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GRADIENT ESTIMATES FOR SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS DEGENERATING AT INFINITY

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Abstract. For $p \in (1, +\infty)$ we derive a weighted L^p estimate for the (spatial) gradient of the solution u of a degenerate parabolic differential equation. Here the underlying domain $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is unbounded and the equation may degenerate only at infinity along some unbounded branch of Ω. Our estimate is strictly related with the still-open problem of giving a concrete characterization of the interpolation space between $W^{2,p}(\Omega)$ and $L^p(\Omega)$ to which the (spatial) gradient of u belongs.

1. Introduction

When dealing with direct or inverse problems related to degenerate parabolic equations of the type

$$D_t[m(x)u(t,x)] - A(x; D_x)u(t,x) = f(t,x), \qquad (t,x) \in [0,T] \times Q, \qquad (1.1)$$

where $m \geq 0$ almost everywhere in Q, $A(x; D_x)$ is a linear elliptic second-order differential operator in divergence form, Q is a (possibly unbounded) open subset of $\mathbf{R}^n$ and $f \in C([0,T]; L^p(Q))$, it is often necessary to know the order $\gamma \in (0,1)$ of the interpolation space between $W^{2,p}(Q)$ and $L^p(Q)$ the gradient $\nabla_x u$ belongs to. In general, what happens is that this order is in a one-to-one relation with the exponent γ appearing in an estimate of the form $\|m\nabla_x u\|_{L^p(Q)} \leq C|\lambda|^{-\gamma}\|f\|_{L^p(Q)}$, where $\operatorname{Re}\lambda$ is large enough and the constant C is independent of u and λ. Usually, such an estimate is expected to be implied by the spectral equation $\lambda m u - A(x; D_x)u = f$, $f \in L^p(Q)$,

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associated with (1.1) through the (formal) Laplace transformation with respect to the variable t. However, to the authors' knowledge, at the moment for degenerate parabolic equations no estimates of this kind are available for $m\nabla_x u$, but only for mu and $A(x;D_x)u$ (cf. [8] and [9]). Therefore, for equations of type (1.1), the question of which interpolation space the (spatial) gradient of u belongs to is still open.

In contrast to the degenerate case, the non-degenerate one (corresponding to a continuous and positive function m which is bounded away from 0 in $\overline{Q}$) has been widely studied and, for it, the situation is nowadays well-understood, at least when Q is bounded or coinciding with $\mathbf{R}^n$ or $\mathbf{R}^n_+$. It is known that $\nabla_x u \in W^{1,p}(Q)$, $p \in (1,+\infty)$, and, for $\operatorname{Re}\lambda$ large enough, it satisfies the estimate $\|\nabla_x u\|_{L^p(Q)} \le C|\lambda|^{-1/2}\|f\|_{L^p(Q)}$ (see, for instance, [10, Theorem 3.1.3]). In other words, for $p \in (1,+\infty)$, $\nabla_x u$ belongs to the classical real interpolation space of order 1/2 between $L^p(Q)$ and $W^{2,p}(Q)$, i.e., the space $(L^p(Q), W^{2,p}(Q))_{1/2,+\infty}$ (cf. [10, Chapter 1] for the definition of $(L^p(Q), W^{2,p}(Q))_{\delta,+\infty}$, $\delta \in (0,1)$). Unfortunately, in the degenerate case the standard interpolation theory does not work, and the spaces $(L^p(Q), W^{2,p}(Q))_{\delta,+\infty}$ have to be replaced with the spaces $L^p_{\delta,A,m}(Q)$ introduced in [9, pp. 25, 26]. Until now, for these latter spaces, no concrete characterizations have been given and only relations of inclusion between them and the spaces $(L^p(Q), W^{2,p}(Q))_{\delta,+\infty}$ are known (cf. [9, Theorem 1.12]).

It is our aim, here, to give a contribution in this field providing λ-dependent L^p-estimates for $\nabla_x u$ when the underlying domain Q in (1.1) is unbounded (possibly with several branches extending to infinity) and satisfying the internal cone property in the sense of [6]. The choice of unbounded domains is due to technical reasons forcing us to require that the function m should satisfy the estimates $|\nabla m(x)| \le Cm(x)$, $x \in Q$. If we further require that m may vanish, this implies that Q must be unbounded (cf. the following Remark 3.5) and that the zeros of m may occur only at infinity along some unbounded branch of Q.

The plan of the paper is the following. In Section 2, we introduce the definitions of some spaces which are natural when dealing with unbounded domains. Section 3 is devoted to recalling an *a priori* estimate for solutions of the Dirichlet problem for elliptic equations in unbounded domains (cf. [7, Theorem 5.1]) and a λ-dependent L^p-estimate for mu proved in [8]. In Section 4, we state our main Theorem 4.4, which will be proved in Section 5, using the two basic Lemmas 5.1 and 5.2 and combining them with the preliminary estimates of Section 3. Finally, Section 6 is devoted to solving the initial problem related to the singular equation (1.1).

2. The spaces BMO, VMO and $M^{p,\lambda}$

We recall here the definitions of three function spaces we will need later to apply a special case of Theorem 5.1 in [7], i.e., the space BMO of the functions of *bounded mean oscillation*, the space VMO of all functions of *vanishing mean oscillation* and the spaces of Morrey type $M^{p,\lambda}$. Even though these spaces were introduced, respectively, in the papers [11], [13] and [5], we prefer here to follow the treatment given in [15] in order to simplify our notation and make it uniform with that of [7].

Let G be a (possibly unbounded) open subset of $\mathbf{R}^N$, $N \in \mathbf{N}$. As usual, for every $x \in \mathbf{R}^N$ and every $r > 0$, $B_N(x,r)$ denotes the open ball of $\mathbf{R}^N$ with center at x and radius r. Then we suppose that G satisfies the following condition which goes back to [6]:

$$\alpha_1 := \sup_{\substack{0<\rho\leq 1\\ x\in G}} \frac{|B_N(x,\rho)|}{|G\cap B_N(x,\rho)|} < +\infty, \tag{2.1}$$

where $|A|$, A being a Lebesgue-measurable subset of $\mathbf{R}^N$, denotes the N-dimensional Lebesgue measure of A. We recall that assumption (2.1) implies that G is not "too narrow," and it is clearly satisfied by any subset G having the *internal* cone property.

Condition (2.1) can be generalized as follows: *for a fixed* $\beta \in \mathbf{R}_+$ *there exists* $r \in \mathbf{R}_+$ *such that*

$$\alpha_r := \sup_{\substack{0<\rho\leq r\\ x\in G}} \frac{|B_N(x,\rho)|}{|G\cap B_N(x,\rho)|} \leq \beta. \tag{2.2}$$

Observe that $0 < r_1 < r_2$ with $\alpha_{r_1} \leq \beta$ and $\alpha_{r_2} \leq \beta$ implies $\alpha_{r_1} \leq \alpha_{r_2}$. Then we set $r_\beta = \sup\{r \in \mathbf{R}_+ : \alpha_r \leq \beta\}$ and observe that α_r is defined in the whole of $(0, r_\beta]$. Indeed, if $r_0 \in (0, r_\beta)$, then there exists $r_1 \in (r_0, r_\beta)$ such that $\alpha_{r_0} \leq \alpha_{r_1} \leq \beta$. Of course, assuming (2.1) is equivalent to saying $r_\beta \geq 1$.

We denote by $BMO(G)$ the space of all functions $g \in L^1_{loc}(\overline{G})$ such that

$$[g]_{BMO(G)} = \sup_{\substack{0<\rho\leq r_\beta\\ x\in G}} \frac{1}{|G\cap B_N(x,\rho)|} \int_{G\cap B_N(x,\rho)} |g(\xi) - g_{G\cap B_N(x,\rho)}|\,\mathrm{d}\xi < +\infty,$$

where $g_{G\cap B_N(x,\rho)} = \frac{1}{|G\cap B_N(x,\rho)|}\int_{G\cap B_N(x,\rho)} g(\zeta)\,\mathrm{d}\zeta$. As is well known, if we denote by $BMO(G,r)$, $r\in(0,+\infty]$, the space defined as $BMO(G)$ with r instead of r_β, then $g\in BMO(G)$ if and only if $g\in BMO(G,r)$ for every $r\in(0,r_\beta]$ and, moreover, $[g]_{BMO(G)} = \lim_{r\to r_\beta}[g]_{BMO(G,r)}$. Then, we denote by $VMO(G)$ the subspace of $BMO(G)$ consisting of those functions $g\in BMO(G)$ such that

$$\eta[g,G](r) := [g]_{BMO(G,r)} \to 0 \quad \text{as} \quad r\to 0^+.$$

We call a modulus of continuity of $g\in VMO(G)$ a function $\eta:(0,r_\beta]\to\mathbf{R}_+$ such that

$$\eta[g,G](r)\le \eta(r), \quad \text{for every } r\in(0,r_\beta], \quad \eta(r)\to 0 \ \text{ as } \ r\to 0^+.$$

Finally, G being a Lebesgue-measurable subset of $\mathbf{R}^N$ not necessarily satisfying (2.1), we denote by $M^{p,\lambda}(G,r)$, $p\in[1,+\infty)$, $\lambda\in[0,N)$, $r\in\mathbf{R}_+$, the subset of $L^p_{loc}(\overline{G})$ consisting of those functions g such that

$$\|g\|_{M^{p,\lambda}(G,r)} := \sup_{\substack{\rho\in(0,r]\\ x\in G}} \rho^{-\lambda/p}\|g\|_{L^p(G\cap B_N(x,\rho))} < +\infty.$$

Endowed with such a norm, $M^{p,\lambda}(G,r)$ turns out to be a Banach space. In particular, we set $M^{p,\lambda}(G) = M^{p,\lambda}(G,1)$ and define $\widetilde{M}^{p,\lambda}(G)$ as the closure of $L^\infty(G)$ in $M^{p,\lambda}(G)$. Denoting by $\Sigma(G)$ the σ-algebra of all Lebesgue-measurable subsets of G, $\widetilde{M}^{p,\lambda}(G)$ can be characterized as the space of the functions $g\in M^{p,\lambda}(G)$ such that

$$\sigma[g,G](\tau) := \sup_{\substack{E\in\Sigma(G)\\ \sup_{y\in G}|E\cap B_N(y,1)|\le\tau}} \|\chi_E g\|_{M^{p,\lambda}(G)} \to 0 \quad \text{as} \quad \tau\to 0^+,$$

χ_E being the characteristic function of E. We call a modulus of continuity of $g\in\widetilde{M}^{p,\lambda}(G)$ a function $\sigma:(0,1]\to\mathbf{R}_+$ such that

$$\sigma[g,G](\tau)\le\sigma(\tau), \quad \text{for every } \tau\in(0,1], \quad \sigma(\tau)\to 0 \ \text{ as } \ \tau\to 0^+.$$

Later on in the proof of our main result, following a well-known procedure due to Agmon [1], we will need to increase the dimension from n to $n+1$, $n\in\mathbf{N}$. For this purpose, we now show two preliminary lemmata. In the first we show under which additional conditions functions g belonging to $BMO(G)\cap C(\overline{G})$ belong also to $VMO(G\times\mathbf{R})$ with an appropriate modulus of continuity. Similarly, in the second lemma, for functions $g\in L^\infty(G)$ we characterize their moduli of continuity in $\widetilde{M}^{p,\lambda}(G\times\mathbf{R})$.

Lemma 2.1. *Let G be a (possibly unbounded) open subset of $\mathbf{R}^N$ satisfying (2.1), and let $g \in BMO(G)$ be a function such that*

$$|g(\xi) - g(\zeta)| \le \varphi(|\xi - \zeta|), \quad \text{for every } \xi, \zeta \in \overline{G}, \tag{2.3}$$

where the modulus of continuity φ satisfies

$$\varphi \text{ is continuous at } s = 0, \text{ non-decreasing in } (0, 2r_\beta] \text{ and } \varphi(0) = 0. \tag{2.4}$$

Then $g \in VMO(G)$ with modulus of continuity $\eta(r) = C_1\varphi(2r)$ and $g \in VMO(G \times \mathbf{R})$ with modulus of continuity $\eta(r) = C_2\varphi(2r)$, C_1 and C_2 being positive constants depending only on α_1 and N.

Proof. First of all, due to (2.2), there exists a positive constant C such that

$$|G \cap B_N(x,\rho)| \ge C|B_N(x,\rho)|, \quad \text{for every } x \in G \text{ and } \rho \in (0, r_\beta], \tag{2.5}$$

where $r_\beta \ge 1$ due to (2.1). Therefore, taking into account (2.3) and (2.4), for every $r \in (0, r_\beta]$ we easily get

$$\begin{aligned}
[g]_{BMO(G,r)} &\le \sup_{\substack{0<\rho\le r\\ x\in G}} \frac{1}{|G \cap B_N(x,\rho)|^2} \int_{G\cap B_N(x,\rho)} \mathrm{d}\xi \int_{G\cap B_N(x,\rho)} \varphi(|\xi-\zeta|)\,\mathrm{d}\zeta \\
&\le C^{-2} \sup_{\substack{0<\rho\le r\\ x\in G}} \frac{1}{|B_N(x,\rho)|^2} \int_{B_N(x,\rho)} \mathrm{d}\xi \int_{B_N(x,\rho)} \varphi(2\rho)\,\mathrm{d}\zeta \;\le\; C^{-2}\varphi(2r).
\end{aligned} \tag{2.6}$$

Passing to the limit as $r \to 0^+$ in (2.6) we immediately deduce that $g \in VMO(G)$ with modulus of continuity $\eta(r) = C^{-2}\varphi(2r)$.

Now, observe that for any $(x,t) \in G \times \mathbf{R}$ and $\rho \in (0, r_\beta]$ we have $(G \times \mathbf{R}) \cap B_{N+1}((x,t),\rho) = \{(\xi,\tau) \in \mathbf{R}^N \times \mathbf{R} : \tau \in (t-\rho, t+\rho),\ \xi \in G \cap B_N(x, (\rho^2 - |t-\tau|^2)^{1/2})\}$. Hence, using (2.5) with $B_N(x,\rho)$ being replaced by $B_N(x, (\rho^2 - |t-\tau|^2)^{1/2})$ and setting $\omega_k = |B_k(0,1)|$, $k \in \mathbf{N}$, we deduce

$$\begin{aligned}
|(G \times \mathbf{R}) \cap B_{N+1}((x,t),\rho)| &= \int_{t-\rho}^{t+\rho} \mathrm{d}\tau \int_{G\cap B_N(x,(\rho^2-|t-\tau|^2)^{1/2})} \mathrm{d}\xi \\
&\ge C \int_{t-\rho}^{t+\rho} |B_N(x,(\rho^2-|t-\tau|^2)^{1/2})|\,\mathrm{d}\tau = C\omega_N \int_{t-\rho}^{t+\rho} (\rho^2 - |t-\tau|^2)^{N/2}\,\mathrm{d}\tau \\
&= 2C\omega_N \rho^{N+1} \int_0^1 (1-\nu^2)^{N/2}\,\mathrm{d}\nu = \widetilde{C}|B_{N+1}((x,t),\rho)|,
\end{aligned} \tag{2.7}$$

where $\widetilde{C} = 2C\omega_N\omega_{N+1}^{-1}\int_0^1(1-\nu^2)^{N/2}\,\mathrm{d}\nu$. Indeed, the change of variable $\tau = t+\rho\nu$ implies

$$\int_{t-\rho}^{t+\rho}(\rho^2-|t-\tau|^2)^{N/2}\,\mathrm{d}\tau = 2\int_t^{t+\rho}(\rho^2-|t-\tau|^2)^{N/2}\,\mathrm{d}\tau = 2\rho^{N+1}\int_0^1(1-\nu^2)^{N/2}\,\mathrm{d}\nu.$$

Consequently, using (2.3), (2.4) and (2.7), for every $r\in(0,r_\beta]$ we easily derive

$$\begin{aligned}[g]_{BMO(G\times\mathbf{R},r)} &\le \sup_{\substack{0<\rho\le r\\(x,t)\in\overline{G}\times\mathbf{R}}}\frac{1}{|(G\times\mathbf{R})\cap B_{N+1}((x,t),\rho)|^2}\\ &\quad\times\int_{(G\times\mathbf{R})\cap B_{N+1}((x,t),\rho)}\mathrm{d}\xi\int_{(G\times\mathbf{R})\cap B_{N+1}((x,t),\rho)}\varphi(|\xi-\zeta|)\,\mathrm{d}\zeta\\ &\le \widetilde{C}^{-2}\sup_{\substack{0<\rho\le r\\(x,t)\in\overline{G}\times\mathbf{R}}}\frac{1}{|B_{N+1}((x,t),\rho)|^2}\int_{B_{N+1}((x,t),\rho)}\mathrm{d}\xi\int_{B_{N+1}((x,t),\rho)}\varphi(2\rho)\,\mathrm{d}\zeta\\ &\le \widetilde{C}^{-2}\varphi(2r). \end{aligned}\tag{2.8}$$

Passing to the limit as $r\to 0^+$ in (2.8) we deduce that $g\in VMO(G\times\mathbf{R})$ with modulus of continuity $\eta(r)=\widetilde{C}^{-2}\varphi(2r)$, and this completes the proof. □

Corollary 2.2. *Let* $g\in C_b^1(\overline{G}) = \{h\in C^1(\overline{G}) : h\in C_b(\overline{G}),\ D_{x_j}h\in C_b(\overline{G}),\ j=1,\dots,N\}$. *Then* g *satisfies* Lemma 2.1 *with* $\varphi(s)=Ms$, $M=(\sum_{j=1}^N\|D_{x_j}h\|^2_{C(\overline{G})})^{1/2}$. *In particular, endowing* $C_b^1(\overline{G})$ *with the norm*

$$\|h\|_{C_b^1(\overline{G})} = \|h\|_{C_b(\overline{G})} + \sum_{j=1}^N\|D_{x_j}h\|_{C_b(\overline{G})},$$

we can take $M=\sqrt{N}\|g\|_{C_b^1(\overline{G})}$.

Lemma 2.3. *Let* G *be a* (*possibly unbounded*) *open subset of* $\mathbf{R}^N$ *and let* $g\in L^\infty(G)$. *Then, for every* $p\in[1,+\infty)$ *and* $\lambda\in[0,N)$ ($\lambda\in[0,N+1)$), $g\in\widetilde{M}^{p,\lambda}(G)$ ($g\in\widetilde{M}^{p,\lambda}(G\times\mathbf{R})$) *with modulus of continuity*

$$\sigma(\tau)=\|g\|_{L^\infty(G)}\omega_N^{\lambda/(pN)}\tau^{(N-\lambda)/(pN)},$$

$(\sigma(\tau)=\|g\|_{L^\infty(G)}\omega_{N+1}^{\lambda/[p(N+1)]}\tau^{(N+1-\lambda)/[p(N+1)]})$.

Proof. The inclusion $L^\infty(G) \subset \widetilde{M}^{p,\lambda}(G)$ follows from the definition of $\widetilde{M}^{p,\lambda}(G)$ as closure of $L^\infty(G)$ in $M^{p,\lambda}(G)$. Similarly, since $L^\infty(G) \subset L^\infty(G\times \mathbf{R})$, we have $L^\infty(G) \subset \widetilde{M}^{p,\lambda}(G \times \mathbf{R})$. Concerning instead the modulus of continuity of $g \in L^\infty(G)$ in $\widetilde{M}^{p,\lambda}(G)$ we easily get

$$\begin{aligned}\sigma[g,G](\tau) &\le \|g\|_{L^\infty(G)} \sup_{\substack{E\in\Sigma(G)\\ \sup_{y\in G}|E\cap B_N(y,1)|\le\tau}} \sup_{\substack{\rho\in(0,1]\\ x\in G}} \rho^{-\lambda/p}|G\cap E\cap B_N(x,\rho)|^{1/p}\\ &\le \|g\|_{L^\infty(G)} \sup_{\substack{E\in\Sigma(G)\\ \sup_{y\in G}|E\cap B_N(y,1)|\le\tau}} \sup_{\substack{\rho\in(0,1]\\ x\in G}} \rho^{-\lambda/p}|E\cap B_N(x,\rho)|^{1/p}. \quad (2.9)\end{aligned}$$

Now, for any $E \in \Sigma(G)$ such that $|E \cap B_N(y,1)| \le \tau$ for every $y \in G$ we have both the inequalities $|E\cap B_N(x,\rho)| \le \tau$ and $|E\cap B_N(x,\rho)| \le \omega_N\rho^N$, $\rho \in (0,1]$, $x \in G$. Therefore, via interpolation, we derive

$$|E\cap B_N(x,\rho)| \le \tau^\varepsilon(\omega_N\rho^N)^{1-\varepsilon}, \quad \varepsilon\in(0,1), \quad \rho\in(0,1]. \qquad (2.10)$$

Choosing $\varepsilon = (N-\lambda)/N$ and replacing (2.10) in (2.9) we derive the assertion for $\widetilde{M}^{p,\lambda}(G)$. The proof for $\widetilde{M}^{p,\lambda}(G\times\mathbf{R})$ follows from (2.9) and (2.10) simply by replacing G, y, x and N with $G\times\mathbf{R}$, (y,s), (x,t) and $N+1$, respectively. □

3. Two preliminary estimates

In this section we report two basic estimates, quite different in their character, that we will need later in the proof of our main theorem. The first is an *a priori* estimate for the solutions of the Dirichlet problem concerning elliptic equations in unbounded domains. The latter is an estimate for the resolvent of a spectral problem connected with degenerate parabolic differential equations in the sense of [9]. In our brief exposition of such estimates, we will refer to [7] and to [8] for the proofs of the first and the latter estimate, respectively.

We begin by recalling some notation. As usual, for any Lebesgue-measurable open subset E of $\mathbf{R}^N$, $N \in \mathbf{N}$, using a standard multi-index notation, we denote by $W^{l,p}(E)$, $l\in\mathbf{N}$, $p\in[1,+\infty)$, the Sobolev space of all functions $v \in L^p(E)$ whose distributional derivatives $D^\beta v$, $|\beta| \le l$, belong to $L^p(E)$. $W^{l,p}(E)$ is endowed with the usual norm $\|v\|_{W^{l,p}(E)} := (\sum_{|\beta|\le l}\|D^\beta v\|^p_{L^p(E)})^{1/p}$. Then, we denote by $W^{l,p}_0(E)$ the closure of $C^\infty_0(E)$ in $W^{l,p}(E)$ and by $W^{l,p}_{loc}(E)$ (respectively $\widetilde{W}^{l,p}_{loc}(E)$) the space of functions

$v : E \to \mathbf{R}$ such that $\zeta v \in W^{l,p}(E)$ (respectively $\zeta v \in W_0^{l,p}(E)$) for any $\zeta \in C_0^\infty(E)$.

Let now G be an unbounded open subset of $\mathbf{R}^n$, $n \geq 3$, satisfying the following assumptions:

i) there exist $\delta,\ M \in \mathbf{R}_+,\ k \in \mathbf{N}$, a locally finite open covering $(U_i)_{i\in\mathbf{N}}$ of ∂G and sequence of diffeomorphisms $\Phi_i : U_i \to B_n(0,i)$ of class $C^{1,1}$, $i \in \mathbf{N}$, such that $\{x \in G : \mathrm{dist}(x, \partial G) < \delta\} \subset \cup_{i\in\mathbf{N}} \Phi_i^{-1}\big(B_n(0,1/4)\big)$;

ii) any intersection of $k+1$ sets of the collection $(U_i)_{i\in\mathbf{N}}$ is empty;

iii) $\Phi_i(U_i \cap G) = \{x \in B_n(0,i) : x_n > 0\}$, for every $i \in \mathbf{N}$;

iv) the $C^{1,1}$-norms of Φ_i and Φ_i^{-1}, $i \in \mathbf{N}$, are uniformly bounded by M.

We introduce in G the linear second-order differential operator in divergence form,

$$A(x; D_x) = \sum_{i,j=1}^{n} D_{x_i}\big(a_{i,j}(x) D_{x_j}\big) - a_0(x)\,, \tag{3.1}$$

where the coefficients $a_{i,j}$ and a_0 are assumed to satisfy the following properties:

$$a_{i,j} = a_{j,i} \in C_b^1(\overline{G}), \quad i,j = 1, \ldots, n, \quad a_0 \in C_b(\overline{G}), \quad a_0 \geq \nu_1 \ \text{ in } \overline{G}, \tag{3.2}$$

$$\nu_2|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq \nu_3|\xi|^2, \quad \text{for every } (x,\xi) \in \overline{G} \times \mathbf{R}^n, \tag{3.3}$$

ν_k, $k = 1,2,3$, being three positive constants.

Notice that, under (3.2), Corollary 2.2 implies that the $a_{i,j}$'s belong to $VMO(G)$ with modulus of continuity $\eta_{i,j}(s) = \sqrt{n}\|a_{i,j}\|_{C_b^1(\overline{G})} s$, whereas Lemma 2.3 implies that $b_0 := a_0$ and $b_i := \sum_{j=1}^n D_{x_i} a_{i,j}$, $i = 1, \ldots, n$, belong to $\widetilde{M}^{p,\lambda}(G)$, $p \in [1, +\infty)$, $\lambda \in [0, n)$, with moduli of continuity $\sigma_j(\tau) = \|b_j\|_{C(\overline{G})} \omega_n^{\lambda/(pn)} \tau^{(n-\lambda)/(pn)}$, $j = 0, \ldots, n$. We set $\eta = (\sum_{i,j=1}^n \eta_{i,j}^2)^{1/2}$ and $H = \max_{i,j=1,\ldots,n}\{\|a_{i,j}\|_{VMO(G)}, \|b_i\|_{C(\overline{G})}, \|b_0\|_{C(\overline{G})}\}$. Then, the following theorem, which is a particular case of Theorem 5.1 in [7], holds true.

Theorem 3.1. *Let $p \in (1, +\infty)$ and $p_i \in [1, +\infty)$, $p_i \leq p$, $i = 0, 1$. Then, for all u such that $u \in W_{loc}^{2,p_1}(G) \cap \widetilde{W}_{loc}^{1,p_1}(G) \cap L^{p_0}(G)$ and $A(x; D_x)u \in L^p(G)$, we have $u \in W^{2,p}(G)$ and the following estimate holds:*

$$\|u\|_{W^{2,p}(G)} \leq c_1[\|A(x; D_x)u\|_{L^p(G)} + \|u\|_{L^{p_0}(G)}], \tag{3.4}$$

where the constant c_1 *depends on* n, p, ν_2, η, σ_j, $j = 0, 1, \dots, n$, H *and* G, *only.*

Remark 3.2. We stress that, if the coefficients of $A(x; D_x)$ are complex-valued, then Theorem 3.1 remains true under the ellipticity condition

$$\Big|\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j\Big| \geq \nu|\xi|^2, \quad \text{a.e. in } G, \text{ for every } \xi \in \mathbf{R}^n, \text{ and some } \nu \in \mathbf{R}_+. \tag{3.5}$$

Indeed, in contrast to what happens for systems where a stronger assumption on the real part of the associated quadratic form is required (cf. [3, pp. 38–44]), for a single equation condition (3.5) is sufficient for the Dirichlet boundary conditions to be complementing in the sense of [2], or, equivalently, for the well-known Shapiro-Lopatinskii conditions to be satisfied (cf. [1, pp. 5–11]). We refer to [14, Theorem 6.1] as an example of how Theorem 3.1 can be shown under the weak ellipticity condition (3.5). Actually, in [14], Theorem 3.1 is shown in a suitable class of weighted Sobolev spaces which, however, reduce to the classical ones if we choose as a weight function the constant function 1.

Having recalled the *a priori* estimate of [7], we can now pass to recalling the *resolvent* estimate of [8]. Let Q be a (possibly unbounded) subset of $\mathbf{R}^n$, $n \in \mathbf{N}$, having a boundary ∂Q of class C^2 and let (3.1)–(3.3) be satisfied, with G being replaced by Q. Moreover, let $m \in C_b^1(\overline{Q})$ be a non-negative function, which need not to be bounded away from 0, satisfying

$$|\nabla m(x)| \leq K_0[m(x)]^{\mu}, \quad \text{for every } x \in \overline{Q}, \quad \text{for some } \mu \in (0,1), \tag{3.6}$$

where $|\nabla m| = \big(\sum_{i=1}^{n}(D_{x_i}m)^2\big)^{1/2}$ and K_0 is a positive constant. Then the following theorem holds true (cf. Theorems 3.1 and 4.1 in [8]).

Theorem 3.3. *Let assumptions* (3.1)–(3.3) *and* (3.6) *be satisfied. In addition, let* $k(p)$ *be a positive constant depending on* p *and chosen so small as to satisfy*

$$h(p) := 1 - k(p)\frac{|p-2|}{2\sqrt{p-1}} > 0, \quad \text{for every } p \in (1, +\infty),$$

and let $\delta := k(p)[1 - h(p)]^{-1} > 0$. *Then, for any* $p \in [2, +\infty)$ *and* $\lambda \in \Sigma_1 = \{z \in \mathbf{C} : \operatorname{Re} z + \frac{k(p)}{2}|\operatorname{Im} z| + \frac{\nu_1}{2\|m\|_{C_b(\bar{Q})}} \geq 0\}$, *the spectral equation* $\lambda m u - A(x; D_x)u = f$, $f \in L^p(Q)$, *admits a unique solution* $u \in W^{2,p}(Q) \cap$

$W_0^{1,p}(Q)$ *satisfying the estimates*

$$\|u\|_{L^p(Q)} \leq C_6(p)\|f\|_{L^p(Q)}, \tag{3.7}$$

$$\|mu\|_{L^p(Q)} \leq C_7(p)|\lambda|^{-2/[p(2-\mu)]}\|f\|_{L^p(Q)}, \quad \lambda \in \Sigma_1. \tag{3.8}$$

Here, $\mu \in (0,1)$ *whereas* $C_j(p)$, $j = 6, 7$, *denote positive constants depending on* p, *only. Similarly, for any* $p \in (1,2)$ *and* $\lambda \in \Sigma_2 = \{z \in \mathbf{C} : \mathrm{Re}\, z + \frac{\sigma}{2}|\mathrm{Im}\, z| + \frac{\nu_1}{2\|m\|_{C_b(\bar{Q})}} \geq 0\}$, $\sigma \in (0,\delta)$, *the spectral equation* $\lambda mu - A(x; D_x)u = f$, $f \in L^p(Q)$, *admits a unique solution* $u \in W^{2,p}(Q) \cap W_0^{1,p}(Q)$ *satisfying the estimates* (3.7) *and*

$$\|mu\|_{L^p(Q)} \leq C_8(p)|\lambda|^{-1/(2-\mu)}\|f\|_{L^p(Q)}, \quad \lambda \in \Sigma_2. \tag{3.9}$$

Here $\mu \in (2-p, 1)$ *whereas* $C_8(p)$ *denotes positive constants depending on* p, *only.*

Remark 3.4. Actually, in [8], the previous theorem is shown under the assumption that Q should be a bounded domain of $\mathbf{R}^n$. However, looking at the quoted paper and at the computations therein, one easily convinces oneself that Theorem 3.3 remains true even without the boundedness assumption on Q. Indeed, in [8], the authors limit themselves to the bounded case essentially for two reasons. Firstly, because they are interested in applying their results to degenerate parabolic equations in bounded domains. Secondly, because in performing their estimates they take advantage of a preliminary result in [12] (cf. formulae (2.3) and (2.4) in [8]), where the underlying domain Ω is assumed to be bounded. The point, here, is that the results in [12] *do not depend* on the boundedness of Ω, since only integrations by parts are used in the computations. As a matter of fact, in [12], Remark 4 in page 706 recalls for the reader that the results can be generalized also to the case $\Omega = \mathbf{R}^n$, suggesting that the boundedness assumption on Ω can be removed.

Notice that in Theorem 3.3 the case $\mu = 1$ in (3.6) has been excluded. We give here a brief explanation about the reasons for this exclusion, stressing that it essentially depends on the boundedness assumption on Q required in [8].

Let V be a (possibly unbounded) connected open subset of $\mathbf{R}^n$ with the property that for each $R > 0$ there exists a $L_0(R) > 0$ such that every pair of points $x, y \in \overline{V} \cap \overline{B_n(0,R)}$ can be joined by a piecewise-C^1-curve $\gamma(x,y)$ satisfying

i) $\gamma(x,y) \subset \overline{V} \cap \overline{B_n(0,R)}$;

ii) $\operatorname{length}(\gamma(x,y)) \leq L_0(R)$.

Now, let $m \in C^1(\overline{V})$, $m \geq 0$, be a non-null function such that $|\nabla m(x)| \leq K_0 m(x)$, $x \in V$, i.e., a function satisfying (3.6) with $\mu = 1$. Since $m \neq 0$, there exists $x_0 \in V$ such that $m(x_0) > 0$. Moreover, the inequality

$$|\nabla m(x)| \leq K_0[m(x) + \varepsilon], \quad x \in V \cap B_n(0,R), \quad \varepsilon > 0,$$

implies that the function $w_\varepsilon(x) = \log[m(x)+\varepsilon]$ belongs to $C^1(\overline{V})$ and satisfies $|\nabla w_\varepsilon(x)| \leq K_0$, $x \in V$. Therefore, for every $y \in \overline{V} \cap \overline{B_n(0,R)}$, $R > |x_0|$, we have

$$\begin{aligned} |w_\varepsilon(y) - w_\varepsilon(x_0)| &= \Big| \int_{\gamma(x_0,y)} \nabla w_\varepsilon(z)\,\mathrm{d}z \Big| \\ &\leq \int_{\gamma(x_0,y)} |\nabla w_\varepsilon(z)|\,\mathrm{d}z \leq K_0 L_0(R). \end{aligned} \tag{3.10}$$

From (3.10) and the definition of w_ε it immediately follows that

$$\mathrm{e}^{-K_0 L_0(R)} \leq \frac{m(y) + \varepsilon}{m(x_0) + \varepsilon} \leq \mathrm{e}^{K_0 L_0(R)}, \tag{3.11}$$

for $y \in \overline{V} \cap \overline{B_n(0,R)}$ and $\varepsilon > 0$. Suppose now that m has a zero in $y_0 \in \overline{V}$. Then, from (3.11) we deduce

$$\frac{\varepsilon}{m(x_0) + \varepsilon} \geq \mathrm{e}^{-K_0 L_0(R)}, \quad \text{for every } \varepsilon > 0,$$

which is of course false for small enough ε. This means that m should be strictly positive on $\overline{V} \cap \overline{B_n(0,R)}$ for all $R > 0$, i.e., on $\overline{V}$, whereas the case the authors in [8] have in mind is related to degenerate parabolic equations in which m may vanish at some points.

Remark 3.5. We stress that every open subset U of $\mathbf{R}^n$ such that *ii*) holds with $L_0(R) = L_0$ independent of R is bounded with $\operatorname{diam} U \leq L_0$. Indeed, $|x - y| \leq \operatorname{length}(\gamma(x,y)) \leq L_0$. Obviously, all convex bounded open subsets of $\mathbf{R}^n$ satisfy *i*) and *ii*). More generally, if U is an open bounded subset of $\mathbf{R}^n$ star-shaped with respect to $x_0 \in U$ and $m \in C^1(\overline{U})$ is such that $|\nabla m(x)| \leq K_0 m(x)$, $x \in \overline{U}$, $m(x_0) > 0$, then, from the previous discussion, it follows that m is positive in $\overline{U}$.

Remark 3.6. Let Q be an unbounded open set with properties *i*) and *ii*) and let $m \in C_b^1(\overline{Q})$ be a non-null function satisfying (3.6) with $\mu = 1$. Then $m(x) > 0$ for every $x \in \overline{Q}$.

4. Main result

As we have seen, Theorem 3.3 provides a λ-dependent L^p-estimate for mu, but it does not provide any for ∇u or $m\nabla u$. Our aim, here, is just to prove an estimate of this type for $m\nabla u$ when the underlying domain Q is unbounded and (possibly) extends to infinity with several branches. For this purpose we need to introduce the class of our admissible unbounded domains $\Omega \subset \mathbf{R}^n$, $n \geq 2$, and of our admissible pair $(A(x; D_x), m)$.

Definition 4.1. Let Ω be an unbounded open subset of $\mathbf{R}^n$, $n \geq 2$, having a boundary $\partial\Omega$ of class C^2. Ω is called an *admissible* domain if and only if $\Omega \times \mathbf{R}$ satisfies assumptions *i)*–*iv)* of Section 3 with G, ∂G, n, $x \in G$, $\{x \in B_n(0, i) : x_n > 0\}$ being replaced, respectively, by $\Omega \times \mathbf{R}$, $\partial\Omega \times \mathbf{R}$, $n+1$, $(x, t) \in \Omega \times \mathbf{R}$, $\{(x, t) \in B_{n+1}(0, i) : t > 0\}$.

Definition 4.2. Let Ω be an admissible domain in the sense of Definition 4.1. We say that the pair $(A(x; D_x), m)$ is *admissible* if: *a)* $A(x; D_x)$ is the linear second-order differential operator in divergence form defined by (3.1) whose coefficients satisfy (3.2) and (3.3) with G being replaced by Ω; *b)* $m \in C_b^2(\overline{\Omega})$ is a non-negative function, which need not to be bounded away from 0, such that

$$|\nabla m(x)| \leq K_1 m(x), \quad \text{for every } x \in \overline{\Omega}, \tag{4.1}$$

K_1 being a positive constant.

Remark 4.3. We stress that (4.1) is the case of (3.6) with $\mu = 1$, i.e., just the case excluded in Theorem 3.3. The difference, here, is that Ω is unbounded and may have several branches extending to infinity, for instance, if $\Omega \cap (\mathbf{R}^n \backslash \overline{B_n(0, R)}) = \bigcup_{j=1}^k \Omega_j$ for some $R > 0$, where $k \in \mathbf{N}$, Ω_j is unbounded for every $j = 1, \ldots, k$, and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. In such a case, examples of functions m having zeros and satisfying (4.1) can be easily found, at least in the class of functions vanishing at infinity along some unbounded branch Ω_j of Ω. Indeed, all functions m of the form $m = \exp[v]$, with $v \in C^{0,1}(\overline{\Omega})$ and $\sup_{x \in \Omega} v(x) < +\infty$, satisfy (4.1) with $K_1 = \|\nabla v\|_{L^\infty(\Omega)}$ and, by virtue of the discussion before Remark 3.5, may have zeros only when $|x| \to +\infty$ along some unbounded branch Ω_j of Ω, but only if $\inf_{x \in \overline{\Omega}_j} v(x) = -\infty$.

Now, let Ω and the pair $(A(x; D_x), m)$ be admissible according to Definitions 4.1 and 4.2. Then, we can apply Theorem 3.3 to the spectral equation

$$[\lambda m(x) - A(x; D_x)]u(x) = f(x), \quad x \in \Omega, \quad f \in L^p(\Omega), \quad \lambda \in \Sigma\,, \tag{4.2}$$

where (cf. Theorem 3.3 for the definitions of σ and $k(p)$) we have set

$$\Sigma = \begin{cases} \{z \in \mathbf{C} : \operatorname{Re} z + \dfrac{\sigma}{2}|\operatorname{Im} z| + \dfrac{\nu_1}{2\|m\|_{C_b(\bar{Q})}} \geq 0\}, & \text{if } p \in (1,2), \\ \{z \in \mathbf{C} : \operatorname{Re} z + \dfrac{k(p)}{2}|\operatorname{Im} z| + \dfrac{\nu_1}{2\|m\|_{C_b(\bar{Q})}} \geq 0\}, & \text{if } p \in [2,+\infty). \end{cases} \tag{4.3}$$

Now, we increase the dimension from n to $n+1$ and with the unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of equation (4.2) we associate the complex-valued function

$$v(x,t) = \zeta(t)m(x)u(x)\mathrm{e}^{irt}, \quad (x,t) \in \Omega \times \mathbf{R}, \quad r \in \mathbf{R}_+, \tag{4.4}$$

where $\zeta \in C_0^\infty(\mathbf{R})$ is a real-valued function such that $\zeta(t) \equiv 1$ for $|t| < 1/2$ and $\zeta(t) \equiv 0$ for $|t| \geq 1$. Then we consider the linear second-order differential operator

$$L_\theta(x; D_x, D_t) = -A(x; D_x) - \mathrm{e}^{i\theta}[1 + m(x)]D_t^2, \tag{4.5}$$

$\theta \in [-\theta_0, \theta_0]$, $\theta_0 \in (\pi/2, \pi)$, related to the cylinder $\Omega \times \mathbf{R}$. Hence, denoting by $A^0(x,\xi)$, $(x,\xi) \in \Omega \times \mathbf{R}^n$, the sum $\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j$, from (3.3) and the fact that m is non-negative, we deduce the following inequality, where $\nu_4 = \min(\nu_2^2, 1)$ and $\xi' = (\xi, \xi_{n+1}) \in \mathbf{R}^{n+1}$:

$$\begin{aligned} &\left|A^0(x,\xi) + \mathrm{e}^{i\theta}[1 + m(x)]\xi_{n+1}^2\right|^2 \\ &= [A^0(x,\xi)]^2 + 2A^0(x,\xi)[1 + m(x)]\xi_{n+1}^2 \cos\theta + [1 + m(x)]^2\xi_{n+1}^4 \\ &\geq 2\cos^2(\theta_0/2)\{[A^0(x,\xi)]^2 + [1 + m(x)]^2\xi_{n+1}^4\} \\ &\geq 2\cos^2(\theta_0/2)\{\nu_2^2|\xi|^4 + \xi_{n+1}^4\} \\ &\geq \nu_4 \cos^2(\theta_0/2)|\xi'|^4, \quad \text{for every } (x,\xi',\theta) \in \overline{\Omega} \times \mathbf{R}^{n+1} \times [-\theta_0, \theta_0]. \end{aligned}$$

Therefore, the weak ellipticity condition (3.5) holds true for $L_\theta(x; D_x, D_t)$. Moreover, by virtue of Corollary 2.2 and Lemma 2.3, it is easy to show that $L_\theta(x; D_x, D_t)$ satisfies also the other remaining assumptions needed to apply Theorem 3.1 with G being replaced by $\Omega \times \mathbf{R}$. Hence, for $i,j = 1,\ldots,n+1$, it makes sense to introduce the numbers $\tau_{i,j}$, μ_i, μ_0, τ and K, which have the same meanings (in dimension $n+1$) as the numbers $\eta_{i,j}$, σ_i, σ_0 and H (cf. the notation before Theorem 3.1), but with $a_{n+1,j}$, $j \neq n+1$, $a_{n+1,n+1}$, b_{n+1} and G being replaced, respectively, by 0, $\mathrm{e}^{i\theta}[1 + m(x)]$, 0 and $\Omega \times \mathbf{R}$.

Theorem 4.4. *Let the unbounded domain* Ω *and the pair* $(A(x; D_x), m)$ *be admissible in the sense of* Definitions 4.1 *and* 4.2. *Then, for every* $p \in$

$(1,+\infty)$, *there exists a large enough* $\omega_p \geq 0$ *such that if* $\lambda \in \omega_p + \Sigma$, Σ *being defined by* (4.3), *the spectral equation* $\lambda mu - A(x;D_x)u = f$, $f \in L^p(\Omega)$, *admits a unique solution* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *satisfying the estimates*

$$\|mu\|_{L^p(\Omega)} \leq C_9|\lambda|^{-\beta}\|f\|_{L^p(\Omega)}\,, \tag{4.6}$$

$$\|m\nabla u\|_{L^p(\Omega)} \leq C_{10}|\lambda|^{-\beta+1/2}\|f\|_{L^p(\Omega)}\,, \tag{4.7}$$

$$\Big(\sum_{j,k=1}^{n}\|mD_{x_j}D_{x_k}u\|_{L^p(\Omega)}^p\Big)^{1/p} \leq C_{11}|\lambda|^{-\beta+1}\|f\|_{L^p(\Omega)}\,. \tag{4.8}$$

Here

$$\beta = \begin{cases} 1, & \text{if } p \in (1,2), \\ 2/p, & \text{if } p \in [2,+\infty)\,, \end{cases} \tag{4.9}$$

whereas C_j, $j = 9, 10, 11$, *denote three positive constants depending on* n, p, ν_2, τ, μ_i, $i = 0, 1, \ldots, n$, K, Ω, K_1, *and the norms* $\|m\|_{C_b^2(\overline{\Omega})}$, $\|a_{i,j}\|_{C_b^1(\overline{\Omega})}$ *and* $\|a_0\|_{C_b(\overline{\Omega})}$.

Remark 4.5. Estimate (4.6) extends estimates (3.8) and (3.9) to the limit case $\mu = 1$, but for unbounded open sets, only. We stress also that when $p \in [2,+\infty)$ the exponents for $|\lambda|$ in (4.7) and (4.8) are non-positive if and only if $p \in [2,4]$ and $p = 2$, respectively. Otherwise, the right-hand sides go to $+\infty$ when $|\lambda| \to +\infty$. Moreover, when $p = 2$ estimates (4.6)–(4.8) coincide with those proved in [10, Theorem 3.1.3] for $p \in (1,+\infty)$ in the non-degenerate case, i.e., when $m = 1$ in (1.1). In this sense our estimates are the sharpest possible.

5. Proof of Theorem 4.4

We begin the proof of Theorem 4.4 by proving two basic lemmas which provide an estimate from above for the norm $\|L_\theta(x;D_x,D_t)v\|_{L^p(\Omega\times\mathbf{R})}$ and an estimate from below for the norm $\|v\|_{W^{2,p}(\Omega\times\mathbf{R})}$, v and $L_\theta(x;D_x,D_t)$ being defined by (4.4) and (4.5), respectively. Then, combining these lemmas with the previous Theorems 3.1 and 3.3 we can prove Theorem 4.4. We stress that, by virtue of Remarks 3.4 and 4.3, we will use Theorem 3.3 with $\mu = 1$ in an unbounded domain Ω, admissible in the sense of Definition 4.1. Indeed, Remark 3.4 ensures that estimates (3.7)–(3.9) remain true in unbounded domains, whereas Remark 4.3 establishes that for such domains functions m having zeros at infinity and satisfying (3.6) with $\mu = 1$, i.e., (4.1), *do* exist.

Lemma 5.1. *Let the unbounded domain Ω and the pair $(A(x;D_x),m)$ be admissible in the sense of* Definitions 4.1 *and* 4.2, *and let $p\in(1,+\infty)$. Then, when v and $L_\theta(x;D_x,D_t)$ are defined by* (4.4) *and* (4.5)*, respectively, the following estimate holds for every $r\geq 1$ and $\theta\in[-\theta_0,\theta_0]$, $\theta_0\in(\pi/2,\pi)$:*

$$\|L_\theta(x;D_x,D_t)v\|_{L^p(\Omega\times\mathbf{R})}\leq c_2\big\{\|[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u\|_{L^p(\Omega)}+\|u\|_{L^p(\Omega)}$$
$$+\ \||\nabla m||\nabla u|\|_{L^p(\Omega)}+(r+r^2)\|mu\|_{L^p(\Omega)}\big\}. \tag{5.1}$$

Here c_2 denotes a positive constant depending only on p, ν_3, the norm $C_b^2(\overline{\Omega})$ of m and the norms $C_b^1(\overline{\Omega})$ of the $a_{i,j}$'s (cf. (5.15)*).*

Proof. First, using (3.1) and (4.4) it is not difficult to show that

$$\mathrm{e}^{i\theta}[1+m(x)]D_t^2v(x,t) \tag{5.2}$$
$$=\mathrm{e}^{i(\theta+rt)}[1+m(x)][\zeta''(t)+2ir\zeta'(t)-r^2\zeta(t)]m(x)u(x)\,,$$
$$A(x;D_x)v(x,t)=\zeta(t)\mathrm{e}^{irt}\Big\{u(x)A_0(x;D_x)m(x)+m(x)A(x;D_x)u(x)$$
$$+\ 2\sum_{i,j=1}^n a_{i,j}(x)D_{x_i}m(x)D_{x_j}u(x)\Big\}, \tag{5.3}$$

where $A_0(x;D_x)$ denotes the principal part of $A(x;D_x)$. Therefore, recalling (4.5) and rearranging the terms in (5.2) and (5.3) we get

$$L_\theta(x;D_x,D_t)v(x,t)=\zeta(t)\mathrm{e}^{irt}m(x)\big\{\big[r^2\mathrm{e}^{i\theta}m(x)-A(x;D_x)\big]u(x)\big\}$$
$$-\,\zeta(t)\mathrm{e}^{irt}u(x)A_0(x;D_x)m(x)-2\zeta(t)\mathrm{e}^{irt}\sum_{i,j=1}^n a_{i,j}(x)D_{x_i}m(x)D_{x_j}u(x)$$
$$-\,\mathrm{e}^{i(\theta+rt)}[\zeta''(t)+2ir\zeta'(t)-r^2\zeta(t)]m(x)u(x)$$
$$-\,\mathrm{e}^{i(\theta+rt)}[\zeta''(t)+2ir\zeta'(t)][m(x)]^2u(x)=:\sum_{k=1}^5 J_k(\zeta,m,u)(x,t)\,. \tag{5.4}$$

Now, since the Minkowski inequality implies

$$\|L_\theta(x;D_x,D_t)v\|_{L^p(\Omega\times\mathbf{R})}\leq\sum_{k=1}^5\|J_k(\zeta,m,u)\|_{L^p(\Omega\times\mathbf{R})}\,, \tag{5.5}$$

to complete the proof we only need to estimate each norm

$$\|J_k(\zeta,m,u)\|_{L^p(\Omega\times\mathbf{R})},\qquad k=1,\dots,5,$$

and to show that (5.1) holds true. First of all, from the definition of $J_1(\zeta, m, u)$ in (5.4), we immediately find

$$\|J_1(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} \le \|\zeta\|_{L^p(\mathbf{R})}\|m\|_{L^\infty(\Omega)}\|[r^2 e^{i\theta} m - A(x; D_x)]u\|_{L^p(\Omega)}\,. \tag{5.6}$$

Similarly, taking advantage of the assumptions $a_{i,j} \in C_b^1(\overline{\Omega})$, $i, j = 1, \dots, n$, and $m \in C_b^2(\overline{\Omega})$, we deduce that $A_0(x; D_x)m$ belongs to $L^\infty(\Omega)$ and

$$\begin{aligned}\|J_2(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} &\le \|A_0(x; D_x)m\|_{L^\infty(\Omega)}\|\zeta\|_{L^p(\mathbf{R})}\|u\|_{L^p(\Omega)} \\ &\le c(a_{i,j}, m)\|\zeta\|_{L^p(\mathbf{R})}\|u\|_{L^p(\Omega)}\,,\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}c(a_{i,j}, m) = &\Big(\sum_{i,j=1}^n \|D_{x_i} a_{i,j}\|^2_{C_b(\overline{\Omega})}\Big)^{1/2}\Big(\sum_{i=1}^n \|D_{x_i} m\|^2_{C_b(\overline{\Omega})}\Big)^{1/2} \\ &+ \Big(\sum_{i,j=1}^n \|a_{i,j}\|^2_{C_b(\overline{\Omega})}\Big)^{1/2}\Big(\sum_{i,j=1}^n \|D_{x_i} D_{x_j} m\|^2_{C_b(\overline{\Omega})}\Big)^{1/2}.\end{aligned} \tag{5.8}$$

Now, (3.3) and the Cauchy-Schwarz inequality imply

$$\Big|\sum_{i,j=1}^n a_{i,j}(x) D_{x_i} m(x) D_{x_j} u(x)\Big| \le \nu_3 |\nabla m(x)||\nabla u(x)|, \quad x \in \Omega. \tag{5.9}$$

Therefore, from (5.4) and (5.9) it follows that

$$\|J_3(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} \le 2\nu_3 \|\zeta\|_{L^p(\mathbf{R})}\||\nabla m||\nabla u|\|_{L^p(\Omega)}\,. \tag{5.10}$$

Concerning J_k, $k = 4, 5$, we first observe that

$$\begin{aligned}|\zeta''(t) + 2ir\zeta'(t) - r^2\zeta(t)| &\le |\zeta''(t)| + 2r|\zeta'(t)| + r^2|\zeta(t)|\,, \\ |\zeta''(t) + 2ir\zeta'(t)| &\le |\zeta''(t)| + 2r|\zeta'(t)|\,.\end{aligned}$$

As a consequence, from the definitions of J_4 and J_5 in (5.4) we easily deduce

$$\begin{aligned}\|J_4(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} &\le (\|\zeta''\|_{L^p(\mathbf{R})} + 2r\|\zeta'\|_{L^p(\mathbf{R})} + r^2\|\zeta\|_{L^p(\mathbf{R})})\|mu\|_{L^p(\Omega)} \\ &\le c_p(1+r)^2\|mu\|_{L^p(\Omega)}\,,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\|J_5(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} &\le (\|\zeta''\|_{L^p(\mathbf{R})} + 2r\|\zeta'\|_{L^p(\mathbf{R})})\|m\|_{L^\infty(\Omega)}\|mu\|_{L^p(\Omega)} \\ &\le c_p(1+2r)\|m\|_{L^\infty(\Omega)}\|mu\|_{L^p(\Omega)}\,,\end{aligned} \tag{5.12}$$

where $c_p = \max_{k=0,1,2} \|\zeta^{(k)}\|_{L^p(\mathbf{R})}$. Therefore, taking $r \geq 1$, (5.11) and (5.12) yield

$$\|J_4(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} \leq 4c_p r^2 \|mu\|_{L^p(\Omega)}\,, \tag{5.13}$$

$$\|J_5(\zeta, m, u)\|_{L^p(\Omega\times\mathbf{R})} \leq 3c_p r \|m\|_{L^\infty(\Omega)} \|mu\|_{L^p(\Omega)}\,. \tag{5.14}$$

Finally, setting

$$c_2 = c_p \max\{c(a_{i,j}, m), 2\nu_3, 4, 3\|m\|_{L^\infty(\Omega)}\}, \tag{5.15}$$

from (5.5)–(5.7), (5.10), (5.13) and (5.14) we derive (5.1), and the proof is complete. □

We now want to estimate from below the norm $\|\cdot\|_{W^{2,p}(\Omega\times\mathbf{R})}$ of v, the function v being defined by (4.4). For this purpose, we first recall that for every a, $b \in \mathbf{R}$ and $p \in (1, +\infty)$ the following inequality holds:

$$|a - b|^p \geq \varkappa_p(|a|^p + |b|^p) - |a||b|^{p-1}\,, \tag{5.16}$$

where

$$0 < \varkappa_p := 2^{-1}\Big[\Big(\frac{1}{p}\Big)^{p/(p-1)} + 1 - \Big(\frac{1}{p}\Big)^{1/(p-1)}\Big] < \frac{1}{2}, \tag{5.17}$$

for every $p \in (1, +\infty)$. Without giving a detailed proof of (5.16) we limit ourselves only to noticing that it can be proved by studying the minima of the function

$$h(s) := \frac{|s-1|^p + |s|}{|s|^p + 1}\,, \quad s \in \mathbf{R}\,,$$

and then choosing $s = a/b$ if $b \neq 0$. Notice that, replacing a and b respectively with $a - b$ and $c + d$, from (5.16) we obtain the further inequality

$$|a - b - c - d|^p \geq \varkappa_p(|a - b|^p + |c + d|^p) - |a - b||c + d|^{p-1}, \tag{5.18}$$

$a, b, c, d \in \mathbf{R}$. Now, having inequalities (5.16) and (5.18) at hand and applying to them the Young inequality $a_1 a_2 \leq (\varepsilon/q)a_1^q + [1/(q'\varepsilon^{q'/q})]a_2^{q'}$, $q \in (1, +\infty)$, $q' = q/(q-1)$, a_1, $a_2 \geq 0$, $\varepsilon > 0$, we deduce

$$|a - b|^p \;\geq\; \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]|a|^p + \Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]|b|^p, \tag{5.19}$$

$$|a - b - c - d|^p \;\geq\; \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]|a - b|^p + \Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]|c + d|^p. \tag{5.20}$$

In particular, if we choose $0 < \varepsilon < \min\{p\varkappa_p, (p'\varkappa_p)^{-p/p'}\}$, we get

$$\varkappa_p - \frac{\varepsilon}{p} > 0\,, \qquad \varkappa_p - \frac{1}{p'\varepsilon^{p'/p}} < 0\,, \tag{5.21}$$

so that, using (5.19) and inequality $|c+d|^p \le 2^{p-1}(|c|^p+|d|^p)$, from (5.20) we obtain

$$|a-b-c-d|^p \ \ge \ \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]^2 |a|^p + \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]|b|^p + 2^{p-1}\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big](|c|^p+|d|^p). \tag{5.22}$$

As we will see in a moment, (5.19) and (5.22) will be the keystone in the proof of the following Lemma 5.2.

Lemma 5.2. *Let Ω be an unbounded domain admissible in the sense of* Definition 4.1. *Let $p \in (1,+\infty)$ and let $0 < \varepsilon < \min\{p\varkappa_p, (p'\varkappa_p)^{-p/p'}\}$, where $\varkappa_p$ is defined by* (5.17) *and p' denotes the conjugate exponent of p. Then, when v is defined by* (4.4), *the following estimate holds for every $r \in \mathbf{R}_+$:*

$$\begin{aligned}\|v\|^p_{W^{2,p}(\Omega\times\mathbf{R})} \ge\ & (1+r^p+r^{2p})\|mu\|^p_{L^p(\Omega)} + \frac{(1+r^p)}{c_4(p,n)}\Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\|m\nabla u\|^p_{L^p(\Omega)}\\ &+ \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]^2 \sum_{j,k=1}^n \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}\\ &+ \Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]\Big\{\frac{(1+r^p)}{c_3(p,n)}\|u\nabla m\|^p_{L^p(\Omega)} + \frac{2^p}{[c_3(p,n)]^2}\||\nabla m||\nabla u|\|^p_{L^p(\Omega)}\\ &+ \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\sum_{j,k=1}^n \|uD_{x_j}D_{x_k}m\|^p_{L^p(\Omega)}\Big\}.\end{aligned} \tag{5.23}$$

Here $c_j(p,n)$, $j=3,4$, denote positive constants depending only on p and n (cf. (5.29)).

Proof. Recalling that the function ζ in the definition (4.4) of v has been chosen so that $\zeta(t) \equiv 1/2$ if $|t|<1$ and $\zeta(t)\equiv 0$ if $|t| \ge 1$, we have

$$\begin{aligned}&\|v\|^p_{W^{2,p}(\Omega\times\mathbf{R})} \ge \|v\|^p_{W^{2,p}(\Omega\times(-1/2,1/2))}\\ &= \int_{-\frac12}^{\frac12}\mathrm{d}t\int_\Omega |m(x)u(x)\mathrm{e}^{irt}|^p\,\mathrm{d}x + \int_{-\frac12}^{\frac12}\mathrm{d}t\int_\Omega \big|D_t\big(m(x)u(x)\mathrm{e}^{irt}\big)\big|^p\,\mathrm{d}x\\ &+ \int_{-\frac12}^{\frac12}\mathrm{d}t\int_\Omega\sum_{j=1}^n \big|D_{x_j}\big(m(x)u(x)\mathrm{e}^{irt}\big)\big|^p\,\mathrm{d}x\end{aligned} \tag{5.24}$$

$$+\int_{-\frac{1}{2}}^{1/2} \mathrm{d}t \int_\Omega \left|D_t^2\big(m(x)u(x)\mathrm{e}^{irt}\big)\right|^p \mathrm{d}x$$
$$+\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}t \int_\Omega \sum_{j=1}^n \left|D_t D_{x_j}\big(m(x)u(x)\mathrm{e}^{irt}\big)\right|^p \mathrm{d}x$$
$$+\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}t \int_\Omega \sum_{j,k=1}^n \left|D_{x_j} D_{x_k}\big(m(x)u(x)\mathrm{e}^{irt}\big)\right|^p \mathrm{d}x =: \sum_{k=1}^6 I_k\,.$$

Therefore, we are now interested in estimating from below each integral I_k, $k = 1, \ldots, 6$, on the right-hand side of (5.24). Since

$$I_1 = \|mu\|^p_{L^p(\Omega)}\,, \qquad I_2 = r^p\|mu\|^p_{L^p(\Omega)}\,, \qquad I_4 = r^{2p}\|mu\|^p_{L^p(\Omega)}\,, \tag{5.25}$$

we turn our attention to I_3, I_5 and I_6. We have

$$I_3 = \int_{-1/2}^{1/2} \mathrm{d}t \int_\Omega \sum_{j=1}^n |m(x)D_{x_j}u(x) + u(x)D_{x_j}m(x)|^p\, \mathrm{d}x\,. \tag{5.26}$$

Now, using (5.19) with $0 < \varepsilon < \min\{p\varkappa_p, (p'\varkappa_p)^{-p/p'}\}$, we get, for every $j = 1, \ldots, n$,

$$\begin{aligned} &|m(x)D_{x_j}u(x) + u(x)D_{x_j}m(x)|^p \\ \geq &\left[\varkappa_p - \frac{\varepsilon}{p}\right]|m(x)|^p|D_{x_j}u(x)|^p + \left[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\right]|D_{x_j}m(x)|^p|u(x)|^p\,. \end{aligned} \tag{5.27}$$

Of course, here, by virtue of our choice for ε we have that (5.21) is satisfied. Before we proceed any further, we recall that for every $g \in W^{1,p}(\Omega)$, $p \in (1, +\infty)$, the following inequality holds for each $x \in \Omega$:

$$c_3(p,n)\sum_{j=1}^n |D_{x_j}g(x)|^p \leq |\nabla g(x)|^p \leq c_4(p,n)\sum_{j=1}^n |D_{x_j}g(x)|^p, \tag{5.28}$$

where

$$\begin{aligned} c_3(p,n) &= \begin{cases} n^{(p-2)/2}, & \text{if } p \in (1,2), \\ 1, & \text{if } p \in [2,+\infty), \end{cases} \\ c_4(p,n) &= \begin{cases} 1, & \text{if } p \in (1,2), \\ n^{(p-2)/2}, & \text{if } p \in [2,+\infty). \end{cases} \end{aligned} \tag{5.29}$$

Notice that (5.28) follows from the inequality

$$\sum_{j=1}^{n} a_j^q \le (\sum_{j=1}^{n} a_j)^q \le n^{q/q'} \sum_{j=1}^{n} a_j^q,$$

$a_j \ge 0$, $j = 1,\dots,n$, $q \in [1,+\infty)$. It suffices to take $a_j = |D_{x_j} g(x)|^2$ and $q = p/2$ when $p \in [2,+\infty)$ and, correspondingly, $a_j = |D_{x_j} g(x)|^p$ and $q = 2/p$ when $p \in (1,2)$. Hence, from (5.26)–(5.28) it follows that

$$I_3 \ge \frac{1}{c_4(p,n)}\Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\|m\nabla u\|^p_{L^p(\Omega)} + \frac{1}{c_3(p,n)}\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]\|u\nabla m\|^p_{L^p(\Omega)}. \tag{5.30}$$

Moreover, due to the fact that $I_5 = r^p I_3$ (cf. (5.24)), (5.30) provides

$$I_5 \ge \frac{r^p}{c_4(p,n)}\Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\|m\nabla u\|^p_{L^p(\Omega)} + \frac{r^p}{c_3(p,n)}\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]\|u\nabla m\|^p_{L^p(\Omega)}. \tag{5.31}$$

As far as I_6 is concerned, we first observe that, for every $j,k = 1,\dots,n$, we have

$$\begin{aligned}\big|D_{x_j}D_{x_k}\big(m(x)u(x)\mathrm{e}^{irt}\big)\big|^p &= \big|m(x)D_{x_j}D_{x_k}u(x) + u(x)D_{x_j}D_{x_k}m(x)\\ &\quad + [D_{x_j}m(x)][D_{x_k}u(x)] + [D_{x_k}m(x)][D_{x_j}u(x)]\big|^p.\end{aligned} \tag{5.32}$$

Now, recalling that our choice of ε implies (5.21), we are allowed to apply (5.22) to the right-hand side of (5.32). Therefore, for every j, $k = 1,\dots,n$, we get

$$\begin{aligned}\big|D_{x_j}D_{x_k}\big(m(x)u(x)\mathrm{e}^{irt}\big)\big|^p &\ge \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]^2 |m(x)|^p|D_{x_j}D_{x_k}u(x)|^p\\ &+ \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]|u(x)|^p|D_{x_j}D_{x_k}m(x)|^p\\ &+ 2^{p-1}\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]\big(|D_{x_j}m(x)|^p|D_{x_k}u(x)|^p + |D_{x_k}m(x)|^p|D_{x_j}u(x)|^p\big).\end{aligned} \tag{5.33}$$

Consequently, using (5.28) and (5.33), from the definition of I_6 in (5.24) we deduce

$$\begin{aligned}I_6 &\ge \Big[\varkappa_p - \frac{\varepsilon}{p}\Big]^2 \sum_{j,k=1}^{n} \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}\\ &\quad + \frac{2^p}{[c_3(p,n)]^2}\Big[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\Big]\||\nabla m||\nabla u|\|^p_{L^p(\Omega)}\end{aligned} \tag{5.34}$$

$$+\left[\varkappa_p - \frac{\varepsilon}{p}\right]\left[\varkappa_p - \frac{1}{p'\varepsilon^{p'/p}}\right]\sum_{j,k=1}^{n}\|uD_{x_j}D_{x_k}m\|^p_{L^p(\Omega)}.$$

Finally, rearranging (5.25), (5.30), (5.31) and (5.34), from (5.24) we derive (5.23), and the proof is complete. □

Now, combining Lemmas 5.1 and 5.2 with Theorems 3.1 and 3.3, we can finally prove Theorem 4.4.

Proof of Theorem 4.4. First, let the unbounded domain Ω and the pair $(A(x;D_x),m)$ be admissible according to Definitions 4.1 and 4.2. Then, Σ being defined by (4.3), the existence of a unique solution $u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ of the spectral equation $\lambda mu - A(x;D_x)u = f$, $f \in L^p(\Omega)$, $\lambda \in \Sigma$, is ensured by Theorem 3.3, taking Remarks 3.4 and 4.3 into account. Now, with the pair $(u, A(x;D_x))$ we associate the pair $(v, L_\theta(x;D_x,D_t))$, where v is the function defined by (4.4) with $r \in \mathbf{R}_+$ to be suitably chosen later on, and $L_\theta(x;D_x,D_t)$, $\theta \in [-\theta_0,\theta_0]$, $\theta_0 \in (\pi/2,\pi)$, is the differential operator defined by (4.5). As observed in Section 4, $L_\theta(x;D_x,D_t)$ enjoys all the necessary assumptions in order to apply Theorem 3.1 with G being replaced by $\Omega\times\mathbf{R}$. Therefore, taking $p_0 = p$ in (3.4) and using $(a+b)^p \le 2^{p-1}(a^p+b^p)$, a, $b \ge 0$, we have

$$\|v\|^p_{W^{2,p}(\Omega\times\mathbf{R})} \le 2^{p-1}c_1^p\left[\|L_\theta(x;D_x,D_t)v\|^p_{L^p(\Omega\times\mathbf{R})} + \|v\|^p_{L^p(\Omega\times\mathbf{R})}\right]. \quad (5.35)$$

Now, let us take $r \ge 1$. Then, since $\|v\|_{L^p(\Omega\times\mathbf{R})} = \|\zeta\|_{L^p(\mathbf{R})}\|mu\|_{L^p(\Omega)}$, using (5.1), (4.1) and the inequality

$$\Big(\sum_{i=1}^{5} a_i\Big)^p \le 5^{p-1}\sum_{i=1}^{5} a_i^p, \quad a_i \ge 0, \quad i = 1,\dots,5,$$

from (5.35) it follows that

$$\|v\|^p_{W^{2,p}(\Omega\times\mathbf{R})} \le c_5\big\{\|[r^2e^{i\theta}m - A(x;D_x)]u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + K_1^p\|m\nabla u\|^p_{L^p(\Omega)} + (1+r^p+r^{2p})\|mu\|^p_{L^p(\Omega)}\big\}, \quad (5.36)$$

where $c_5 = 10^{p-1}(c_1c_2)^p$. Hence, taking $r \ge r_0$, $r_0 = [(1+\sqrt{5})/2]^{1/p}$, so that $1 + r^p \le r^{2p}$, using (5.23) and again (4.1), from (5.36) we easily derive

$$r^{2p}\|mu\|^p_{L^p(\Omega)} + \frac{r^p}{c_4(p,n)}\big[\varkappa_p - \frac{\varepsilon}{p}\big]\|m\nabla u\|^p_{L^p(\Omega)} + \big[\varkappa_p - \frac{\varepsilon}{p}\big]^2\sum_{j,k=1}^{n}\|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}$$
$$\le c_5\big\{\|[r^2e^{i\theta}m - A(x;D_x)]u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + K_1^p\|m\nabla u\|^p_{L^p(\Omega)} + 2r^{2p}\|mu\|^p_{L^p(\Omega)}\big\}$$

$$+\frac{r^{2p}K_1^p}{c_3(p,n)}\Big[\frac{1}{p'\varepsilon^{p'/p}}-\varkappa_p\Big]\|mu\|^p_{L^p(\Omega)}+\frac{2^pK_1^p}{[c_3(p,n)]^2}\Big[\frac{1}{p'\varepsilon^{p'/p}}-\varkappa_p\Big]\|m\nabla u\|^p_{L^p(\Omega)}$$
$$+\Big[\frac{1}{p'\varepsilon^{p'/p}}-\varkappa_p\Big]\Big[\varkappa_p-\frac{\varepsilon}{p}\Big]\sum_{j,k=1}^n\|uD_{x_j}D_{x_k}m\|^p_{L^p(\Omega)}\,, \tag{5.37}$$

where all the constants on both sides are positive by virtue of (5.17) and the choice $0<\varepsilon<\min\{p\varkappa_p,(p'\varkappa_p)^{-p/p'}\}$. Now, recalling that $m\in C_b^2(\overline{\Omega})$, we set

$$c_6=\frac{1}{c_4(p,n)}\big[\varkappa_p-\frac{\varepsilon}{p}\big],\quad c_7=\big[\varkappa_p-\frac{\varepsilon}{p}\big]^2,$$
$$c_8=K_1^p\{c_5+\frac{2^p}{[c_3(p,n)]^2}\big[\frac{1}{p'\varepsilon^{p'/p}}-\varkappa_p\big]\},$$
$$c_9=c_5+n^2\big[\frac{1}{p'\varepsilon^{p'/p}}-\varkappa_p\big]\big[\varkappa_p-\frac{\varepsilon}{p}\big]\|m\|^p_{C_b^2(\overline{\Omega})},$$
$$c_{10}=2c_5+\frac{K_1^p}{c_3(p,n)}\big[\frac{1}{p'\varepsilon_1^{p'/p}}-\varkappa_p\big].$$

Then, from (5.37) we deduce

$$\begin{aligned}&r^{2p}\|mu\|^p_{L^p(\Omega)}+c_6r^p\|m\nabla u\|^p_{L^p(\Omega)}+c_7\sum_{j,k=1}^n\|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}\\&\le c_5\|[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u\|^p_{L^p(\Omega)}+c_8\|m\nabla u\|^p_{L^p(\Omega)}\\&+c_9\|u\|^p_{L^p(\Omega)}+c_{10}r^{2p}\|mu\|^p_{L^p(\Omega)}.\end{aligned}\tag{5.38}$$

From now on, we set $\lambda=r^2\mathrm{e}^{i\theta}$, with $|\lambda|=r^2\ge r_0^2$ to be chosen large enough so that $\lambda\in\Sigma$ and estimates (3.7)–(3.9) hold true. As we said at the beginning of the section, due to Remarks 3.4 and 4.3 such estimates are valid also when $\mu=1$, i.e., in our case. In particular, since $f=[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u$ and $|\lambda|=r^2$, (3.7)–(3.9) can be rewritten in the following form:

$$\|u\|_{L^p(\Omega)}\le C_6(p)\|[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u\|_{L^p(\Omega)},\quad\text{for every }\ p\in(1,+\infty),\tag{5.39}$$

$$\|mu\|_{L^p(\Omega)}\le\begin{cases}C_7(p)r^{-4/p}\|[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u\|_{L^p(\Omega)}, & \text{if } p\in[2,+\infty),\\ C_8(p)r^{-2}\|[r^2\mathrm{e}^{i\theta}m-A(x;D_x)]u\|_{L^p(\Omega)}, & \text{if } p\in(1,2).\end{cases}\tag{5.40}$$

Due to (5.40) we now split the rest of the proof into two cases. First, let $p\in[2,+\infty)$. Then, taking r also greater than $\omega_p=(2c_8/c_6)^{1/p}$, so that

$c_6 - r^{-p}c_8 \geq c_6/2$, from (5.38), (5.39) and the first inequality in (5.40) we obtain

$$\begin{aligned} & r^{2p}\|mu\|^p_{L^p(\Omega)} + \frac{c_6}{2}r^p\|m\nabla u\|^p_{L^p(\Omega)} + c_7\sum_{j,k=1}^n \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)} \\ & \leq c_{11}r^{2p-4}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|^p_{L^p(\Omega)}\,, \end{aligned} \tag{5.41}$$

where $c_{11} = c_5 + c_9[C_6(p)]^p + c_{10}[C_7(p)]^p$. Finally, from (5.41) we derive

$$\|mu\|_{L^p(\Omega)} \leq c_{12}r^{-4/p}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.42}$$

$$\|m\nabla u\|_{L^p(\Omega)} \leq c_{13}r^{(p-4)/p}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.43}$$

$$\Big(\sum_{j,k=1}^n \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}\Big)^{1/p} \leq c_{14}r^{(2p-4)/p}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.44}$$

where $c_{12} = (c_{11})^{1/p}$, $c_{13} = (2c_{11}/c_6)^{1/p}$ and $c_{14} = (c_{11}/c_7)^{1/p}$. Recalling that $\lambda = r^2\mathrm{e}^{i\theta}$ from (5.42)–(5.44) we deduce (4.6)–(4.8) for the case $p \in [2,+\infty)$. Now, let $p \in (1,2)$. Then, taking again $r \geq \omega_p$, from (5.38), (5.39) and the second inequality in (5.40) we obtain

$$\begin{aligned} & r^{2p}\|mu\|^p_{L^p(\Omega)} + \frac{c_6}{2}r^p\|m\nabla u\|^p_{L^p(\Omega)} + c_7\sum_{j,k=1}^n \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)} \\ & \leq c_{15}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|^p_{L^p(\Omega)}\,, \end{aligned} \tag{5.45}$$

where c_{15} is defined as c_{11}, but with $C_7(p)$ being replaced by $C_8(p)$. Consequently, from (5.45) we get

$$\|mu\|_{L^p(\Omega)} \leq c_{16}r^{-2}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.46}$$

$$\|m\nabla u\|_{L^p(\Omega)} \leq c_{17}r^{-1}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.47}$$

$$\Big(\sum_{j,k=1}^n \|mD_{x_j}D_{x_k}u\|^p_{L^p(\Omega)}\Big)^{1/p} \leq c_{18}\|[r^2\mathrm{e}^{i\theta}m - A(x;D_x)]u\|_{L^p(\Omega)}\,, \tag{5.48}$$

where c_j, $j = 16, 17, 18$, are defined as c_{j-4}, but with c_{11} being replaced by c_{15}. Recalling $\lambda = r^2\mathrm{e}^{i\theta}$, (5.46)–(5.48) imply (4.6)–(4.8) for the case $p \in (1,2)$ and the proof is complete.

6. Solving a singular initial problem

Taking the spectral Theorem 4.3 into account, from Theorem 3.26 in [9] we can easily derive the following existence and uniqueness result. For this purpose we need to introduce the following family of interpolation spaces:

$$L^p_{\delta,A,M}(\Omega) = \left\{ g \in L^p(\Omega) : \sup_{t\geq 1} t^\delta \|A(tM-A)^{-1}g\|_{L^p(\Omega)} < +\infty \right\},$$

where $p \in (1,+\infty)$, $\delta \in (0,1)$, $Mu = mu$ and $A = A(x, D_x)$, the domain Ω and the pair $(A(x;D_x), m)$ being admissible according to Definitions 4.1 and 4.2. In particular, since $m \in C^2_b(\overline{\Omega})$, $g = mh$ belongs to $L^p_{\delta,A,M}(\Omega)$ if $h \in W^{2,p}(\Omega) \cap W^{1,p}(\Omega)$. Introduce now the family of linear operators $\{R(t)\}_{t>0} \subset \mathcal{L}(L^p(\Omega))$ defined by the Dunford integral

$$R(t) = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} M(\lambda M - A)^{-1} \, \mathrm{d}\lambda,$$

where Γ is a piecewise-smooth curve in Σ (cf. (4.3)) joining $\infty\, e^{-i\theta}$ and $\infty\, e^{i\theta}$, $\theta \in [-\theta_0, \theta_0]$, $\theta_0 \in (\pi/2, \pi)$. Of course, defining also $R(0) = 1$, $\{R(t)\}_{t\geq 0}$ is a semigroup on $L^p(\Omega)$, $p \in (1,+\infty)$. Moreover, according to Theorem 4.3 and the spectral equation $\lambda mu + Au = f$, we easily deduce that $R \in C^\infty(\mathbf{R}_+; \mathcal{L}(L^p(\Omega))$ and satisfies the following estimates:

$$\|R^{(k)}(t)\|_{\mathcal{L}(L^p(\Omega))} \leq \widetilde{c}_k t^{\beta-(k+1)}, \quad t \in \mathbf{R}_+, \quad k \in \mathbf{N} \cup \{0\}, \tag{6.1}$$

$$\|\nabla R(t)\|_{\mathcal{L}(L^p(\Omega))} \leq C_{12}(t^{\beta-1} + t^{\beta-3/2}), \quad t \in \mathbf{R}_+, \tag{6.2}$$

where β is defined by (4.9). Notice that $\nabla R \in C((0,+\infty); \mathcal{L}(L^p(\Omega))$ for every $p \in (1,+\infty)$ and that (6.2) is a consequence of the following inequalities (cf. (4.1), (4.6) and (4.7)):

$$\begin{aligned}\|\nabla(mu)\|_{L^p(\Omega)} &\leq \|u\nabla m\|_{L^p(\Omega)} + \|m\nabla u\|_{L^p(\Omega)} \leq K_1\|mu\|_{L^p(\Omega)} + \|m\nabla u\|_{L^p(\Omega)} \\ &\leq (K_1 C_9 |\lambda|^{-\beta} + C_{10}|\lambda|^{-\beta+1/2})\|f\|_{L^p(\Omega)}.\end{aligned}$$

Theorem 6.1. *Let $p \in (1,+\infty)$ and let Ω and the pair $(A(x;D_x), m)$ be admissible according to* Definitions 4.1 *and* 4.2*. Further, let $u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, $f \in C^\gamma([0,T]; L^p(\Omega))$, $\gamma \in (1-\beta, 1)$, $T > 0$, and $-Au_0 + mf(0,\cdot) = g_0$, $g_0 \in L^p_{\delta,A,M}(\Omega)$. Then the initial problem*

$$\begin{cases} D_t(mu(t)) + Au(t) = f(t), & t \in (0,T], \\ mu(t) \to mu_0 \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad t \to 0+, \end{cases}$$

has a unique solution

$$u \in C^{\gamma+\beta-1}([0,T]; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)), \qquad mu \in C^{\gamma+\beta}([0,T]; L^p(\Omega)),$$

admitting the representations

$$mu(t) = R(t)mu_0 + \int_0^t R(t-s)f(s)\,ds, \qquad t \in (0,T].$$

$$Au(t) = R(t)Au_0 + A\int_0^t M^{-1}R(t-s)f(s)\,ds, \qquad t \in (0,T].$$

Moreover, if $mu_0 \in W^{1,p}(\Omega) \cap L^p_{\delta_0,A,M}(\Omega)$, $\delta_0 \in (0,1]$, *then* $\nabla(mu) \in C^\rho([0,T]; L^p(\Omega))$ *for any* $\rho \in (0, \min\{\delta_0, \beta - 1/2\})$.

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