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CRITICAL GROWTH BIHARMONIC ELLIPTIC PROBLEMS UNDER STEKLOV-TYPE BOUNDARY CONDITIONS

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Abstract. We study the fourth-order nonlinear critical problem $\Delta^2 u = u^{2^*-1}$ in a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 5$, subject to the boundary conditions $u = \Delta u - du_\nu = 0$ on $\partial\Omega$. We provide estimates for the range of parameters $d \in \mathbb{R}$ for which this problem admits a positive solution. If the domain is the unit ball, we obtain an almost complete description.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be a smooth, bounded domain, let $2^* = \frac{2n}{n-4}$ denote the critical Sobolev exponent, and let $d \in \mathbb{R}$. The present paper is concerned with the following fourth-order elliptic problem with purely critical growth and Steklov-type boundary conditions:

$$\begin{cases} \Delta^2 u = u^{2^*-1}, \quad u > 0 & \text{in } \Omega \\ u = 0, \quad \Delta u - du_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here u_ν denotes the outer normal derivative of u on $\partial\Omega$. It is already evident from the well-studied second-order case that nonlinear equations with critical growth terms present highly interesting phenomena concerning the

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existence/nonexistence of positive solutions; see the seminal paper by Brezis-Nirenberg [6] and also [25, Chapter III] for a survey. For fourth-order equations the existence/nonexistence problem is even more challenging, since the available techniques strongly depend on the imposed boundary conditions. The present paper is motivated by its applications to conformal geometry (see e.g. [18, Section 2.2]) and by the growing interest in recent years in the corresponding Dirichlet boundary value problem

$$\Delta^2 u = u^{2^*-1}, \quad u > 0 \quad \text{in } \Omega, \qquad u = u_\nu = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and Navier boundary value problem

$$\Delta^2 u = u^{2^*-1}, \quad u > 0 \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We point out that (1.3) corresponds to $d = 0$ in (1.1), whereas (1.2) should be seen as the limit case $d = -\infty$ in (1.1). We wish to show that the existence (respectively nonexistence) of solutions to (1.1) depends in a subtle way on the parameter d; thus, we highlight aspects of the equation $\Delta^2 u = u^{2^*-1}$ which cannot be observed by considering just (1.2) and (1.3).

Let us recall that if the domain Ω is strictly star-shaped, then neither (1.2) nor (1.3) admit solutions; see [19, 20, 27]. A first natural question then arises: do these nonexistence results really depend on the geometry of the domain? The answer is positive. For instance, (1.3) has a solution on every domain Ω with nontrivial topology; see [7]. This result is not available for (1.2), but it is shown in [2] that a solution exists on domains with small holes. Moreover, in [12] it is proved that both (1.2) and (1.3) have solutions in some contractible (non-star-shaped) domains. It should be mentioned that the existence results for (1.2) are for *nontrivial* solutions, not necessarily positive; this is due to the possible lack of the positivity preserving property for Δ^2 in certain domains.

A second natural question which arises is the following: do the above-mentioned nonexistence results depend on the particular nonlinearity (pure power) considered? Also for this question, the answer is positive since subcritical perturbations of the pure power term may lead to existence results: we refer to [8, 10, 14] for the Dirichlet case (1.2) and to [4, 13, 29] for the Navier case (1.3). At this point, a third natural question arises: do the nonexistence results also depend on the boundary conditions considered? As far as we are aware, this question has not been raised previously, and it is precisely one of the purposes of the present paper to give some answer to it. In other words, unlike in the present paper, in all the just-mentioned references modifications of the *domain* or of the *equation* were considered. In particular, modifications of the equation turned out to be quite sensitive to the space dimension, and this led to the study of the so-called critical dimensions [24]. According to [8, Theorem 1.1] and [24, Theorem 3] (respectively [29, Theorem 1] and [12, Theorem 3]) it is known that the only critical dimensions for the biharmonic operator Δ^2 under Dirichlet (respectively Navier) boundary conditions are $n = 5, 6, 7$. Here we study modifications of the *boundary conditions,* and we show that these have a quite a different effect that seems "almost independent" of the space dimension. More precisely, when $\Omega = B$ (the unit ball) for any $n \geq 5$ we find the threshold $d = 4$ for nonexistence results relative to (1.1). Moreover, a suitable modification of the Brezis-Nirenberg technique [6] (for the existence results) seems to show that the critical dimensions for the Steklov problem might be different, namely $n = 5, 6$.

2. Main results

To present our results concerning (1.1), we recall some facts about the boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = 0, \quad \Delta u - d u_\nu = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{2.1}$$

This problem was studied by Kuttler [17] and Payne [21] more than 30 years ago, whereas the recent paper [3] contains extensions and new results, relating in particular (2.1) to the positivity-preserving properties of Δ^2 under the Steklov-type boundary conditions. Let $\mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ endowed with the norm $\|\Delta u\|_2$ for all $u \in \mathcal{H}(\Omega)$. The smallest (positive) eigenvalue σ of (2.1) is characterized variationally as

$$\sigma := \inf_{u \in \mathcal{H}(\Omega)} \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} u_\nu^2} = \inf_{u \in \mathcal{H}(\Omega)} \frac{\|\Delta u\|_2^2}{\|u\|_{\partial_\nu}^2}. \tag{2.2}$$

Here and in the following, we denote by $\| \cdot \|_p$ the usual $L^p(\Omega)$-norm ($1 \leq p \leq \infty$), and we put

$$\|u\|_{\partial_\nu}^2 = \int_{\partial\Omega} u_\nu^2 \qquad \text{for } u \in H^2 \cap H^1_0(\Omega).$$

Hence, σ is the largest constant satisfying

$$\|\Delta u\|_2^2 \geq \sigma \|u\|_{\partial_\nu}^2 \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega) \tag{2.3}$$

and $\sigma^{-1/2}$ is the norm of the compact linear operator $H^2 \cap H^1_0(\Omega) \to L^2(\partial\Omega)$, $u \mapsto u_\nu$. It is known (see [17] for $n = 2$ and [3] for $n \geq 3$) that, up to a multiplicative constant, there exists a unique eigenfunction $\phi^1 \in \mathcal{H}(\Omega)$

corresponding to the eigenvalue σ, and $-\Delta\phi^1 \geq 0$ in Ω (so that $\phi^1 > 0$ in Ω and $\phi^1_\nu < 0$ on $\partial\Omega$).

We say that a function $u \in H^2 \cap H^1_0(\Omega)$ is a weak solution of (1.1) if $u > 0$ almost everywhere in Ω and

$$\int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} u_\nu v_\nu = \int_\Omega u^{2^*-1} v \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega)\ . \tag{2.4}$$

It can be shown that a weak solution in this sense is in fact a strong (classical) solution; see [3, Proposition 23] and also [28].

Our first result is concerned with *least-energy solutions* (or mountain-pass solutions according to the variational characterization of [1]) that minimize the functional

$$Q_d : H^2 \cap H^1_0(\Omega) \setminus \{0\} \to \mathbb{R}, \qquad Q_d(u) = \frac{\|\Delta u\|_2^2 - d\|u\|^2_{\partial_\nu}}{\|u\|^2_{2^*}}\ . \tag{2.5}$$

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be a smooth, bounded domain, and let σ be as in* (2.2). *Then*

(i) If $d \geq \sigma$, then (1.1) *admits no solution.*

(ii) There exists $\sigma^* < \sigma$ such that if $\sigma^* < d < \sigma$, then (1.1) *admits a least-energy solution u_d; these solutions satisfy*

$$u_d \to 0 \quad \text{in } H^2(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad \frac{u_d}{\|\Delta u_d\|_2} \to \phi^1 \quad \text{in } H^2(\Omega) \quad \text{as} \quad d \to \sigma\ , \tag{2.6}$$

where ϕ^1 is the first positive eigenfunction of (2.1) *such that $\|\Delta\phi^1\|_2 = 1$.*

In contrast to the cases $d = 0$ (Navier boundary conditions) and $d = -\infty$ (Dirichlet boundary conditions), the existence statement of Theorem 1 is independent of geometrical assumptions on the domain Ω. The proof of Theorem 1 (*ii*) is variational and relies on a compactness argument; see Proposition 13 below.

Next, we restrict our attention to the case where $\Omega = B$, the unit ball. In this case, it is known that $\sigma = n$; see [3]. Then, we prove

Theorem 2. *Assume that $\Omega = B$ (the unit ball) and that $d \leq 4$. Then* (1.1) *admits no solution.*

The proof of this result consists in two steps. First we use an identity for arbitrary $C^4(\overline{\Omega})$-functions noted by Mitidieri [19] to derive a Pohozaev-type identity (in the spirit of [22, 23]) for solutions of (1.1). More precisely, we obtain the following boundary integral equality for solutions u of (1.1) (see Section 6 below):

$$\int_{\partial\Omega} \Big[2\,(x\cdot\nabla\Delta u)\;-\;d^2\,(x\cdot\nu)\,u_\nu + nd\,u_\nu\Big]\,u_\nu \;=\;0\;. \tag{2.7}$$

In the particular case where $\Omega = B$, we have $x = \nu$ on ∂B, and (2.7) reduces to

$$\int_{\partial B} [2(\Delta u)_\nu + d(n-d)u_\nu]u_\nu = 0\;. \tag{2.8}$$

The last identity still seems unrelated to the statement of Theorem 2. However, choosing an auxiliary function in a careful way, we can use (2.8) and the Hopf boundary lemma to complete the proof. We point out that it is unclear how to use (2.7) on more general domains. In fact, even on a star-shaped domain nothing seems to be known about the term $x\cdot\nabla\Delta u$. Note however that in the case of Navier boundary conditions, i.e., $d = 0$, two terms in (2.7) disappear and $x\cdot\nabla\Delta u$ reduces to $(x\cdot\nu)(\Delta u)_\nu$ because Δu vanishes on $\partial\Omega$.

As a consequence of Theorem 2, we obtain the following Sobolev inequality with remainder term:

Corollary 3. *For all* $u \in H^2\cap H^1_0(B)$ *we have*

$$\|\Delta u\|_2^2 \geq S\|u\|_{2^*}^2 + 4\|u\|_{\partial_\nu}^2, \tag{2.9}$$

where S *is the Sobolev constant for the embedding* $H^2\cap H^1_0(B)\subset L^{2^*}(B)$.

Let us recall that the constant S in (2.9) is independent of the domain; see [28]. It is clear that (2.9) has no analogue in the first-order space H^1_0. Inequality (2.9) should also be compared with Theorem 5 in [12].

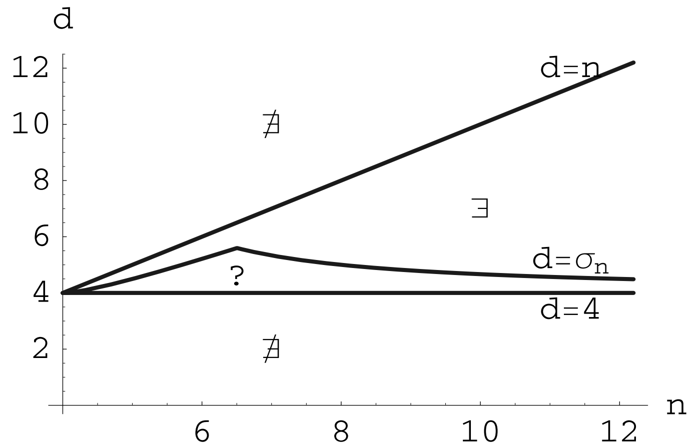
Next we discuss the range of parameters d for which we can ensure existence of solutions to (1.1) on $\Omega = B$. For $n \geq 5$ we define the number

$$\sigma_n = \begin{cases} n-(n-4)(n^2-4)\dfrac{\Gamma(\frac{n}{2})}{2^{\frac{8}{n}+1}}\left(\dfrac{n\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{4}{n}}\left(\dfrac{\Gamma(\frac{2n}{n-4})}{\Gamma(\frac{n^2}{2(n-4)})}\right)^{1-\frac{4}{n}} & \text{if } n=5 \text{ or } n=6\\ \frac{4(n-3)}{n-4} & \text{if } n\geq 7. \end{cases}$$

In particular, $\sigma_5 \approx 4.5$ and $\sigma_6 \approx 5.2$; see [30]. Then, we prove

Theorem 4. *Assume that* $\Omega = B$ *(the unit ball). Then for all* $d \in (\sigma_n, n)$ *problem* (1.1) *admits a radial solution. Moreover, the solution is superharmonic in* $\overline{B}$.

In the next picture, we represent the existence/nonexistence regions for (1.1) (when $\Omega = B$) according to Theorems 1, 2 and 4. Note that $\sigma_n \to 4$ as $n \to \infty$; namely, σ_n tends to become "optimal." The region with a question mark "?" represents the region $4 < d \le \sigma_n$, which is not covered by our results.



Conjecture 5. When $\Omega = B$, (1.1) admits a (radial) solution u_d if and only if $4 < d < n$. Moreover, as $d \to 4^+$, u_d tends to concentrate, namely $u_d(0) \to +\infty$ and $u_d(x) \to 0$ for $0 < |x| \le 1$.

In the next section we collect some further results providing evidence in favour of this conjecture. If it were true, we would have a lower bound for d *independent* of the dimension n: by scaling, in a ball of radius R the existence interval would then be $d \in (\frac{4}{R}, \frac{n}{R})$.

Remark 6. If $d \ge 0$, we may relax the requirement $u > 0$ in (1.1) with $u \ge 0$ and $u \not\equiv 0$. Indeed a solution u satisfies $u_\nu \le 0$ on $\partial\Omega$ so that, using the boundary condition, we also infer that $-\Delta u \ge 0$ on $\partial\Omega$. Since $-\Delta(-\Delta u) = \Delta^2 u = u^{2^*-1} \ge 0$ in Ω, by the maximum principle for $-\Delta$, we deduce $-\Delta u > 0$ in Ω. In turn, this implies $u > 0$ in Ω and

$$u_\nu < 0 \qquad \text{on } \partial\Omega\ . \tag{2.10}$$

The remainder of this paper is organized as follows. In Section 3 we collect some further results, some of them related to Conjecture 5 above. In particular, we provide some numerical evidence for this conjecture. We also state some open problems. In Section 4 we prove a compactness result, which is the crucial step in the proof of the existence statements in Theorems 1 and 4. Sections 4–7 contain the proofs of our main results. In Section 8 we prove Theorem 7 below.

3. Further results and open problems

3.1. **Three variational identities for radial solutions.** Throughout this section we assume that $\Omega = B$ (the unit ball) and we consider radially symmetric solutions. In this case, if we put $r = |x|$, then (1.1) reads

$$u^{iv}(r)+\frac{2(n-1)}{r}u'''(r)+\frac{(n-1)(n-3)}{r^2}u''(r)-\frac{(n-1)(n-3)}{r^3}u'(r)=u^{\frac{n+4}{n-4}}(r) \tag{3.1}$$

$r \in [0,1)$, while the boundary conditions become

$$u(1)=0\ , \qquad u''(1)+(n-1-d)u'(1)=0\ . \tag{3.2}$$

Moreover, every nontrivial solution satisfies $u'(1) < 0$ by the nonexistence result for (1.2). Hence the identity (2.8) yields the additional boundary condition

$$\begin{aligned} 0 &= (\Delta u)'(1) + \frac{d(n-d)}{2}u'(1) \\ &= u'''(1) + (n-1)u''(1) + \Big(\frac{d(n-d)}{2} - (n-1)\Big)u'(1). \end{aligned} \tag{3.3}$$

Using this and a change of variables introduced in [11], we will prove

Theorem 7. *Let $u_d = u_d(r)$ be a positive solution to (3.1)–(3.2) for some d. Then,*

$$\frac{(d-n)(d-4)}{2}u_d'(1) = \int_0^1 r^{n+1}u_d^{2^*-1}(r)\,dr\ ,$$

$$\frac{d(d-n)}{2}u_d'(1) = \int_0^1 r^{n-1}u_d^{2^*-1}(r)\,dr\ ,$$

$$d(d-4)(d-n)(d+n-4)|u_d'(1)|^2 = \frac{32(n+4)}{(n-4)^2}\int_0^1 r^{n+2}u_d^{2^*-3}(r)[u_d'(r)]^3dr$$

$$+\frac{48(n+4)}{n-4}\int_0^1 r^{n+1}u_d^{2^*-2}(r)|u_d'(r)|^2\,dr - 8(n^2-16)\int_0^1 r^{n-1}u_d^{2^*}(r)\,dr\ .$$

Moreover, if a solution u_d to problem (3.1)–(3.2) exists for all $4 < d < n$, then as $d \to 4^+$ we necessarily have that $u_d'(1)$ remains bounded and $u_d(r) \to 0$ for all $r > 0$.

Since $2^* - 3 > -1$ for any n and u vanishes of order 1 at $r = 1$, the third identity in Theorem 7 makes sense. Note that the first identity in Theorem 7 immediately yields a weaker version of Theorem 2, namely the nonexistence of positive *radial* solutions of (1.1) in B when $d \notin (4, n)$. The identities of Theorem 7 suggest that the correct existence interval is $d \in (4, n)$, and we feel that they could help to complete the proof of Conjecture 5 above.

3.2. **Numerical results.** Here we present some numerical experiments with Mathematica that give another strong hint that Conjecture 5 should hold. Let us briefly describe the procedure we followed.

First, we fixed $d \in (4, n)$, and we set the Cauchy problem for (3.1) in $r = 1$ by choosing $u_d(1) = 0$, $u_d'(1) < 0$ as a parameter and, according to (3.2) and (3.3), by taking

$$u_d''(1) = (d + 1 - n)u_d'(1) \ , \qquad u_d'''(1) = (n - d)\Big(n - 1 - \frac{d}{2}\Big)u_d'(1) \ .$$

Then, we asked Mathematica to plot the solution on the interval $(0, 1]$. We tried different values of $u_d'(1)$. If it was too negative, the solution remained positive and blew up to $+\infty$ before reaching (going backwards!) $r = 0$. If it was negative but too close to 0, the solution attained a maximum, changed sign and blew up to $-\infty$ before reaching $r = 0$. By dichotomy, we chose intermediate values of $u_d'(1)$ until we reached an equilibrium. We reached only one equilibrium for every d. This suggests

Problem 8. When $\Omega = B$, does there exist a *unique* radial solution to (1.1) for all $d \in (4, n)$?

For d very close to 4 and to n, the program was quite unstable and it was not so clear that uniqueness of the solution was ensured. For large values of n (below we consider the case $n = 12$) one should not completely trust the numerical results, both because very large numbers appear and because even if the "correct" shooting derivative $u'(1)$ was of the order of 10^3, small perturbations of order 10^{-6} gave rise to quite different results. In the tables below, we enclose what we obtained in our experiments.

Table 1. Numerical results in the case $n = 5$, $\frac{n+4}{n-4} = 9$.

d	4.999999	4.9999	4.99	4.8
$u_d'(1)$	-0.96	-1.71	-3.02	-3.85
$u_d(0)$	0.48	0.85	1.52	2.38

d	4.6	4.5	4.25	4.01	4.0001
$u'_d(1)$	-3.57	-3.34	-2.56	-0.97	-0.3
$u_d(0)$	2.85	3.11	4	10	32

Table 2. Numerical results in the case $n = 6$, $\frac{n+4}{n-4} = 5$.

d	5.99999	5.9999	5.999	5.9	5.75
$u'_d(1)$	-0.75	-1.34	-2.38	-7.15	-8.29
$u_d(0)$	0.35	0.67	1.2	3.9	5.2

d	5.5	5	4.5	4.1	4.01
$u'_d(1)$	-8.49	-7.01	-4.49	-1.72	-0.52
$u_d(0)$	6.8	11	18	46	145

Table 3. Numerical results in the case $n = 12$, $\frac{n+4}{n-4} = 2$.

d	11.9	11	10	9	8
$u'_d(1)$	-776.62	-5429.32	-6949.15	-6235.15	-4521.8
$u_d(0)$	425	6500	$2 \cdot 10^4$	$5.2 \cdot 10^4$	$1.3 \cdot 10^5$

d	7	6	5	4.1
$u'_d(1)$	-2657.56	-1159.79	-268.86	-2.38
$u_d(0)$	$2.9 \cdot 10^5$	$7.5 \cdot 10^5$	$3.5 \cdot 10^6$	$2 \cdot 10^8$

In order to test the procedure, when $n = 5$ we also tried the values $d = 3$ and $d = 6$, which are out of the range $(4, n)$. In both cases, regardless of the choice of $u'_d(1)$, the solution blew up to $+\infty$ before reaching $r = 0$. The blow-up seemed monotonic; namely, for $|u'_d(1)|$ decreasing, the blow-up time was also decreasing.

Finally, note that our numerical results seem to show that $u'_d(1) \to 0$ and $u_d(0) \to +\infty$ as $d \to 4^+$.

3.3. **Nodal radially symmetric solutions in the ball.** We consider here radial sign-changing solutions of (1.1) when $\Omega = B$. More precisely, for $r \in [0, 1)$ we want to solve

$$u^{iv}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = |u(r)|^{\frac{8}{n-4}}u(r) \ , \tag{3.4}$$

with boundary conditions (3.2) and

$$u'(0) = u'''(0) = 0 \ . \tag{3.5}$$

This problem admits no solution, as stated in

Proposition 9. *For any* $d \in \mathbb{R}$*,* (3.4) *with boundary conditions* (3.2)–(3.5) *admits no sign-changing solution.*

Proof. This follows by the arguments developed in [15]; see also [11]. It is shown there that any solution of (3.4)–(3.5) (with $u(0) > 0$) that attains 0 in finite time, remains then negative and blows up to $-\infty$ in finite time. □

Let us also recall the following consequence of the comparison principle due to McKenna-Reichel [18], which applies to *any* solution of (3.4):

Proposition 10. *Any solution* u *of* (3.4)–(3.2)–(3.5) *with* $u(0) > 0$ *satisfies* $u'(r) < 0$*,* $\Delta u(r) < 0$*,* $(\Delta u)'(r) > 0$ *for all* $r \in (0, 1]$*.*

Proof. Let $\alpha = u(0) > 0$, and consider the equation $\Delta^2 v = v^{2^*-1}$ in $\mathbb{R}^n$. It is well-known [26] that it admits a unique positive entire radial solution v vanishing at infinity and satisfying $v(0) = \alpha$. Moreover, this solution satisfies $v'(r) < 0$ and $\Delta v(r) < 0$ for all $r > 0$. In view of the comparison principle in [18] we first deduce that $\Delta u(0) < \Delta v(0)$ and, subsequently, that $u'(r) < v'(r) < 0$ and $\Delta u(r) < \Delta v(r) < 0$ for all $r \in (0, 1]$. This proves the first two inequalities. The third inequality follows by integrating the equation $\left\{r^{n-1} \left[\Delta u(r)\right]'\right\}' = r^{n-1} u^{2^*-1}(r)$ over $[0, r]$ for $r \in (0, 1]$. □

3.4. **Low- and high-energy solutions in general domains.** In view of Theorem 1 and Proposition 14 below, we know that there exists an interval $I = I(\Omega) \subset (0, \sigma)$ such that (1.1) admits a *least-energy* solution if and only if $d \in I(\Omega)$. The results in [28] show that there exists no least-energy solution of (1.1) when $d = 0$. Hence $d_0(\Omega) := \inf I(\Omega) \geq 0$. Note also that $d_0(B) \geq 4$ by Theorem 2. This leads to the following question:

Problem 11. Which geometrical properties of Ω determine the value of $d_0(\Omega)$?

As already mentioned, when $d = 0$ (Navier boundary conditions) the existence of *higher-energy* positive solutions of (1.1) depends on the geometry of Ω. This suggests the following question:

Problem 12. For which domains Ω and parameters d does (1.1) have a higher-energy positive solution? Is it possible to choose $d > d_0(\Omega)$ and to get multiple positive solutions?

4. A compactness result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ be a smooth bounded domain. We first recall the characterization of the Sobolev constant for the embedding $H^2 \cap H^1_0(\Omega) \subset L^{2^*}(\Omega)$:

$$S \;=\; \inf_u \frac{\|\Delta u\|_2^2}{\|u\|_{2^*}^2} \;,$$

where the infimum is taken over all $u \in H^2 \cap H^1_0(\Omega) \setminus \{0\}$. It was shown in [28] that S is never achieved if $\Omega \neq \mathbb{R}^n$ and that S does not depend on the domain. Moreover, we have

$$S = \pi^2(n-4)(n^2-4)n\Big(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\Big)^{4/n} \; ; \tag{4.1}$$

see also [26]. Consider now the following minimization problem, where Q_d is defined in (2.5):

$$\Sigma_d(\Omega) \;:=\; \inf_{u \in H^2 \cap H^1_0(\Omega) \setminus \{0\}} \; Q_d(u) \; . \tag{4.2}$$

The purpose of this section is to prove the following:

Proposition 13. *Assume that* $0 < d < \sigma$. *Then if* $\Sigma_d(\Omega) < S$, *the infimum in* (4.2) *is achieved. Moreover, up to a change of sign, any minimizer of* (4.2) *is strictly superharmonic in* $\overline{\Omega}$. *Finally, up to a Lagrange multiplier, any minimizer is a positive solution of* (1.1).

Proof. Let $\{u_m\}_{m \geq 0}$ be a minimizing sequence for $\Sigma_d(\Omega)$ such that

$$\|u_m\|_{2^*}^2 = 1. \tag{4.3}$$

Then,

$$\|\Delta u_m\|_2^2 - d\|u_m\|_{\partial_\nu}^2 = \Sigma_d(\Omega) + o(1) \qquad (m \to +\infty). \tag{4.4}$$

Moreover, recalling (2.3) we have

$$\|\Delta u_m\|_2^2 = \Sigma_d(\Omega) + d\|u_m\|_{\partial_\nu}^2 + o(1) \leq \Sigma_d(\Omega) + \frac{d}{\sigma}\|\Delta u_m\|_2^2 + o(1)$$

so that $\{u_m\}_{m \geq 0}$ is bounded in $H^2 \cap H^1_0(\Omega)$. Hence, $\{\nabla u_m\}_{m \geq 0}$ is bounded in $H^1(\Omega)$. Exploiting the compactness of the embeddings $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ and $H^2 \cap H^1_0(\Omega) \subset L^2(\Omega)$, we deduce that there exists $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\begin{gathered} u_m \rightharpoonup u \quad \text{in } H^2 \cap H^1_0(\Omega), \qquad (u_m)_\nu \to u_\nu \quad \text{in } L^2(\partial\Omega), \\ u_m \to u \quad \text{in } L^2(\Omega), \end{gathered} \tag{4.5}$$

up to a subsequence. That is, if we set $v_m := u_m - u$, then

$$\begin{gathered} v_m \rightharpoonup 0 \quad \text{in } H^2 \cap H^1_0(\Omega), \qquad (v_m)_\nu \to 0 \quad \text{in } L^2(\partial\Omega), \\ v_m \to 0 \quad \text{in } L^2(\Omega). \end{gathered} \tag{4.6}$$

On the other hand, by (4.3) we infer that $\|\Delta u_m\|_2^2 \geq S$, so that from (4.4) we also obtain

$$d\|u_m\|_{\partial_\nu}^2 = \|\Delta u_m\|_2^2 - \Sigma_d(\Omega) + o(1) \geq S - \Sigma_d(\Omega) + o(1),$$

which remains bounded away from 0 since $\Sigma_d(\Omega) < S$. From this fact we deduce that $u \neq 0$.

In view of (4.5)–(4.6) we may rewrite (4.4) as

$$\|\Delta u\|_2^2 + \|\Delta v_m\|_2^2 - d\|u\|_{\partial_\nu}^2 = \Sigma_d(\Omega) + o(1). \tag{4.7}$$

Moreover, by (4.3) and the Brezis-Lieb lemma [5], we have

$$1 = \|u + v_m\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|v_m\|_{2^*}^{2^*} + o(1) \leq \|u\|_{2^*}^2 + \|v_m\|_{2^*}^2 + o(1)$$

$$\leq \|u\|_{2^*}^2 + \frac{1}{S}\|\Delta v_m\|_2^2 + o(1) \ ,$$

where we also used the fact that both $\|u\|_{2^*}$ and $\|v_m\|_{2^*}$ do not exceed 1. Since $\Sigma_d(\Omega) \geq 0$ for every $0 < d < \sigma$, this last inequality gives

$$\Sigma_d(\Omega) \leq \Sigma_d(\Omega)\|u\|_{2^*}^2 + \frac{\Sigma_d(\Omega)}{S}\|\Delta v_m\|_2^2 + o(1).$$

By combining this inequality with (4.7), we obtain

$$\|\Delta u\|_2^2 - d\|u\|_{\partial_\nu}^2 = \Sigma_d(\Omega) - \|\Delta v_m\|_2^2 + o(1)$$

$$\leq \Sigma_d(\Omega)\|u\|_{2^*}^2 + \Big(\frac{\Sigma_d(\Omega)}{S} - 1\Big)\|\Delta v_m\|_2^2 + o(1) \leq \Sigma_d(\Omega)\|u\|_{2^*}^2 + o(1),$$

which shows that $u \neq 0$ is a minimizer for (4.2). This proves the first part of Proposition 13.

Consider now a minimizer u for (4.2) and assume for contradiction that it is not superharmonic (nor subhamonic) in Ω. Then, define $w \in H^2 \cap H^1_0(\Omega)$ as the unique solution of

$$\begin{cases} -\Delta w = |\Delta u| & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle for superharmonic functions it follows that $w > 0$ in Ω and $w_\nu < 0$ on $\partial\Omega$.

Moreover, both $w \pm u$ are superharmonic (but not harmonic!) in Ω and vanish on $\partial\Omega$. This proves that

$$|u| < w \quad \text{in } \Omega\ , \qquad |u_\nu| < |w_\nu| \quad \text{on } \partial\Omega\ .$$

In turn, these inequalities (and $-\Delta w = |\Delta u|$) prove that $Q_d(w) < Q_d(u)$, which contradicts the assumption that u minimizes (4.2).

Therefore, any minimizer u for (4.2) is superharmonic (and positive) in Ω. By the standard Lagrange-multiplier method, it is readily seen that a multiple of u is a positive solution of (1.1). Since it is superharmonic in Ω it also satisfies $u_\nu < 0$ on $\partial\Omega$. Hence, since $-\Delta u = -du_\nu$ on $\partial\Omega$, we finally infer that $-\Delta u > 0$ in $\overline{\Omega}$. □

Remark 14. In view of [28] we have that $\Sigma_0(\Omega) = S$. Moreover, using the first eigenfunction ϕ^1 in (2.5)–(4.2), we also have $\Sigma_\sigma(\Omega) = 0$. As a consequence of Proposition 13, one can show that the map $d \mapsto \Sigma_d(\Omega)$ is continuous on $[0, \sigma]$ and is strictly decreasing in the range where $\Sigma_d(\Omega) < S$.

5. Proof of Theorem 1

Let ϕ^1 be a positive eigenfunction of (2.1). Taking $v = \phi^1$ as test function in (2.4) we obtain

$$\int_\Omega \Delta u \Delta \phi^1 - d\int_{\partial\Omega} u_\nu \phi^1_\nu = \int_\Omega u^{2^*-1}\phi^1\ . \tag{5.1}$$

Two integrations by parts yield

$$\begin{aligned}\int_\Omega \Delta u \Delta\phi^1 &= -\int_\Omega \nabla u \nabla \Delta \phi^1 + \int_{\partial\Omega} \Delta\phi^1 u_\nu \\ &= \int_\Omega u\Delta^2\phi^1 + \sigma\int_{\partial\Omega} u_\nu\phi^1_\nu = \sigma\int_{\partial\Omega} u_\nu \phi^1_\nu,\end{aligned} \tag{5.2}$$

where we took into account that ϕ^1 solves (2.1) and u satisfies the boundary conditions in (1.1). Plugging (5.2) into (5.1) yields

$$(\sigma - d)\int_{\partial\Omega} u_\nu\phi^1_\nu = \int_\Omega u^{2^*-1}\phi^1 > 0\ .$$

In view of (2.10), this shows that $d < \sigma$. We have so shown that if (1.1) admits a solution, then necessarily $d < \sigma$. This proves statement (i) of Theorem 1.

Consider again the first eigenfunction ϕ^1, and let

$$\sigma^* := \frac{\|\Delta\phi^1\|_2^2 - S\|\phi^1\|_{2^*}^2}{\|\phi^1\|_{\partial_\nu}^2}\ .$$

Then, $\sigma^* < \sigma$ and for all $d > \sigma^*$ we have

$$\Sigma_d(\Omega) \leq \frac{\|\Delta\phi^1\|_2^2 - d\|\phi^1\|_{\partial_\nu}^2}{\|\phi^1\|_{2^*}^2} < S \ .$$

The existence part of statement (ii) then follows from Proposition 13.

We now prove the first of (2.6). To this end, we remark that in view of the characterization of ϕ^1 in (4.2), we have

$$\Sigma_d = \Sigma_d(\Omega) \leq \frac{\|\Delta\phi^1\|_2^2 - d\|\phi^1\|_{\partial_\nu}^2}{\|\phi^1\|_{2^*}^2} = \frac{1-\frac{d}{\sigma}}{\|\phi^1\|_{2^*}^2} \to 0 \quad \text{as } d \to \sigma \ . \tag{5.3}$$

Since u_d is a least-energy solution of (1.1), we have

$$\frac{\|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2}{\|u_d\|_{2^*}^2} = \Sigma_d \ . \tag{5.4}$$

Moreover, by taking $v = u_d$ in (2.4), we have

$$\|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2 = \|u_d\|_{2^*}^{2^*} \ . \tag{5.5}$$

Identities (5.4)–(5.5) readily imply that $\|u_d\|_{2^*} = \Sigma_d^{(n-4)/8}$. In turn, this and (5.3) show that

$$u_d \to 0 \quad \text{in } L^{2^*}(\Omega) \qquad \text{as } d \to \sigma \ . \tag{5.6}$$

We endow $H^2 \cap H_0^1(\Omega)$ with the scalar product

$$(v,w)_d := \int_\Omega \Delta v \Delta w - d\int_{\partial\Omega} v_\nu w_\nu \ . \tag{5.7}$$

We write u_d according to the decomposition $H^2 \cap H_0^1(\Omega) = [\text{span}\{\phi^1\}] \oplus [\text{span}\{\phi^1\}]^\perp$, where orthogonality is intended with respect to the scalar product in (5.7). In this way, for all $d \in (\sigma^*, \sigma)$ we obtain $\alpha_d \in \mathbb{R}$ and $\psi_d \in [\text{span}\{\phi^1\}]^\perp$ such that

$$u_d = \alpha_d\phi^1 + \psi_d \ . \tag{5.8}$$

Using (5.5) and (5.6), we infer (as $d \to \sigma$)

$$o(1) \geq (u_d, u_d)_d = \alpha_d^2(\phi^1,\phi^1)_d + (\psi_d,\psi_d)_d \geq \alpha_d^2\frac{\sigma-d}{\sigma} + \frac{\sigma_2-d}{\sigma_2}\|\Delta\psi_d\|_2^2 \ , \tag{5.9}$$

where $\sigma_2 > \sigma$ denotes the second Steklov eigenvalue (see [9]). The above inequality implies at once that $\|\Delta\psi_d\|_2 \to 0$ so that also $\|\psi_d\|_{2^*} \to 0$. Together with (5.6), this implies that $\alpha_d \to 0$ and finally that

$$u_d \to 0 \quad \text{in } H^2(\Omega) \qquad \text{as } d \to \sigma \ . \tag{5.10}$$

The $L^p(\Omega)$ convergence for any $p < \infty$ follows now by the same argument used in [28, Lemma B1]. Moreover, we obtain uniform convergence with the same argument as [28, Lemma B3] and recalling that the boundary conditions in (1.1) satisfy the complementing condition; see [3]. This proves the first part of (2.6).

In order to prove the second part of (2.6), we note that by (5.3), (5.4) and (5.9), we infer

$$o(1) = \frac{\|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2}{\|u_d\|_{2^*}^2} \geq \frac{\sigma_2 - d}{2\sigma_2} \frac{\|\Delta\psi_d\|_2^2}{\alpha_d^2\|\phi^1\|_{2^*}^2 + \|\psi_d\|_{2^*}^2} \ ,$$

where we also used the inequality

$$\|u_d\|_{2^*}^2 \leq (\|\alpha_d\phi^1\|_{2^*} + \|\psi_d\|_{2^*})^2 \leq 2(\alpha_d^2\|\phi^1\|_{2^*}^2 + \|\psi_d\|_{2^*}^2) \ .$$

Therefore, we obtain

$$o(1) \geq \frac{\sigma_2 - d}{2\sigma_2} \frac{\alpha_d^{-2}\|\Delta\psi_d\|_2^2}{\|\phi^1\|_{2^*}^2 + \alpha_d^{-2}\|\psi_d\|_{2^*}^2} \ . \tag{5.11}$$

For contradiction, if $\alpha_d^{-2}\|\psi_d\|_{2^*}^2 \to +\infty$, then we may neglect $\|\phi^1\|_{2^*}^2$ in the previous inequality so that we obtain

$$o(1) \geq \frac{\sigma_2 - d}{2\sigma_2} \frac{\|\Delta\psi_d\|_2^2}{\|\psi_d\|_{2^*}^2} \ ,$$

contradicting Sobolev's inequality. This contradiction shows that $\alpha_d^{-2}\|\psi_d\|_{2^*}^2$ remains bounded. Hence, (5.11) implies that $\alpha_d^{-2}\|\Delta\psi_d\|_2^2 \to 0$ as $d \to \sigma$. In particular, this means that $\alpha_d > 0$ (recall $u_d > 0$) and $\alpha_d^{-2}\|\Delta u_d\|_2^2 \to 1$ as $d \to \sigma$. Therefore, we finally obtain

$$\left\|\frac{\Delta u_d}{\|\Delta u_d\|_2} - \Delta\phi^1\right\|_2^2 = 2 - \frac{2}{\|\Delta u_d\|_2} \int_\Omega \Delta\phi^1(\alpha_d\Delta\phi^1 + \Delta\psi_d) \to 0,$$

which proves (2.6) and completes the proof of Theorem 1.

6. Proof of Theorem 2 and Corollary 3

First we show that, on an arbitrary smooth, bounded domain Ω, any solution u of (1.1) satisfies the boundary integral identity (2.7). Our starting point is the Rellich-type identity [19, (2.6)] of Mitidieri for arbitrary $C^4(\overline{\Omega})$-functions, which can be written as

$$\int_\Omega (\Delta^2 u)\, x\cdot\nabla u\, dx - \frac{n}{2}\int_\Omega (\Delta u)^2\, dx - (n-2)\int_\Omega \nabla\Delta u\cdot\nabla u\, dx$$

$$= -\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2 x\cdot\nu\, ds$$

$$+\int_{\partial\Omega}\Big((\Delta u)_\nu(x\cdot\nabla u)+u_\nu(x\cdot\nabla\Delta u)-\nabla\Delta u\cdot\nabla u\,(x\cdot\nu)\Big)\,ds. \tag{6.1}$$

For nonnegative u satisfying the first boundary condition $u=0$ on $\partial\Omega$, we have $\nu|\nabla u| = -\nabla u$ and therefore $\nabla\Delta u\cdot\nabla u\,(x\cdot\nu) = (\Delta u)_\nu(x\cdot\nabla u)$ on $\partial\Omega$, so that (6.1) reduces to

$$\int_\Omega(\Delta^2 u)\,x\cdot\nabla u - \frac{n}{2}\int_\Omega(\Delta u)^2 + (n-2)\int_\Omega(\Delta^2 u)u$$
$$= -\frac{1}{2}\int_{\partial\Omega}(\Delta u)^2 x\cdot\nu + \int_{\partial\Omega}u_\nu(x\cdot\nabla\Delta u). \tag{6.2}$$

Here we integrated by parts to get the third term in (6.2). Now, for solutions of (1.1), we can use (2.4) to rewrite the left-hand side of (6.2) as

$$\begin{aligned}
&\int_\Omega(\Delta^2 u)\,x\cdot\nabla u - \frac{n}{2}\int_\Omega(\Delta u)^2 + (n-2)\int_\Omega(\Delta^2 u)u\\
&\quad= \int_\Omega u^{2^*-1}(x\cdot\nabla u) - \frac{n}{2}\left(\int_\Omega u^{2^*} + d\int_{\partial\Omega}u_\nu^2\right) + (n-2)\int_\Omega u^{2^*}\\
&\quad= \frac{n-4}{2n}\int_\Omega x\cdot\nabla(u^{2^*}) + \frac{n-4}{2}\int_\Omega u^{2^*} - \frac{nd}{2}\int_{\partial\Omega}u_\nu^2\\
&\quad= \frac{n-4}{2n}\int_\Omega \operatorname{div}(u^{2^*}x) - \frac{nd}{2}\int_{\partial\Omega}u_\nu^2\\
&\quad= \frac{n-4}{2n}\int_{\partial\Omega}u^{2^*}x\cdot\nu - \frac{nd}{2}\int_{\partial\Omega}u_\nu^2 \;=\; -\frac{nd}{2}\int_{\partial\Omega}u_\nu^2.
\end{aligned}\tag{6.3}$$

Combining (6.2) and (6.3) and using that $\Delta u = du_\nu$ on $\partial\Omega$, we get (2.7). Next we consider $\Omega = B$, and we prove Theorem 2. Assume for contradiction that u is a solution of (1.1) for some $d \le 4$, and consider the auxiliary function $\phi\in C^2(\overline{B})$ defined by

$$\phi(x) = (4-d+d|x|^2)\Delta u(x) - 4dx\cdot\nabla u(x) + d(8-2n)u(x), \qquad x\in\overline{B}.$$

Then $\phi = 0$ on ∂B, since $u=0$ and $\Delta u = du_\nu$ on ∂B. A short computation shows

$$\begin{aligned}
\Delta\phi &= 2dn\Delta u + 4dx\cdot\nabla\Delta u + (4-d+d|x|^2)\Delta^2 u\\
&\quad - 4d(2\Delta u + x\cdot\nabla\Delta u) + d(8-2n)\Delta u\\
&= (4-d+d|x|^2)u^{2^*-1}.
\end{aligned}$$

If $u > 0$ solves (1.1), then $\Delta\phi > 0$, since $d \le 4$. By the maximum principle we conclude that $\phi < 0$ in B, and $\phi_\nu > 0$ on ∂B by the Hopf boundary lemma. But on ∂B we also get by direct computation (using also the second boundary condition)

$$\begin{aligned}\phi_\nu &= 2d\Delta u + 4(\Delta u)_\nu - 4d(u_\nu + u_{\nu\nu}) + d(8-2n)u_\nu \\ &= 2d\Delta u + 4(\Delta u)_\nu - 4d(u_\nu + \Delta u - (n-1)u_\nu) + d(8-2n)u_\nu \\ &= 2\Big(2(\Delta u)_\nu + d(n-d)u_\nu\Big),\end{aligned}$$

so that $2(\Delta u)_\nu + d(n-d)u_\nu > 0$ on ∂B. Since $u > 0$ in B we have $u_\nu \le 0$ on ∂B. Then, the last inequality combined with identity (2.8) yields $u_\nu = 0$ on ∂B. But then u would be a solution of the *Dirichlet* problem (1.2) in B, which is known to have no positive solutions [20]. This contradiction concludes the proof of Theorem 2. □

Corollary 3 is a direct consequence of the definition of Σ_d combined with Theorem 2 and the fact that, in view of Proposition 13, $\Sigma_4(B) = S$. □

7. Proof of Theorem 4

Throughout this section we denote by Σ_d the number $\Sigma_d(B)$ defined in (4.2). Moreover, we will use some properties of the Gamma and Beta functions, for which we refer to [30, 31]. Let us recall that

$$\omega_n := |\partial B| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \ . \tag{7.1}$$

In order to prove Theorem 4, we apply Proposition 13. More precisely, we show that if d lies in the range specified by Theorem 4, then $\Sigma_d < S$. And this is obtained by constructing a suitable radial function $u \in H^2 \cap H^1_0(B)$ for which $Q_d(u) < S$. To this end, we have to distinguish between "high" space dimensions $n \ge 7$ and "low" space dimensions $n = 5, 6$. In the first case we prove

Lemma 15. *Assume that $n \ge 7$. Then, $\Sigma_d < S$ for all $4\frac{n-3}{n-4} < d < n$.*

Proof. For all $\varepsilon > 0$ consider the entire function $u_\varepsilon(x) := \frac{1}{(\varepsilon^2+|x|^2)^{\frac{n-4}{2}}}$. It is known (see e.g. [26]) that

$$S = \frac{\int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2}{\left(\int_{\mathbb{R}^n} |u_\varepsilon|^{2^*}\right)^{2/2^*}} \qquad \text{for all } \varepsilon > 0 \ , \tag{7.2}$$

where S is as in (4.1). We now briefly recall some basic facts about u_ε. Firstly, we compute

$$\int_{\mathbb{R}^n} |u_\varepsilon|^{2^*} = \int_{\mathbb{R}^n} \frac{1}{(\varepsilon^2+|x|^2)^n} = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n}$$

$$= \frac{\omega_n}{\varepsilon^n}\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}\,dr = \frac{\omega_n}{2\varepsilon^n}\int_0^\infty \frac{t^{\frac{n}{2}-1}}{(1+t)^n}\,dt = \frac{\omega_n}{2\varepsilon^n}\frac{[\Gamma(\frac{n}{2})]^2}{\Gamma(n)} =: \frac{K_2}{\varepsilon^n}\ . \quad (7.3)$$

Moreover, by (7.2),

$$\int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 = S\ \Big(\int_{\mathbb{R}^n} |u_\varepsilon|^{2^*}\Big)^{2/2^*} = S\frac{K_2^{2/2^*}}{\varepsilon^{n-4}} =: \frac{K_1}{\varepsilon^{n-4}}. \quad (7.4)$$

In particular, (7.3)–(7.4) and (4.1) show that K_1 and K_2 are linked by the following relations:

$$K_1 = n(n-4)(n^2-4)K_2\ , \qquad K_1 = S\,K_2^{2/2^*}\ . \quad (7.5)$$

Consider now the function

$$U_\varepsilon(x) := u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} = \frac{1}{(\varepsilon^2+|x|^2)^{\frac{n-4}{2}}} - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\ .$$

Since $U_\varepsilon \in H^2\cap H^1_0(B)$, we may compute

$$Q_d(U_\varepsilon) = \frac{\int_B |\Delta u_\varepsilon|^2 - d\int_{\partial B} |(u_\varepsilon)_\nu|^2}{\Big(\int_B \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*}\Big)^{2/2^*}}.$$

We first remark that

$$\int_{\partial B} |(u_\varepsilon)_\nu|^2 = \frac{(n-4)^2}{(1+\varepsilon^2)^{n-2}}\,\omega_n \to (n-4)^2\omega_n \qquad \text{as } \varepsilon\to 0\ . \quad (7.6)$$

Next, we claim that as $\varepsilon\to 0$ the following two facts hold:

$$\int_B |\Delta u_\varepsilon|^2 = \frac{K_1}{\varepsilon^{n-4}} - 4(n-4)\,\omega_n + o(1), \quad (7.7)$$

$$\int_B \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*} = \frac{K_2}{\varepsilon^n} - \frac{4\omega_n}{(n-4)(n+2)\varepsilon^4} + o(\varepsilon^{-4}), \quad (7.8)$$

where K_2 and K_1 are defined in (7.3)–(7.4). Postponing their proofs, from (7.5)–(7.6)–(7.7)–(7.8) we get as $\varepsilon\to 0$

$$Q_d(U_\varepsilon) = \frac{\frac{K_1}{\varepsilon^{n-4}}\Big(1 - \frac{\varepsilon^{n-4}}{K_1}\omega_n(n-4)(4+d(n-4)) + o(\varepsilon^{n-4})\Big)}{\frac{K_2^{2/2^*}}{\varepsilon^{n-4}}\Big(1 - \frac{4\omega_n}{(n-4)(n+2)K_2}\varepsilon^{n-4} + o(\varepsilon^{n-4})\Big)^{2/2^*}}$$

$$= S\Big(1 - \omega_n(n-4)(4+d(n-4))\frac{\varepsilon^{n-4}}{K_1} + o(\varepsilon^{n-4})\Big)$$

$$\times\Big(1 + \frac{4\omega_n\varepsilon^{n-4}}{n(n+2)K_2} + o(\varepsilon^{n-4})\Big)$$

$$= S\Big[1 - \varepsilon^{n-4}\omega_n\Big(\frac{(n-4)(4+d(n-4))}{K_1} - \frac{4}{n(n+2)K_2}\Big) + o(\varepsilon^{n-4})\Big].$$

Hence, if

$$\frac{(n-4)(4+d(n-4))}{K_1} - \frac{4}{n(n+2)K_2} > 0 \tag{7.9}$$

then $Q_d(U_\varepsilon) < S$ for sufficiently small ε, and the statement follows. But (7.9) also reads

$$d > \frac{4}{n-4}\Big(\frac{K_1}{n(n+2)(n-4)K_2} - 1\Big) = 4\frac{n-3}{n-4}\ ,$$

where the last (striking!) equality follows from (7.5). So we have proved that if (7.7) and (7.8) hold, then $Q_d(U_\varepsilon) < S$ for sufficiently small ε provided that $d > 4\frac{n-3}{n-4}$.

Hence, the proof of the lemma will be complete once we demonstrate the estimates (7.7) and (7.8). By (7.4) we have

$$\int_B |\Delta u_\varepsilon|^2 = \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 - \int_{\mathbb{R}^n\setminus B} |\Delta u_\varepsilon|^2$$

$$= \frac{K_1}{\varepsilon^{n-4}} - (n-4)^2\int_{\mathbb{R}^n\setminus B}\frac{(n\varepsilon^2+2|x|^2)^2}{(\varepsilon^2+|x|^2)^n} = \frac{K_1}{\varepsilon^{n-4}} - 4(n-4)\,\omega_n + o(1),$$

which is (7.7). More delicate is the proof of (7.8). By (7.3) we have

$$\int_B \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*} = \int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*}$$

$$- \int_{\mathbb{R}^n\setminus B} |u_\varepsilon(x)|^{2^*} - \int_B \Big(|u_\varepsilon(x)|^{2^*} - \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*}\Big)$$

$$= \frac{K_2}{\varepsilon^n} - \frac{\omega_n}{n} + o(1) - \int_B \Big(|u_\varepsilon(x)|^{2^*} - \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*}\Big). \tag{7.10}$$

We now decompose the last term in the sum in (7.10) as follows:

$$\int_B \Big(|u_\varepsilon(x)|^{2^*} - \Big|u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}}\Big|^{2^*}\Big)$$

$$= \int_{B_{\varepsilon^{1/n}}} \Big(|u_\varepsilon(x)|^{2^*} - \Big| u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} \Big|^{2^*} \Big)$$

$$+ \int_{B\setminus B_{\varepsilon^{1/n}}} \Big(|u_\varepsilon(x)|^{2^*} - \Big| u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} \Big|^{2^*} \Big). \tag{7.11}$$

We study separately the two terms above. For the first term, we have

$$\int_{B_{\varepsilon^{1/n}}} \Big(|u_\varepsilon(x)|^{2^*} - \Big| u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} \Big|^{2^*} \Big)$$

$$= \int_{B_{\varepsilon^{1/n}}} \frac{1}{(\varepsilon^2+|x|^2)^n} \Big[1 - \Big(1 - \Big(\frac{\varepsilon^2+|x|^2}{\varepsilon^2+1} \Big)^{\frac{n-4}{2}} \Big)^{2^*} \Big]$$

$$= \int_{B_{\varepsilon^{1/n}}} \frac{1}{(\varepsilon^2+|x|^2)^n} \Big[2^* \Big(\frac{\varepsilon^2+|x|^2}{\varepsilon^2+1} \Big)^{\frac{n-4}{2}} + o\Big(\Big(\frac{\varepsilon^2+|x|^2}{\varepsilon^2+1} \Big)^{\frac{n-4}{2}} \Big) \Big]$$

$$= 2^* \omega_n (1+o(1)) \int_0^{\varepsilon^{1/n}} \frac{r^{n-1}}{(\varepsilon^2+r^2)^{\frac{n+4}{2}}}\, dr = \frac{2^*\omega_n}{2\varepsilon^4}(1+o(1)) \int_0^\infty \frac{t^{\frac{n}{2}-1}}{(1+t)^{\frac{n}{2}+2}}\, dt$$

$$= \frac{2^*\omega_n}{2\varepsilon^4} \frac{\Gamma(\frac{n}{2})\Gamma(2)}{\Gamma(\frac{n}{2}+2)}(1+o(1)) = \frac{4\omega_n}{(n-4)(n+2)\varepsilon^4}(1+o(1)).$$

For the second term, we have

$$\int_{B\setminus B_{\varepsilon^{1/n}}} \Big(|u_\varepsilon(x)|^{2^*} - \Big| u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} \Big|^{2^*} \Big)$$

$$= \int_{B\setminus B_{\varepsilon^{1/n}}} \frac{1}{(\varepsilon^2+|x|^2)^n} \Big[1 - \Big(1 - \Big(\frac{\varepsilon^2+|x|^2}{\varepsilon^2+1} \Big)^{\frac{n-4}{2}} \Big)^{2^*} \Big]$$

$$\le \int_{B\setminus B_{\varepsilon^{1/n}}} \frac{1}{(\varepsilon^2+|x|^2)^n} \le \omega_n \int_{\varepsilon^{1/n}}^\infty \frac{dr}{r^{n+1}} = \frac{\omega_n}{n\varepsilon} + o(1) = o(\varepsilon^{-4}).$$

Inserting these two estimates into (7.11) yields

$$\int_B \Big(|u_\varepsilon(x)|^{2^*} - \Big| u_\varepsilon(x) - \frac{1}{(\varepsilon^2+1)^{\frac{n-4}{2}}} \Big|^{2^*} \Big) = \frac{4\omega_n}{(n-4)(n+2)\varepsilon^4} + o(\varepsilon^{-4}).$$

In turn, inserting this estimate into (7.10) proves (7.8). □

The lower bound $4\frac{n-3}{n-4}$ found in Lemma 15 is not smaller than n when $n = 5$ or $n = 6$. Therefore, Lemma 15 does not apply in these dimensions. But here we can prove

Lemma 16. *Assume that* $n = 5$ *or* $n = 6$. *Then,* $\Sigma_d < S$ *for all* $\sigma_n < d < n$, *where*

$$\sigma_n = n - (n-4)(n^2-4)\frac{\Gamma(\frac{n}{2})}{2^{\frac{8}{n}+1}}\Big(\frac{n\Gamma(\frac{n}{2})}{\Gamma(n)}\Big)^{\frac{4}{n}}\Big(\frac{\Gamma(\frac{2n}{n-4})}{\Gamma(\frac{n^2}{2(n-4)})}\Big)^{1-\frac{4}{n}} .$$

Therefore, $\sigma_5 \approx 4.5$ *and* $\sigma_6 \approx 5.2$.

Proof. It is shown in [3] that the first eigenfunction of (2.1) in B can be normalized to be $\phi^1(x) = 1 - |x|^2$. We have

$$\|\Delta\phi^1\|_2^2 = 4n\omega_n \ , \qquad \|\phi^1\|_{\partial_\nu}^2 = 4\omega_n \ ,$$

$$\|\phi^1\|_{2^*}^{2^*} = \omega_n \int_0^1 (1-r^2)^{\frac{2n}{n-4}} r^{n-1}\, dr$$

$$= \frac{\omega_n}{2}\int_0^1 (1-u)^{\frac{2n}{n-4}} u^{\frac{n}{2}-1}\, du = \frac{2\omega_n}{n}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{2n}{n-4})}{\Gamma(\frac{n^2}{2(n-4)})},$$

from which we conclude that

$$Q_d(\phi^1) = 2(n-d)\Big(2\omega_n\Big)^{\frac{4}{n}}\Big(\frac{n\Gamma(\frac{n^2}{2(n-4)})}{\Gamma(\frac{n}{2})\Gamma(\frac{2n}{n-4})}\Big)^{\frac{n-4}{n}} .$$

By combining (4.1) with (7.1) and the just-found value of $Q_d(\phi^1)$, we deduce that $Q_d(\phi^1) < S$ whenever $d > \sigma_n$. This completes the proof of the lemma. □

8. Proof of Theorem 7

With the change of variables

$$u(r) = r^{-\frac{n-4}{2}}\, v(\log r) \quad (0 < r \le 1), \quad v(t) = e^{\frac{n-4}{2}t}u(e^t) \quad (t \le 0), \tag{8.1}$$

equation (3.1) may be rewritten as

$$v^{iv}(t) - K_2 v''(t) + K_1 v(t) = v^{\frac{n+4}{n-4}}(t) \qquad t \in (-\infty, 0) \ , \tag{8.2}$$

where

$$K_1 \ = \ \Big(\frac{n(n-4)}{4}\Big)^2 \ , \qquad K_2 \ = \ \frac{n^2-4n+8}{2} > 0.$$

We now establish some properties of the solution of (8.2). Firstly, we derive an upper bound for v:

Lemma 17. *For all* $t \in (-\infty, 0]$ *we have* $v(t) \le \Big(\frac{(n-4)n^3}{16}\Big)^{(n-4)/8}$.

Proof. For every $t \in (-\infty, 0]$, we define the energy function

$$E(t) := \frac{1}{2^*} v^{2^*}(t) - \frac{K_1}{2} v^2(t) + \frac{K_2}{2} (v'(t))^2 + \frac{1}{2} (v''(t))^2 - v'(t) v'''(t). \quad (8.3)$$

By differentiating and using (8.2), we obtain

$$E'(t) = -[v^{iv}(t) - K_2 v''(t) + K_1 v(t) - v^{\frac{n+4}{n-4}}(t)] v'(t) = 0 \qquad \forall t \in (-\infty, 0).$$

From this, observing that $E(t) \to 0$ as $t \to -\infty$, we conclude that

$$E(t) = 0 \qquad \forall t \in (-\infty, 0).$$

Since $v(t)$ is positive on $(-\infty, 0)$ and vanishes both for $t = 0$ and as $t \to -\infty$, v admits a global maximum over $(-\infty, 0]$. Let $\bar{t}$ be the maximum point of v. Then, $v'(\bar{t}) = 0$ and

$$0 = E(\bar{t}) = \frac{1}{2^*} v^{2^*}(\bar{t}) - \frac{K_1}{2} v^2(\bar{t}) + \frac{1}{2} (v''(\bar{t}))^2 \geq \left[v^{2^*-2}(\bar{t}) - \frac{nK_1}{n-4} \right] \frac{v^2(\bar{t})}{2^*},$$

which proves the statement. □

Next, for $t = 0$, we write higher-order derivatives in terms of the first-order derivative:

Lemma 18. *We have*

$$v(0) = 0, \quad v''(0) = (d-2) v'(0), \quad v'''(0) = \frac{n^2 - 4n + 2d^2 - 8d + 16}{4} v'(0), \quad (8.4)$$

$$v^{iv}(0) = \frac{(n^2 - 4n + 8)(d-2)}{2} v'(0), \quad v^v(0) = A(n, d) v'(0), \quad (8.5)$$

where $16A(n, d) = n^2(n-4)^2 + 16(3n^2 - 12n + 16) + 4d(d-4)(n^2 - 4n + 8) > 0$ *since* $n \geq 5$ *and* $d > 4$.

Proof. In view of the change of variables (8.1), we may "translate" the boundary conditions on u in terms of the boundary conditions on v. To this end, we use the formulas in [11, Section 3] to obtain

$$u'(1) = v'(0) \ , \quad u''(1) = v''(0) - (n-3) v'(0) \ ,$$

$$u'''(1) = v'''(0) - \frac{3}{2}(n-2) v''(0) + \frac{3n^2 - 12n + 8}{4} v'(0).$$

Hence, v satisfies $v(0) = 0$ and $v''(0) + (2-d) v'(0) = 0$. Moreover, (3.3) becomes first $2u'''(1) + 2(n-1) u''(1) + (2(1-n) + d(n-d)) u'(1) = 0$ and subsequently $4v'''(0) + 2(4-n) v''(0) + (2d(n-d) - n^2) v'(0) = 0$. This proves (8.4).

From equation (8.2) we infer $v^{iv}(0) = K_2 v''(0)$ so that (8.4) yields the first of (8.5). Moreover, by differentiating (8.2), we obtain $v^v(t) - K_2 v'''(t) + K_1 v'(t) = \frac{n+4}{n-4} v^{\frac{8}{n-4}}(t) v'(t)$ so that the so-far-proved relations show that also the second of (8.5) holds. □

In order to apply Lemma 18, we prove some identities concerning $v'(0)$:

Lemma 19. *The following identities hold:*

$$d(d-4)(d-n)(d+n-4)|v'(0)|^2 = \frac{32(n+4)}{(n-4)^2} \int_{-\infty}^0 v^{\frac{12-n}{n-4}}(t)[v'(t)]^3\,dt\ , \tag{8.6}$$

$$\frac{(d-n)(d-4)}{2} v'(0) = \int_{-\infty}^0 e^{nt/2} v^{\frac{n+4}{n-4}}(t)\,dt\ , \tag{8.7}$$

$$\frac{d(d-n)}{2} v'(0) = \int_{-\infty}^0 e^{(n-4)t/2} v^{\frac{n+4}{n-4}}(t)\,dt\ . \tag{8.8}$$

Proof. We multiply equation (8.2) by $v'''(t)$ and integrate over $(-\infty, 0)$ to obtain

$$\int_{-\infty}^0 v^{iv}(t)v'''(t)\,dt - K_2 \int_{-\infty}^0 v''(t)v'''(t)\,dt + K_1 \int_{-\infty}^0 v(t)v'''(t)\,dt$$
$$= \int_{-\infty}^0 v^{\frac{n+4}{n-4}}(t)v'''(t)\,dt\ .$$

By (8.1) we know that v and its derivatives vanish as $t \to -\infty$; therefore, with two integrations by parts the previous identity becomes

$$\frac{1}{2}|v'''(0)|^2 - \frac{K_2}{2}|v''(0)|^2 - \frac{K_1}{2}|v'(0)|^2 = -\frac{n+4}{n-4}\int_{-\infty}^0 v^{\frac{8}{n-4}}(t)v'(t)v''(t)\,dt\ . \tag{8.9}$$

Notice that a further integration by parts gives

$$\int_{-\infty}^0 v^{\frac{8}{n-4}}(t)v'(t)v''(t)\,dt$$
$$= -\frac{8}{n-4}\int_{-\infty}^0 v^{\frac{12-n}{n-4}}(t)[v'(t)]^3\,dt - \int_{-\infty}^0 v^{\frac{8}{n-4}}(t)v'(t)v''(t)\,dt\ ,$$

so that

$$\int_{-\infty}^0 v^{\frac{8}{n-4}}(t)v'(t)v''(t)\,dt = -\frac{4}{n-4}\int_{-\infty}^0 v^{\frac{12-n}{n-4}}(t)[v'(t)]^3\,dt\ .$$

By replacing this identity in (8.9) and by using (8.4) we obtain (8.6).

Next, we multiply (8.2) by $e^{nt/2}$. Then, we may rewrite the equation as

$$\frac{d}{dt}\Big[e^{nt/2}\Big(v'''(t)-\frac{n}{2}v''(t)-\frac{(n-4)^2}{4}v'(t)+\frac{n(n-4)^2}{8}v(t)\Big)\Big]$$
$$=e^{nt/2}v^{\frac{n+4}{n-4}}(t)\ . \qquad (8.10)$$

By integrating (8.10) over $(-\infty,0)$ and using (8.4) we obtain (8.7).

Finally, we multiply (8.2) by $e^{(n-4)t/2}$. Then, we may rewrite the equation as

$$\frac{d}{dt}\Big[e^{(n-4)t/2}\Big(v'''(t)-\frac{n-4}{2}v''(t)-\frac{n^2}{4}v'(t)+\frac{n^2(n-4)}{8}v(t)\Big)\Big]$$
$$=e^{(n-4)t/2}v^{\frac{n+4}{n-4}}(t)\ . \qquad (8.11)$$

By integrating (8.11) over $(-\infty,0)$ and using (8.4) we obtain (8.8). □

We now prove the following:

Lemma 20. *Assume that for all $4<d<n$ there exists a solution v_d to (8.2). Then, as $d\to 4^+$, $v_d'(0)$ remains bounded and $v_d(t)\to 0$ for all $t\le 0$.*

Proof. From Lemma 17 and (8.8) we obtain

$$\frac{d(n-d)}{2}|v_d'(0)|=\int_{-\infty}^0 e^{(n-4)t/2}v_d^{\frac{n+4}{n-4}}(t)\,dt\le C\int_{-\infty}^0 e^{(n-4)t/2}\,dt=C'\ .$$

This proves the first statement. Using this fact in (8.7) yields

$$\int_{-\infty}^0 e^{nt/2}v_d^{\frac{n+4}{n-4}}(t)\,dt\to 0 \qquad \text{as } d\to 4^+\ .$$

The proof of the lemma is thus complete. □

We may now complete the proof of Theorem 7. Using (8.1) it is not difficult to rewrite (8.7)–(8.8) as

$$\frac{(d-n)(d-4)}{2}u_d'(1)=\int_0^1 r^{n+1}u_d^{\frac{n+4}{n-4}}(r)\,dr\ ,$$
$$\frac{d(d-n)}{2}u_d'(1)=\int_0^1 r^{n-1}u_d^{\frac{n+4}{n-4}}(r)\,dr\ .$$

Moreover, (8.6) reads

$$d(d-4)(d-n)(d+n-4)|u_d'(1)|^2$$
$$=4(n^2-16)\int_0^1 r^{n-1}u_d^{2^*}(r)\,dr+24(n+4)\int_0^1 r^n u_d^{2^*-1}(r)u_d'(r)\,dr$$
$$+\frac{48(n+4)}{n-4}\int_0^1 r^{n+1}u_d^{2^*-2}(r)|u_d'(r)|^2\,dr$$

$$+\frac{32(n+4)}{(n-4)^2}\int_0^1 r^{n+2}u_d^{2^*-3}[u_d'(r)]^3\,dr\ .$$

The third identity then follows by integrating by parts the second integral.

Finally, Lemma 20 states that, as $d \to 4^+$, $u_d'(1)$ remains bounded and $u_d(r) \to 0$ for all $r > 0$.

Remark 21. Instead of performing the change of variables (8.1) and using the equation (8.2), one could also try to argue directly on equation (3.1). In this case, one should replace the energy functional (8.3) with the one suggested in [16]. More precisely, one can show that

$$2ru'(r)(\Delta u)'(r) + (n-4)u(r)(\Delta u)'(r) +$$

$$+nu'(r)\Delta u(r) - \frac{n-4}{n}ru^{2^*}(r) - r|\Delta u(r)|^2 \equiv 0 \qquad (r \in [0,1])\ .$$

However, with this alternative approach it seems much more difficult to obtain a result like Lemma 17, which is crucial in the proof of Lemma 20. Moreover, although it is not a fundamental identity, (8.6) seems out of reach without using (8.1).

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