if $\lambda < \lambda_1$ *we also have*

$$I_0(u) \ge \frac{1}{p}\Big(1 - \frac{\lambda^+}{\lambda_1}\Big)\|u\|^p, \tag{6.4}$$

where $\lambda^+ = \max\{\lambda, 0\}$ *and* $\lambda^- = \min\{\lambda, 0\}$. Furthermore, $\mathcal{I}_0$ is *weakly lower semicontinuous in* E, being convex, and $\mathcal{I}_0'$ *satisfies*

$$\langle \mathcal{I}_0'(u) - \mathcal{I}_0'(v), u - v\rangle \ge (\|u\|^{p-1} - \|v\|^{p-1})\,(\|u\| - \|v\|) \tag{6.5}$$

for all $u, v \in E$ by Lemma 3.2.

Proposition 6.1. $\mathcal{J}_0'$ *is completely continuous from* E *into* $E^\star$ *and* $\mathcal{J}_0$ *is weakly continuous in* E.

Proof. It is enough to show that $\mathcal{J}_0'$ maps weakly convergent sequences of E into strongly convergent sequences of $E^\star$. Let $(u_n)_n \subset E$ be such that $u_n \rightharpoonup u$ in E. Hence $u_n \to u$ in $L^p(\Omega, f)$, since the embedding $E \hookrightarrow L^p(\Omega, f)$ is compact by (2.1). Denote by $C > 0$ a constant such that $\|\varphi\|_{p,f} \le C\|\varphi\|$ for all $\varphi \in E$.

In particular, $\|u_n\|_{p,f} \to \|u\|_{p,f}$, or equivalently, $\|v_n\|_{p',f} \to \|v\|_{p',f}$, where $v_n = |u_n|^{p-2}u_n$ and similarly $v = |u|^{p-2}u$. We claim that $v_n \to v$ in $L^{p'}(\Omega, f)$. Indeed, fix any subsequence $(v_{n_k})_k$ of $(v_n)_n$. The related subsequence $(u_{n_k})_k$ of $(u_n)_n$ converges in $L^p(\Omega, f)$ and admits a subsequence, say $(u_{n_{k_j}})_j$, converging to u a.e. in Ω. Hence, the corresponding subsequence $(v_{n_{k_j}})_j$ of $(v_{n_k})_k$ converges to v a.e. in Ω. Therefore, since $1 < p' < \infty$, by the Clarkson and Mil'man theorems it follows that $v_{n_{k_j}} \rightharpoonup v$ in $L^{p'}(\Omega, f)$, since the sequence $(\|v_n\|_{p',f})_n$ is bounded, and so by Radon's theorem we get that $v_{n_{k_j}} \to v$ in $L^{p'}(\Omega, f)$, since $\|v_n\|_{p',f} \to \|v\|_{p',f}$. This shows the claim, since the subsequence $(v_{n_k})_k$ of $(v_n)_n$ is arbitrary.

For all $\phi \in E$, with $\|\phi\| = 1$, we have

$$\begin{aligned}|\langle \mathcal{J}_0'(u_n) - \mathcal{J}_0'(u), \phi\rangle| &\le \lambda \int_\Omega f(x)|v_n - v| \cdot |\phi|\, dx \le \lambda \|v_n - v\|_{p',f}\|\phi\|_{p,f} \\ &\le C\|v_n - v\|_{p',f}\end{aligned}$$

by Hölder's inequality. In other words, $\|\mathcal{J}_0'(u_n) - \mathcal{J}_0'(u)\|_{E^\star} \to 0$ as $n \to \infty$. Thus, $\mathcal{J}_0$ is weakly continuous in E by Corollary 41.9 of [40], the space E being reflexive. □

Lemma 6.2. *The functions*

$$\mathfrak{g}(x,s) = \begin{cases} \dfrac{g(x,s)}{w_2(x)|s|^{p-1}}, & x \in \Omega,\ s \ne 0, \\ 0, & x \in \Omega,\ s = 0, \end{cases}$$

$$G_0(x,s) = \begin{cases} \dfrac{G(x,s)}{w_2(x)|s|^{p}}, & x \in \Omega,\ s \ne 0, \\ 0, & x \in \Omega,\ s = 0, \end{cases}$$

$$\mathfrak{h}(x,s) = \begin{cases} \dfrac{h(x,s)}{w_3(x)|s|^{p-1}}, & x \in \partial\Omega,\ s \ne 0, \\ 0, & x \in \partial\Omega,\ s = 0, \end{cases}$$

$$H_0(x,s) = \begin{cases} \dfrac{H(x,s)}{w_3(x)|s|^{p}}, & x \in \partial\Omega,\ s \ne 0, \\ 0, & x \in \partial\Omega,\ s = 0, \end{cases}$$

are of Carathéodory type, and the Nemytskii operators

$$\mathcal{N}_{\mathfrak{g}}, \mathcal{N}_{G_0} : L^r(\Omega, w_2) \to L^{r/(r-p)}(\Omega, w_2),$$
$$\mathcal{N}_{\mathfrak{h}}, \mathcal{N}_{H_0} : L^q(\partial\Omega, w_3) \to L^{q/(q-p)}(\partial\Omega, w_3)$$

are continuous and bounded, with $\mathcal{N}_{\mathfrak{g}}(0)=\mathcal{N}_{G_0}(0)=\mathcal{N}_{\mathfrak{h}}(0)=\mathcal{N}_{H_0}(0)=0$.

Proof. By (g_1) and $(g_2)'$ along with (h_1) and $(h_2)'$ it is clear that $\mathfrak{g}$, G_0, $\mathfrak{h}$, and H_0 are well-defined and of Carathéodory type. Moreover, they also imply that

$$|\mathfrak{g}(x,s)| \le C_{\mathfrak{g}}(\tilde{g}_0(x)+|s|^{r-p}) \text{ and } 0 \le G_0(x,s) \le C_{G_0}(\tilde{g}_0(x)+|s|^{r-p})$$

for a.a. $x \in \Omega$, and

$$|\mathfrak{h}(x,s)| \le C_{\mathfrak{h}}(\tilde{h}_0(x)+|s|^{q-p}) \text{ and } 0 \le H_0(x,s) \le C_{H_0}(\tilde{h}_0(x)+|s|^{q-p})$$

for a.a. $x \in \partial\Omega$, where $C_{\mathfrak{g}}$, C_{G_0}, $C_{\mathfrak{h}}$, and C_{H_0} are suitable positive constants and $\tilde{g}_0 = g_0/w_2$, while $\tilde{h}_0 = h_0/w_3$. By (g_1) and (h_1) we have $\tilde{g}_0 \in L^{r/(r-p)}(\Omega, w_2)$ and $\tilde{h}_0 \in L^{q/(q-p)}(\partial\Omega, w_3)$, so that the result follows at once by applying Lemma 2.2-(i), with $A=\Omega$, $d=N$, $w=w_2$, $\alpha=r$, and $\beta=r/(r-p)$, and Lemma 2.2-(i), with $A=\partial\Omega$, $d=N-1$, $w=w_3$, $\alpha=q$, and $\beta=q/(q-p)$. □

Lemma 6.3. *We have* $\lim\limits_{\|u\|\to 0}\dfrac{\|I'(u)\|_{E^\star}}{\|u\|^{p-1}}=0$ *and* $\lim\limits_{\|u\|\to 0}\dfrac{I(u)}{\|u\|^p}=0$.

Proof. We first show that $\|I_1'(u)\|_{E^\star}=o\left(\|u\|^{p-1}\right)$ as $\|u\|\to 0$. Fix $u \in E$, with $\|u\|>0$. Using the notation of Lemma 6.2, we have

$$\frac{\|I_1'(u)\|_{E^\star}}{\|u\|^{p-1}} \le \sup_{\|\varphi\|=1}\|u\|^{1-p}\int_\Omega |g(x,u(x))|\cdot|\varphi(x)|\,dx = \sup_{\|\varphi\|=1}\|u\|^{1-p}J(u,\varphi),$$

where

$$J(u,\varphi)=\int_\Omega |\mathfrak{g}(x,u(x))|w_2(x)|u(x)|^{p-1}|\varphi(x)|\,dx.$$

By Hölder's inequality, we obtain for all $\varphi \in E$, with $\|\varphi\|=1$,

$$\begin{aligned} J(u,\varphi) &\le \Big(\int_\Omega w_2(x)|\mathfrak{g}(x,u(x))|^{r/(r-p)}\,dx\Big)^{(r-p)/r}\|u\|_{r,w_2}^{p-1}\|\varphi\|_{r,w_2} \\ &\le C\Big(\int_\Omega w_2(x)|\mathfrak{g}(x,u(x))|^{r/(r-p)}\,dx\Big)^{(r-p)/r}\|u\|^{p-1}, \end{aligned}$$

where C is a suitable constant related to the Sobolev constant of the continuous embedding $E \hookrightarrow L^r(\Omega, w_2)$. In conclusion, we have shown that

$$\frac{\|I_1'(u)\|_{E^\star}}{\|u\|^{p-1}} \le C\Big(\int_\Omega w_2(x)|\mathfrak{g}(x,u(x))|^{r/(r-p)}\,dx\Big)^{(r-p)/r} \to 0$$

as $\|u\| \to 0$ by Lemma 6.2. Furthermore,

$$\frac{|I_1(u)|}{\|u\|^p} \le C\Big(\int_\Omega w_2(x)|G_0(x,u(x))|^{r/(r-p)}\,dx\Big)^{(r-p)/r} \to 0$$

as $\|u\| \to 0$, again by Lemma 6.2. Similarly, $\|I_2'(u)\|_{E^\star} = o\left(\|u\|^{p-1}\right)$ and $I_2(u) = o\left(\|u\|^p\right)$ as $\|u\| \to 0$. This completes the proof. □

Lemma 6.4. *The following properties hold.*

(α) $s \mapsto s^{1-p}\langle I'(su), u\rangle$ *is strictly increasing for* $u \neq 0$ *and* $s > 0$;
(β) I' *is completely continuous in* E.

Proof. (α) follows directly from $(g_3)'$ and $(h_3)'$.
(β) Condition (g_1) implies that $|g(x,s)| \le g_0(x) + K_g|s|^{r-1}$ for a.a. $x \in \Omega$ and $s \in \mathbb{R}$, where $K_g = (1+C_g)C_g$. Hence, Lemma 2.2-(i), with $A = \Omega$, $d = N$, $w = w_2$, $\alpha = r$, and $\beta = r'$, imply that the Nemytskii operator $\mathcal{N}_g$ associated to g is bounded and continuous from $L^r(\Omega, w_2)$ to $L^{r'}(\Omega, w_2)$, since $g_0 \in L^{r'}(\Omega, w_2)$. Moreover, I_1' can be viewed as the composition

$$I_1' : E \hookrightarrow L^r(\Omega, w_2) \stackrel{\mathcal{N}_g}{\to} L^{r'}(\Omega, w_2) \hookrightarrow E^\star,$$

where $E \hookrightarrow L^r(\Omega, w_2)$ is completely continuous by Theorem 1 of [31] and (g_1). Hence, I_1' is completely continuous in E, being a composition of continuous operators, with the first completely continuous.

In the same way, we prove that I_2' is completely continuous by virtue of (h_1). Indeed, condition (h_1) forces that $|h(x,s)| \le h_0(x) + K_h|s|^{q-1}$ for a.a. $x \in \partial\Omega$ and all $u \in \mathbb{R}$, with $K_h = (1+C_h)C_h$, so that Lemma 2.2-(i) can be applied with $A = \partial\Omega$, $d = N-1$, $w = w_3$, $\alpha = q$, and $\beta = q'$, since $h_0 \in L^{q'}(\partial\Omega, w_3)$ by (h_1). In conclusion, the operator I' is completely continuous in E. □

Lemma 6.5. $\lim_{s\to\infty} s^{-p} I(su) = \infty$ *uniformly for* u *on weakly compact subsets of* $E \setminus \{0\}$.

Proof. Let $\mathcal{W} \subset E \setminus \{0\}$ be a weakly compact subset of $E \setminus \{0\}$, and fix $(u_n)_n \subset \mathcal{W}$. It is enough to show that if $s_n \to \infty$ as $n \to \infty$, then there exists a subsequence of $(s_n^{-p} I(s_n u_n))_n$ which diverges to ∞. Now, there exists a subsequence of $(u_n)_n$, still labeled $(u_n)_n$, and $u \in E \setminus \{0\}$, such that

$$\begin{gathered} u_n \rightharpoonup u \text{ in } E, \quad u_n \to u \text{ in } L^p(\Omega, w_1), \quad u_n \to u \text{ a.e. in } \Omega, \\ u_n \to u \text{ in } L^p(\partial\Omega, w_4), \quad u_n \to u \text{ a.e. on } \partial\Omega, \\ w_4(x) = (1+|x|)^{-\alpha_4},\ x \in \partial\Omega, \quad p-1 < \alpha_4 < N, \end{gathered} \tag{6.6}$$

by Theorem 1 of [31], where w_1 is defined in (2.1). Hence, either $|u| > 0$ in a subset A of Ω, with $\text{meas}(A) > 0$, or $|u| > 0$ in a subset Γ of $\partial\Omega$, with $\text{meas}_{N-1}(\Gamma) > 0$, since $\|u\| > 0$ and $w_1 > 0$ in Ω, and $w_4 > 0$ in $\partial\Omega$

Case $|u| > 0$ *in* A, $\text{meas}(A) > 0$. For n sufficiently large

$$\frac{G(x, s_n u_n(x))}{s_n^p} = \frac{G(x, s_n u_n(x))}{g_0(x)|s_n u_n(x)|^p}\, g_0(x)|u_n(x)|^p \quad \text{a.e. in } A.$$

Hence, $s_n^{-p}G(x, s_n u_n(x)) \to \infty$ a.e. in A by (g_2), since $|s_n u_n(x)| \to \infty$ as $n \to \infty$. Thus, by (g_4), which continues to hold, and the Fatou lemma,

$$\frac{I_1(s_n u_n)}{s_n^p} = \int_\Omega \frac{G(x, s_n u_n(x))}{g_0(x)|s_n u_n(x)|^p}\, g_0(x)|u_n(x)|^p\, dx \to \infty \quad \text{as } n \to \infty.$$

Case $|u| > 0$ *in* Γ, $\text{meas}_{N-1}(\Gamma) > 0$. Similarly, using $(h_2)'$ in place of $(g_2)'$ and arguing as above, by (h_4) we obtain

$$\frac{I_2(s_n u_n)}{s_n^p} = \int_{\partial\Omega} \frac{H(x, s_n u_n)}{h_0(x)|s_n u_n(x)|^p}\, h_0(x)|u_n(x)|^p\, dx \to \infty \quad \text{as } n \to \infty.$$

This shows that $s_n^{-p}I(s_n u_n) \to \infty$ as $n \to \infty$ and completes the proof, $(s_n)_n$ being an arbitrary sequence diverging to ∞. □

7. Existence in the Szulkin–Weth case

We are going to apply the main results of Section 5 to problem (1.1) under the assumptions of Section 6, with E now the real infinite-dimensional uniformly convex Banach space defined in Section 2, and the Euler–Lagrange functional Φ associated to (1.1), given in (4.1), being thought of as in (6.1). Hence $\Psi(z) = \Phi(u)$, $u = m(z)$, for all $z \in S_E$, that is, when $M = S_E$. Of course S_E is a complete metric space.

Consider $\mathfrak{g}(t) = t^{p-1}$, which is continuous and strictly increasing in $\mathbb{R}_0^+$, and the corresponding duality map $J_{\mathfrak{g}}$, defined by

$$J_{\mathfrak{g}}(u) = \{u^\star \in E^\star : \langle u^\star, u\rangle = \mathfrak{g}(\|u\|)\|u\| = \|u\|^p,\ \|u^\star\|_{E^\star} = \mathfrak{g}(\|u\|) = \|u\|^{p-1}\}$$

for all $u \in E$. The functional $\mathcal{G} : E \to \mathbb{R}$, given by

$$\mathcal{G}(u) = \int_0^{\|u\|} \mathfrak{g}(t)\, dt = \frac{1}{p}\|u\|^p,$$

is of class $C^1(E)$, and so $\mathcal{G}'(u) = J_{\mathfrak{g}}(u)$ for all $u \in E$ by Theorem 1-(iii) of [15]; that is, for all $u, v \in E$

$$\langle \mathcal{G}'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\, dS = \langle J_{\mathfrak{g}}(u), v\rangle.$$

In particular, $J_{\mathfrak{g}} : E \to E^\star$ is a single-valued continuous map. Furthermore, $J_{\mathfrak{g}}$ maps bounded sets of E into bounded sets of $E^\star$ and

$$S_E = \{z \in E : \langle J_{\mathfrak{g}}(z), z\rangle = 1\}.$$

Hence S_E is a complete C^1 manifold of E. In particular, the tangent space $T_z(S_E)$ of S_E at $z \in S_E$ can be written in the form

$$T_z(S_E) = \{v \in E : \langle J_{\mathfrak{g}}(z), v\rangle = 0\} = \mathrm{Ker} J_{\mathfrak{g}}(z).$$

Thus, S_E has a *Finsler structure* on the bundle $T(S_E)$; see [29, Section 2] and [28, Corollary at page 120]. In other words, S_E is a *complete* C^1 *Finsler manifold.*

As an application of Theorem 5.6 we are now able to prove the main result of the section.

Theorem 7.1. *Suppose that* (g_1), $(g_2)'$–$(g_3)'$, (h_1), *and* $(h_2)'$–$(h_3)'$ *hold and that* $\lambda < \lambda_1$. *Then we have the following:*

(a) *Problem* (1.1) *has a ground-state solution, that is, a nontrivial solution of minimal energy. Moreover, if* g *and* h *are odd in* u, *then* (1.1) *has infinitely many pairs of solutions.*

(b) *Problem* (1.1) *admits a least-energy sign-changing solution.*

Proof of Theorem 7.1-(a)**.** Whenever $\lambda < \lambda_1$, the functional $\Phi = I_0 - I$, defined in (6.1), satisfies all the assumptions (i)–(vi) of Theorem 5.6. Indeed, Lemmas 6.3–6.5 assure the validity of (i)–(iv). Moreover, I_0 is trivially positively homogeneous of degree p and the main inequality in (v) is satisfied with $c_0 = (1 - \lambda^+/\lambda_1)/p > 0$ and $C_0 = (1 - \lambda^-/\lambda_1)/p \geq c_0$; cf. (6.3)–(6.4). Hence, also (v) holds. Finally, as noted in Section 6, also (vi) is satisfied by virtue of Lemma 3.2 and Proposition 6.1.

Therefore, by Theorem 5.6 problem (1.1) admits a ground-state solution in E. Moreover, if g and h are odd in u, then Φ is even, and so by Theorem 5.6 problem (1.1) has infinitely many pairs of solutions in E. This completes the proof of Theorem 7.1-(a). □

In order to prove the existence of least-energy sign-changing solutions of problem (1.1), we put

$$\mathcal{N}_{sc} := \{u \in E : u^+, u^- \in \mathcal{N}\} \quad \text{and} \quad c_{sc} := \inf_{u \in \mathcal{N}_{sc}} \Phi(u).$$

First observe that if $u \in E$, then $u^+, u^- \in E$ and $\Phi(u) = \Phi(u^+) + \Phi(u^-)$. Moreover, if u is a solution of (1.1), taking $v = u^+$ in (6.2), we obtain that $0 = \langle \Phi'(u), u^+ \rangle = \langle \Phi'(u^+), u^+ \rangle$ and, similarly, taking $v = u^-$ in (6.2), we get $0 = \langle \Phi'(u), u^- \rangle = \langle \Phi'(u^-), u^- \rangle$. Hence, $u^+, u^- \in \mathcal{N}$; that is, every sign-changing solution $u \in E$ of $\Phi'(u) = 0$ lies in $\mathcal{N}_{sc}$.

Proof of Theorem 7.1-(b). We claim that (1.1) admits a least-energy sign-changing solution $u \in \mathcal{N}_{sc}$, with $\Phi(u) = c_{sc} \geq 2c$, where c is given in Lemma 5.1-(e).

Let $(u_n)_n \subset \mathcal{N}_{sc}$ be a minimizing sequence; that is, $\Phi(u_n) \to c_{sc}$ as $n \to \infty$. Now, $\Phi(u_n) = \Phi(u_n^+) + \Phi(u_n^-)$, with $\Phi(u_n^+) \geq c$ and $\Phi(u_n^-) \geq c$ for all n. In particular, $(\Phi(u_n^+))_n$ and $(\Phi(u_n^-))_n$ are bounded. Hence, by Lemma 5.2-(a) there exists a subsequence, still labeled $(u_n)_n$, such that $u_n^+ \rightharpoonup v \neq 0$, $u_n^- \rightharpoonup z \neq 0$, $u_n^+ \to v$, and $u_n^- \to z$ a.e. in Ω. Without loss of generality we may also assume $(\Phi(u_n^+))_n$ and $(\Phi(u_n^-))_n$ are converging. Therefore, $0 = u_n^+ u_n^- \to vz = 0$ a.e. in Ω. Moreover, again by Lemma 5.2-(a), there exist $s_v > 0$ and $s_z > 0$ such that $u = s_v v + s_z z = u^+ + u^- \in \mathcal{N}_{sc}$ and

$$\begin{aligned} 2c \leq \Phi(u) = \Phi(s_v v) + \Phi(s_z z) &\leq \liminf_{n\to\infty} \Phi(u_n^+) + \liminf_{n\to\infty} \Phi(u_n^-) \\ &= \lim_{n\to\infty} \Phi(u_n) = c_{sc}. \end{aligned}$$

In particular, $\Phi(u) = c_{sc} \geq 2c$, as claimed.

The final delicate step consists in proving that $\Phi'(u) = 0$, using the argument of the proof of Theorem 18 of [38], which we repeat here for the sake of clarity. For all $s, t > 0$, with at least one of them different from 1, by Lemma 5.1-(a) we have

$$\Phi(su^+ + tu^-) = \Phi(su^+) + \Phi(tu^-) < \Phi(u^+) + \Phi(u^-) = \Phi(u) = c_{sc}. \quad (7.1)$$

Assume for the sake of contradiction that $\Phi'(u) \neq 0$. Then, there exist $\delta > 0$ and $\mu > 0$ such that $\|\Phi'(v)\|_{E^\star} \geq \mu$ for all $v \in E$, with $\|v - u\| \leq 3\delta$. Put $\phi(s,t) = su^+ + tu^-$ for all $(s,t) \in D = I \times I$, $I = [1/2, 3/2]$. Clearly, the definition of ϕ and (7.1) imply that $\Phi(\phi(s,t)) = c_{sc}$ if and only if $s = t = 1$ and $\Phi(\phi(s,t)) < c_{sc}$ otherwise. In particular,

$$c_m = \max_{(s,t)\in\partial D} \Phi(\phi(s,t)) < c_{sc}. \quad (7.2)$$

The deformation Lemma 2.3 of [39], with $\varepsilon = \min\{(c_{sc} - c_m)/4,\ \delta\mu/8\}$ and S the closed ball of E with center u and radius δ, yields the existence of a continuous deformation $\wp : [0,1] \times E \to E$ such that

(α) $\wp(1, v) = v$ for all $v \notin \Phi^{-1}([c_{sc} - 2\varepsilon, c_{sc} + 2\varepsilon])$;

(β) $\Phi(\wp(1,v)) \le c_{sc} - \varepsilon$ for every $v \in E$, with $\|v-u\| \le \delta$ and $\Phi(v) \le c_{sc} + \varepsilon$;

(γ) $\Phi(\wp(1,v)) \le \Phi(v)$ for all $v \in E$.

Property (γ) gives at once that $\max_{(s,t)\in D} \Phi(\wp(1,\phi(s,t))) < c_{sc}$.

The next step is to prove that $\wp(1,\phi(D)) \cap \mathcal{N}_{sc} \ne \emptyset$, which contradicts the definition of c_{sc} and completes the proof. Define for all $(s,t) \in D$ the continuous maps

$$\Psi_0(s,t) = (\langle \Phi'(su^+), u^+\rangle, \langle \Phi'(tu^-), u^-\rangle),$$
$$\Psi_1(s,t) = \Big(\frac{1}{s}\langle \Phi'(\wp(1,\phi(s,t))^+), \wp(1,\phi(s,t))^+\rangle, \frac{1}{t}\langle \Phi'(\wp(1,\phi(s,t))^-), \wp(1,\phi(s,t))^-\rangle\Big).$$

The $C^1(\mathbb{R}^+)$ function $\psi(s) = \Phi(su^+)$ admits a unique strict maximum point at $s_{u^+} = 1$, since $\psi'(1) = \langle \Phi'(u^+), u^+\rangle = 0$ and $u^+ \in \mathcal{N}$; see the proof of Lemma 5.1-(a). Moreover, $\psi'(s) = \langle \Phi'(su^+), u^+\rangle > 0$ for $s \in (0,1)$, while $\psi'(s) = \langle \Phi'(su^+), u^+\rangle < 0$ for $s > 1$. In particular, the continuous function $I \ni s \mapsto \langle \Phi'(su^+), u^+\rangle$ is positive if $s = 1/2$ and negative if $s = 3/2$, so that the formula (3.15) of Proposition 3.4.1 in [16] implies that $\deg(\langle \Phi'(su^+), u^+\rangle, I, 0) = -1$. Similarly, using the above argument, with u^- in place of u^+, we get that the continuous function $I \ni t \mapsto \langle \Phi'(tu^-), u^-\rangle$ is positive if $t = 1/2$ and negative if $t = 3/2$, so that again $\deg(\langle \Phi'(tu^-), u^-\rangle, I, 0) = -1$. The cartesian product property in Theorem 3.16-7 of [36] gives that $\deg(\Psi_0, D, (0,0)) = 1$. By (7.2) and ($\alpha$) we deduce that $\phi(s,t) = \wp(1,\phi(s,t))$ for all $(s,t) \in \partial D$. Thus, $\Psi_0(s,t) = \Psi_1(s,t)$ for all $(s,t) \in \partial D$, and the dependence only on the boundary-values property in Theorem 3.16-2 of [36] yields $\deg(\Psi_1, D, (0,0)) = \deg(\Psi_0, D, (0,0)) = 1$. Therefore, Corollary 3.4.1. of [16] assures the existence of some $(s,t) \in D$ such that $\Psi_1(s,t) = (0,0)$; that is, $\wp(1,\phi(s,t))^+, \wp(1,\phi(s,t))^- \in \mathcal{N}$; in other words, $\wp(1,\phi(s,t)) \in \mathcal{N}_{sc}$. Hence, $\wp(1,\phi(D)) \cap \mathcal{N}_{sc} \ne \emptyset$. This contradiction shows that $\Phi'(u) = 0$, as claimed. □

In the last part of the section we complete the investigation on the existence of solutions of (1.1), considering the case $\lambda \ge \lambda_1$, under conditions (g_1), $(g_2)'$–$(g_3)'$, (h_1), and $(h_2)'$–$(h_3)'$. Let us start with a preliminary monotonicity result.

Proposition 7.2. *The function $s \mapsto \mathcal{G}(x,s) = g(x,s)s - pG(x,s)$ is non-decreasing in $\mathbb{R}^+$ for a.a. $x \in \Omega$. Similarly, for a.a. $x \in \partial\Omega$ the function $s \mapsto \mathcal{H}(x,s) = h(x,s)s - pH(x,s)$ is non-decreasing in $\mathbb{R}^+$.*

Proof. Let $0 < s \le t$. By $(g_3)'$, for all $x \in \Omega \setminus \mathcal{N}$, $\text{meas}(\mathcal{N}) = 0$, we have

$$\begin{aligned}
\mathcal{G}(x,t) - \mathcal{G}(x,s) &= p\Big\{\frac{1}{p}[g(x,t)t - g(x,s)s] - [G(x,t) - G(x,s)]\Big\}\\
&= p\Big\{\int_0^t \frac{g(x,t)}{t^{p-1}}\tau^{p-1}\,d\tau - \int_0^s \frac{g(x,s)}{s^{p-1}}\tau^{p-1}\,d\tau - \int_s^t \frac{g(x,\tau)}{\tau^{p-1}}\tau^{p-1}\,d\tau\Big\}\\
&= p\Big\{\int_s^t \Big[\frac{g(x,t)}{t^{p-1}} - \frac{g(x,\tau)}{\tau^{p-1}}\Big]\tau^{p-1}\,d\tau + \int_0^s \Big[\frac{g(x,t)}{t^{p-1}} - \frac{g(x,s)}{s^{p-1}}\Big]\tau^{p-1}\,d\tau\Big\}\\
&\ge 0.
\end{aligned}$$

Similarly, using assumption $(h_3)'$ in place of $(g_3)'$, we obtain the analogous conclusion for the function $\mathcal{H}$, as stated. □

Lemma 7.3. *The functional Φ satisfies the $(PS)_\ell$ condition in E for all $\lambda \ge 0$ and $\ell \in \mathbb{R}$.*

Proof. Let $(u_n)_n \subset E$ be a $(PS)_\ell$ sequence; that is, $\Phi(u_n) \to \ell$ and $\Phi'(u_n) \to 0$ as $n \to \infty$. First we claim that $(u_n)_n$ is bounded. Proceed by contradiction and suppose that $0 < \|u_n\| \to \infty$ as $n \to \infty$, up to a subsequence. Hence, using the notation of Proposition 7.2, we find that

$$\int_\Omega \mathcal{G}(x,u_n)\,dx + \int_{\partial\Omega} \mathcal{H}(x,u_n)\,dS = p\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle \to p\,\ell \tag{7.3}$$

as $n \to \infty$. Define $v_n = u_n/\|u_n\|$ for all n. By Proposition A.2, (2.1), (g_1), and (h_1) there exists $v \in E$ such that, up to a subsequence,

$$\begin{gathered}
v_n \rightharpoonup v \text{ in } E, \quad v_n \to v \text{ in } L^p(\Omega, f),\\
v_n \to v \text{ in } L^r(\Omega, w_2), \quad v_n \to v \text{ in } L^q(\partial\Omega, w_3).
\end{gathered}$$

We first assume for the sake of contradiction that $v = 0$. Let s_n be the smallest value of $t \in [0,1]$ such that

$$\Phi(tu_n) = \max_{s\in[0,1]} \Phi(su_n). \tag{7.4}$$

Now, fix $k \in \mathbb{N}$ and define $z_n = \sqrt[p]{2pk}\,v_n$. Hence, $z_n \to 0$ in $L^r(\Omega, w_2)$ as $n \to \infty$. Similarly, also $z_n \to 0$ in $L^q(\partial\Omega, w_3)$ as $n \to \infty$. Consequently, by (g_4) and (h_4) (which continue to hold), the Hölder inequality, and Lemma 6.2

$$\begin{aligned}
0 &\le \int_\Omega G(x,z_n)\,dx \le \Big(\int_\Omega w_2(x)G_0(x,z_n)^{r/(r-p)}\,dx\Big)^{(r-p)/r}\|z_n\|^p_{r,w_2} = o(1),\\
0 &\le \int_{\partial\Omega} H(x,z_n)\,dS \le \Big(\int_{\partial\Omega} w_3(x)H_0(x,z_n)^{q/(q-p)}\,dS\Big)^{(q-p)/q}\|z_n\|^p_{q,w_3,\partial\Omega} = o(1)
\end{aligned}$$

as $n \to \infty$. Therefore, by (7.4) and the fact $\|z_n\|_{p,f} \to 0$, we get

$$\Phi(s_n u_n) \geq \Phi(z_n) = 2k - \lambda \|z_n\|_{p,f}^p - \int_\Omega G(x, z_n)\, dx - \int_{\partial\Omega} H(x, z_n)\, dS \geq k$$

for all n sufficiently large. That is, $\Phi(s_n u_n) \to \infty$ as $n \to \infty$. Moreover, $\Phi(0) = 0$ and $\Phi(u_n) \to \ell \in \mathbb{R}$, so that $s_n \in (0, 1)$ for all n sufficiently large, and in turn

$$\begin{aligned} s_n^p(\|u_n\|^p - \lambda\|u_n\|_{p,f}^p) - s_n \Big\{ \int_\Omega g(x, s_n u_n) u_n\, dx - \int_{\partial\Omega} h(x, s_n u_n) u_n\, dS \Big\} \\ = \langle \Phi'(s_n u_n), s_n u_n \rangle = s_n \frac{d}{ds} \Phi(s u_n)\Big|_{s=s_n} = 0. \end{aligned}$$

Thus, by Proposition 7.2, it follows that

$$\begin{aligned} \int_\Omega \mathcal{G}(x, u_n)\, dx + \int_{\partial\Omega} \mathcal{H}(x, u_n)\, dS &\geq \int_\Omega \mathcal{G}(x, s_n u_n)\, dx + \int_{\partial\Omega} \mathcal{H}(x, s_n u_n)\, dS \\ &= s_n^p(\|u_n\|^p - \lambda\|u_n\|_{p,f}^p) - p \left\{ \int_\Omega G(x, s_n u_n)\, dx + \int_{\partial\Omega} H(x, s_n u_n)\, dS \right\} \\ &= p\Phi(s_n u_n) \to \infty \end{aligned}$$

as $n \to \infty$, which contradicts (7.3). This means that the case $v = 0$ cannot occur.

Let us then suppose $v \neq 0$. Since $\lambda \geq 0$, condition (6.3) implies that $I_0(u) \leq \|u\|^p/p$ for all $u \in E$. Therefore,

$$\frac{\Phi(u_n)}{\|u_n\|^p} \leq \frac{1}{p} - \frac{I(\|u_n\| v_n)}{\|u_n\|^p} \to -\infty$$

by Lemma 6.5, with $\mathcal{W} = \{v_n : n \in \mathbb{N}\} \cup \{v\} \subset E \setminus \{0\}$. Hence also the case $v \neq 0$ is impossible.

In conclusion, the sequence $(u_n)_n$ must be bounded in E. Thus, by Proposition A.2, we have that $u_n \rightharpoonup u$ in E, up to a subsequence. By (6.2)

$$\langle \Phi'(u_n), u_n - u \rangle = \langle \mathcal{I}_0'(u_n), u_n - u \rangle - \langle \mathcal{J}_0'(u_n), u_n - u \rangle - \langle I'(u_n), u_n - u \rangle \to 0,$$

since $(u_n)_n$ is a $(PS)_\ell$ sequence. As shown in Proposition 6.1, the Fréchet derivative $\mathcal{J}_0' : E \to E^\star$ is completely continuous, and this implies in particular that $\langle \mathcal{J}_0'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$. Moreover, Lemma 6.4-(β) implies that $\langle I'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$. Consequently, $\langle \mathcal{I}_0'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$. Therefore, by (6.5) in particular $\|u_n\| \to \|u\|$ as $n \to \infty$. By Proposition A.2 it follows at once that $u_n \to u$ as $n \to \infty$ in E, as claimed. $\square$

We can follow the mini-max construction of Section 3. As shown in Proposition 3.4, the mini-max eigenvalues defined in (3.5)–(3.6), where now $\mathcal{J} = p\mathcal{I}_0$, are such that

$$\lambda_1 = \min_{u\in\mathcal{M}} p\mathcal{I}_0(u), \quad \lambda_k \in \mathbb{R}^+, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

$$i(\{u \in E \setminus \{0\} : \mathcal{I}_0(u) \le \lambda_k \|u\|_{p,f}^p\}) = i(\{u \in E : \mathcal{I}_0(u) < \lambda_{k+1} \|u\|_{p,f}^p\}) = k$$

for all k, with $\lambda_k < \lambda_{k+1}$. In particular, λ_1 coincides with the first eigenvalue of (3.1) given in (3.2).

Theorem 7.4. *Under* (g_1), $(g_2)'$–$(g_3)'$, (h_1), *and* $(h_2)'$–$(h_3)'$, *problem* (1.1) *admits a nontrivial solution for all* $\lambda \ge \lambda_1$.

Proof. Since $\lambda_1 < \lambda_2 \le \cdots \le \lambda_k \to \infty$ and $\lambda \ge \lambda_1$, there exists $k \ge 1$ such that $\lambda_k \le \lambda < \lambda_{k+1}$. Define the two symmetric closed cones

$$C_- = \left\{ u \in E : \mathcal{I}_0(u) \le \lambda_k \|u\|_{p,f}^p \right\}, \quad C_+ = \left\{ u \in E : \mathcal{I}_0(u) \ge \lambda_{k+1} \|u\|_{p,f}^p \right\}.$$

Clearly, $C_- \cap C_+ = \{0\}$ and $i(C_- \setminus \{0\}) = i(E \setminus C_+) = k$, as noted above. Furthermore, for all $u \in C_+$ inequality (4.3) continues to hold. Now, Lemma 6.3 implies that $\|u\|^{-p} I(u) \to 0$ as $\|u\| \to 0$. Therefore, there exists a number $r_+ > 0$ such that for all $u \in C_+$, with $\|u\| = r_+$, we have $\Phi(u) \ge \alpha$, where $\alpha = r_+^p (1 - \lambda/\lambda_{k+1})/2p > 0$. On the other hand, $\Phi(u) \le (\lambda_k - \lambda)\|u\|_{p,f}^p/p \le 0$ for all $u \in C_-$, since (g_4) and (h_4) are still valid.

The embeddings $E \hookrightarrow L^p(\Omega, f)$ and $E \hookrightarrow L^p(\partial\Omega, w_4)$ are compact by Theorem 1 of [31], where w_4 is defined in (6.6). Hence, the cone C_- is also closed in the real normed space $E = (E, |||\cdot|||)$, where $|||\cdot||| = \|\cdot\|_{p,f} + \|\cdot\|_{p,w_4,\partial\Omega}$. Taking $e \in E \setminus C_-$ and $t > 0$, we easily see, as in proof of Theorem 4.3, that (4.4) holds.

We claim that there exists $r_- > r_+$ such that $\Phi(u) \le 0$ for all u in $C_- + \mathbb{R}^+ e$, with $\|u\| \ge r_-$. Otherwise, there would exist a sequence $(u_n)_n$ in $C_- + \mathbb{R}^+ e \subset E$, with $\|u_n\| \to \infty$, such that $\Phi(u_n) > 0$. In particular, $0 < \|u_n\| \le \kappa\, |||u_n||| = \kappa\, (\|u_n\|_{p,f} + \|u_n\|_{p,w_4,\partial\Omega})$. Hence, up to a subsequence, we have $v_n = u_n/\|u_n\| \in S_E$ for all n and $v_n \rightharpoonup v \ne 0$, since $1 \le \kappa \lim_n (\|v_n\|_{p,f} + \|v_n\|_{p,w_4,\partial\Omega}) = \kappa\, (\|v\|_{p,f} + \|v\|_{p,w_4,\partial\Omega}) = \kappa\, |||v|||$. Put $\mathcal{W} = \{v_n : n \in \mathbb{N}\} \cup \{v\} \subset E \setminus \{0\}$. Then,

$$0 < \Phi(u_n) \le \frac{1}{p}\Big(1 + \frac{\lambda}{\lambda_1}\Big)\|u_n\|^p - I(\|u_n\| v_n) \sim -I(\|u_n\| v_n) \to -\infty$$

as $n \to \infty$ by Lemma 6.5. This contradiction proves the claim.

The geometrical construction of Theorem 2.7 is completed, so that the corresponding sets satisfy the assertion. In particular, $(Q, D_- \cup H)$ links

S_+ cohomologically in dimension $k+1$ over $\mathbb{Z}_2$ and inequalities (4.5) remain valid also in this setting. Finally, $\Phi \in C^1(E)$ satisfies the $(PS)_\ell$ condition by Lemma 7.3 for all $\lambda \geq \lambda_1$ and $\ell \in \mathbb{R}$. Therefore, problem (1.1) admits a weak nontrivial solution $u \in E$, with $\Phi(u) \geq \alpha$ by virtue of Theorem 2.7. □

APPENDIX A

Following some ideas contained in [4, Proposition A.9] and [15, Theorem 6], we show that the Banach space $E = (E, \|\cdot\|)$, defined in the Introduction, is uniformly convex. To do this, we first prove the validity of the Clarkson inequalities in the space $\mathbb{R}^N$ endowed with the Euclidean norm $|\cdot|$. The original proof of Clarkson in $\mathbb{C}$ can be found in [1, Lemma 2.27].

Lemma A.1. *If $1 < p \leq 2$, then for all x, $y \in \mathbb{R}^N$*

$$\left|\frac{x+y}{2}\right|^{p'} + \left|\frac{x-y}{2}\right|^{p'} \leq \Big(\frac{1}{2}|x|^p + \frac{1}{2}|y|^p\Big)^{1/(p-1)}. \tag{A.1}$$

If $p \geq 2$ then for all x, $y \in \mathbb{R}^N$

$$\left|\frac{x+y}{2}\right|^{p} + \left|\frac{x-y}{2}\right|^{p} \leq \frac{1}{2}|x|^p + \frac{1}{2}|y|^p. \tag{A.2}$$

Proof. First observe that (A.1) and (A.2) coincide when $p = 2$, and in this case they hold with the equality sign, being exactly the parallelogram law. On the other hand, they also hold with the equality sign if either $x = 0$ or $y = 0$. Hence, in the sequel of the proof, we consider only the nontrivial case.

Let $1 < p < 2$. Condition (A.1) is equivalent to

$$\big(\,|x+y|^{p'} + |x-y|^{p'}\,\big)^{1/p'} \leq 2^{1/p'}\,(|x|^p + |y|^p)^{1/p}\,. \tag{A.3}$$

Let us distinguish two cases.

Case $|x| \geq |y| > 0$. Put for convenience $w = x/|x|$ and $v = y/|x|$, so that $|w| = 1$ and $0 < |v| \leq 1$. Dividing (A.3) by $|x| > 0$, we get

$$|w+v|^{p'} + |w-v|^{p'} \leq 2(1+|v|^p)^{1/(p-1)}. \tag{A.4}$$

For brevity, let $a = \sum_{i=1}^N w_i v_i$, so that $|a| \leq c \leq 1$, where $c = |v| \in (0,1]$. Thus, (A.4) is equivalent to

$$(1+2a+c^2)^{p'/2} + (1-2a+c^2)^{p'/2} \leq 2(1+c^p)^{1/(p-1)}. \tag{A.5}$$

The function

$$f_c(a) = (1+2a+c^2)^{p'/2} + (1-2a+c^2)^{p'/2}$$

is well–defined and non–negative in $[-c,c]$. Moreover, f_c is even and admits its maximum value in $a=\pm c$, with $f_c(\pm c)=(1+c)^{p'}+(1-c)^{p'}$. Hence, (A.5) reduces to

$$(1+c)^{p'}+(1-c)^{p'}\le 2(1+c^p)^{1/(p-1)},$$

which is exactly the original formula (7) of Theorem 2 of Clarkson in [8].

Case $0<|x|<|y|$. Due to the symmetry of (A.3), we can repeat the proof of the first case, interchanging x with y, so that now $v=x/|y|$ and $|v|\in(0,1)$.

Let $p>2$. Clearly $1<p'<2$, and so (A.1) becomes for p'

$$\begin{aligned}\Big|\frac{x+y}{2}\Big|^p+\Big|\frac{x-y}{2}\Big|^p\le\frac{1}{2}|x|^p+\frac{1}{2}|y|^p&\le\Big(\frac{1}{2}|x|^{p'}+\frac{1}{2}|y|^{p'}\Big)^{p/p'}\\&\le\frac{2^{p/p'-1}}{2^{p/p'}}(|x|^p+|y|^p),\end{aligned}$$

that is (A.2) holds. □

Proposition A.2. *The Banach space $E=(E,\|\cdot\|)$, defined in the Introduction, is uniformly convex.*

Proof. Let us first consider the case $p\ge 2$. Fix $\varepsilon\in(0,2)$ and let $u,v\in E$ be such that $\|u\|=\|v\|=1$ and $\|u-v\|\ge\varepsilon$. Using Lemma A.1, we have

$$\begin{aligned}&\Big\|\frac{u+v}{2}\Big\|^p+\Big\|\frac{u-v}{2}\Big\|^p\\&=\int_\Omega a(x)\Big\{\Big|\frac{\nabla u+\nabla v}{2}\Big|^p+\Big|\frac{\nabla u-\nabla v}{2}\Big|^p\Big\}dx+\int_{\partial\Omega}b(x)\Big\{\Big|\frac{u+v}{2}\Big|^p+\Big|\frac{u-v}{2}\Big|^p\Big\}dS\\&\le\frac{1}{2}\int_\Omega a(x)\Big\{|\nabla u|^p+|\nabla v|^p\Big\}dx+\frac{1}{2}\int_{\partial\Omega}b(x)\Big\{|u|^p+|v|^p\Big\}dS\\&=\frac{1}{2}\Big(\|u\|^p+\|v\|^p\Big)=1.\end{aligned}$$

Therefore,

$$\Big\|\frac{u+v}{2}\Big\|^p\le 1-\Big(\frac{\varepsilon}{2}\Big)^p,$$

and so, taking $\delta=\delta(\varepsilon)$ such that $1-(\varepsilon/2)^p=(1-\delta)^p$, we obtain that $\|u+v\|\le 2(1-\delta)$.

Suppose now $1<p<2$. Fix $\varepsilon\in(0,2^{2/p})$ and let $u,v\in E$ be such that $\|u\|=\|v\|=1$ and $\|u-v\|\ge\varepsilon$. First note that for any $\varphi\in E$, we have that $|\nabla\varphi|^{p'}\in L^{p-1}(\Omega,a)$, since

$$\||\nabla\varphi|^{p'}\|_{p-1,a}=\Big(\int_\Omega a(x)|\nabla\varphi|^{p'(p-1)}\,dx\Big)^{\frac{1}{p-1}}$$

$$= \Big(\int_\Omega a(x) |\nabla \varphi|^p \, dx \Big)^{\frac{1}{p-1}} = \|\nabla \varphi\|_{p,a}^{p'}.$$

Hence, the reverse Minkowski inequality proved in Proposition A.5 of [4] yields

$$\begin{aligned}
&\Big\| \frac{\nabla u + \nabla v}{2} \Big\|_{p,a}^{p'} + \Big\| \frac{\nabla u - \nabla v}{2} \Big\|_{p,a}^{p'} \\
&\qquad = \Big\| \Big| \frac{\nabla u + \nabla v}{2} \Big|^{p'} \Big\|_{p-1,a} + \Big\| \Big| \frac{\nabla u - \nabla v}{2} \Big|^{p'} \Big\|_{p-1,a} \\
&\qquad \le \Big\| \Big| \frac{\nabla u + \nabla v}{2} \Big|^{p'} + \Big| \frac{\nabla u - \nabla v}{2} \Big|^{p'} \Big\|_{p-1,a} \\
&\qquad = \Big[\int_\Omega a(x) \Big(\Big| \frac{\nabla u + \nabla v}{2} \Big|^{p'} + \Big| \frac{\nabla u - \nabla v}{2} \Big|^{p'} \Big)^{p-1} dx \Big]^{1/(p-1)} \\
&\qquad \le \frac{1}{2} \Big(\|\nabla u\|_{p,a}^p + \|\nabla v\|_{p,a}^p \Big)^{1/(p-1)},
\end{aligned}$$

where in the last step we have used the inequality Lemma A.1. Hence

$$\Big\| \frac{\nabla u + \nabla v}{2} \Big\|_{p,a}^{p'} + \Big\| \frac{\nabla u - \nabla v}{2} \Big\|_{p,a}^{p'} \le \Big(\frac{1}{2} \|\nabla u\|_{p,a}^p + \frac{1}{2} \|\nabla v\|_{p,a}^p \Big)^{1/(p-1)}. \quad \text{(A.6)}$$

Similarly $\|u\|_{p,b,\partial\Omega}^{p'} = \||u|^{p'}\|_{p-1,b,\partial\Omega}$ and $\|v\|_{p,b,\partial\Omega}^{p'} = \||v|^{p'}\|_{p-1,b,\partial\Omega}$, so that, proceeding exactly as before and, applying again the reverse Minkowski inequality proved in Proposition A.5 of [4], we obtain

$$\Big\| \frac{u+v}{2} \Big\|_{p,b,\partial\Omega}^{p'} + \Big\| \frac{u-v}{2} \Big\|_{p,b,\partial\Omega}^{p'} \le \Big(\frac{1}{2} \|u\|_{p,b,\partial\Omega}^p + \frac{1}{2} \|v\|_{p,b,\partial\Omega}^p \Big)^{1/(p-1)}. \quad \text{(A.7)}$$

Therefore, combining (A.6) with (A.7), we get

$$\begin{aligned}
&\Big\| \frac{\nabla u + \nabla v}{2} \Big\|_{p,a}^{p'} + \Big\| \frac{\nabla u - \nabla v}{2} \Big\|_{p,a}^{p'} + \Big\| \frac{u+v}{2} \Big\|_{p,b,\partial\Omega}^{p'} + \Big\| \frac{u-v}{2} \Big\|_{p,b,\partial\Omega}^{p'} \\
&\quad \le \Big(\frac{1}{2} \|\nabla u\|_{p,a}^p + \frac{1}{2} \|\nabla v\|_{p,a}^p \Big)^{1/(p-1)} + \Big(\frac{1}{2} \|u\|_{p,b,\partial\Omega}^p + \frac{1}{2} \|v\|_{p,b,\partial\Omega}^p \Big)^{1/(p-1)} \\
&\quad \le \Big(\frac{1}{2} \|\nabla u\|_{p,a}^p + \frac{1}{2} \|\nabla v\|_{p,a}^p + \frac{1}{2} \|u\|_{p,b,\partial\Omega}^p + \frac{1}{2} \|v\|_{p,b,\partial\Omega}^p \Big)^{1/(p-1)} \\
&\quad = \Big(\frac{1}{2} \|u\| + \frac{1}{2} \|v\| \Big)^{1/(p-1)},
\end{aligned}$$

since $1/(p-1) > 1$; in other words,

$$\Big\| \frac{\nabla u + \nabla v}{2} \Big\|_{p,a}^{p'} + \Big\| \frac{u+v}{2} \Big\|_{p,b,\partial\Omega}^{p'} \quad \text{(A.8)}$$

$$\leq \Big(\frac{1}{2}\|u\| + \frac{1}{2}\|v\|\Big)^{1/(p-1)} - \Big\{\Big\|\frac{\nabla u - \nabla v}{2}\Big\|_{p,a}^{p'} + \Big\|\frac{u-v}{2}\Big\|_{p,b,\partial\Omega}^{p'}\Big\}.$$

Now, since $\|u-v\| \geq \varepsilon$, it follows that

$$\begin{aligned}\varepsilon^{p'} &\leq \Big(\|\nabla u - \nabla v\|_{p,a}^{p} + \|u-v\|_{p,b,\partial\Omega}^{p}\Big)^{1/(p-1)}\\ &\leq 2^{2/(p-1)}\Big(\Big\|\frac{\nabla u - \nabla v}{2}\Big\|_{p,a}^{p'} + \Big\|\frac{u-v}{2}\Big\|_{p,b,\partial\Omega}^{p'}\Big),\end{aligned}$$

since again $1/(p-1) > 1$. Hence,

$$\Big(\frac{\varepsilon}{2^{2/p}}\Big)^{p'} \leq \Big\|\frac{\nabla u - \nabla v}{2}\Big\|_{p,a}^{p'} + \Big\|\frac{u-v}{2}\Big\|_{p,b,\partial\Omega}^{p'}.$$

Thus, choosing $\delta = \delta(\varepsilon)$ such that $1 - \varepsilon/2^{2/p} = 2^{(p-2)/(p-1)}(1-\delta)^{p'}$, from (A.8), we obtain

$$\Big\|\frac{\nabla u + \nabla v}{2}\Big\|_{p,a}^{p'} + \Big\|\frac{u+v}{2}\Big\|_{p,b,\partial\Omega}^{p'} \leq 2^{(p-2)/(p-1)}(1-\delta)^{p'}.$$

In conclusion, since again $1/(p-1) > 1$,

$$\Big\|\frac{u+v}{2}\Big\|^{p'} \leq 2^{(2-p)/(p-1)}\Big(\Big\|\frac{\nabla u + \nabla v}{2}\Big\|_{p,a}^{p'} + \Big\|\frac{u+v}{2}\Big\|_{p,b,\partial\Omega}^{p'}\Big) \leq (1-\delta)^{p'};$$

that is, $\|u+v\| \leq 2(1-\delta)$, as required. □

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