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SADDLE-TYPE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider a class of semilinear elliptic equations of the form

$$-\Delta u(x,y) + W'(u(x,y)) = 0, \quad (x,y) \in \mathbb{R}^2 \tag{0.1}$$

where $W : \mathbb{R} \to \mathbb{R}$ is modeled on the classical two well Ginzburg-Landau potential $W(s) = (s^2-1)^2$. We show, via variational methods, that for any $j \geq 2$, the equation (0.1) has a solution $v_j \in C^2(\mathbb{R}^2)$ with $|v_j(x,y)| \leq 1$ for any $(x,y) \in \mathbb{R}^2$ satisfying the following symmetric and asymptotic conditions: setting $\tilde{v}_j(\rho,\theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$, there results

$$\tilde{v}_j(\rho, \tfrac{\pi}{2}+\theta) = -\tilde{v}_j(\rho, \tfrac{\pi}{2}-\theta) \text{ and } \tilde{v}_j(\rho, \theta+\tfrac{\pi}{j}) = -\tilde{v}_j(\rho,\theta),\ \forall (\rho,\theta) \in \mathbb{R}^+ \times \mathbb{R}$$
$$\text{and } \tilde{v}_j(\rho,\theta) \to 1 \text{ as } \rho \to +\infty \text{ for any } \theta \in [\tfrac{\pi}{2} - \tfrac{\pi}{2j}, \tfrac{\pi}{2}).$$

1. Introduction

We consider semilinear elliptic equations of the form

$$-\Delta u(x,y) + W'(u(x,y)) = 0 \tag{1.1}$$

for $(x,y) \in \mathbb{R}^2$, where we assume

(W) $W \in \mathcal{C}^2(\mathbb{R})$ satisfies $W(-s) = W(s)$, $W(s) \geq 0$ for any $s \in \mathbb{R}$, $W(s) > 0$ for any $s \in (-1,1)$, $W(\pm 1) = 0$ and $W''(\pm 1) > 0$.

Examples of potentials W satisfying (W) are the Ginzburg-Landau potential, $W(s) = (s^2-1)^2$, and the Sine-Gordon potential, $W(s) = 1 + \cos(\pi s)$, used to study various problems in phase transitions and condensed state physics. In these models, the global minima of W represent energetically favorite *pure phases* of the material and the solutions u of (1.1) pointwise describe the possible stationary states of the system.

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The problem of existence and multiplicity of entire bounded solutions of (1.1) has been widely investigated in the mathematical literature.

A long-standing problem concerning (1.1) (or its analog in greater dimension) is to characterize the set of the solutions $u \in C^2(\mathbb{R}^2)$ (or $u \in C^2(\mathbb{R}^n)$ with $n > 2$) of (1.1) satisfying $|u(x,y)| \le 1$, $\partial_x u(x,y) \ge 0$ and the asymptotic condition

$$\lim_{x\to\pm\infty} u(x,y) = \pm 1, \qquad y \in \mathbb{R} \text{ (or } y \in \mathbb{R}^{n-1}). \tag{1.2}$$

This problem was pointed out by Ennio De Giorgi in [14], where he conjectured that, at least when $n \le 8$ and $W(s) = (s^2-1)^2$, the whole set of solutions of (1.1)–(1.2) can be obtained by the action of the group of space roto-translations on the unique solution $q_+ \in C^2(\mathbb{R})$, of the one-dimensional problem

$$-\ddot{q}(x) + W'(q(x)) = 0, \qquad q(0) = 0 \text{ and } q(\pm\infty) = \pm 1. \tag{1.3}$$

The conjecture was first proved in the planar case by Ghoussoub and Gui in [16] also for a general (not necessarily even) potential W satisfying (W). We refer also to [9], [10] and [15], where a weaker version of the De Giorgi conjecture, known as the Gibbons conjecture, has been solved for all the dimensions n and in more general settings. The De Giorgi conjecture has been proved for a general potential W in dimension $n = 3$ in [8] (see also [2]), and for the Ginzburg-Landau potential in dimension $n \le 8$ in [20], papers to which we refer also for an extensive bibliography on the argument.

A different and related problem concerning equation (1.1) is the existence of *saddle solutions*, which was first studied by Dang, Fife and Peletier in [13]. In that paper the authors consider potentials W satisfying

- $(W1)$ $W \in \mathcal{C}^3([-1,1])$ satisfies $W(-s) = W(s)$, $W'(\pm 1) = W'(0) = 0$, $W''(\pm 1) > 0 > W''(0)$ and the function $W'(u)/u$ is strictly increasing on $(0,1)$.

They prove that if $(W1)$ is satisfied then (1.1) has a unique solution $u \in C^2(\mathbb{R}^2)$ such that

$$\begin{aligned} &u(x,-y) = -u(x,y) \text{ and } u(-x,y) = -u(x,y) \text{ on } \mathbb{R}^2, \\ &0 < u(x,y) < 1 \text{ if } x > 0 \text{ and } y > 0. \end{aligned} \tag{1.4}$$

By (1.4), the solution u has the same sign of the function xy and is called a saddle solution. The monotonicity of the function $W'(u)/u$ allows the authors to prove their result by the use a supersolution-subsolution method.

Moreover, we refer to a work by Schatzman [21] where the stability of the saddle solution is studied and to a recent paper by Cabré and Terra [12] where, in the case of the Ginzburg-Landau potential, the existence and stability of saddle solutions is studied in dimension greater than 2. A vectorial version of the result in [13] has been obtained by Alama, Bronsard and Gui in [1], where systems of equations of the type (1.1) have been studied.

We finally mention a work by Shi, [23], where the result in [13] has been generalized to the case in which $W \in \mathcal{C}^3([-1,1])$ is a more general potential satisfying

$(W2)$ $\exists\, \alpha \in (-1,1)$ such that $W(\pm 1) = W'(\pm 1) = W'(\alpha) = 0$, $W''(\pm 1) > 0 > W''(\alpha)$ and $W'(u)(u-\alpha) < 0$ if $u \in (-1,1) \setminus \{\alpha\}$.

In [23], using a bifurcation and a blow-up argument already developed in [24], Shi proves that for these potentials (1.1) has a unique saddle solution $u \in C^2(\mathbb{R}^2)$ satisfying

$$\begin{aligned} &u(x,y) = \alpha \text{ if } xy = 0,\ (u(x,y)-\alpha)xy > 0 \text{ if } xy \neq 0,\\ &|u(x,y)| < 1,\ u(x,y) = u(y,x) \text{ and } u(x,y) = u(-y,-x) \text{ on } \mathbb{R}^2. \end{aligned} \tag{1.5}$$

In both the papers [13] and [23] it is moreover showed that the saddle solution enjoys the following asymptotic property:

for any $m > 0$ there results $\lim_{x\to+\infty} u(x, mx) = 1$.

Gathering these results, specializing the one in [23] to the case of even potentials, we recognize that the saddle solution u satisfies the following symmetric and asymptotic conditions: setting $\tilde{u}(\rho,\theta) = u(\rho\cos(\theta), \rho\sin(\theta))$ there results

$(S2)$ $\tilde{u}(\rho, \frac{\pi}{2}+\theta) = -\tilde{u}(\rho, \frac{\pi}{2}-\theta)$ and $\tilde{u}(\rho, \theta+\frac{\pi}{2}) = -\tilde{u}(\rho,\theta)$ for any $(\rho,\theta) \in \mathbb{R}^+ \times \mathbb{R}$.
Moreover, $\tilde{u}(\rho,\theta) \to 1$ as $\rho \to +\infty$ for any $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$.

In other words, in the angle variable, the saddle solution is odd (with respect to $\frac{\pi}{2}$) and $\frac{\pi}{2}$ is an antiperiodic. Consequently, the half-lines $\theta = \frac{\pi}{2} + k\frac{\pi}{2}$ $(k = 0,\dots,3)$ are nodal lines for u. The asymptotic behavior of u between two contiguous nodal lines is characterized moreover by the fact that for $k = 0,\dots,3$ there results $\tilde{u}(\rho,\theta) \to (-1)^{k+1}$ as $\rho \to +\infty$ whenever $\theta \in (\frac{\pi}{2} + \frac{k\pi}{2}, \frac{\pi}{2} + \frac{(k+1)\pi}{2})$.

In the present paper we generalize the property $(S2)$ to define what we can call *saddle-type solutions* of (1.1); i.e., solutions $u \in C^2(\mathbb{R}^2)$ of (1.1) such that for a certain $j \in \mathbb{N}$ there results

(Sj) $\tilde{u}(\rho, \frac{\pi}{2}+\theta) = -\tilde{u}(\rho, \frac{\pi}{2}-\theta)$ and $\tilde{u}(\rho, \theta+\frac{\pi}{j}) = -\tilde{u}(\rho,\theta)$ for any $(\rho,\theta) \in \mathbb{R}^+ \times \mathbb{R}$.
Moreover, $\tilde{u}(\rho,\theta) \to 1$ as $\rho \to +\infty$ for any $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$.

A saddle-type solution satisfying (Sj) is antisymmetric with respect to the half-line $\theta = \frac{\pi}{2}$, and $\frac{2\pi}{j}$-periodic in the angle variable. It has $2j$ nodal lines since $\tilde{u}(\rho, \frac{\pi}{2} + \frac{k\pi}{j}) = 0$ for $\rho \geq 0$, $k = 0, \ldots, 2j-1$, and its asymptotic behavior is characterized by the fact that for $k = 0, \ldots, 2j-1$ there results $\tilde{u}(\rho, \theta) \to (-1)^{k+1}$ as $\rho \to +\infty$ whenever $\theta \in (\frac{\pi}{2} + \frac{k\pi}{j}, \frac{\pi}{2} + \frac{(k+1)\pi}{j})$.

Our main result is the following.

Theorem 1.1. *If* (W) *holds true, then for any* $j \geq 2$ *there exists* $u \in C^2(\mathbb{R}^2)$ *a solution of* (1.1) *satisfying* (Sj) *and such that* $|u(x,y)| \leq 1$ *for any* $(x,y) \in \mathbb{R}^2$.

We remark that the validity of Theorem 1.1 was already conjectured by Shi in [22] where the author named the saddle-type solutions as "*pizza*" solutions.

We note moreover that the same kind of symmetry has already been considered by Van Groesen ([25]) and by Alessio and Dambrosio ([3]) in looking for nonradial solutions of radially symmetric elliptic equations on the unit disc in $\mathbb{R}^2$.

Since, as one plainly recognizes, the one-dimensional solution $u(x,y) = q_+(x)$ of (1.1) satisfies the condition $(S1)$, by Theorem 1.1 we see that for any $j \in \mathbb{N}$ there exists a saddle-type solution of (1.1). Moreover, the asymptotic conditions characterizing a saddle-type solution guarantee that if u and v respectively satisfy (Sj) and (Sk), with $j \neq k$, then u and v are geometrically distinct; i.e., one is not the rotation of the other. Then, Theorem 1.1 gives rise to the existence of infinitely many, nonradial and geometrically distinct, bounded entire solutions of (1.1).

We note that we do not require any sign condition on W', as in $(W1)$ and $(W2)$, and in this sense our result, specialized to the case $j = 2$, generalizes the ones in [13] and [23]. On the other hand, the evenness of the potential, which we need in our proof to get sufficient compactness in the problem, is not required in $(W2)$, and it should be interesting to understand whether it is possible to establish an analog of Theorem 1.1 without that assumption.

Our proof of Theorem 1.1 is linked to but different from the one used by Alama, Bronsard and Gui in [1]. While in [1] variational arguments are used to find the saddle solution by an approximation procedure, using bounded planar domains, in Section 3 we develop a direct variational procedure, inspired by the one introduced in [4], which allows us to find the saddle-type solutions as minima of suitable renormalized action functionals (for the use of renormalized functionals in different contexts we also refer to [17, 18, 19, 6, 7]).

Our approach leads to a very simple proof. For the sake of completeness, we prove most of the intermediate results even if already present in the literature. In Section 2, as a preliminary study, we recall and organize a list of properties of the one-dimensional problem associated to (1.1). In Section 3 we build up the variational principle and then prove Theorem 1.1.

2. THE ONE-DIMENSIONAL PROBLEM

In this section we recall some results concerning the one-dimensional equation associated with (1.1). In fact, given $L > 0$, possibly $L = +\infty$, we focus our study on some variational properties of the solutions to the problem

$$\begin{cases} -\ddot{q}(x) + W'(q(x)) = 0, & x \in (-L, L), \\ q(-x) = -q(x), & x \in (-L, L). \end{cases} \tag{2.1}$$

Remark 2.1. We make precise some basic consequences of the assumptions on W, fixing some constants that will remain unchanged in the rest of the paper.

First, we note that since $W \in \mathcal{C}^2(\mathbb{R})$ and $W''(\pm 1) > 0$, there exists $\bar{\delta} \in (0, \frac{1}{4})$ and $\overline{w} > \underline{w} > 0$ such that

$$\overline{w} \geq W''(s) \geq \underline{w} \text{ for any } |s| \in [1 - 2\bar{\delta}, 1 + 2\bar{\delta}]. \tag{2.2}$$

In particular, since $W(\pm 1) = W'(\pm 1) = 0$, setting $\chi(s) = \min\{|1-s|, |1+s|\}$, we have that

$$\frac{\underline{w}}{2}\chi(s)^2 \leq W(s) \leq \frac{\overline{w}}{2}\chi(s)^2 \text{ and } |W'(s)| \leq \overline{w}\chi(s), \ \forall |s| \in [1 - 2\bar{\delta}, 1 + 2\bar{\delta}]. \tag{2.3}$$

We consider the space

$$\Gamma = \{q \in H^1_{loc}(\mathbb{R}) : q(x) = -q(-x) \text{ for any } x \in \mathbb{R}\},$$

and the functional

$$F(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}(x)|^2 + W(q(x))\, dx.$$

Moreover, if I is an interval in $\mathbb{R}$, we set

$$F_I(q) = \int_I \frac{1}{2}|\dot{q}(x)|^2 + W(q(x))\, dx.$$

Remark 2.2. We note that F and F_I, for any given interval $I \subset \mathbb{R}$, are well defined on $H^1_{loc}(\mathbb{R})$ with values in $[0, +\infty]$ and weakly lower semicontinuous with respect to the $H^1_{loc}(\mathbb{R})$ topology.

We are interested in the minimal properties of F on Γ, and we set

$$c = \inf_{\Gamma} F \qquad \text{and} \quad \mathcal{K} = \{q \in \Gamma : F(q) = c\}.$$

Remark 2.3. We note that if $q \in \Gamma$ is such that $W(q(x)) \geq \omega > 0$ for any $x \in (\sigma, \tau) \subset \mathbb{R}$, then

$$F_{(\sigma,\tau)}(q) \geq \tfrac{1}{2(\tau-\sigma)}|q(\tau) - q(\sigma)|^2 + \omega(\tau - \sigma) \geq \sqrt{2\omega}\ |q(\tau) - q(\sigma)|. \tag{2.4}$$

In particular, if $q \in \Gamma$ and $\delta > 0$ are such that $|q(x)| \leq 1 - \delta$ for every $x \in (\sigma, \tau) \subset \mathbb{R}$, then

$$F_{(\sigma,\tau)}(q) = \int_{\sigma}^{\tau} \tfrac{1}{2}|\dot{q}|^2 + W(q)\, dx \geq \omega_\delta(\tau - \sigma),$$

where

$$\omega_\delta = \min_{|s| \leq 1-\delta} W(s) > 0, \qquad \delta \in (0, 1). \tag{2.5}$$

Finally, we will denote

$$\lambda_\delta = \min\{1; \sqrt{2\omega_{\delta/2}}\frac{\delta}{4}; \sqrt{\underline{w}}\tfrac{\delta^2}{2}\} \text{ and in particular } \bar{\lambda} = \lambda_{\bar{\delta}}, \tag{2.6}$$

where $\overline{\delta}$ was fixed in Remark 2.1.

Lemma 2.1. *If $q \in \Gamma$ is such that $F(q) \leq c + \bar{\lambda}$, then $\|q\|_{L^\infty(\mathbb{R})} \leq 1 + 2\overline{\delta}$.*

Proof. Let $q \in \Gamma$ and assume for the sake of contradiction that there exists x_0 such that $q(x_0) > 1 + 2\overline{\delta}$. Up to reflection, we can assume that $x_0 > 0$. Since $q(0) = 0$ and q is continuous, there exist $x_1, \sigma, \tau \in \mathbb{R}$ with $0 < x_1 < \sigma < \tau < x_0$ such that $q(x_1) = 1$, $q(\sigma) = 1 + \overline{\delta}$, $q(\tau) = 1 + 2\overline{\delta}$ and $1 + \overline{\delta} < q(x) < 1 + 2\overline{\delta}$ for any $x \in (\sigma, \tau)$. By (2.3), $W(q(x)) \geq \frac{\underline{w}}{2}\overline{\delta}^2$ for any $x \in (\sigma, \tau)$, and then, by (2.4) and (2.6), we obtain $F_{(\sigma,\tau)}(q) \geq 2\bar{\lambda}$. Moreover, since $q \in \Gamma$ and $q(x_1) = 1$, we have also $F_{(-x_1,x_1)}(q) \geq c$. Then, we reach the contradiction $c + \bar{\lambda} \geq F(q) \geq F_{(-x_1,x_1)}(q) + F_{(\sigma,\tau)}(q) \geq c + 2\bar{\lambda}$. □

Note that, as a consequence of Lemma 2.1, using Remark 2.3 and the fact that $W(s) > 0$ for any $|s| \leq 1 + 2\overline{\delta}$, $|s| \neq 1$, one plainly recognizes that

$$\text{if } q \in \Gamma \text{ is such that } F(q) < c + \bar{\lambda} \text{ then } |q(x)| \to 1 \text{ as } x \to \pm\infty. \tag{2.7}$$

Moreover, again by Lemma 2.1, we easily derive the following first compactness result.

Lemma 2.2. *Let $(q_n) \subset \Gamma$ be such that $F(q_n) \leq c + \bar{\lambda}$ for all $n \in \mathbb{N}$. Then, there exists $q \in \Gamma$ such that, along a subsequence, $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$ and $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$. Moreover, $F(q) \leq \liminf_{n\to\infty} F(q_n)$.*

Proof. Since $F(q_n) \leq c + \bar{\lambda}$ for any $n \in \mathbb{N}$, by Lemma 2.1 we find that $\|q_n\|_{L^\infty(\mathbb{R})} \leq 1 + 2\bar{\delta}$. Since $\|\dot{q}_n\| \leq 2(c + \bar{\lambda})$ for any $n \in \mathbb{N}$, we obtain that there exists $q \in H^1_{loc}(\mathbb{R})$ such that, along a subsequence, $q_n \to q$ weakly in $H^1_{loc}(\mathbb{R})$, so in $L^\infty_{loc}(\mathbb{R})$, and $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$. Moreover, since $q_n(-x) = -q_n(x)$ for any $x \in \mathbb{R}$, $n \in \mathbb{N}$, by pointwise convergence we obtain $q(-x) = -q(x)$ for all $x \in \mathbb{R}$ and so $q \in \Gamma$. Then, the lemma follows by Remark 2.2. □

By Lemma 2.2, the Weierstrass theorem tells us that the functional F attains its infimum value on Γ; i.e., $\mathcal{K} \neq \emptyset$. Since $W(s) = W(-s)$, it is classical to derive that if $q \in \mathcal{K}$ then q satisfies the equation $-\ddot{q}(x) + W'(q(x)) = 0$, $x \in \mathbb{R}$, and moreover, by (2.7), we have that $|q(x)| \to \pm 1$ as $x \to \pm\infty$. A simple comparison argument shows moreover that $\|q\|_{L^\infty(\mathbb{R})} \leq 1$. As proved e.g. in [5], Lemma 2.2, we know that the equation $-\ddot{q}(x) + W'(q(x)) = 0$ admits, modulo translations, a unique solution on $\mathbb{R}$ satisfying the conditions $\|q\|_{L^\infty(\mathbb{R})} \leq 1$ and $\lim_{x\to\pm\infty} q(x) = \pm 1$. In addition, that solution is increasing on $\mathbb{R}$. This information and the symmetry of our problem allow us to conclude the following.

Proposition 2.1. *There exists a unique $q^+ \in \Gamma$ such that $F(q^+) = c$ and $q^+(x)x > 0$ for all $x \in \mathbb{R}$. Moreover, $q^+ \in C^2(\mathbb{R})$ satisfies $-\ddot{q}^+(x) + W'(q^+(x)) = 0$ for all $x \in \mathbb{R}$ with $q^+(x) \to 1$ as $x \to +\infty$. Finally, setting $q^-(x) = -q^+(x)$, there results $\mathcal{K} = \{q^+, q^-\}$.*

For our purposes, we need to better characterize the compactness properties of F. In Lemma 2.3 below we first describe concentration properties of the functions in the sublevels of F.

Remark 2.4. By Remark 2.3, for every $d > 0$ there exists $\ell_d > 0$ such that if $F_{(\sigma,\tau)}(q) \leq c + 1$ and $|q(x)| \leq 1 - d$ for every $x \in (\sigma, \tau)$, then $\tau - \sigma \leq \ell_d$. Given $\delta \in (0, \bar{\delta}]$ we fix $d(\delta) \in (0, \frac{\delta}{2})$ such that $(1 + \bar{w})d(\delta)^2 < \lambda_\delta$. We denote

$$L_\delta = \ell_{d(\delta)} \text{ and in particular } \bar{L} = L_{\bar{\delta}}.$$

Lemma 2.3. *Let $\delta \in (0, \bar{\delta}]$, $q \in \Gamma$ and $L \geq L_\delta$ be such that $F_{(-L,L)}(q) \leq c + \lambda_\delta$. Then $|q(x)| \geq 1 - \delta$ for all $x \in [L_\delta, L]$.*

Proof. By the choice of L_δ in Remark 2.4, since

$$F_{(0,L_\delta)}(q) \leq F_{(-L,L)}(q) \leq c + \lambda_\delta \leq c + 1,$$

there exists $\bar{x} \in [0, L_\delta]$ such that $|q(\bar{x})| \geq 1 - d(\delta)$. Up to a reflection, we can assume $q(\bar{x}) \geq 1 - d(\delta)$ and we set

$$\bar{q}(x) = \begin{cases} -1 & \text{if } x \leq -\bar{x} - 1, \\ -q(\bar{x}) + (1 - q(\bar{x}))(x + \bar{x}) & \text{if } x \in [-\bar{x} - 1, -\bar{x}], \\ q(x) & \text{if } x \in [-\bar{x}, \bar{x}], \\ q(\bar{x}) + (1 - q(\bar{x}))(x - \bar{x}) & \text{if } x \in [\bar{x}, \bar{x} + 1], \\ 1 & \text{if } x \geq \bar{x} + 1. \end{cases}$$

Since by (2.3) we have

$$W(\bar{q}(x)) \leq \tfrac{\overline{w}}{2}(1 - \bar{q}(x))^2 \leq \tfrac{\overline{w}}{2}(1 - q(\bar{x}))^2 \leq \tfrac{\overline{w}}{2} d(\delta)^2 \text{ for any } x \in [\bar{x}, \bar{x} + 1],$$

a direct estimate tells us that $F_{[\bar{x}, \bar{x}+1]}(\bar{q}) \leq \frac{1}{2}(1 + \overline{w}) d(\delta)^2$. Hence, since $\bar{q} \in \Gamma$, by symmetry and the choice of $d(\delta)$ in Remark 2.4, we obtain

$$c \leq F(\bar{q}) \leq F_{[-\bar{x}, \bar{x}]}(q) + (1 + \overline{w}) d(\delta)^2 \leq F_{[-\bar{x}, \bar{x}]}(q) + \lambda_\delta. \tag{2.8}$$

Assume now for the sake of contradiction that there exists $\xi \in [L_\delta, L]$ such that $q(\xi) \leq 1 - \delta$. Then, by continuity there exists an interval $(\sigma, \tau) \subset (\bar{x}, \xi)$ such that $q(\sigma) = 1 - \frac{\delta}{2}$, $q(\tau) = 1 - \delta$ and $1 - \delta \leq q(x) \leq 1 - \frac{\delta}{2}$ for all $x \in (\sigma, \tau)$. By (2.4) and (2.6), we obtain

$$F_{(\bar{x}, L)}(q) \geq F_{(\sigma, \tau)}(q) \geq \sqrt{2\omega_{\delta/2}} \tfrac{\delta}{2} \geq 2\lambda_\delta,$$

and then, by symmetry and (2.8), $c + \lambda_\delta \geq F_{(-L, L)}(q) = F_{(-\bar{x}, \bar{x})}(q) + 2F_{(\bar{x}, L)}(q) \geq c - \lambda_\delta + 4\lambda_\delta = c + 3\lambda_\delta$, a contradiction. □

Remark 2.5. Note that if $q \in \Gamma$ and $F(q) \leq c + \bar{\lambda}$, then $F_{(-L,L)}(q) \leq c + \bar{\lambda}$ for any $L \geq \bar{L}$. Then, by Lemma 2.3, we have $|q(x)| \geq 1 - \bar{\delta}$ for all $x \geq \bar{L}$.

By Lemma 2.3 we derive the following compactness property of F.

Lemma 2.4. *If $(q_n) \subset \Gamma$ satisfies $F(q_n) \to c$, then* $\mathsf{dist}_{H^1(\mathbb{R})}(q_n, \mathcal{K}) \to 0$.

Proof. To show that $\mathsf{dist}_{H^1(\mathbb{R})}(q_n, \mathcal{K}) \to 0$ we prove that given any subsequence of q_n we can extract from it a sub-subsequence along which

$$\mathsf{dist}_{H^1(\mathbb{R})}(q_n, \mathcal{K}) \to 0.$$

Fixed any subsequence of (q_n), still denoted (q_n), let $\bar{n} \in \mathbb{N}$ be such that $F(q_n) \leq c + \bar{\lambda}$ for all $n \geq \bar{n}$. By Remark 2.5 we obtain that

$$|q_n(x)| \geq 1 - \bar{\delta} \text{ for all } x \geq \bar{L},\ n \geq \bar{n}. \tag{2.9}$$

Moreover, Lemma 2.2 implies that there exists $q \in \Gamma$ and a subsequence of (q_n), still denoted (q_n), such that $q_n \to q$ in $L^\infty_{loc}(\mathbb{R})$, $\dot{q}_n \to \dot{q}$ weakly in

$L^2(\mathbb{R})$ and $F(q) = c$. In particular, $q \in \mathcal{K}$, and by the pointwise convergence, $|q(x)| \geq 1 - \bar{\delta}$ for all $x \geq \bar{L}$.

Up to a reflection, we can assume that along this subsequence there results $q_n(x) \geq 1 - \bar{\delta}$ for all $x \geq \bar{L}$, $n \geq \bar{n}$. This implies that $q(x) \geq 1 - \bar{\delta}$ for all $x \geq \bar{L}$, and since $q \in \mathcal{K} = \{q^-, q^+\}$ we derive $q \equiv q^+$. The lemma will follow once we show that $\|q_n - q^+\|_{H^1(\mathbb{R})} \to 0$.

Let us first prove that $\dot{q}_n \to \dot{q}^+$ in $L^2(\mathbb{R})$. Since $\dot{q}_n \to \dot{q}^+$ weakly in $L^2(\mathbb{R})$, it is sufficient to derive that $\|\dot{q}_n\|_{L^2(\mathbb{R})} \to \|\dot{q}^+\|_{L^2(\mathbb{R})}$. By the weak semicontinuity of the norm, we have that $\liminf_{n\to+\infty} \|\dot{q}_n\|_{L^2(\mathbb{R})} \geq \|\dot{q}^+\|_{L^2(\mathbb{R})}$. Moreover, since $W(s) \geq 0$ for all $s \in \mathbb{R}$, by pointwise convergence and Fatou's lemma we obtain

$$\liminf_{n\to+\infty} \int_{\mathbb{R}} W(q_n)\ dx \geq \int_{\mathbb{R}} W(q^+)\ dx.$$

Hence,

$$\begin{aligned}
&\limsup_{n\to+\infty} \|\dot{q}_n\|_{L^2(\mathbb{R})} = \limsup_{n\to+\infty} 2(F(q_n) - \int_{\mathbb{R}} W(q_n)\ dx) \\
&= 2c - 2\liminf_{n\to+\infty} \int_{\mathbb{R}} W(q_n)\ dx \leq 2c - 2\int_{\mathbb{R}} W(q^+)\ dx = \|\dot{q}^+\|_{L^2(\mathbb{R})},
\end{aligned}$$

proving as we claimed that $\dot{q}_n \to \dot{q}^+$ in $L^2(\mathbb{R})$.

Let us now show that $q_n - q^+ \to 0$ in $L^2(\mathbb{R})$. First note that by the weak convergence in $H^1_{loc}(\mathbb{R})$, there results $\|q_n - q^+\|_{L^2((-L,L))} \to 0$ for any $L > 0$. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} W(q_n(x))\ dx &= F(q_n) - \frac{1}{2}\|\dot{q}_n\|_{L^2(\mathbb{R})} \to F(q) - \frac{1}{2}\|\dot{q}^+\|_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} W(q^+(x))\ dx,
\end{aligned}$$

and since for any $L > 0$

$$\int_L^{+\infty} W(q^+)\ dx \leq \liminf_{n\to+\infty} \int_L^{+\infty} W(q_n)\ dx,$$

we derive that

$$\int_L^{+\infty} W(q_n)\ dx \to \int_L^{+\infty} W(q^+)\ dx \quad \text{for all } L > 0.$$

Then, given any $\varepsilon > 0$, let $L \geq \bar{L}$ be such that $\int_L^{+\infty} W(q^+)\ dx < \varepsilon$ and let $n_0 \geq \bar{n}$ be such that $\int_L^{+\infty} W(q_n)\ dx < 2\varepsilon$ for all $n \geq n_0$.

Now, by Lemma 2.1 and (2.9), we have that $1-\bar\delta \le q_n(x) \le 1+2\bar\delta$ for all $n \ge \bar n$, $x \ge \bar L$, and hence, by (2.3), $W(q_n(x)) \ge \frac{\underline{w}}{2}(1-q_n(x))^2$ for all $x \ge \bar L$, $n \ge \bar n$. Then, for all $n \ge n_0$ we obtain

$$\int_L^{+\infty} (1-q_n(x))^2\, dx \le \tfrac{2}{\underline{w}} \int_L^{+\infty} W(q_n(x))\, dx \le \tfrac{4\varepsilon}{\underline{w}}.$$

Moreover, by pointwise convergence, we obtain also $1-\bar\delta \le q^+(x) \le 1+2\bar\delta$ for all $x \ge \bar L$, and therefore, as above, we derive

$$\int_L^{+\infty} (1-q^+(x))^2\, dx \le \tfrac{2}{\underline{w}} \int_L^{+\infty} W(q^+(x))\, dx \le \tfrac{2\varepsilon}{\underline{w}}.$$

Then, for all $n \ge n_0$ we conclude

$$\begin{aligned} \|q_n - q^+\|^2_{L^2(\mathbb{R})} &= \|q_n - q^+\|^2_{L^2((-L,L))} + 2\|q_n - q^+\|^2_{L^2([L,+\infty))} \\ &= o(1) + 2\|q_n - q^+\|^2_{L^2([L,+\infty))} \\ &\le o(1) + 4(\|q_n - 1\|^2_{L^2([L,+\infty))} + \|1-q^+\|^2_{L^2([L,+\infty))}) \\ &\le o(1) + \tfrac{24}{\underline{w}}\varepsilon, \end{aligned}$$

where $o(1) \to 0$ as $n \to +\infty$, and then, as we claimed, $\|q_n - q^+\|_{L^2(\mathbb{R})} \to 0$. □

Note that Lemma 2.4 is actually equivalent to saying that for all $r > 0$ there exists $\mu_r > 0$ such that

$$\text{if } q \in \Gamma \text{ satisfies } \mathsf{dist}_{H^1(\mathbb{R})}(q, \mathcal{K}) \ge r, \text{ then } F(q) \ge c + \mu_r. \tag{2.10}$$

Fixing any $L > 0$, we now consider the functional $F_L = F_{(-L,L)}$ on the space

$$\Gamma_L = \{q \in H^1((-L,L)) : q(x) = -q(-x) \text{ for any } x \in (-L,L)\},$$

and we set $c_L = \inf_{\Gamma_L} F_L$ and $\mathcal{K}_L = \{q \in \Gamma_L : F_L(q) = c_L\}$, noting that $c \ge c_L > 0$ for any $L > 0$.

Proposition 2.2. *For every $L > 0$ there results $\mathcal{K}_L \ne \emptyset$, and if $q \in \mathcal{K}_L$, then $q \in C^2((-L,L))$ satisfies $-\ddot q(x) + W'(q(x)) = 0$ for all $x \in (-L,L)$ and $\dot q(\pm L) = 0$. Moreover, if $L \ge \bar L$, then $1-\bar\delta \le |q(x)|$ for all $x \in [\bar L, L]$ and*

$$0 \le 1 - q(x) \le \bar\delta\sqrt{2}e^{\sqrt{\frac{\underline{w}}{2}}(\bar L - x)} \quad \forall x \in [\bar L, L].$$

Proof. Fixing $L > 0$, let $(q_n) \subset \Gamma_L$ be such that $F_L(q_n) \to c_L$. Note that it is not restrictive to assume that $\|q_n\|_{L^\infty((-L,L))} \le 1$ for all $n \in \mathbb{N}$. Indeed, setting $\bar q_n(x) = \max\{-1; \min\{1; q_n(x)\}\}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we obtain that $\bar q_n \in \Gamma_L$, $F_L(\bar q_n) \le F_L(q_n)$ and $\|\bar q_n\|_{L^\infty((-L,L))} \le 1$.

Since $\|q_n\|_{L^\infty((-L,L))} \le 1$ and $\|\dot q_n\|_{L^2((-L,L)} \le 2(c_L+1)$ for any $n \in \mathbb{N}$ sufficiently large, we obtain that there exists $q \in H^1((-L,L))$ such that, along a subsequence, $q_n \to q$ weakly in $H^1((-L,L))$ and hence strongly in $L^\infty((-L,L))$. Then, $q(-x) = -q(x)$ for all $x \in (-L,L)$, and hence $q \in \Gamma_L$ and $F_L(q) \ge c_L$. By Remark 2.2, we conclude that $q \in \mathcal{K}_L$. Then, it is standard to show that

$$\int_{-L}^{L} \dot q\dot\psi + W'(q)\psi \; dx = 0, \quad \forall \psi \in C^\infty((-L,L))$$

and hence that $q \in C^2((-L,L))$ satisfies $-\ddot q(x) + W'(q(x)) = 0$ for all $x \in (-L,L)$ and $\dot q(\pm L) = 0$.

Note that if $L \ge \bar L$ and $q \in \mathcal{K}_L$, then $F_L(q) = c_L \le c < c + \bar\lambda$, and by Lemma 2.3 we derive that $|q(x)| \ge 1 - \bar\delta$ for all $x \in [\bar L, L]$.

To complete the proof we have now to show the exponential estimate. Consider the function

$$v(x) = \begin{cases} q(x) & \text{if } x \in [\bar L, L] \\ q(2L-x) & \text{if } x \in [L, 2L-\bar L]. \end{cases}$$

Observe that, since $\dot q(L) = 0$, we have $v \in C^1([\bar L, 2L-\bar L])$ and, by definition, $1-\bar\delta \le |v(x)|$ for all $x \in [\bar L, 2L-\bar L]$. Moreover, $-\ddot v(x) + W'(v(x)) = 0$ for all $x \in (\bar L, L) \cup (L, 2L-\bar L)$, and we find that $v \in C^2([\bar L, 2L-\bar L])$ solves the equation on the entire interval $(\bar L, 2L-\bar L)$.

We set $\phi(x) = (1-v(x))^2$, $x \in [\bar L, 2L-\bar L]$. Then we have $0 \le \phi(x) \le \bar\delta^2$ for all $x \in [\bar L, 2L-\bar L]$, and, by (2.3),

$$\begin{aligned} \ddot\phi(x) &= -2(1-v(x))\ddot v(x) + 2\dot v^2(x) \ge -2W'(v(x))(1-v(x)) \\ &= 2(W'(1) - W'(v(x))(1-v(x)) \ge 2\underline{w}(1-v(x))^2 = 2\underline{w}\phi(x). \end{aligned}$$

Defining $\psi(x) = \bar\delta^2 \frac{\cosh(\sqrt{2\underline{w}}(x-L))}{\cosh(\sqrt{2\underline{w}}(L-\bar L))}$, for $x \in (\bar L, 2L-\bar L)$, noting that $\psi(\bar L) = \psi(2L-\bar L) = \bar\delta^2$ and that $\ddot\psi(x) = 2\underline{w}\psi(x)$ for all $x \in (\bar L, 2L-\bar L)$, one recognizes that the function $\eta(x) = \psi(x) - \phi(x)$ satisfies

$$\begin{cases} \ddot\eta(x) \le 2\underline{w}\eta(x), & x \in (\bar L, 2L-\bar L), \\ \eta(\bar L) = \eta(2L-\bar L) \ge 0. \end{cases}$$

Thus, $\eta(x) \ge 0$; i.e., $\psi(x) \ge \phi(x)$, for all $x \in [\bar L, 2L-\bar L]$, and so

$$0 \le 1 - v(x) \le \bar\delta \left(\frac{\cosh(\sqrt{2\underline{w}}(x-L))}{\cosh(\sqrt{2\underline{w}}(L-\bar L))} \right)^{1/2} \text{ for all } x \in [\bar L, 2L-\bar L].$$

Since for $x \in [\bar{L}, L]$ there results $v(x) = q(x)$ and $2\cosh(\sqrt{2\underline{w}}(x-L)) \le 2e^{\sqrt{2\underline{w}}(L-x)}$, and since $2\cosh(\sqrt{2\underline{w}}(L-\bar{L})) \ge e^{\sqrt{2\underline{w}}(L-\bar{L})}$, we find

$$0 \le 1 - q(x) \le \bar{\delta}\sqrt{2}e^{\sqrt{\frac{\underline{w}}{2}}(\bar{L}-x)}, \quad x \in [\bar{L}, L]$$

and the lemma follows. □

Thanks to Proposition 2.2 we can better characterize the behavior of the function $L \mapsto c_L$.

Lemma 2.5. *The function $L \mapsto c_L$ is monotone increasing with $c_L \to c$ as $L \to +\infty$ and precisely*

$$0 \le c - c_L \le Ce^{-\sqrt{2\underline{w}}L}, \quad \forall L > 0.$$

Proof. To prove the monotonicity of $L \mapsto c_L$, fix $L_1 \le L_2$ and let $q \in \mathcal{K}_{L_2}$; one has $c_{L_1} \le F_{L_1}(q) \le F_{L_2}(q) = c_{L_2}$. Analogously $c_L \le c$ for any $L > 0$ since $c_L \le F_L(q^+) \le F(q^+) = c$.

Let us prove now the exponential estimate. For $q \in \mathcal{K}_L$, $L > \bar{L}$, assuming (without restriction) that $q(x) \ge 1 - \bar{\delta}$ for $x \in [\bar{L}, L]$, we set

$$\tilde{q}(x) = \begin{cases} -1 & \text{if } x \le -L-1 \\ -1 + (q(-L)+1)(L+1+x) & \text{if } x \in [-L-1, -L] \\ q(x) & \text{if } x \in (-L, L) \\ 1 + (q(L)-1)(L+1-x) & \text{if } x \in [L, L+1] \\ 1 & \text{if } x \ge L+1, \end{cases}$$

noting that $\tilde{q} \in \Gamma$ and so $F(\tilde{q}) \ge c$. Moreover, by symmetry, $F(\tilde{q}) = c_L + 2F_{(L,L+1)}(\tilde{q})$, and we deduce that $c - c_L \le 2F_{(L,L+1)}(\tilde{q})$.

To evaluate $F_{(L,L+1)}(\tilde{q})$ we simply note that

$$F_{(L,L+1)}(\tilde{q}) = \int_L^{L+1} \frac{1}{2}|1-q(L)|^2 + W(\tilde{q})\, dx$$

and that by (2.3)

$$W(\tilde{q}(x)) \le \frac{\overline{w}}{2}(1-\tilde{q}(x))^2 \le \frac{\overline{w}}{2}(1-q(L))^2, \text{ for all } x \in [L, L+1].$$

By Proposition 2.2 we have $1 - q(L) \le \bar{\delta}\sqrt{2}e^{\sqrt{\frac{\underline{w}}{2}}(\bar{L}-L)}$, and so $F_{(L,L+1)}(\tilde{q}) \le (1+\overline{w})\bar{\delta}^2 e^{\sqrt{2\underline{w}}(\bar{L}-L)}$, from which we conclude

$$0 \le c - c_L \le 2(1+\overline{w})\bar{\delta}^2 e^{\sqrt{2\underline{w}}\bar{L}} e^{-\sqrt{2\underline{w}}L}, \quad \forall L \ge \bar{L}.$$

Since $c - c_L \le c$, we then obtain the existence of a constant $C > 0$ such that, as we stated, $0 \le c - c_L \le Ce^{-\sqrt{2\underline{w}}L}$ for any $L > 0$. □

To proceed to study the elliptic problem on $\mathbb{R}^2$, we finally need to state a further compactness property concerning the functionals F_L.

Lemma 2.6. *Let $y_n \to +\infty$ and $(q_n) \subset \Gamma_{y_n}$ be such that $F_{y_n}(q_n) - c_{y_n} \to 0$ as $n \to +\infty$. Then,*

$$\mathsf{dist}_{H^1((-y_n y_n))}(q_n, \mathcal{K}) \to 0.$$

Proof. As in Lemma 2.4 we show that given any subsequence of q_n we can extract from it a sub-subsequence along which $\mathsf{dist}_{H^1((-y_n,y_n))}(q_n, \mathcal{K}) \to 0$. So, we fix a subsequence of (q_n), denoted again (q_n).

Fixing a sequence $\delta_k \to 0$, let $\bar{L}_{\delta_k}$ be given by Remark 2.4 and λ_{δ_k} by (2.6). Since $y_n \to +\infty$ and since $F_{y_n}(q_n) - c_{y_n} \to 0$, there exists an increasing sequence $(n_k) \subset \mathbb{N}$ such that for any $k \in \mathbb{N}$ there results $y_{n_k} \geq \bar{L}_{\delta_k}$ and $F_{y_{n_k}}(q_{n_k}) \leq c_{y_{n_k}} + \lambda_{\delta_k}$. Then, by Lemma 2.3, we have that $|q_{n_k}(y_{n_k})| \geq 1 - \delta_k$. Assuming, up to reflection, that $q_{n_k}(y_{n_k}) \geq 1 - \delta_k$, we set

$$\tilde{q}_{n_k}(x) = \begin{cases} -1 & \text{if } x \leq -y_{n_k} - 1 \\ -1 + (q(-y_{n_k}) + 1)(L_{n_k} + 1 + x) & \text{if } x \in [-y_{n_k} - 1, -y_{n_k}] \\ q_{n_k}(x) & \text{if } x \in [-y_{n_k}, y_{n_k}] \\ 1 + (q(y_{n_k}) - 1)(y_{n_k} + 1 - x) & \text{if } x \in [y_{n_k}, y_{n_k} + 1] \\ 1 & \text{if } x \geq y_{n_k} + 1, \end{cases}$$

noting that $\tilde{q}_{n_k} \in \Gamma$ and hence $F(\tilde{q}_{n_k}) \geq c$ for any $k \in \mathbb{N}$. Moreover, by (2.3),

$$\begin{aligned} F(\tilde{q}_{n_k}) &= F_{y_{n_k}}(\tilde{q}_{n_k}) + \int_{y_{n_k} \leq |x| \leq y_{n_k}+1} \frac{1}{2}|\dot{\tilde{q}}_{n_k}|^2 + W(\tilde{q}_{n_k})\, dx \\ &\leq F_{y_{n_k}}(q_{n_k}) + (1+\overline{w})\delta_k^2 \leq c_{y_{n_k}} + \lambda_{\delta_k} + (1+\overline{w})\delta_k^2. \end{aligned}$$

Since $c_{y_{n_k}} \to c$, we have $F(\tilde{q}_{n_k}) \to c$ and so, by Lemma 2.4, $\mathsf{dist}_{H^1(\mathbb{R})}(\tilde{q}_{n_k}, \mathcal{K}) \to 0$. In particular, $\mathsf{dist}_{H^1((-y_{n_k}, y_{n_k}))}(q_{n_k}, \mathcal{K}) \leq \mathsf{dist}_{H^1(\mathbb{R})}(\tilde{q}_{n_k}, \mathcal{K}) \to 0$. $\square$

By the previous lemma we obtain in particular that for all $r > 0$ there exist $\nu_r > 0$ and $M_r > 0$ such that for all $L > M_r$,

$$\text{if } q \in \Gamma_L \text{ satisfies } \mathsf{dist}_{H^1((-L,L))}(q, \mathcal{K}) > r \text{ then } F_L(q) > c_L + \nu_r. \tag{2.11}$$

3. Saddle-type solutions

We fix $j \in \mathbb{N}$, $j \geq 2$. Setting $a_j = \tan(\frac{\pi}{2j})$, from now on, given $y > 0$, with abuse of notation we denote $I_y = (-a_j y, a_j y)$, $\Gamma_y = \Gamma_{a_j y}$, $c_y = c_{a_j y}$, and for

$q \in \Gamma_y$, we set $F_y(q) = F_{a_j y}(q)$. We define

$$T = \{(x, y) \in \mathbb{R}^2 : x \in I_y,\ y > 0\},$$
$$\mathcal{M} = \{u \in H^1_{loc}(T) : u(x, y) = -u(-x, y) \text{ for a.e. } (x, y) \in T\}.$$

We consider on $\mathcal{M}$ the functional

$$\varphi(u) = \int_0^{+\infty} \|\partial_y u(\cdot, y)\|^2_{L^2(I_y)} + (F_y(u(\cdot, y)) - c_y)\, dy.$$

Note that if $u \in \mathcal{M}$ then $u(\cdot, y) \in \Gamma_y$ for almost every $y > 0$, and so $F_y(u(\cdot, y)) - c_y \geq 0$ for almost every $y \in \mathbb{R}$. Hence we find that φ is well defined on $\mathcal{M}$ with values in $[0, +\infty]$. Moreover, as in [4], Lemma 3.1, one can prove that φ is weakly lower semicontinuous with respect to the $H^1_{loc}(T)$ topology. We apply the direct method of the calculus of variations to look for a minimum of φ on $\mathcal{M}$. This problem is meaningful since, as a consequence of the following simple lemma, there results

$$m \equiv \inf_{\mathcal{M}} \varphi < +\infty.$$

Lemma 3.1. *Setting $u^+(x, y) = q^+(x)$ for any $(x, y) \in T$, there results $u^+ \in \mathcal{M}$ and $\varphi(u^+) < +\infty$, where q^+ is defined in Proposition* 2.1.

Proof. Note that, trivially, $u^+ \in \mathcal{M}$ and $\|\partial_y u^+(\cdot, y)\|_{L^2(I_y)} = 0$ for any $y > 0$. Then, by Lemma 2.5

$$\varphi(u^+) = \int_0^{+\infty} F_y(u^+(\cdot, y)) - c_y\, dy \leq \int_0^{+\infty} F(u^+(\cdot, y)) - c_y\, dy$$
$$= \int_0^{+\infty} c - c_y\, dy \leq C \int_0^{+\infty} \mathrm{e}^{-\sqrt{2\underline{w}}a_j y}\, dy < +\infty,$$

and the lemma follows. □

We remark that if $u \in \mathcal{M}$ and $(\sigma, \tau) \subset \mathbb{R}^+$, then defining $Q_{(\sigma,\tau)} = I_\sigma \times (\sigma, \tau)$, we have $u \in H^1(Q_{(\sigma,\tau)})$. This implies that for almost every $x \in I_\sigma$ the function $u(x, \cdot)$ is absolutely continuous on $[\sigma, \tau]$. In particular,

$$|u(x, \tau) - u(x, \sigma)|^2 = \Big| \int_\sigma^\tau \partial_y u(x, y)\, dy \Big|^2 \leq (\tau - \sigma) \int_\sigma^\tau |\partial_y u(x, y)|^2\, dy,$$

and so integrating on I_σ we obtain

$$\|u(\cdot, \tau) - u(\cdot, \sigma)\|^2_{L^2(I_\sigma)} \leq (\tau - \sigma) \|\partial_y u\|^2_{H^1(Q_{(\sigma,\tau)})}. \tag{3.1}$$

By (3.1) we find in particular that given any bounded interval $I \subset \mathbb{R}$ and any $u \in \mathcal{M}$, if $\bar{y} > 0$ is such that $I \subset I_{\bar{y}}$ then the function $y > \bar{y} \mapsto u(\cdot, y) \in L^2(I)$ is continuous.

Another important observation for our construction is an estimate concerning the functional φ, analogous to the one we gave in (2.4) for the one-dimensional functional F. Given $(\sigma,\tau)\subset\mathbb{R}_+$ and $u\in\mathcal{M}$ we let

$$\varphi_{(\sigma,\tau)}(u)=\int_\sigma^\tau \|\partial_y u(\cdot,y)\|^2_{L^2(I_y)}+(F_y(u(\cdot,y))-c_y)\,dy.$$

Note that, if $u\in\mathcal{M}$ is such that $F_y(u(\cdot,y))-c_y\geq\mu>0$ for almost every $y\in(\sigma,\tau)\subset\mathbb{R}$, then

$$\begin{aligned}\varphi_{(\sigma,\tau)}(u) &\geq \frac{1}{2(\tau-\sigma)}\|u(\cdot,\tau)-u(\cdot,\sigma)\|^2_{L^2(I_\sigma)}+\mu(\tau-\sigma)\\ &\geq \sqrt{2\mu}\|u(\cdot,\tau)-u(\cdot,\sigma)\|_{L^2(I_\sigma)}.\end{aligned}\tag{3.2}$$

The estimate (3.2), together with Lemma 2.6, allows us to characterize the asymptotic behavior, as $y\to+\infty$, of the functions $u\in\mathcal{M}$ such that $\varphi(u)<+\infty$. Precisely,

Lemma 3.2. *If $u\in\mathcal{M}$ and $\varphi(u)<+\infty$, then, fixing any bounded interval $I\subset\mathbb{R}$, we have*

$$\mathsf{dist}_{L^2(I)}(u(\cdot,y),\mathcal{K})\to 0\qquad \textit{as } y\to+\infty.$$

Proof. Since $\varphi(u)<+\infty$ and $F_y(u(\cdot,y)-c_y\geq 0$ for almost every $y>0$, we plainly derive that there exists an increasing sequence $y_n\to+\infty$ such that $F_{y_n}(u(\cdot,y_n))-c_{y_n}=0$. Fixing any bounded interval $I\subset\mathbb{R}$, by Lemma 2.6 we obtain that $\mathsf{dist}_{L^2(I)}(u(\cdot,y_n),\mathcal{K})\to 0$ as $n\to+\infty$. Possibly considering the function $-u$, it is not restrictive to assume that along a subsequence, still denoted (y_n), we have $\|u(\cdot,y_n)-q^+\|_{L^2(I)}\to 0$ as $n\to+\infty$. We claim that in fact $\|u(\cdot,y)-q^+\|_{L^2(I)}\to 0$ as $y\to+\infty$. Indeed, arguing by contradiction, by (3.1) we obtain the existence of a sequence of intervals (σ_n,τ_n), a positive number $r_0>0$ and a positive integer $\bar n_1\in\mathbb{N}$ such that $I\subset I_{y_{\bar n_1}}$ and for $n\geq\bar n_1$ there results

i) $(\sigma_n,\tau_n)\subset(y_n,y_{n+1})$,
ii) $\|u(\cdot,\tau_n)-u(\cdot,\sigma_n)\|_{L^2(I)}=r_0$,
iii) $\mathsf{dist}_{L^2(I)}(u(\cdot,y),\mathcal{K})\geq r_0$, for any $y\in(\sigma_n,\tau_n)$.

By (2.11) and (iii) we find that there exists $\nu_0>0$ and $\bar n_2\geq\bar n_1$ such that $F_y(u(\cdot,y))-c_y\geq\nu_0$ for any $y\in(\sigma_n,\tau_n)$ and $n\geq\bar n_2$. Using now (3.2) and (ii) we find that $\varphi_{(\sigma_n,\tau_n)}(u)\geq\sqrt{2\nu_0}r_0>0$ for any $n\geq\bar n_2$ and so, by (i), we conclude $\varphi(u)\geq\sum_{n\geq\bar n_2}\varphi_{(\sigma_n,\tau_n)}(u)=+\infty$, a contradiction which proves the lemma. □

We can now prove the following existence result.

Proposition 3.1. *There exists $\bar u \in \mathcal{M}$ such that $\varphi(\bar u) = m$, $\|\bar u\|_{L^\infty(T)} \le 1$ and $\bar u(\cdot, y) - q^+ \to 0$ in $L^2_{loc}(\mathbb{R})$ as $y \to +\infty$.*

Proof. Let $(u_n) \subset \mathcal{M}$ be a minimizing sequence for φ. Note that, if we consider the sequence $w_n(x,y) = \max\{-1; \min\{1; u_n(x,y)\}\}$ we have that $w_n \in \mathcal{M}$ and $\varphi(w_n) \le \varphi(u_n)$. Then it is not restrictive to assume that $\|u_n\|_{L^\infty(T)} \le 1$ for any $n \in \mathbb{N}$.

It is not difficult to recognize that, fixing any $r > 0$, if $T_r = T \cap \{y < r\}$, then (u_n) is a bounded sequence on $H^1(T_r)$.

Indeed, since $\|u_n\|_{L^\infty(T)} \le 1$ for any $n \in \mathbb{N}$ we have $\|u_n\|_{L^2(T_r)} \le |T_r| < +\infty$ for any $n \in \mathbb{N}$. Moreover, since $\|\partial_y u_n\|^2_{L^2(T_r)} \le 2\varphi(u_n) = 2m + o(1)$ and since

$$\begin{aligned}\|\partial_x u_n\|^2_{L^2(T_r)} &= \int_0^r \int_{I_y} |\partial_x u_n(x,y)|^2 \,dx\,dy \\ &\le 2\varphi_{(0,r)}(u_n) + 2\int_0^r c_y\,dy \le 2\varphi(u_n) + 2cr \le 2(m + cr) + o(1),\end{aligned}$$

there exists a constant $C_r > 0$, depending on r, for which $\|\nabla u_n\|^2_{L^2(T_r)} \le C_r$ for any $n \in \mathbb{N}$, and our claim follows.

Thus, by a classical diagonal argument, there exists $\bar u \in H^1_{loc}(T)$ and a subsequence of (u_n), still denoted (u_n), such that $u_n - \bar u \to 0$ weakly in $H^1_{loc}(T)$ and for almost every $(x,y) \in T$. By pointwise convergence, since $u_n(x,y) = -u_n(-x,y)$ and $\|u_n\|_{L^\infty(T)} \le 1$ for any $n \in \mathbb{N}$ and for almost every $(x,y) \in T$, we find that also $\bar u(x,y) = -\bar u(-x,y)$ for almost every $(x,y) \in T$ and $\|\bar u\|_{L^\infty(T)} \le 1$. Then $\bar u \in \mathcal{M}$ and by the weak lower semicontinuity property of φ we obtain that $\varphi(\bar u) \le m$ and then $\varphi(\bar u) = m$.

Moreover, by Lemma 3.2 it follows that fixing any bounded interval $I \subset \mathbb{R}$, we have

$$\mathsf{dist}_{L^2(I)}(\bar u(\cdot, y), \mathcal{K}) \to 0 \text{ as } y \to +\infty.$$

Then, by Proposition 2.1 and (3.1) we have either $\bar u(\cdot, y) - q^+ \to 0$ or $\bar u(\cdot, y) - q^- \to 0$ in $L^\infty_{loc}(\mathbb{R})$. If the second case occurs, the lemma follows considering the function $-u$. □

By Lemma 3.1 we have that if $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfies $\psi(x,y) = -\psi(-x,y)$ then $\varphi(\bar u + \psi) \ge \varphi(\bar u)$. This is sufficient to show, as stated in the next lemma, that in fact $\bar u$ is a weak solution on T of the equation $-\Delta u + W'(u) = 0$.

Lemma 3.3. *For any $\psi \in C_0^\infty(\mathbb{R}^2)$ we have*

$$\int_T \nabla \bar u \cdot \nabla \psi + W'(\bar u)\psi = 0.$$

Proof. Letting $\psi \in C_0^\infty(\mathbb{R}^2)$ we set

$$\psi_o(x,y) = \tfrac{1}{2}(\psi(x,y) - \psi(-x,y)) \text{ and } \psi_e(x,y) = \tfrac{1}{2}(\psi(x,y) + \psi(-x,y)).$$

Since the functions $\nabla\bar{u}\cdot\nabla\psi_e$ and $\nabla\psi_o\cdot\nabla\psi_e$ are odd in the variable x we have that for any $t>0$ there results

$$\begin{aligned}\tfrac{1}{t}(\varphi(\bar{u}+t\psi)-\varphi(\bar{u})) &= \tfrac{1}{t}(\varphi(\bar{u}+t\psi_o)-\varphi(\bar{u}))\\ &\quad+\tfrac{1}{t}\int_T \tfrac{t^2}{2}|\nabla\psi_e|^2 + W(\bar{u}+t\psi) - W(\bar{u}+t\psi_o)\,dx\,dy.\end{aligned}$$

Since $\bar{u}$ is a minimum point for φ on $\mathcal{M}$ and since $\bar{u}+t\psi_o\in\mathcal{M}$ we then find

$$\tfrac{1}{t}(\varphi(\bar{u}+t\psi)-\varphi(\bar{u})) \ge \tfrac{1}{t}\int_T W(\bar{u}+t\psi)-W(\bar{u})+W(\bar{u})-W(\bar{u}+t\psi_o)\,dx\,dy.$$

By using the dominated convergence theorem, since the function $W'(\bar{u})\psi_e$ is odd in the variable x, we finally obtain

$$\begin{aligned}&\int_T \nabla\bar{u}\cdot\nabla\psi + W'(\bar{u})\psi\,dx\,dy = \lim_{t\to0^+}\tfrac{1}{t}(\varphi(\bar{u}+t\psi)-\varphi(\bar{u}))\\ &\ge \lim_{t\to0^+}\int_T \tfrac{W(\bar{u}+t\psi)-W(\bar{u})}{t} + \tfrac{W(\bar{u})-W(\bar{u}+t\psi_o)}{t}\,dx\,dy = \int_T W'(\bar{u})\psi_e\,dx\,dy = 0.\end{aligned}$$

This proves that

$$\int_T \nabla\bar{u}\cdot\nabla\psi + W'(\bar{u})\psi\,dx\,dy \ge 0$$

for any $\psi\in C_0^\infty(\mathbb{R}^2)$, which is actually equivalent to the statement of the lemma. □

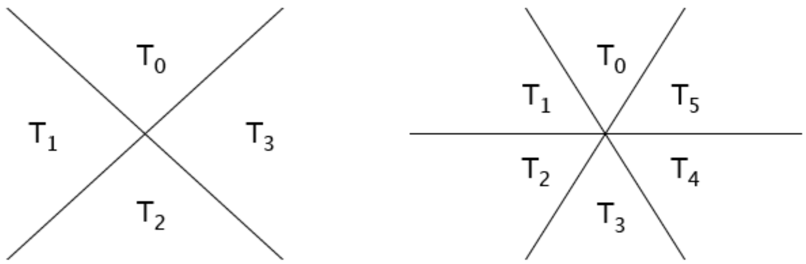
We are now able to construct, from the function $\bar{u}: T\to\mathbb{R}$ given by Proposition 3.1, a function $v_j:\mathbb{R}^2\to\mathbb{R}$ solution to (1.1) satisfying the conditions (Sj) and then to conclude the proof of Theorem 1.1. Indeed, setting $\theta_j = \frac{\pi}{j}$, we consider the rotation matrix

$$A_j = \left\| \begin{matrix} \cos(\theta_j) & \sin(\theta_j)\\ -\sin(\theta_j) & \cos(\theta_j)\end{matrix} \right\|$$

and for $k=0,\dots,2j-1$, we denote by T_k the $k\theta_j$-rotated of T; i.e., $T_k = A_j^k T$. Note that we have $\mathbb{R}^2 = \cup_{k=0}^{2j-1} T_k$, and that if $k_1\ne k_2$, then $\mathrm{int}(T_{k_1})\cap\mathrm{int}(T_{k_2})=\emptyset$ (see Figure 1).

If $j\ge2$ and $0\le k\le 2j-1$, we have $A_j^{-k}T_k = T$, and so we can define

$$v_j(x,y) = (-1)^k\bar{u}(A_j^{-k}(x,y)),\quad \forall (x,y)\in T_k.$$



$j = 2$ $j = 3$

FIGURE 1. The families $\{T_k \,/\, k = 0, \dots, 2j-1\}$ for $j = 2, 3$

Note that $v_j|_{T_1}$ is the reflection of $v_j|_{T_0}$ with respect to the axis which separates T_0 from T_1 and, in general, $v_j|_{T_k}$ is the reflection of $v_j|_{T_{k-1}}$ with respect to the axis separating T_{k-1} from T_k, for any $k \in \{1, \dots, 2j-1\}$. From the properties of the reflection operator (see e.g. [11], Lemma IX.2.), since $\bar{u} \in H^1_{loc}(T_0)$, we find that $v_j \in H^1_{loc}(\mathbb{R}^2)$. Moreover, note that if $\psi \in C_0^\infty(\mathbb{R}^2)$ and $k \in \{1, \dots, 2j-1\}$ then, trivially, $\psi \circ A_j^k \in C_0^\infty(\mathbb{R}^2)$ and so by Lemma 3.3 we obtain

$$\int_{T_k} \nabla v_j \cdot \nabla \psi + W'(v_j)\psi \, dx\, dy = (-1)^k \int_{T_0} \nabla \bar{u} \cdot \nabla \psi \circ A_j^k + W'(\bar{u})\psi \circ A_j^k \, dx\, dy = 0.$$

Hence, for any $\psi \in C_0^\infty(\mathbb{R}^2)$, we find

$$\int_{\mathbb{R}^2} \nabla v_j \cdot \nabla \psi + W'(v_j)\psi \, dx\, dy = \sum_{k=0}^{2j-1} \int_{T_k} \nabla v_j \cdot \nabla \psi + W'(v_j)\psi \, dx\, dy = 0;$$

i.e., v_j is a weak, and so, by standard bootstrap arguments, a classical $C^2(\mathbb{R}^2)$ solution of equation (1.1).

Moreover, setting $\tilde{v}_j(\rho, \theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$, since we know that $\bar{u}(x, y) = -\bar{u}(-x, y)$, by the definition of v_j it follows that $\tilde{v}_j$ satisfies the symmetric requirements in conditions (Sj):

$$\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta) \text{ and } \tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho, \theta), \; \forall (\rho, \theta) \in \mathbb{R}_+ \times \mathbb{R}.$$

Finally, we note that, since $\|v_j\|_{L^\infty} \leq 1$, by local Schauder estimates we have $\|v_j\|_{C^2(\mathbb{R}^2)} < +\infty$. Since $\varphi(v_j|_{T_0}) = m$, this allows us to show in the next Lemma that even the asymptotic requirement in conditions (Sj) is satisfied.

Lemma 3.4. *Let* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$. *Then* $\tilde{v}_j(\rho,\theta) \to 1$ *as* $\rho \to +\infty$.

Proof. Assume for the sake of contradiction that there exists $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$, a sequence $\rho_n \to +\infty$ and a positive number η_0 such that

$$1 - \bar{u}(\rho_n \cos(\theta), \rho_n \sin(\theta)) \geq 2\eta_0.$$

Since $\|v_j\|_{C^2(\mathbb{R}^2)} < +\infty$ this implies that setting

$$(x_n, y_n) = (\rho_n \cos(\theta), \rho_n \sin(\theta)),$$

there exists $r_0 > 0$ such that $1 - \bar{u}(x,y) \geq \eta_0$ for any $(x,y) \in T_0 \cap (\cup_{n=1}^{+\infty} B_{2r_0}(x_n,y_n))$. Since $q^+(x) \to 1$ as $x \to +\infty$, it follows that there exists $\eta_1 > 0$ and $\bar{n} \in \mathbb{N}$ such that $\|\bar{u}(\cdot,y) - q^+\|_{H^1(I_y)} \geq \eta_1$ for $y \in \cup_{n\geq\bar{n}}(y_n - r_0, y_n + r_0)$. By (2.11) this implies that there exists $\nu > 0$ and $\bar{n}_1 > \bar{n}$ such that $F_y(\bar{u}(\cdot,y)) - c_y \geq \nu$ for any $y \in \cup_{n\geq\bar{n}_1}(y_n - r_0, y_n + r_0)$, which gives rise to the contradiction $\varphi(\bar{u}) = +\infty$. □

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