Advances in Differential Equations Volume 12, Number 6 (2007), 669–720

STURMIAN NODAL SET ANALYSIS FOR HIGHER-ORDER PARABOLIC EQUATIONS AND APPLICATIONS

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Abstract. We describe the local pointwise structure of multiple zeros of solutions of $2m$th-order linear uniformly parabolic equations,

$$u_t = \sum_{|\beta| \le 2m} a_\beta(x,t) D_x^\beta u \quad \text{in } \mathbf{R}^N \times [-1,1] \quad (m \ge 2), \tag{0.1}$$

with bounded and Lipschitz-continuous (for $|\beta| = 2m$) coefficients, in the existence-uniqueness class $\{|u(x,t)| \le B e^{b|x|^\alpha}\}$, where $B, b > 0$ are constants and $\alpha = \frac{2m}{2m-1}$. Assuming that $u(0,0) = 0$ and using the *Sturmian backward continuation blow-up variable* $y = x/(-t)^{\frac{1}{2m}} \quad (t < 0)$, we perform a classification of all possible types of formation as $t \to 0^-$ of multiple spatial zeros of the solutions $u(x,t)$. We show that there exists a countable family of multiple zeros evolving as $t \to 0^-$ according to the nodal sets of *polynomial* eigenfunctions of a non-self-adjoint operator $\mathbf{B}^*$ associated with that in (0.1).

Next, we show that other related *polynomial* solutions occur in the collapse of multiple zeros as $t \to 0^+$, which is described in terms of the *forward continuation variable* $y = x/t^{\frac{1}{2m}} \quad (t > 0)$.

For the 1D second-order ($m = 1$) parabolic equation with smooth coefficients,

$$u_t = a(x,t) u_{xx} + q(x,t) u \quad (a(x,t) \ge a_0 > 0),$$

this two-step analysis is known as Sturm's Second Theorem on zero sets, established by C. Sturm in 1836, [32]. His more famous First Theorem (*the number of zeros of solutions is non-increasing with time*) was derived as a consequence of the second one. In the last thirty years these PDE ideas of Sturm found new applications, generalizations and extensions in various areas of general parabolic theory, stability and orbital connection problems, unique continuation and Poincaré–Bendixson theorems, mean curvature and curve shortening flows, symplectic geometry, etc.

Accepted for publication: December 2006.
AMS Subject Classifications: 35K55, 35K65.
Research supported by RTN network HPRN-CT-2002-00274 and CERN-INTAS00-0136.

Using such a local classification of multiple zeros, we establish a unique continuation theorem for higher-order parabolic PDEs and inequalities, and estimate the Hausdorff dimension of nodal sets of solutions. It turns out that some common features of multiple zeros formation can be observed for solutions of other PDEs including linear dispersion and wave equations,

$$u_t = u_{xxx} \quad \text{and} \quad u_{tt} = u_{xx},$$

and we present a discussion on this issue.

1. Introduction: scaling approach to multiple zeros and results

1.1. $2m$th-order linear parabolic PDEs. Consider a general $2m$th-order linear parabolic equation ($m \ge 2$ is an arbitrary integer)

$$u_t = \sum_{|\beta| \le 2m} a_\beta(x,t) D^\beta u \quad \text{in} \;\; Q_1 = \mathbf{R}^N \times [-1,1], \tag{1.1}$$

where the coefficients $\{a_\beta\}$ are real and bounded for all $|\beta| \le 2m$, locally Lipschitz continuous for $|\beta| = 2m$, and satisfy the *parabolicity condition*: there exists a constant $\delta > 0$ such that

$$(-1)^m \sum_{|\beta|=2m} a_\beta(x,t)\xi^\beta \le -\delta|\xi|^{2m} \quad \text{for all} \;\; (x,t) \in Q_1 \text{ and } \xi \in \mathbf{R}^N. \tag{1.2}$$

Let $u(x,t)$ be a classical, $C^{2m,1}_{x,t}$, solution of (1.1) in the existence-uniqueness class of locally measurable functions

$$\mathcal{U} = \big\{ |u(x,t)| \le B \mathrm{e}^{b|x|^\alpha} \big\}, \quad \text{with the exponent} \;\; \alpha = \tfrac{2m}{2m-1}, \tag{1.3}$$

where B and b are positive constants; see classic parabolic theory, [9, 13].

1.2. Sturmian backward blow-up rescaling and "micro-structure" of PDEs. What is micro-structure of evolution equations? The main goal of the paper is to show how to detect the local pointwise structure of the solutions at any fixed internal point in Q_1 using the *optimal parabolic blow-up* $\{x,t\}$-scaling. This leads to delicate and difficult asymptotic problems. Even among canonical linear PDEs, there are just a few examples admitting such an analysis. Most of them, starting from Sturm's pioneering paper [32] published in 1836, are associated with the *heat equation* in $\mathbf{R}^N$,

$$u_t = \Delta u \equiv \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}, \tag{1.4}$$

which has known pointwise structure of solutions; see references given below and comments in the final section concerning PDEs of other types.

The results on local solution structure play a special role in the theory of linear and non-linear PDEs. In general, once a local pointwise structure of possible solutions is known, including description of admitted "singularities," existence-uniqueness theory is produced by fixing those functional classes and settings which exclude structures and singularities violating the desired "regularity" or uniqueness of solutions. The pointwise approach to singularity formation phenomena in PDEs is often referred as *blow-up* theory. The pointwise analysis of singularities differs from several other (and often much more common and famous for classic PDEs) directions of the theory devoted to various *a priori* bounds, estimates, integral and entropy inequalities, etc., up to statistical, stochastic, and even chaotic ("turbulent") properties of general solution subsets, where probability and averaging methods apply. These do not have straightforward connections with the micro-scale pointwise structure of solutions and cannot reveal it in principle.

In mechanics and physics, dealing with continuous media like fluids, gases, or porous media, the questions of the local pointwise behaviour of parameters are often called problems of "micro-structure" (or the "turbulent" molecular structure in fluid dynamics) of the medium. This local mechanism of turbulence is often very difficult, but takes responsibility for the eventual evolutionary development of global *coherent macro-patterns*. In our present case, the artificial "medium" is represented and created by solutions of the PDE under consideration. Loosely speaking, we are actually going to study the internal "atomic-molecular-turbulent" structure of the given class of PDEs.

On two mathematical blow-up challenges of the 21st century. For a lot of non-linear evolution PDEs and systems, the basic questions of their micro-structure are extremely difficult and lead to problems that remain open and will not be solved for a long time. Many of these "local" questions are often the most difficult in PDE theory (it seems that this is not still well recognized).

For instance, one of the most famous related areas is characterized by the *Navier–Stokes equations* in $\mathbf{R}^3$ (and other similar PDE systems in $\mathbf{R}^N$, with $N \geq 3$), where the unknown (existent or not) "turbulent" micro-structure has generated a lot of speculations, discussions, and fundamental open theoretical concepts. It seems that a positive (a negative one needs, e.g., a suitable new and, clearly, very hard a priori estimate) answer to those issues on turbulent behaviour are not possible without a detailed analysis of the blow-up micro-structure, which becomes very difficult for such special PDE systems.

Another PDE area, not less famous nowadays, is the the study of blow-up structure of *Ricci flows* in $\mathbf{R}^2$. *Perel'man's new monotonicity formulae* made it possible to guarantee getting a unique symmetric continuation beyond blow-up singularity in such second-order geometric flows, and hence to solve the *Poincaré Conjecture* (a closed connected 3D manifold is homeomorphic to $\mathbf{S}^3$); see [5] for history, references, and recent development.

Indeed, it is curious and, in fact, not that surprising, that these two, possibly the most famous mathematical problems that remained open during an essential part of the 20th century, are directly attributed to blow-up theory (micro-structure) of non-linear PDEs.

Sturmian blow-up rescaling: a "zoom" to micro-structure. We return to the linear parabolic PDEs (1.1) that represent a rather general object, a medium as suggested, with a difficult internal micro-structure to be described. To this end, we will study the finite-time *blow-up* asymptotic behaviour of arbitrary solutions $u(x,t)$ of (1.1) at a fixed point, which is the origin $(0,0)$ for definiteness. We use the *Sturmian backward continuation blow-up variable*, which for $m = 1$, was first introduced by Sturm himself [32] (see comments below),

$$y = x/(-t)^{\frac{1}{2m}}, \quad (x,t) \in Q_1^- = \mathbf{R}^N \times [-1,0), \tag{1.5}$$

so blow-up occurs at $t = 0^-$. We next introduce the corresponding new time variable

$$\tau = -\ln(-t) \to +\infty \quad \text{as } \ t \to 0^-. \tag{1.6}$$

As usual in blow-up problems, the main feature and the fundamental sense of the blow-up rescaling (1.5), (1.6) is that, as $t \to 0^-$, i.e., as $\tau \to +\infty$, the behaviour of solutions on compact subsets in y, $|y| \le C = \text{const}$, gives a parabolic *"zoom"* embracing fast shrinking subsets in the original spatial variable x, on which

$$\{|y| \le C\} \quad \Longrightarrow \quad \big\{|x| \le C(-t)^{\frac{1}{2m}} \equiv C\mathrm{e}^{-\frac{1}{2m}\tau} \to 0\big\}. \tag{1.7}$$

The rescaled variable (1.5) is purely dimensional for every $2m$th-order linear or quasi-linear uniformly parabolic equation (depending on the highest order $2m$ only). It turns out that, on smaller compact subsets in $\{y,\tau\}$, solution structures are less interesting and rather poor. Therefore, the rescaled variables (1.5), (1.6) define the optimal, minimally possible scales of the non-trivial "turbulent" micro-behaviour available in the PDEs (1.1) with sufficiently smooth coefficients.

Multiple zeros via two limits: Sturm's approach. In order to describe the micro-structure provided by the PDE, we study formation and collapse of *multiple zeros* of a given classical solution $u(x,t)$ of (1.1) assuming that

$$u(0,0) = 0. \tag{1.8}$$

We then need to perform a suitable asymptotic analysis of the behaviour of the solution that includes two limits:

(I) Formation of multiple zeros: the *blow-up limit* as $t \to 0^-$, and, as the next step,

(II) Collapse of multiple zeros: the *extension limit* as $t \to 0^+$ establishing the behaviour beyond "singularity" occurred at $t = 0^-$.

In fact, these are the concepts of Sturm's approach to multiple zeros [32], who solved both asymptotic problems (I) and (II) for $m = 1$ in 1D. This comprises Sturm's Second Theorem in parabolic PDE theory; see further details and references in [14, Chapter 1].

The first limit (I) is of crucial importance and actually determines all possible types of multiple zeros which can occur in the parabolic equation (1.1). This gives a countable variety of such micro-patterns and hence explains the degree of local "turbulence" available. The second limit (II) then uniquely depends on the first one and reduces to a standard asymptotic analysis of sufficiently smooth extension of solutions beyond the singularity occurring at $t = 0^-$.

1.3. **Perturbed rescaled equation.** We next need to explain the nature and the origin of the micro-structure of solutions on subsets (1.7). Since by (1.5)

$$D_x^\beta u \equiv (-t)^{-\frac{|\beta|}{2m}} D_y^\beta u,$$

in terms of the new independent variables $\{y, \tau\}$ given in (1.5), (1.6), the solution $u = u(y,\tau)$ satisfies the following perturbed *rescaled equation*:

$$u_\tau = \mathbf{B}^* u + \mathbf{C}(\tau) u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \tag{1.9}$$

where $\mathbf{B}^*$ is the $2m$th-order elliptic operator with the *unbounded* coefficient y,

$$\mathbf{B}^* = \sum_{|\beta|=2m} A_\beta D_y^\beta - \frac{1}{2m}\, y \cdot \nabla_y, \quad \text{with } A_\beta = a_\beta(0,0). \tag{1.10}$$

For convenience, $\mathbf{B}^*$ is written as adjoint to another operator $\mathbf{B}$ to be presented later on. The higher-order principle counterpart in (1.10),

$$\mathbf{B}_0 = \sum_{|\beta|=2m} A_\beta D_y^\beta,$$

is a symmetric homogeneous $2m$th-order elliptic operator with constant coefficients.

The time-dependent perturbation $\mathbf{C}(\tau)$ in (1.9) is given by

$$\mathbf{C}(\tau) = \sum_{|\beta|=2m} R_\beta(y,\tau) D_y^\beta + \sum_{|\beta|<2m} \mathrm{e}^{-\frac{2m-|\beta|}{2m}\tau} a_\beta\big(y\mathrm{e}^{-\frac{1}{2m}\tau}, -\mathrm{e}^{-\tau}\big) D_y^\beta, \quad (1.11)$$

where $R_\beta(y,\tau) \equiv a_\beta\big(y\mathrm{e}^{-\frac{1}{2m}\tau}, -\mathrm{e}^{-\tau}\big) - a_\beta(0,0)$.

Therefore, $\mathbf{C}(\tau)$ is *exponentially small* for coefficients $\{a_\beta\}$ that are locally Lipschitz continuous for all $|\beta| = 2m$ and are uniformly bounded for all $|\beta| \le 2m$. As we mentioned, these are the main assumptions on the PDE coefficients of (1.1). Then, as $\tau \to \infty$, uniformly on compact subsets in y,

$$R_\beta(y,\tau) = O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big), \quad (1.12)$$

which is enough for further analysis. In fact, the assumption on Lipschitz continuity for $|\beta| = 2m$ can be weakened. It follows that on smooth solutions, $\mathbf{C}(\tau)u$ in (1.9) is an exponentially small perturbation satisfying as $\tau \to \infty$, uniformly on compact subsets,

$$|\mathbf{C}(\tau)u| = O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big). \quad (1.13)$$

Further estimates of perturbations are to be performed in the weighted Sobolev spaces associated with operator (1.10) and the adjoint one to be introduced next.

1.4. **Sections 2–5: linear non-self-adjoint operators.** It follows from equation (1.9) that, first, one needs to study spectral and other properties of the linear operator $\mathbf{B}^*$ and of the adjoint one $\mathbf{B}$. Section 2 is devoted to some preliminaries concerning the fundamental solutions of operators $\frac{\partial}{\partial t} - \mathbf{B}$ and $\frac{\partial}{\partial t} - \mathbf{B}^*$, semigroups $\mathrm{e}^{\mathbf{B}\tau}$ and $\mathrm{e}^{\mathbf{B}^*\tau}$, and resolvents of $\mathbf{B}$ and $\mathbf{B}^*$. Here we study the unperturbed homogeneous parabolic equations

$$w_\tau = \mathbf{B}w \quad \text{and} \quad w_\tau = \mathbf{B}^*w \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+. \quad (1.14)$$

In Section 3, we describe the spectral properties of the adjoint operator

$$\mathbf{B} = \sum_{|\beta|=2m} A_\beta D_y^\beta + \frac{1}{2m}\, y \cdot \nabla_y + \frac{N}{2m} I, \quad (1.15)$$

in the weighted space $L^2_\rho = L^2_\rho(\mathbf{R}^N)$, where

$$\rho(y) = \mathrm{e}^{a|y|^\alpha} > 0 \quad \text{and } a > 0 \text{ is a sufficiently small constant.} \quad (1.16)$$

Throughout the paper, we will use the notation

$$\langle \cdot, \cdot \rangle \text{ is the (dual, see below) scalar product in } L^2;$$

$\langle \cdot, \cdot \rangle_\rho$ and $\| \cdot \|_\rho$ are scalar product and induced norm in $L^2_\rho(\mathbf{R}^N)$.

We show that $\mathbf{B}$ has the point spectrum only, $\sigma(\mathbf{B}) = \big\{\lambda_\beta = -\frac{|\beta|}{2m}, \, |\beta| \geq 0\big\}$, and the eigenfunctions $\Phi = \{\psi_\beta\}$ form a complete subset in L^2_ρ. In Section 4, we study spectral properties of the adjoint operator $\mathbf{B}^*$ in $L^2_{\rho^*}(\mathbf{R}^N)$ with $\rho^* = \frac{1}{\rho}$ and describe the complete subset $\Phi^* = \{\psi^*_\beta\}$ of its *polynomial* eigenfunctions. In Section 5, we introduce subspaces in which the eigenfunction subsets Φ and Φ^* are closed.

1.5. **Section 6: main results on formation and collapse of multiple zeros.** Using eigenfunction expansions, we show that multiple zeros at the origin $(0,0)$ of any suitable solution $u(x,t) \not\equiv 0$ of (1.9) has a local structure corresponding to stable subspace of $\mathbf{B}^*$. Namely, in Section 6, we establish that for any such solution, there exists a finite $l > 0$ such that as $\tau \to \infty$,

$$u(y,\tau) = \mathrm{e}^{-\frac{l}{2m}\tau}[\varphi^*_l(y) + o(1)] \quad \text{uniformly on compact subsets,} \tag{1.17}$$

where φ^*_l is a polynomial eigenfunction of $\mathbf{B}^*$,

$$\varphi^*_l(y) = \sum_{|\beta|=l} c_\beta \psi^*_\beta(y) \not\equiv 0, \tag{1.18}$$

corresponding to the eigenvalue $\lambda_l = -\frac{l}{2m} < 0$. Therefore, (1.17) describes all possible types of formation of multiple zeros for uniformly parabolic PDEs (1.1). Hence, any blow-up formation of an lth-order multiple zero at $(0,0)$ after rescaling (1.5) is driven as $t \to 0^-$ by zero surfaces of an lth-order polynomial eigenfunction (1.18) of $\mathbf{B}^*$. Such polynomial distribution of zero surfaces and other related spatial features is actually the micro-structure (a turbulence mechanism) of the PDEs (1.1). It has important applications to be revealed later.

We next study the easier counterpart limit $t \to 0^+$ that extends the multiple zeros evolution for $t > 0$ and describes their *collapse*. This regular limit is performed via the *forward continuation* independent variables

$$y = x/t^{\frac{1}{2m}}, \quad \tau = \ln t. \tag{1.19}$$

In this case, polynomial functions associated with the operator $\mathbf{B}$ in (1.15) occur.

1.6. **Second-order parabolic equations: on Sturm's classification of zeros.** This zero formation-collapse analysis is classical for $m = 1$. In one dimension, it was performed by Sturm [32] for C^∞ solutions of linear parabolic equations such as

$$u_t = a(x,t)u_{xx} + q(x,t)u \quad (a(x,t) \ge a_0 > 0).$$

Actually, Sturm considered more general equations such as[1]

$$gu_t = (ku_x)_x - lu, \tag{1.20}$$

where g, k, and l are smooth functions of x, as well as t [32, p. 431]. See [14, pp. 3–5], where these basic Sturm's calculations are presented, and [14, Chapter 1] for history, key references, and main applications of two Sturm's theorems in general PDE theory. As a consequence of his analysis of multiple zeros for (1.20) showing that zero curves can only disappear in evolution, famous Sturm's First Theorem was formulated [32, p. 431] establishing that the number of zeros of solutions cannot increase with time.

For instance, for the *heat equation* (1.4) in Q_1^- $(m = 1)$, the multiple zeros classification is as follows. The corresponding Sturm variable (1.5),

$$y = x/\sqrt{-t},$$

yields the rescaled equation (1.14) with the second-order symmetric operator

$$\mathbf{B}^* = \Delta - \frac{1}{2} y \cdot \nabla \equiv \frac{1}{\rho^*} \nabla \cdot (\rho^* \nabla), \quad \text{where } \rho^*(y) = \mathrm{e}^{-\frac{1}{4}|y|^2}. \tag{1.21}$$

In this case, $\mathbf{B}^*$ is known to be self-adjoint in $L^2_{\rho^*}(\mathbf{R}^N)$ with the domain $H^2_{\rho^*}(\mathbf{R}^N)$. It has the discrete spectrum

$$\sigma(\mathbf{B}^*) = \big\{ \lambda_\beta = -\tfrac{|\beta|}{2},\, |\beta| \ge 0 \big\},$$

and the resolvent is a compact integral operator. The eigenfunctions $\{\psi_\beta\}$, which are separable Hermite polynomials $c_\beta H_\beta$ in $\mathbf{R}^N$ [3] (c_β are normalization multipliers), given by the generating formula

$$D^\beta \mathrm{e}^{-\frac{1}{4}|y|^2} = H_\beta(y) \mathrm{e}^{-\frac{1}{4}|y|^2},$$

form an orthonormal basis in $L^2_{\rho^*}$. Since for $m = 1$, (1.21) is self-adjoint, the classical Agmon-Ogawa estimates apply to the corresponding perturbed rescaled equations like (1.9) to ensure the convergence (1.17). We refer to a detailed analysis for $m = 1$ in [6].

[1] These are actual Sturm's notations.

1.7. **Sections 7 and 8: applications to unique backward continuation and global properties of nodal sets for parabolic PDEs.** Using the optimal characterization (1.17) of arbitrary multiple zeros for solutions $u(x,t) \not\equiv 0$, in Section 7 we establish a unique backward continuation theorem for $2m$th-order parabolic equations.

In Section 8 we study some global properties of the nodal set

$$Z_t[u] = \{x \in \mathbf{R}^N : \ u(x,t) = 0\}, \ \ t \in (-1,1), \tag{1.22}$$

of non-trivial solutions to (1.1). We prove that the Hausdorff dimension of $Z_t[u]$ satisfies

$$\dim_{\mathcal{H}} Z_t[u] \le N - 1. \tag{1.23}$$

1.8. **Section 9: four final comments on multiple zeros in other PDEs.** It is interesting that definite influence of *polynomial eigenfunction* formation of multiple zeros as the result of micro-scale *blow-up* asymptotic analysis are traced out in other, not necessarily parabolic, PDEs, exhibiting different natures of evolution. Rigorous results here are very difficult and still not completed. Nevertheless, we feel that a short discussion concerning recent results in this related area would be appropriate, at least, in order to compare with more mathematically consistent analysis of parabolic PDEs.

(i) First, it is natural to touch some quasi-linear and degenerate parabolic equations.

(ii) Second, it is surprising that polynomial eigenfunctions occur in multiple zeros analysis for *odd-order linear dispersion* PDEs such as (1.1) with $2m \mapsto 2m+1$, e.g., for the third-order PDE

$$u_t = u_{xxx} \quad \text{in} \quad Q_1 = [-1,1] \times \mathbf{R} \quad (m=1, \ N=1). \tag{1.24}$$

The corresponding spectral theory has some distinct features in comparison with that in the parabolic case.

(iii) Third, polynomial eigenfunctions of *quadratic pencils* of linear operators appear in formation of multiple zeros generated by solutions $u(x,t)$ of the linear *wave equation*

$$u_{tt} = u_{xx} \quad \text{in} \quad Q_1. \tag{1.25}$$

A similar polynomial eigenfunction theory can be produced for the fourth-order (and higher-order) *hyperbolic* PDEs such as

$$u_{tt} = -\Delta^2 u \quad \text{in} \quad Q_1 = [-1,1] \times \mathbf{R}^N, \tag{1.26}$$

though, obviously, a rigorous justification becomes more difficult.

(iv) Eigenfunction analysis of multiple zeros applies to a wide class of *integral evolution equations*. This class comprises all the above PDEs and many others that do not admit a purely differential (but pseudo-differential) interpretation. Such integral evolution contains necessary spectral and polynomial-like characteristics for a similar multiple zero analysis.

2. PRELIMINARIES: FUNDAMENTAL SOLUTIONS, SEMIGROUPS, RESOLVENTS

2.1. **Fundamental solution.** Consider the Cauchy problem for the homogeneous $2m$th-order parabolic equation with constant coefficients

$$u_t = \mathbf{B}_0 u \equiv \sum_{|\beta|=2m} A_\beta D_x^\beta u \quad \text{in } \mathbf{R}^N \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \in \mathcal{U} \cap \{t=0\}. \tag{2.1}$$

Let $b(x,y)$ be the fundamental solution of the operator $\frac{\partial}{\partial t} - \mathbf{B}_0$, [9, 13]. It has the self-similar form

$$b(x,t) = t^{-\frac{N}{2m}} F(y), \quad y = x/t^{\frac{1}{2m}}, \tag{2.2}$$

where F is a unique solution of the linear elliptic equation

$$\mathbf{B}F \equiv \mathbf{B}_0 F + \tfrac{1}{2m} y \cdot \nabla_y F + \tfrac{N}{2m} F = 0 \quad \text{in } \mathbf{R}^N, \quad \text{with} \int_{\mathbf{R}^N} F = 1. \tag{2.3}$$

The following estimates holds [9]:

$$|F(y)| \le D\mathrm{e}^{-d|y|^\alpha} \quad \text{in } \mathbf{R}^N, \quad \alpha = \tfrac{2m}{2m-1}, \tag{2.4}$$

where D and d are positive constants. The unique solution of (2.1) is given by the convolution

$$u(x,t) = b(t) * u_0 \equiv t^{-\frac{N}{2m}} \int_{\mathbf{R}^N} F\big((x-z)t^{-\frac{1}{2m}}\big) u_0(z)\, \mathrm{d}z. \tag{2.5}$$

Taking an extra weight via (1.16) and assuming that

$$\int \rho^\nu |u_0|^2 < \infty,$$

for arbitrarily small constant $\nu > 0$, yields that (2.5) determines the global solution.

2.2. **Semigroup with infinitesimal generator B in (1.15).** The rescaled solution

$$w(y,\tau) = t^{\frac{N}{2m}} u\big(y t^{\frac{1}{2m}}, t\big), \quad \text{where } \tau = \ln t \in \mathbf{R}, \tag{2.6}$$

satisfies the parabolic equation

$$w_\tau = \mathbf{B}w. \tag{2.7}$$

One can see that $w(y,\tau)$ satisfies the Cauchy problem for (2.7) in $\mathbf{R}^N \times \mathbf{R}_+$ with initial data at $\tau = 0$ (i.e., at $t = 1$)

$$w_0(y) = u(y,1) \equiv b(y-\cdot,1) * u_0(\cdot). \tag{2.8}$$

Rescaling convolution (2.5) yields the following explicit representation of the semigroup with the infinitesimal generator $\mathbf{B}$:

$$w(y,\tau) = \mathrm{e}^{\mathbf{B}\tau} u(y,1) \equiv \int_{\mathbf{R}^N} F\big(y - z\mathrm{e}^{-\frac{1}{2m}\tau}\big) u_0(z)\,\mathrm{d}z \quad \text{for} \quad \tau \ge 0. \tag{2.9}$$

Performing another rescaling $w(y,\tau) = (1+t)^{\frac{N}{2m}} u\big(y(1+t)^{\frac{1}{2m}}, t\big)$ with the new time variable $\tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+$, we obtain the solution $w(y,\tau)$ of the Cauchy problem for equation (2.7) with initial data $w_0(y) \equiv u_0(y)$. Rescaling (2.5), we deduce a more complicated, but standard (without the relation (2.8)), representation of the semigroup

$$w(y,\tau) = \mathrm{e}^{\mathbf{B}\tau} u_0 \equiv (1-\mathrm{e}^{-\tau})^{-\frac{N}{2m}} \int F\big((y - z\mathrm{e}^{-\frac{1}{2m}\tau})(1-\mathrm{e}^{-\tau})^{-\frac{1}{2m}}\big) u_0(z)\,\mathrm{d}z. \tag{2.10}$$

2.3. **Semigroup with the adjoint infinitesimal generator $\mathbf{B}^*$ in (1.10).** In order to construct the semigroup with the infinitesimal generator $\mathbf{B}^*$, we use the rescaled variables corresponding to blow-up as $t \to 1^-$,

$$u(x,t) = w(y,\tau), \quad y = x/(1-t)^{\frac{1}{2m}}, \quad \tau = -\ln(1-t) : (0,1) \to \mathbf{R}_+.$$

Then w solves the problem

$$w_\tau = \mathbf{B}^* w \quad \text{for} \ \ \tau > 0, \quad w(0) = u_0. \tag{2.11}$$

Rescaling (2.5), we obtain the following explicit representation of the semigroup:

$$w(y,\tau) = \mathrm{e}^{\mathbf{B}^*\tau} u_0 \equiv (1-\mathrm{e}^{-\tau})^{-\frac{N}{2m}} \int F\big((y\mathrm{e}^{-\frac{1}{2m}\tau} - z)(1-\mathrm{e}^{-\tau})^{-\frac{1}{2m}}\big) u_0(z)\,\mathrm{d}z. \tag{2.12}$$

2.4. **Resolvents.** By the descent method for constructing of resolvents, [9], fixing $\lambda \in \mathbb{C}$, we consider an auxiliary non-homogeneous problem,

$$w_\tau = \mathbf{B}w - \mathrm{e}^{\lambda\tau} g \quad \text{for} \quad \tau > 0 \ \ \text{with} \quad w(0) = 0.$$

Here we assume that g belongs to the weighted L^2-space $L^2_\rho(\mathbf{R}^N)$ introduced in the next section. Performing formal computations and setting $w = \mathrm{e}^{\lambda\tau} v$ yields the equation

$$v_\tau = (\mathbf{B} - \lambda I)v - g,$$

and hence

$$v(\tau) = -\int_0^\tau e^{(\mathbf{B}-\lambda I)(\tau-s)} g \, \mathrm{d}s.$$

Setting $\tau - s = \eta$ and letting $\tau \to \infty$ yields that there exists a limit

$$v(\infty) = -\int_0^\infty e^{(\mathbf{B}-\lambda I)\eta} g \, \mathrm{d}\eta \equiv (\mathbf{B}-\lambda I)^{-1} g,$$

provided that the integral converges. Using the semigroup representation (2.10) and changing the variable $e^{-\eta} = z \in (0,1)$ give the integral operator

$$\begin{aligned} &(\mathbf{B}-\lambda I)^{-1} g = \textstyle\int_{\mathbf{R}^N} K(y,\zeta) g(\zeta) \, \mathrm{d}\zeta, \quad \text{with the kernel} \\ &K(y,\zeta) = -\textstyle\int_0^1 z^{\lambda-1} (1-z)^{-\frac{N}{2m}} F\big((y-\zeta z^{\frac{1}{2m}})(1-z)^{-\frac{1}{2m}}\big) \, \mathrm{d}z. \end{aligned} \tag{2.13}$$

Similarly, representation of the resolvent of the adjoint operator $\mathbf{B}^*$ is

$$\begin{aligned} &(\mathbf{B}^*-\lambda I)^{-1} g = \textstyle\int_{\mathbf{R}^N} K^*(y,\zeta) g(\zeta) \mathrm{d}\zeta, \quad \text{where} \\ &K^*(y,\zeta) = -\textstyle\int_0^1 z^{\lambda-1} (1-z)^{-\frac{N}{2m}} F\big((y z^{\frac{1}{2m}}-\zeta)(1-z)^{-\frac{1}{2m}}\big) \, \mathrm{d}z. \end{aligned} \tag{2.14}$$

Both operators in (2.13) and (2.14) are compact for $\lambda \notin \sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$; see below.

3. SPECTRAL PROPERTIES OF B

We begin with spectral properties of the operator in (2.3) which for $m > 1$ is not symmetric and does not admit a self-adjoint extension in any weighted space $L^2_\rho = L^2_\rho(\mathbf{R}^N)$ for arbitrary weights $\rho > 0$. As a differential operator with smooth coefficients, it is closable [20]. We consider $\mathbf{B}$ in the weighted space L^2_ρ with the exponentially growing weight function (1.16), where a is a small positive constant and, at least,

$$a \in (0, 2d) \quad \big(d \text{ is as in (2.4)}\big). \tag{3.1}$$

The case $m = 1$ is the only one where the operator adjoint to (1.21) is symmetric in a weighted L^2-space and admits a unique Friedrichs self-adjoint extension [3]. Then

$$\mathbf{B} = \Delta + \frac{1}{2} y \cdot \nabla + \frac{N}{2} I \equiv \frac{1}{\rho} \nabla \cdot (\rho \nabla) + \frac{N}{2} I, \tag{3.2}$$

where $\rho(y) = e^{|y|^2/4}$ is the inverse Gaussian kernel, i.e., (1.16) with $\alpha = 2$ and $a = \frac{1}{4}$. Note that if, according to (3.1), $a < \frac{1}{2}$, but $a \neq \frac{1}{4}$, then (3.2) is not self-adjoint in L^2_ρ, but nevertheless enjoys a number of good spectral properties which, as we will show, remain valid for any $m > 1$.

The crucial spectral and various other properties of eigenvalues and eigenfunctions of $\mathbf{B}$ can be obtained directly from the explicit representation of the (analytic) semigroup (2.9) or (2.10). Indeed, this is an important advantage of the spectral analysis.

3.1. **Domain of the operator.** Let us introduce a weighted Sobolev space, which is a Hilbert space H^{2m}_ρ of functions, with the inner product and the norm

$$\langle v,w\rangle_{2m,\rho}=\int_{\mathbf{R}^N}\rho\sum_{k=0}^{2m}D^k v\,D^k w\,\mathrm{d}y\quad\text{and}\quad\|v\|^2_{2m,\rho}=\int_{\mathbf{R}^N}\rho\sum_{k=0}^{2m}|D^k v|^2\,\mathrm{d}y, \tag{3.3}$$

where $D^k v$ denote vectors $\{D^\beta v,\ |\beta|=k\}$. Then $H^{2m}_\rho\subset L^2_\rho\subset L^2$.

Proposition 3.1. $\mathbf{B}: H^{2m}_\rho\to L^2_\rho$ *is a bounded linear operator.*

Proof. It follows from (2.3) that $\mathbf{B}v\in L^2_\rho$ for any $v\in H^{2m}_\rho$ provided that

$$\int_{\mathbf{R}^N}\rho|y\cdot\nabla v|^2\,\mathrm{d}y\le C\|v\|^2_{2m,\rho}\quad\text{for any}\ \ v\in H^{2m}_\rho, \tag{3.4}$$

with a constant $C>0$. The proof follows the lines of a similar analysis in [8, Section 2]. □

Embeddings like (3.4) are associated with the well-known general estimates in weighted spaces (see Maz'ja [30, p. 40] and Heinig [22, Lemma 2.1]), which go back to the Hardy classic inequality (1920), [21].

3.2. **Discrete spectrum.**

Lemma 3.1. (i) *The spectrum of* $\mathbf{B}$ *consists of real eigenvalues only,*

$$\sigma(\mathbf{B})=\big\{\lambda_\beta=-\tfrac{|\beta|}{2m},\ \ |\beta|=0,1,2,\dots\big\}, \tag{3.5}$$

and eigenvalues λ_β *have finite multiplicity with eigenfunctions*

$$\psi_\beta(y)=\tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}D^\beta F(y), \tag{3.6}$$

where F *is the analytic rescaled kernel of the fundamental solution* (2.2).
(ii) *The eigenfunction subset* $\Phi=\{\psi_\beta\}$ *is complete in* L^2 *and in* L^2_ρ.
(iii) *Resolvent* $(\mathbf{B}-\lambda I)^{-1}$ *is compact in* L^2_ρ.

As we have mentioned in the introduction, in the case $m=1$, for the operator (3.2), we have that F is the positive rescaled Gaussian kernel $F(y)=(4\pi)^{-\frac{N}{2}}\,\mathrm{e}^{-\frac14|y|^2}$, and the eigenfunctions are

$$\psi_\beta(y)=c_\beta\mathrm{e}^{-\frac{|y|^2}{4}}\,H_\beta(y),\quad H_\beta(y)\equiv H_{\beta_1}(y_1)\cdots H_{\beta_N}(y_N),$$

where the H_β denote separable Hermite polynomials in $\mathbf{R}^N$. The operator $\mathbf{B}$ with domain H^2_ρ, where $\rho = \mathrm{e}^{|y|^2/4}$, is self-adjoint and the eigenfunctions form an orthonormal basis in L^2_ρ; see [3, p. 48]. For $m > 1$, the eigenfunctions are orthogonal to the adjoint ones in the topology of the dual L^2-space.

Proof of Lemma 3.1**.** (i) Spectrum and eigenfunctions. Fix an $l = |\beta| \geq 0$. The existence of such eigenvalues and eigenfunctions follows by applying D^β to the elliptic equation (2.3)

$$D^\beta \mathbf{B} F \equiv \mathbf{B} D^\beta F + \tfrac{|\beta|}{2m}\, D^\beta F = 0. \tag{3.7}$$

Using semigroups. In order to show that $\mathbf{B}$ admits no other eigenvalues, we consider the explicit semigroup representation (2.9). We use Taylor's power series of the analytic kernel

$$F\big(y - z\mathrm{e}^{-\frac{1}{2m}\tau}\big) = \sum_{(\beta)} \mathrm{e}^{-\frac{|\beta|}{2m}\tau}\, \frac{(-1)^{|\beta|}}{\beta!} D^\beta F(y) z^\beta \equiv \sum_{(\beta)} \mathrm{e}^{-\frac{|\beta|}{2m}\tau}\, \psi_\beta(y)\, \frac{1}{\sqrt{\beta!}}\, z^\beta, \tag{3.8}$$

where $z^\beta \equiv z_1^{\beta_1} \cdots z_N^{\beta_N}$. In Section 5, we show that such a series converges uniformly on any compact subset and, obviously, hence defines the known analytic rescaled kernels of fundamental solutions. Substituting (3.8) into (2.9), we arrive at the following eigenfunction expansion of the solution:

$$w(y,\tau) = \sum_{(\beta)} \mathrm{e}^{-\frac{|\beta|}{2m}\tau} M_\beta(u_0) \psi_\beta(y), \tag{3.9}$$

where $\lambda_\beta = -\frac{|\beta|}{2m}$ for any β and $\psi_\beta(y)$ are the eigenvalues (the point spectrum) and eigenfunctions of $\mathbf{B}$. Here

$$M_\beta(u_0) = \frac{1}{\sqrt{\beta!}} \int_{\mathbf{R}^N} z^\beta u_0(z)\, \mathrm{d}z \tag{3.10}$$

are the moments of the initial datum w_0 (recall the relation (2.8) between w_0 and u_0). Thus, (3.9) demonstrates a definite "trace" z^β of polynomial eigenfunctions $\psi^*_\beta(z)$ of the adjoint operator $\mathbf{B}^*$, but in view of (2.8) cannot give its actual form.

To this end, using the actual representation of the semigroup (2.10), instead of (3.8), we have to apply the full expansion of the kernel

$$F\big((y - z\mathrm{e}^{-\frac{1}{2m}\tau})(1-\mathrm{e}^{-\tau})^{-\frac{1}{2m}}\big) = \sum_{(\gamma)} \frac{1}{\gamma!}\, D^\gamma F(0) \big(y - z\mathrm{e}^{-\frac{1}{2m}\tau}\big)^\gamma\, (1-\mathrm{e}^{-\tau})^{-\frac{|\gamma|}{2m}}, \tag{3.11}$$

and utilize the binomial expansion,

$$\big(y - z\,\mathrm{e}^{-\frac{1}{2m}\,\tau}\big)^{\gamma} = \sum_{(0 \le \delta \le \gamma)} C_{\gamma}^{\delta}\,\mathrm{e}^{-\frac{|\delta|}{2m}\,\tau}\,y^{\gamma-\delta}\,(-z)^{\delta}, \tag{3.12}$$

where the partial order of multi-indices, $0 \le \delta = (\delta_1, \dots, \delta_N) \le \gamma = (\gamma_1, \dots, \gamma_N)$, means that $0 \le \delta_1 \le \gamma_1, \dots, 0 \le \delta_N \le \gamma_N$, and

$$C_{\gamma}^{\delta} = C_{\gamma_1}^{\delta_1} \cdots C_{\gamma_N}^{\delta_N} = \frac{\gamma!}{\delta!(\gamma-\delta)!}, \tag{3.13}$$

are the corresponding binomial coefficients. Substituting (3.11) and (3.12) into (2.10), we obtain an extra multiplier that also needs simple Taylor's expansion,

$$\begin{aligned}(1-\mathrm{e}^{-\tau})^{-\frac{|\gamma|+N}{2m}} &= 1 + \frac{|\gamma|+N}{2m}\,\mathrm{e}^{-\tau} + \dots + \frac{1}{s!}\,\frac{|\gamma|}{2m}\big(\frac{|\gamma|+N}{2m}+1\big)\cdots \\ &\times \big(\frac{|\gamma|+N}{2m}+s-1\big)\,\mathrm{e}^{-s\tau} + \dots \equiv \sum_{s \ge 0} \kappa_s\big(\frac{|\gamma|+N}{2m}\big)\,\mathrm{e}^{-s\tau}.\end{aligned} \tag{3.14}$$

Finally, we obtain a *full* eigenfunction expansion of the solution (2.10), which is convenient to write in the following form:

$$\begin{aligned}\mathrm{e}^{\mathbf{B}\tau}u_0 &= \sum_{s \ge 0}\sum_{(\gamma)}\sum_{(0 \le \delta \le \gamma)} \mathrm{e}^{-(\frac{|\delta|}{2m}+s)\,\tau} \\ &\times \frac{1}{\delta!}D_{\zeta}^{\delta}\big[\frac{1}{(\gamma-\delta)!}D_{\zeta}^{\gamma-\delta}F(\zeta)y^{\gamma-\delta}\big]\big|_{\zeta=0}\kappa_s\big(\frac{|\gamma|+N}{2m}\big)(-1)^{|\delta|}\int z^{\delta}u_0(z)\mathrm{d}z,\end{aligned} \tag{3.15}$$

where we have substituted C_{γ}^{δ} from (3.13). This representation also confirms the point spectrum λ_β of $\mathbf{B}$ obtained earlier in (3.9). In addition, (3.15) shows the formation of polynomial eigenfunctions $\psi_\beta^*(z)$ of $\mathbf{B}^*$. In Section 4 we derive those polynomials directly from the linear elliptic PDE, so we will not use a complicated formula (3.15) in detail. Note also that the expansion (3.15) has a pure discrete nature (infinite sums and no integral terms) of rather arbitrary solutions of the analytic parabolic equation (2.1). This indicates that $\mathbf{B}$ has a pure discrete spectrum and does not have a continuous one, for which the spectral decomposition contains integral members; see further analysis and another justification below. Thus, in (3.10),

$$M_\beta(u_0) = \frac{1}{\sqrt{\beta!}}\,\langle u_0, z^\beta\rangle \equiv \langle u_0, \psi_\beta^*\rangle,$$

where ψ_β^* are polynomial eigenfunctions of the adjoint operator $\mathbf{B}^*$ to be described in the next section.

It follows from the asymptotic analysis of expansion (3.9) as $\tau \to \infty$ that no other eigenfunctions exist, all eigenvalues are real and are given in (3.5).

(ii) Completeness: L^2-space. Let us show that the system of the eigenfunctions $\{D^\alpha F\}$ is complete in L^2. By the Riesz–Fischer theorem, we have to show that, given a function $G \in L^2$, the equalities

$$\int D^\alpha F(x) G(x) \mathrm{d}x = 0 \quad \text{for any } \alpha \tag{3.16}$$

imply that $G = 0$.

Let $\hat{F}(\xi) = \mathcal{F}(F)$ and $\hat{G}(\xi) = \mathcal{F}(G)$ be the Fourier transforms (FTs). The FT of $D^\alpha F$ is then proportional to $\xi^\alpha \hat{F}$, so substituting $G = \mathcal{F}^{-1}(\hat{G})$ and integrating by parts yields

$$\int D^\alpha F(x) G(x) \, \mathrm{d}x \sim \int \xi^\alpha \hat{F}(\xi) \hat{G}(-\xi) \, \mathrm{d}\xi = 0 \quad \text{for any } \alpha.$$

Applying the Fourier transform to equation (2.3) leads to the first-order PDE

$$P_{2m}(\xi) \hat{F} - \frac{1}{2m} \, \xi \cdot \nabla \hat{F} = 0,$$

where by the parabolicity condition (1.2),

$$P_{2m}(\xi) = \sum_{|\beta|=2m} A_\beta (\mathrm{i}\xi)^\beta = (-1)^m \sum_{|\beta|=2m} A_\beta \xi^\beta \le -\delta |\xi|^{2m} \quad \text{in } \mathbf{R}^N. \tag{3.17}$$

Since $P_{2m}(\xi)$ is a homogeneous $2m$th-order polynomial, by Euler's formula

$$\xi \cdot \nabla P_{2m}(\xi) = 2m P_{2m}(\xi),$$

we find that (recall that $b(x,0) = \delta(x)$)

$$\hat{F}(\xi) \equiv \mathcal{F}(F(\cdot))(\xi) = \mathrm{e}^{P_{2m}(\xi)} \implies \int \xi^\alpha \mathrm{e}^{P_{2m}(\xi)} \hat{G}(-\xi) \, \mathrm{d}\xi = 0, \tag{3.18}$$

for any α. By the parabolicity condition (1.2), the function

$$M(z) = \int \xi^\alpha \mathrm{e}^{P_{2m}(\xi)} \hat{G}(-\xi) \mathrm{e}^{\mathrm{i}z \cdot \xi} d\xi,$$

is entire analytic in $\mathbb{C}^N$ (since $|\mathrm{e}^{\mathrm{i}z\cdot\xi}| \le \mathrm{e}^{|\mathrm{Im}\, z||\xi|}$). Equality (3.18) means that $D^\alpha M(0) = 0$ for any β. Therefore, $M(z) \equiv 0$. Thus, $\hat{G}(-\xi) = 0$ almost everywhere and $G = 0$.

Completeness: L^2_ρ-space. In order to prove completeness in L^2_ρ, as in [8, Section 2], we suppose that a function $G \in L^2_\rho$ is orthogonal relative to the inner product in L^2_ρ to all eigenfunctions; i.e.,

$$\int \rho(y) D^\alpha F(y) G(y)\, \mathrm{d}y = 0 \quad \text{for all } \alpha.$$

Since F is analytic, it implies that

$$\int \rho(y) F(y-x) G(y)\, \mathrm{d}y = 0 \quad \text{for all } x \in \mathbf{R}^N.$$

Consider the Cauchy problem for the linear parabolic equation (2.1) with initial data

$$u_0(x) = \rho(x) G(x) \quad \text{in} \quad \mathbf{R}^N.$$

One can see from the Poisson-type integral (2.5) and (2.4) by using Eidelman's estimate [9, Lemma 5.1] (see also an extension for integrals over $\mathbf{R}^N$ in [8, Proposition 4.1]) that the solution exists for all $t \in (0,1]$ provided that the exponent $a > 0$ in the weight (1.16) satisfies (3.1). Then $u(x,t)$ is analytic in x. We have

$$u(x,1) = \int F(x-y) G(y) \rho(y)\, \mathrm{d}y.$$

Therefore, $u(x,1) \equiv 0$. It follows by the uniqueness theorem for the inverse parabolic equation [13, p. 181] that $u(x,0) = 0$, and $G = 0$.

(iii) Compact resolvent. We next need to deduce that $(\mathbf{B} - I)^{-1}$ is an integral compact operator and has a point spectrum only. The proof is similar to that in [8, Theorem 2.2]. A simpler compactness analysis in a subspace of L^2_ρ is presented in Section 5. As an alternative, this also follows from the compact embedding of the domain,

$$H^{2m}_\rho \subset L^2_\rho, \tag{3.19}$$

which is a typical question of well-developed theory of weighted Sobolev spaces; see e.g., Maz'ja [30]. In particular, for $2m > N$, the result (3.19) is immediate, since the functions $v(y) \in H^{2m}_\rho$ must satisfy $w(y) \to 0$ as $y \to \infty$. Therefore, this compact embedding follows from the standard Ascoli-Arzelá theorem for functions on arbitrary compact subsets in $\mathbf{R}^N$. The analysis for $2m \le N$ exhibits the same peculiarities as in the classic compact embedding $H^{2m} \subset L^2$ in a bounded smooth domain. On the other hand, for compactness the explicit resolvent representation (2.13) can be used. Note that checking the L^2-property of the kernel is rather tricky since it is not uniformly exponentially small at infinity. This completes the proof of Lemma 3.1. $\square$

4. Discrete spectrum and polynomial eigenfunctions of $\mathbf{B}^*$

Let us describe the eigenfunctions of the adjoint operator (1.10). We consider $\mathbf{B}^*$ in the weighted space $L^2_{\rho^*}$ with the exponentially decaying weight function

$$\rho^*(y) \equiv \tfrac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0, \tag{4.1}$$

and ascribe to $\mathbf{B}^*$ the domain $H^{2m}_{\rho^*}$ dense in $L^2_{\rho^*}$. Then $\mathbf{B}^*: \; H^{2m}_{\rho^*} \to L^2_{\rho^*}$ is adjoint to $\mathbf{B}$,

$$\langle \mathbf{B} v, w\rangle = \langle v, \mathbf{B}^* w\rangle \quad \text{for any } \; v \in H^{2m}_\rho \; \text{ and } \; w \in H^{2m}_{\rho^*}, \tag{4.2}$$

and hence $\mathbf{B}^*$ is a bounded closed linear operator, [25, Chapter 4].

4.1. **Using explicit representation of the semigroup.** We first predict several key properties of the adjoint operator $\mathbf{B}^*$ by using the semigroup representation (2.12).

Comparing semigroups (2.12) and (2.10), we see that the only difference is in the argument of the rescaled kernel $F(\cdot)$. Therefore, instead of (3.11), we use the following expansion:

$$F\big((y\,\mathrm{e}^{-\frac{1}{2m}\tau} - z)(1-\mathrm{e}^{-\tau})^{-\frac{1}{2m}}\big) = \sum_{(\gamma)} \frac{1}{\gamma!}\, D^\gamma F(0) \big(y\mathrm{e}^{-\frac{1}{2m}\tau} - z\big)^\gamma \,(1-\mathrm{e}^{-\tau})^{-\frac{|\gamma|}{2m}}, \tag{4.3}$$

where, instead of (3.12),

$$\big(y\,\mathrm{e}^{-\frac{1}{2m}\tau} - z\big)^\gamma = \sum_{(0\le\delta\le\gamma)} C^\delta_\gamma\, \mathrm{e}^{-\frac{|\gamma-\delta|}{2m}\tau}\, y^{\gamma-\delta}\,(-z)^\delta. \tag{4.4}$$

Then, using both (4.3) and (4.4) in (2.12), as well as (3.14), yields a different expansion in comparison with that in (3.15),

$$\begin{aligned} \mathrm{e}^{\mathbf{B}^*\tau} u_0 = \sum_{s\ge0}\sum_{(\gamma)}\sum_{(0\le\delta\le\gamma)} &\mathrm{e}^{-(\frac{|\gamma-\delta|}{2m}+s)\,\tau}(-1)^{|\gamma|}\, y^{\gamma-\delta}\kappa_s\big(\frac{|\gamma|+N}{2m}\big) \\ &\times \frac{1}{(\gamma-\delta)!}\, D^{\gamma-\delta}_\zeta\Big[\int \big(\frac{1}{\delta!}\, D^\delta F(\zeta) z^\delta\big)\, u_0(z)\,\mathrm{d}z\Big]\Big|_{\zeta=0}, \end{aligned} \tag{4.5}$$

where the binomial coefficient C^δ_γ is cancelled according to (3.13). This can be viewed as an expansion over the point spectrum of $\mathbf{B}^*$,

$$w(y,\tau) = \sum_{(\beta)} \mathrm{e}^{-\frac{|\beta|}{2m}\tau} M^*_\beta(u_0)\psi^*_\beta(y), \tag{4.6}$$

where $\psi^*_\beta(y)$ are finite polynomial eigenfunctions (see their direct derivations below) and the expansion coefficients are $M^*_\beta(u_0) = \langle u_0, \psi_\beta\rangle$. Similarly, to the case of the operator $\mathbf{B}$ at the beginning of the proof of Lemma 3.1, in view of standard regularity properties of linear parabolic flows such as (2.11), the semigroup expansion (4.6) reveals some key spectral properties of $\mathbf{B}^*$ to be fixed below in greater detail:

(i) the point spectrum is $\sigma_p(\mathbf{B}^*) = \{\lambda_\beta = -\frac{|\beta|}{2m}\}$ having finite multiplicity,

(ii) there is no continuous spectrum (it seems that the pure discrete nature of the expansion (4.6) can be directly connected with typical properties of compact resolvent),

(iii) polynomial eigenfunctions $\{\psi_*(y)\}$ are closed in $L^2_{\rho^*}$, etc.

4.2. **Spectrum, polynomial eigenfunctions, and other properties.** Let us fix the main spectral properties of $\mathbf{B}^*$.

Lemma 4.1. (i) *The spectrum of* $\mathbf{B}^*$ *is discrete,*

$$\sigma(\mathbf{B}^*) = \sigma(\mathbf{B}) = \bigl\{\lambda_\beta = -\tfrac{|\beta|}{2m},\ \ |\beta| = 0,1,2,...\bigr\}, \tag{4.7}$$

and eigenfunctions $\{\psi^*_\beta(y)\}$ *are polynomials of order* $|\beta|$,

$$\psi^*_\beta(y) = \frac{1}{\sqrt{\beta!}}\Bigl[y^\beta + \sum_{j=1}^{[\frac{|\beta|}{2m}]} \frac{1}{j!}(-\mathbf{B}_0)^j y^\beta\Bigr]. \tag{4.8}$$

(ii) *The eigenfunction subset* $\Phi^* = \{\psi^*_\beta\}$ *is complete in* $L^2_{\rho^*}$.

(iii) *Resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *is compact in* $L^2_{\rho^*}$.

Proof. (i) Spectrum and polynomial eigenfunctions. Firstly, since the adjoint to a compact operator is compact [25, Chapter 4], we conclude from Lemma 3.1(iii) that spectra are discrete and coincide, $\sigma(\mathbf{B}) = \sigma(\mathbf{B}^*)$. Secondly, let us detect the explicit representation of the finite polynomials $\{\psi^*_\beta\}$. Let

$$\hat{\psi}^*(\xi) = \mathcal{F}(\psi^*) \equiv \int \psi^*(y)\mathrm{e}^{-\mathrm{i}y\cdot\xi}\,\mathrm{d}y,$$

be the Fourier transform of an eigenfunction satisfying $\mathbf{B}^*\psi^* = \lambda\psi^*$. Then $\hat{\psi}^*$ solves the first-order equation

$$\tfrac{1}{2m}\,\xi\cdot\nabla\hat{\psi}^* + \bigl[\tfrac{N}{2m} + P_{2m}(\xi)\bigr]\hat{\psi}^* = \lambda\hat{\psi}^* \quad \text{in } \mathbf{R}^N. \tag{4.9}$$

The general solution is given by

$$\hat{\psi}^*(\xi) = \Phi\ |\xi|^{2m\lambda - N}\mathrm{e}^{-P_{2m}(\xi)} \quad \text{in} \quad \mathbf{R}^N \setminus \{0\}, \tag{4.10}$$

where $\Phi = \Phi(\frac{\xi}{|\xi|})$ is an arbitrary smooth function on the unit sphere $S_{N-1} = \{|\xi| = 1\}$ in $\mathbf{R}^N$. In view of the parabolicity assumption (1.2), we obtain in

(4.10) an exponentially growing factor,

$$\left|e^{-P_{2m}(\xi)}\right| \geq e^{\delta|\xi|^{2m}} \quad \text{as} \quad \xi \to \infty.$$

Therefore, the only distributions satisfying equation (4.9) correspond to $\Phi = 0$ on S_1, i.e., are those having supports concentrated at the origin $\xi = 0$. Hence, $\hat{\psi}^*$ must have the following form (see e.g., Vladimirov [33, Section 8]): there exists finite k and constants $\{C_\alpha\}$ such that

$$\hat{\psi}^*(y) = \sum_{|\alpha|=0}^{k} C_\alpha D_y^\alpha \delta(y).$$

Therefore, taking $\mathcal{F}^{-1}$ yields that $\psi^*(y)$ must be a polynomial of degree k. Next, we have that

$$\psi^*(y) = \sum_{j=0}^{s} P_j(y) \quad \text{with} \;\; s = \left[\tfrac{k}{2m}\right], \tag{4.11}$$

where $P_j(y)$ are homogeneous polynomials of degree $k - 2mj$. Since by the Euler identity

$$-\frac{1}{2m}\sum_{j=1}^{N} y_j \frac{\partial P_0(y)}{\partial y_j} = -\frac{k}{2m} P_0(y) = \lambda P_0(y),$$

we see that $\lambda = -\frac{k}{2m}$ (thus, once more, we have detected the point spectrum), and, hence, $P_0(y)$ is an arbitrary homogeneous polynomial of degree k. Other polynomials $P_j(y)$ are then defined as follows:

$$P_j(y) = \tfrac{1}{j!}\,(-\mathbf{B}_0)^j P_0(y), \quad j = 1, \ldots, s. \tag{4.12}$$

We fix

$$P_0(y) = \tfrac{1}{\sqrt{\beta!}}\, y^\beta,$$

in (4.11), so that, for eigenfunctions (3.6) of $\mathbf{B}$, the corresponding adjoint eigenfunctions take the form (4.8). As usual, the definition (4.2) of the adjoint operator then guarantees the orthonormality condition,

$$\langle \psi_\beta, \psi_\gamma^* \rangle = \delta_{\beta,\gamma} \quad \text{for any} \;\; \beta \text{ and } \gamma, \tag{4.13}$$

where $\delta_{\beta,\gamma}$ is Kronecker's delta. Note that operators $\mathbf{B}$ and $\mathbf{B}^*$ have zero Morse index and do not have eigenvalues with positive real parts. For $\beta = 0$, the eigenfunctions are

$$\psi_0(y) = F(y), \quad \psi_0^*(y) = 1, \tag{4.14}$$

so that $\langle \psi_0, \psi_0^* \rangle = 1$ by the definition (2.3) of the fundamental solution.

(ii) Completeness. It follows from the well-known facts that polynomials $\{y^\beta\}$, which are higher-order terms in any eigenfunction ψ_β, are complete in all suitably weighted L^p-spaces; see [25, p. 431]. Then (4.11) implies the completeness of Φ^* in $L^2_{\rho^*}$.

(iii) Compact resolvent. As we have mentioned, since $(\cdot)^*$ and $(\cdot)^{-1}$ commute for operators in Banach spaces and the adjoint operator of a compact operator is compact [25, Chapter 4], we have from Lemma 3.1(iii) that $(\mathbf{B}^* - \lambda I)^{-1}$ is compact with the point spectrum only. □

5. EIGENFUNCTION EXPANSIONS AND LITTLE HILBERT SPACES

In this section, we describe some convenient subspaces where the sets Φ and Φ^* are closed, i.e., where eigenfunction expansions are well-defined.

5.1. **Operator B: Hilbert spaces $\tilde{L}^2_\rho$, $\tilde{H}^{2m}_\rho$, l^2_ρ, and h^{2m}_ρ.** Let us first introduce some subspaces in L^2_ρ, where the complete eigenfunction subset Φ of operator $\mathbf{B}$ is closed. As usual, we define the linear subspace $\tilde{L}^2_\rho$ of eigenfunction expansions, i.e.,

$$v \in \tilde{L}^2_\rho \quad \text{if and only if} \quad v = \sum c_\beta \psi_\beta \ \text{ with convergence in } \ L^2_\rho, \tag{5.1}$$

as the closure of the subset of finite sums $\{\sum_{|\beta| \le K} c_\beta \psi_\beta,\ K \in \mathbb{N}\}$ in the L^2_ρ-norm. By the completeness-closure of Φ in $\tilde{L}^2_\rho$ and orthonormality (4.13), the expansion coefficients are

$$c_\beta = \langle v, \psi^*_\beta \rangle \quad \text{for any } \ \beta. \tag{5.2}$$

Φ is not orthonormal in L^2_ρ and hence, in general, one cannot expect the equality $\tilde{L}^2_\rho = L^2_\rho$ to hold (unlike the self-adjoint case $m = 1$ with $a = \frac{1}{4}$), though on radial functions the difference between these spaces (if any) is quite small.

With the normalization (3.6), in $\tilde{L}^2_\rho$ the expansion coefficients can grow exponentially fast, so we characterize this subspace as follows:

Proposition 5.1. *Let, for an arbitrarily small constant $\varepsilon > 0$, as $|\beta| \to \infty$,*

$$c_\beta = o(|\beta|^{|\beta|(\nu - \varepsilon)}), \quad \text{where } \nu = \frac{2-\alpha}{2\alpha} > 0. \tag{5.3}$$

Then $v = \sum c_\beta \psi_\beta \in \tilde{L}^2_\rho$.

Proof. There holds

$$\int_{\mathbf{R}^N} \rho|v|^2 = \int \rho|\sum c_\beta \psi_\beta|^2 \equiv \sum_{(\beta,\gamma)} A_{\beta\gamma} c_\beta c_\gamma, \text{ where } A_{\beta\gamma} = \int \rho \psi_\beta \psi_\gamma. \tag{5.4}$$

Bearing in mind (3.6), it follows from standard kernel estimates [9] (cf. a sharp asymptotic estimate of the rescaled kernel in the right-hand side of (2.4)) that

$$|D^\beta F(y)| \le c^{|\beta|}(1+|y|)^{|\beta|(\alpha-1)} \mathrm{e}^{-d|y|^\alpha} \quad \text{in } \mathbf{R}^N, \tag{5.5}$$

where c is independent of $|\beta|$. Therefore,

$$|A_{\beta\gamma}| \le \frac{c^{|\beta+\gamma|}}{\sqrt{\beta!\gamma!}} \int \mathrm{e}^{-b|y|^\alpha}(1+|y|)^{|\beta+\gamma|(\alpha-1)}, \tag{5.6}$$

where $b = 2d - a > 0$ by the definition of the weight (1.16), (3.1). One can see that, for $|\beta| = l \gg 1$, the right-hand side of (5.6) attains its minimal value for $|\gamma| \ll l$, and then by Stirling's series of the Gamma function, omitting all the lower-order multipliers and keeping only those of the type given in (5.3),

$$\int_{\mathbf{R}^N} \mathrm{e}^{-b|y|^\alpha}(1+|y|)^{|\beta+\gamma|(\alpha-1)} \sim \int_0^\infty z^{N-1} \mathrm{e}^{-bz^\alpha} z^{l(\alpha-1)}\, \mathrm{d}z \sim \Gamma\big(\frac{l(\alpha-1)}{\alpha}\big) \sim l^{\frac{l(\alpha-1)}{\alpha}}.$$

This implies the estimate

$$|A_{\beta\gamma}| \sim \frac{1}{\sqrt{l!}}\, l^{\frac{l(\alpha-1)}{\alpha}} \sim \frac{1}{\sqrt{l^l}}\, l^{\frac{l(\alpha-1)}{\alpha}} = l^{-\frac{l(2-\alpha)}{2\alpha}}. \tag{5.7}$$

The case $|\gamma| \sim |\beta| \sim l$ is similar. Hence, (5.4) converges under assumption (5.3). $\square$

More accurate, using Stirling's series shows that

$$\sum c_\beta \psi_\beta \in \tilde{L}^2_\rho \quad \text{if} \quad c_\beta = O(\delta^l |\beta|^{|\beta|\nu}),$$

with a sufficiently small constant $\delta > 0$. Since estimates of the leading terms in (5.7) are sharp, it follows that

$$v = \sum c_\beta \psi_\beta \in \tilde{L}^2_\rho \implies c_\beta = o(|\beta|^{|\beta|(\nu+\varepsilon)}) \quad \text{with an } \varepsilon > 0, \tag{5.8}$$

for "almost all" $|\beta| \gg 1$.

By $\tilde{H}^{2m}_\rho \subset \tilde{L}^2_\rho$ we denote the dense linear subspace obtained as the closure in the norm of H^{2m}_ρ of the subset of eigenfunction expansions with coefficients satisfying (5.3). $\tilde{H}^{2m}_\rho$ with the scalar product of H^{2m}_ρ becomes a Hilbert space and can be considered as the domain of $\mathbf{B}$ in H^{2m}_ρ. There holds

$$\tilde{H}^{2m}_\rho \subseteq H^{2m}_\rho \cap \tilde{L}^2_\rho. \tag{5.9}$$

Note that (5.3) does not apply for $m = 1$ since then $\alpha = 2$ and hence $\nu = 0$. Actually, a natural optimal analogy of $\tilde{H}^{2m}_\rho$ for $m = 1$ is H^2_ρ, the domain of $\mathbf{B}$ in L^2_ρ.

We will need a subspace of $\tilde{L}^2_\rho$ introduced as a *little* Hilbert space l^2_ρ of functions $v = \sum c_\beta \psi_\beta \in \tilde{L}^2_\rho$ with coefficients satisfying

$$\sum |c_\beta|^2 < \infty, \tag{5.10}$$

where the scalar product and the induced norm are given by

$$(v, w)_0 = \sum c_\beta a_\beta \;\text{ for } w = \sum a_\beta \psi_\beta \in l^2_\rho, \quad \text{and} \quad \|v\|^2_0 = (v, v)_0. \tag{5.11}$$

Obviously, l^2_ρ is isomorphic to the Hilbert space l^2 of sequences $\{c_\beta\}$ with the same inner product, and hence

$$\Phi \quad \text{is orthonormal in } \; l^2_\rho. \tag{5.12}$$

We next define a little Sobolev space h^{2m}_ρ of functions $v \in l^2_\rho$ such that $\mathbf{B}v \in l^2_\rho$, i.e.,

$$\sum |\lambda_\beta c_\beta|^2 < \infty.$$

The scalar product and the induced norm in h^{2m}_ρ are

$$(v, w)_1 = (v, w)_0 + (\mathbf{B}v, \mathbf{B}w)_0 \quad \text{and} \quad \|v\|^2_1 = (v, v)_1 \equiv \sum (1 + |\lambda_\beta|^2)|c_\beta|^2. \tag{5.13}$$

This norm is equivalent to the graph norm induced by the positive operator $(-\mathbf{B} + aI)$ with $a > 0$. Then h^{2m}_ρ is the domain of $\mathbf{B}$ in l^2_ρ. We also have a Sobolev embedding theorem,

$$h^{2m}_\rho \subset l^2_\rho \quad \text{compactly}, \tag{5.14}$$

which follows from the well-known criterion of compactness in l^p: a $T \subset l^p$ is compact if and only if

$$\forall \;\; \varepsilon > 0 \;\; \exists \text{ integer } K = K(\varepsilon) > 0 \text{ such that } \forall \; \{c_\beta\} \in T \implies \sum_{|\beta| \ge K} |c_\beta|^p < \varepsilon; \tag{5.15}$$

see, e.g., [28]. In the self-adjoint case $m = 1$, the little space l^2_ρ coincides with the big one,

$$l^2_\rho = L^2_\rho \quad \text{for } \; m = 1 \;\; \text{if } \; a = \tfrac{1}{4} \;\; \text{in } \; (1.16). \tag{5.16}$$

Then h^2_ρ is the domain H^2_ρ of $\mathbf{B}$ (recall that if $a \neq \frac{1}{4}$, then $\mathbf{B}$ is not self-adjoint in L^2_ρ).

Since the orthonormality of Φ is known to be of importance in operator theory and applications, in some linear and non-linear problems dealing with operators like $\mathbf{B}$, the little space l^2_ρ can play a special role in comparison with the big one L^2_ρ.

It follows from (5.11) that $\mathbf{B}$ is *self-adjoint* in l^2_ρ with the domain h^{2m}_ρ,

$$(\mathbf{B}v, w)_0 = (v, \mathbf{B}w)_0 \quad \text{for all } \; v, w \in h^{2m}_\rho. \tag{5.17}$$

Notice that this *a posteriori* conclusion in a special functional setting is obtained after establishing all the necessary spectral properties of the operator. Let us state other straightforward consequences (this list can be easily extended).

Proposition 5.2. (i) *l^2_ρ is a dense subspace of $\tilde{L}^2_\rho$,* (ii) *$\Phi = \{\psi_\beta\}$ is complete and closed in l^2_ρ in the topology of L^2_ρ,* (iii) *resolvent $(\mathbf{B} - \lambda I)^{-1}$ is compact in l^2_ρ, and* (iv) *$\mathbf{B}$ is sectorial in l^2_ρ.*

Proof. (i) Obviously, $l^2_\rho \subset \tilde{L}^2_\rho$ by Proposition 5.1. Concerning the density of l^2_ρ, we note that given a $v = \sum c_\beta \psi_\beta \in \tilde{L}^2_\rho$, the sequence of truncations $\{\sum_{|\beta| \le K} c_\beta \psi_\beta, K \in \mathbb{N}\} \subset l^2_\rho$ converges to v in the topology of L^2_ρ as $K \to \infty$ by completeness and closure of $\{\psi_\beta\}$.

(ii) Since Φ is orthonormal in l^2_ρ, it follows that the only element orthogonal to $\{\psi_\beta\}$ is 0, and hence completeness of $\{\psi_\beta\}$ in l^2_ρ follows from the Riesz–Fischer theorem. It is closed as a complete orthonormal subset in a separable Hilbert space, [25, Chapter 3].

(iii) For any $v = \sum c_\beta \psi_\beta \in l^2_\rho$ from the unit ball T_1 in l^2_ρ with $\sum |c_\beta|^2 \le 1$, we have $(\mathbf{B} - \lambda I)^{-1} v = \sum b_\beta \psi_\beta$, where

$$b_\beta = \frac{c_\beta}{\lambda_\beta - \lambda} = -\frac{c_\beta}{\frac{|\beta|}{2m} + \lambda} = -\frac{2mc_\beta}{|\beta|}[1 + O(\frac{1}{|\beta|})] \quad \text{for } \; |\beta| \gg 1. \tag{5.18}$$

Therefore, for any $\varepsilon > 0$, there exists $K = K(\varepsilon) > 0$ such that for any $v \in T_1$,

$$\sum_{|\beta| \ge K} |b_\beta|^2 \le \frac{4m^2}{K^2} \sum |c_\beta|^2 \le \frac{4m^2}{K^2} < \varepsilon.$$

By the compactness criterion (5.15) in l^2, $(\mathbf{B} - \lambda I)^{-1}$ maps T_1 onto a compact subset in l^2_ρ.

(iv) Recall that $(\mathbf{B} - \lambda I)^{-1}$ is a meromorphic function having a pole $\sim \frac{1}{\lambda}$ as $\lambda \to 0$ since $\lambda_0 = 0$ has multiplicity one [20]. We then need an extra estimate of the resolvent, which is easy to get in l^2_ρ (it is not easy at all in the big space L^2_ρ). Consider the sector $\Phi_\theta = \{\lambda \in \mathbb{C} : \lambda \ne 0, |\arg \lambda| < \frac{\pi}{2} + \theta\}$, with a $\theta \in (0, \frac{\pi}{2})$. Then, for any $v = \sum c_\beta \psi_\beta \in l^2_\rho$, we apply (5.18) by using that

$$\frac{1}{|\lambda_\beta - \lambda|} \le \frac{1}{|\lambda| \sin\theta} \quad \text{in} \quad \Phi_\theta,$$

and obtain the following estimate:

$$\|(\mathbf{B} - \lambda I)^{-1} v\|_0 = \sqrt{\sum \frac{1}{|\lambda_\beta - \lambda|^2} |c_\beta|^2} \le \frac{1}{\sin\theta} \frac{1}{|\lambda|} \|v\|_0.$$

Since $\mathbf{B}$ is closed and densely defined, it is a sectorial operator in l^2_ρ; see [13]. □

5.2. **Adjoint operator $\mathbf{B}^*$: Hilbert spaces $\tilde{L}^2_{\rho^*}$, $\tilde{H}^{2m}_{\rho^*}$, $l^2_{\rho^*}$, and $h^{2m}_{\rho^*}$.** Similarly, for the adjoint operator $\mathbf{B}^*$, we define the subspace $\tilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}$, where the eigenfunction subset Φ^* is closed (cf. (5.1)), i.e., $v = \sum c_\beta \psi^*_\beta$ with

$$c_\beta = \langle v, \psi_\beta \rangle \quad \text{for any} \;\; \beta. \tag{5.19}$$

Proposition 5.3. *Given an arbitrarily small $\varepsilon > 0$, there holds*

$$c_\beta = o(|\beta|^{-|\beta|(\nu+\varepsilon)}) \implies \sum c_\beta \psi^*_\beta \in \tilde{L}^2_{\rho^*}. \tag{5.20}$$

Proof. Similar to (5.4), we have

$$\int \rho^* |v|^2 = \sum_{(\beta,\gamma)} A^*_{\beta\gamma} c_\beta c_\gamma, \quad \text{with} \quad A^*_{\beta\gamma} = \int \rho^* \psi^*_\beta \psi^*_\gamma, \tag{5.21}$$

where by (4.8) and Stirling's series we first estimate the coefficients for $|\beta| = l$, $|\gamma| \ll 1$,

$$|A^*_{\beta\gamma}| \sim \frac{c^{|\beta|}}{\sqrt{\beta!\gamma!}} \int e^{-a|y|^\alpha} (1+|y|)^{|\beta+\gamma|} \sim \frac{1}{\sqrt{l!}} \Gamma(\frac{l}{\alpha}) \sim l^{-\frac{l}{2}} l^{\frac{l}{\alpha}} = l^{\frac{l(2-\alpha)}{2\alpha}}. \tag{5.22}$$

By a similar analysis for $|\gamma| \sim |\beta| \sim l \gg 1$, we have that (5.20) guarantees convergence of series (5.21). □

It also follows that if $v = \sum c_\beta \psi^*_\beta \in \tilde{L}^2_{\rho^*}$, then for arbitrarily small fixed $\varepsilon > 0$,

$$c_\beta = o(|\beta|^{-|\beta|(\nu-\varepsilon)}) \quad \text{for "almost all"} \;\; |\beta| \gg 1 \quad \big(\nu = \frac{2-\alpha}{2\alpha}\big). \tag{5.23}$$

Remark: on symmetric normalization of eigenfunctions. Conditions (5.3) and (5.20) on the expansion coefficients in l^2_ρ and $l^2_{\rho^*}$ are different and do not exhibit a natural symmetry, unlike the self-adjoint case $m = 1$. The symmetry can be restored by introducing normalization multipliers,

$$(\beta!)^{-\frac{\alpha-1}{\alpha}} \text{ in } (3.6) \quad \text{and} \quad (\beta!)^{-\frac{1}{\alpha}} \text{ in } (4.8), \text{ respectively.}$$

For $m = 1$, where $\alpha = 2$, both are equal to $(\beta!)^{-1/2}$, which we continue to use for $m > 1$.

By $\tilde H^{2m}_{\rho^*}$ we denote the closure in the norm of $H^{2m}_{\rho^*}$ of the linear subspace of eigenfunction expansions with coefficients satisfying (5.23) for some $\varepsilon > 0$. Being equipped with the scalar product (3.3) with $\rho \mapsto \rho^*$, $\tilde H^{2m}_{\rho^*}$ is a Hilbert space becoming the domain of $\mathbf{B}^*$ in $H^{2m}_{\rho^*}$. We have

$$\tilde H^{2m}_{\rho^*} \subseteq H^{2m}_{\rho^*} \cap \tilde L^2_{\rho^*}. \tag{5.24}$$

In view of the fast decay (5.20) of the coefficients, similar to l^2_ρ, we introduce the adjoint little Hilbert space $l^2_{\rho^*}$ of eigenfunction expansions $v = \sum c_\beta \psi^*_\beta \in \tilde L^2_{\rho^*}$ with the scalar product $(\cdot,\cdot)_{0*}$ and the norm $\|\cdot\|_{0*}$ defined as in (5.11). As the domain of $\mathbf{B}^*$ in $l^2_{\rho^*}$, we introduce the corresponding little Sobolev space $h^{2m}_{\rho^*}$ compactly embedded into $l^2_{\rho^*}$. By $(\cdot,\cdot)_{1*}$ and $\|\cdot\|_{1*}$ we denote the scalar product and the induced norm.

Then $\mathbf{B}^*$ is self-adjoint in $l^2_{\rho^*}$, and, obviously, $\tilde L^2_{\rho^*}$ and $\tilde H^{2m}_{\rho^*}$ are dense subspaces of $l^2_{\rho^*}$.

6. Classification of multiple zeros for parabolic PDEs: formation and collapse phenomena

We now return to the perturbed equation (1.9), where the exponential perturbation $\mathbf{C}(\tau)$ includes operators of $2m$th order. Therefore, application of classical estimates on semigroups generated by sectorial operators [13, 23], based on fractional powers of operators, are not straight-forward. We also note that the known detailed study of similar asymptotics of multiple zeros for the second-order parabolic equations [24, 1, 7, 6] essentially uses estimates and other features of the self-adjoint rescaled operator $\mathbf{B}^*$ given in (1.21) ($m = 1$). This approach principally does not apply to the non self-adjoint case $m > 1$, though some other more abstract general ideas and results from [24] and [1] are quite effective and will be used later on.

Assuming that $u(\tau) \in \tilde L^2_{\rho^*}$ for $\tau > 0$, we will use the eigenfunction expansion of such solutions of equation (1.9),

$$u(\tau) = \sum c_\beta(\tau)\psi^*_\beta, \quad \text{with coefficients } c_\beta(\tau) = \langle u(\tau), \psi_\beta\rangle. \tag{6.1}$$

We next impose extra smoothness conditions on the solution and on the coefficients of the equation. We suppose that $u(\tau)$ is uniformly bounded in $H^{2m}_{\rho^*}$,

$$\|u(\tau)\|_{2m,*} \le C \quad \text{for all } \tau > 0, \tag{6.2}$$

a natural *a priori* bound in parabolic theory; see [9, 13]. Thus, assuming that $u(\tau) \in \tilde{L}^2_{\rho^*}$, bearing in mind (5.23), we may suppose that, for a small $\varepsilon > 0$,

$$c_\beta(\tau) = o(|\beta|^{-|\beta|(\nu-\varepsilon)}) \to 0 \quad \text{as } |\beta| \to \infty \quad \text{uniformly in } \tau \in [1, \infty). \tag{6.3}$$

The rate of decay (6.3) is sufficient for performing manipulations with various series, and, indeed, such a condition can be weakened.

Substituting (6.1) into equation (1.9) and multiplying in L^2 by the adjoint eigenfunctions ψ_β, we obtain the following system on the expansion coefficients $\{c_\beta\}$:

$$\dot{c}_\beta = \lambda_\beta c_\beta + J_\beta(\tau), \quad \text{where } J_\beta(\tau) \equiv \big\langle \mathbf{C}(\tau) \sum_{(\gamma)} c_\gamma \psi^*_\gamma, \psi_\beta \big\rangle \quad \text{for any } \beta. \tag{6.4}$$

Using (1.11) and integrating by parts yields the perturbation, consisting of two terms,

$$J_\beta(\tau) = J_{\beta 1}(\tau) + J_{\beta 2}(\tau), \quad \text{where}$$

$$J_{\beta 1}(\tau) = \big\langle \sum_{|\mu|=2m} R_\mu(\tau) D^\mu \sum_{(\gamma)} c_\gamma \psi^*_\gamma, \psi_\beta \big\rangle \equiv \sum_{|\mu|=2m} \sum_{(\gamma)} c_\gamma g_{\mu\gamma\beta}(\tau), \tag{6.5}$$

$$\begin{aligned} J_{\beta 2}(\tau) &= \big\langle \sum_{|\mu|<2m} \mathrm{e}^{-\frac{2m-|\mu|}{2m}\tau} a_\mu(\tau) D^\mu \sum_{(\gamma)} c_\gamma \psi^*_\gamma, \psi_\beta \big\rangle \\ &\equiv \sum_{|\mu|<2m} \mathrm{e}^{-\frac{2m-|\mu|}{2m}\tau} \sum_{(\gamma)} c_\gamma h_{\mu\gamma\beta}(\tau), \end{aligned} \tag{6.6}$$

$$g_{\mu\gamma\beta}(\tau) = \langle R_\mu(\tau) D^\mu \psi^*_\gamma, \psi_\beta \rangle, \quad h_{\mu\gamma\beta}(\tau) = \langle a_\mu(\tau) D^\mu \psi^*_\gamma, \psi_\beta \rangle. \tag{6.7}$$

By (1.12), we set $g_{\mu\gamma\beta}(\tau) = \mathrm{e}^{-\frac{1}{2m}\tau} \tilde{g}_{\mu\gamma\beta}(\tau)$. Since the exponential estimates (1.12) and (1.13) hold on compact subsets only, in the higher-order perturbation term (6.5), one needs to estimate the integrals over $\{|y| \ge r\}$ with $r = r(\tau) \gg 1$. In these integrals, we have that $R_\mu(\tau)$ are uniformly bounded, and hence, similar to (5.22) in the essential "diagonal" cases with $|\beta| \sim |\gamma| \sim l \gg 1$,

$$\Big| \int_{|y| \ge r} R_\mu(\tau) D^\mu \psi^*_\gamma \psi_\beta \Big| \sim \frac{1}{l!} \int_r^\infty \mathrm{e}^{-dz^\alpha} z^{l\alpha+N-1}\, \mathrm{d}z \sim \mathrm{e}^{-\frac{1}{2} dr^\alpha}. \tag{6.8}$$

As usual, we keep the leading multiplier only, which is sufficient for necessary rough estimates. Therefore, choosing $r(\tau) \sim \tau^{1/\alpha}$ for $\tau \gg 1$, we obtain similar factors with, possibly, some extra multipliers with not of more than algebraic growth as $\tau \to \infty$ that will be omitted.

As the necessary hypothesis on the coefficients $a_\beta(x,t)$ of the parabolic equation (1.1), we have that the multipliers $\tilde g_{\mu\gamma\beta}(\tau)$ and $h_{\mu\gamma\beta}(\tau)$ are bounded and, under assumption (6.3), the corresponding series in (6.5) and (6.6) converge sufficiently fast. The regularity hypotheses can be weakened, but are a convenient restriction for further asymptotic analysis.

Summing up the above manipulations, we arrive at the following infinite-dimensional dynamical system on the expansion coefficients:

$$\dot c_\beta = \lambda_\beta c_\beta + \sum_{(\mu,\gamma)} \mathrm{e}^{-\nu_\mu \tau} j_{\mu\gamma\beta}(\tau) c_\gamma, \tag{6.9}$$

where

$$\nu_\mu = \begin{cases} \frac{2m-|\mu|}{2m} & \text{for } \ |\mu| < 2m, \\ \frac{1}{2m} & \text{for } \ |\mu| = 2m, \end{cases}$$

and $j_{\mu\gamma\beta}(\tau)$ are bounded coefficients related to $\tilde g_{\mu\gamma\beta}$ and $h_{\mu\gamma\beta}$.

6.1. **Multiple zeros for the unperturbed equation.** A complete classification of the multiple zeros is straightforward for the unperturbed equation (1.9) with the null operator $\mathbf{C} = 0$. Then (6.4) takes the diagonal form

$$\dot c_\beta = \lambda_\beta c_\beta \quad \Longrightarrow \quad c_\beta(\tau) = C_\beta \mathrm{e}^{\lambda_\beta \tau}, \ \text{ where } C_\beta = c_\beta(0) \text{ for any } \beta.$$

Therefore, this linear homogeneous parabolic equation admits different types of formation of multiple zeros given by the countable subset of patterns

$$u_\beta(y,\tau) = \mathrm{e}^{\lambda_\beta \tau} \psi_\beta^*(y) \quad \text{for any} \quad |\beta| \ge 1. \tag{6.10}$$

The first pattern with $\beta = 0$ and $\lambda_0 = 0$ is excluded, since $\psi_0^* = 1$, and the corresponding pattern $u_0(y,\tau) \equiv 1$ does not vanish. Using (4.8), denote by

$$P_0(y) = \frac{1}{\sqrt{l!}} \sum_{|\beta|=l} C_\beta y^\beta \not\equiv 0,$$

the homogeneous polynomial of lth order. Then, similar to (4.8),

$$\varphi_l^*(y) = P_0(y) + \sum_{j=1}^{[\frac{l}{2m}]} \frac{1}{j!} \, (-\mathbf{B}_0)^j P_0(y). \tag{6.11}$$

Thus, a general structure of zero surfaces of lth-order multiple zero is given by the nodal set of an eigenfunction $\varphi_l^*(y)$, i.e., by a non-trivial linear combination

$$\varphi_l^*(y) = \sum_{|\beta|=l} C_\beta \psi_\beta^*(y) \not\equiv 0. \tag{6.12}$$

Namely, if $C_\beta = 0$ for any $|\beta| < l$ and there exists a $C_\beta \neq 0$ for a $|\beta| = l$, then the asymptotic behaviour of the corresponding solution is as follows:

$$u(y,\tau) = \mathrm{e}^{-\frac{l}{2m}\tau}\big[\varphi_l^*(y) + O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big)\big] \quad \text{as } \tau \to \infty. \tag{6.13}$$

If such a finite l does not exist, then the solution is trivial, $u = 0$. This provides us with the first backward uniqueness result to be extended later to more general equations. Of course, this uniqueness conclusion is trivial for solutions $u(y,\tau)$ analytic in y.

6.2. **Perturbed equation.** We now perform a similar analysis of the perturbed dynamical system (6.9).

<u>STEP 1.</u> We begin with the first equation (6.9) for $c_0(\tau)$ with $\beta = 0$ and $\lambda_0 = 0$, where by (6.3), the right-hand side can be estimated as follows:

$$\dot{c}_0 = \sum_{(\mu,\gamma)} \mathrm{e}^{-\nu_\mu \tau} j_{\mu\gamma 0} c_\gamma = O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big) \quad \text{for } \tau \gg 1, \tag{6.14}$$

meaning that $|\dot{c}_0(\tau)| \le A\mathrm{e}^{-\tau/2m}$ for all $\tau \ge 0$, where $A > 0$ is a constant. Hence, there exists a finite limit $C_0 = c_0(\infty)$, and integrating equation over (τ, ∞) yields

$$c_0(\tau) = C_0 + O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big). \tag{6.15}$$

Let us estimate the coefficients $c_\beta(\tau)$ with $|\beta| \ge 1$. Writing down the equations in the form

$$(c_\beta(\tau)\mathrm{e}^{-\lambda_\beta \tau})' = \mathrm{e}^{-\lambda_\beta \tau} \sum_{(\mu,\gamma)} \mathrm{e}^{-\nu_\mu \tau} j_{\mu\gamma\beta} c_\gamma = O\big(\mathrm{e}^{-(\lambda_\beta + \frac{1}{2m})\tau}\big) \tag{6.16}$$

and integrating over $(0,\tau)$ with $\tau \gg 1$ yields

$$c_\beta(\tau) = c_\beta(0)\mathrm{e}^{\lambda_\beta \tau} + \mathrm{e}^{\lambda_\beta \tau} \int_0^\tau O\big(\mathrm{e}^{-(\lambda_\beta + \frac{1}{2m})s}\big)\,\mathrm{d}s = O\big(\tau \mathrm{e}^{-\frac{1}{2m}\tau}\big), \tag{6.17}$$

where an extra power multiplier τ is taken into account in the resonance case $\lambda_\beta = -\frac{1}{2m}$, i.e., for $|\beta| = 1$. Here and later on, in similar estimates, we omit the dependence of the constants in such estimates on $|\beta|$, which is covered by assumptions of fast convergence of the series involved. It is important that, under the above hypotheses on $j_{\mu\gamma\beta}(\tau)$ and (6.3), estimates (6.16) and (6.17) can be treated as being uniform in β.

Thus, in view of the above hypotheses on the solution, if $C_0 \neq 0$, then one obtains

$$u(y,\tau) = C_0 + o(1), \tag{6.18}$$

i.e., as in the unperturbed case, solutions do not exhibit zero formation as $\tau \to \infty$ on any compact subset in y. The estimate of the reminder $o(1) \sim O(\tau \mathrm{e}^{-\tau/2m})$ obtained via the above manipulations is valid up to an extra not-more-than-algebraically growing multiplier via the perturbation terms in (6.9).

Hence, we assume that $C_0 = 0$. Then by (6.17)

$$c_\beta(\tau) = O\big(\tau \mathrm{e}^{-\frac{1}{2m}\tau}\big) \quad \text{uniformly in } |\beta| \geq 0. \tag{6.19}$$

Substituting these estimates into (6.14) yields a refined estimate of the first coefficient

$$\dot{c}_0 = O\big(\tau \mathrm{e}^{-\frac{1}{m}\tau}\big) \implies c_0(\tau) = O\big(\tau \mathrm{e}^{-\frac{1}{m}\tau}\big). \tag{6.20}$$

<u>STEP 2.</u> Consider next equations with $|\beta| = 1$, $\lambda_\beta = -\frac{1}{2m}$, where we use estimates (6.19) to get that

$$\dot{c}_\beta = -\frac{1}{2m}\, c_\beta + \sum_{(\mu,\gamma)} \mathrm{e}^{-\nu_\mu \tau} j_{\mu\gamma\beta} c_\gamma = -\frac{1}{2m}\, c_\beta + O\big(\tau \mathrm{e}^{-\frac{1}{m}\tau}\big).$$

Multiplying by $\mathrm{e}^{\tau/2m}$,

$$\big(c_\beta(\tau)\mathrm{e}^{\frac{1}{2m}\tau}\big)' = O\big(\tau \mathrm{e}^{-\frac{1}{2m}\tau}\big), \tag{6.21}$$

we have that there exists a finite limit $c_\beta(\tau)\mathrm{e}^{\frac{1}{2m}\tau} \to C_\beta$ as $\tau \to \infty$. Integrating (6.21) over (τ, ∞), we deduce

$$c_\beta(\tau) = C_\beta \mathrm{e}^{-\frac{1}{2m}\tau} + O\big(\tau \mathrm{e}^{-\frac{1}{m}\tau}\big), \quad |\beta| = 1. \tag{6.22}$$

If $C_\beta \neq 0$ for some β with $|\beta| = 1$, then using (6.19), from equations with $|\beta| \geq 2$ we estimate the coefficients as follows:

$$(c_\beta \mathrm{e}^{-\lambda_\beta \tau})' = O\big(\tau \mathrm{e}^{-(\lambda_\beta + \frac{1}{m})\tau}\big), \tag{6.23}$$

and integrating over $(0,\tau)$, we get for $|\beta| \geq 2$

$$c_\beta(\tau) = O\big(\tau^2 \mathrm{e}^{-\frac{1}{m}\tau}\big). \tag{6.24}$$

It follows from (6.20), (6.24), and (6.22) that, for all $|\beta| \neq 1$ and those $|\beta| = 1$ with $C_\beta = 0$, the expansion coefficients satisfy (6.24) uniformly in $|\beta|$. Hence, in this case (6.22) implies the asymptotic behaviour

$$u(y,\tau) = \mathrm{e}^{-\frac{1}{2m}\tau}[\varphi_1^*(y) + o(1)], \tag{6.25}$$

with the eigenfunction φ_1^* given by (6.12) and $o(1) \sim O(\tau^2 \mathrm{e}^{-\tau/2m})$ with, possibly, an extra algebraic factor.

Step l**.** We iterate the dynamical system $l-1$ times assuming that the limits

$$c_\beta(\tau)\mathrm{e}^{-\lambda_\beta \tau} \to C_\beta \quad \text{as } \tau \to \infty, \tag{6.26}$$

are trivial, $C_\beta = 0$, for all $|\beta| = 0, 1, \ldots, l-1$, and there exists a first β, $|\beta| = l$, such that $C_\beta \neq 0$ (as above, existence of such limits follows from the convergence of integrals). Then, similarly to the previous analysis, we derive that

$$c_\beta(\tau) = C_\beta \mathrm{e}^{-\frac{l}{2m}\tau} + O\big(\tau^l \mathrm{e}^{-\frac{l+1}{2m}\tau}\big) \quad \text{for } |\beta| = l \text{ and } C_\beta \neq 0, \quad \text{and} \tag{6.27}$$

$$c_\beta(\tau) = O\big(\tau^{l+1} \mathrm{e}^{-\frac{l+1}{2m}\tau}\big) \quad \text{for } |\beta| \neq l \text{ or } |\beta| = l \text{ and } C_\beta = 0. \tag{6.28}$$

Note again that (6.28) are uniform in $|\beta| \gg 1$. Therefore, the corresponding multiple zero pattern has the form

$$u(y,\tau) = \mathrm{e}^{-\frac{l}{2m}\tau}[\varphi_l^*(y) + o(1)], \tag{6.29}$$

with the eigenfunction (6.12) and $o(1) \sim O(\tau^{l+1}\mathrm{e}^{-\tau/2m})$ with, possibly, an extra factor of the algebraic growth. This completes the classification of finite-order multiple zeros.

Non-existence of non-trivial zeros of infinite order. We next need to prove that infinite-order zeros exist for trivial solutions $u = 0$ only; i.e., the dynamical system (6.9) does not admit non-trivial solutions with a super-exponential decay rate as $\tau \to \infty$. In the abstract form, for linear equations in Hilbert spaces, such results are well established in the mathematical literature; see [1, 6, 7, 24] and earlier references on Agmon's and Ogawa's results therein. Some of the approaches essentially rely on the symmetry of the unperturbed operator $\mathbf{B}^*$ (cf. [6, Section 5]), and therefore cannot be applied here. We follow the lines of the analysis in [1, Appendix], which is formulated for self-adjoint operators, but admits a natural extension to general operators $\mathbf{B}^*$ with real spectrum bounded from above.

Proposition 6.1. *Assume that* $u(\tau) \in \tilde H^{2m}_{\rho^*}$ *is such that*

$$\|u(\tau)\|_{0*} = o(\mathrm{e}^{-K\tau}) \quad \textit{as } \tau \to \infty, \tag{6.30}$$

where K *can be an arbitrarily large constant. Then* $u = 0$.

Proof. We follow the lines and basic notation of the analysis in [1, pp. 434–437]. We have that the operator $\mathbf{A} = -\mathbf{B}^* : E_1 = h^{2m}_{\rho^*} \to l^2_{\rho^*} = E$ has a non-negative discrete spectrum $\sigma(\mathbf{A}) = \{\tilde\lambda_\beta = \frac{|\beta|}{2m}\}$ with the constant spectral half-gaps denoted by $\delta_k \equiv \delta = \frac{1}{4m}$. Then we set $\gamma_k = \frac{2k+1}{4m}$. The spectral projections of $\mathbf{A}$ denoted by $P_k = P(\gamma_k)$ for $k = 1, 2, \dots$ are given by

$$P_k u = \sum_{|\beta| \le k} \langle u, \psi_\beta \rangle \psi_\beta^*, \quad \text{and} \quad Q_k u = (I - P_k) u = \sum_{|\beta| > k} \langle u, \psi_\beta \rangle \psi_\beta^*.$$

We next check conditions (B1) and (B2) in [1, p. 435]. Under given assumptions on the smoothness of coefficients of the parabolic operator, the perturbation operator $\mathbf{C}(\tau)$ in (1.9) is Hölder continuous with respect to the operator norm on $\mathcal{L}(E_1, E)$. Since $\mathbf{C}(\tau)$ is a differential $2m$th-order operator with smooth exponentially small coefficients given in (1.11) and (1.12), we have that $\mathbf{C}(\tau) : E_1 \to E$ is a bounded operator,

$$\|\mathbf{C}(\tau) u\|_{0*} \le \|\mathbf{C}(\tau)\| \, \|u\|_{1*}, \quad \text{where} \ \|\mathbf{C}(\tau)\| = O\big(\mathrm{e}^{-\frac{1}{2m}\tau}\big) \ \text{for} \ \tau \gg 1.$$

By shifting the origin in time, we may assume that

$$M = \sup_{\tau \ge 0} \|\mathbf{C}(\tau)\| < \tfrac{\delta}{2};$$

i.e., $\|\mathbf{C}(\tau)\|$ is small in comparison to the gaps in the spectrum of $\mathbf{A}$. Denote by $u(\tau)$ the unique sufficiently smooth solution of (1.9) with initial data $u_0 \in E$; see [23] and [13]. We next introduce the subspaces

$$V_k = \{u_0 \in E_1 : \quad \mathrm{e}^{\gamma_k \tau} u(\tau) \to 0 \ \text{as} \ \tau \to \infty\},$$

where $V_{k+1} \subset V_k$ for any $k \ge 1$.

We now apply Lemma 5 in [1], which is proved along similar lines without using specific self-adjoint properties of the unperturbed operator $\mathbf{A} = -\mathbf{B}^*$. This part is based on the analysis of the integral equation

$$u(\tau) = \mathrm{e}^{\mathbf{B}^* \tau} Q_k u_0 + \int_0^\tau \mathrm{e}^{\mathbf{B}^*(\tau - s)} Q_k \mathbf{C}(s) u(s) \mathrm{d}s - \int_\tau^\infty \mathrm{e}^{\mathbf{B}^*(\tau - s)} P_k \mathbf{C}(s) u(s) \, \mathrm{d}s.$$

Setting $v(\tau) = \mathrm{e}^{\gamma_k \tau} u(\tau)$ gives the integral equation

$$v(\tau) = \mathrm{e}^{(\gamma_k + \mathbf{B}^*)\tau} Q_k u_0 + \int_0^\infty K(\tau - s) \mathbf{C}(s) v(s) \, \mathrm{d}s \equiv q(\tau) + L v(\tau), \quad (6.31)$$

with the kernel

$$K(\nu) = \big\{ Q_k \mathrm{e}^{(\gamma_k + \mathbf{B}^*)\nu} \ \text{if} \ \nu > 0 \ \text{and} \ -P_k \mathrm{e}^{(\gamma_k + \mathbf{B}^*)\nu} \ \text{if} \ \nu < 0 \big\}.$$

It follows that for $\nu > 0$ and any $w \in h^{2m}_{\rho^*}$,

$$\|K(\nu) w\|_{0*}^2 = \sum_{|\beta| > k} \mathrm{e}^{2(\gamma_k + \lambda_\beta)\nu} |c_\beta|^2 \le \mathrm{e}^{-2\delta\nu} \|w\|_{0*}^2 \le \mathrm{e}^{-2\delta\nu} \|w\|_{1*}^2,$$

where $A^*_{\beta\gamma}$ is estimated as in (5.22), so that in operator norms on $\mathcal{L}(E)$ and $\mathcal{L}(E_1, E)$,

$$\|K(\nu)\| \le \mathrm{e}^{-\delta\nu} \quad \text{for } \nu > 0.$$

By a similar estimate for $\nu < 0$, we conclude that the kernel is exponentially decaying,

$$\|K(\nu)\| \le \mathrm{e}^{-\delta|\nu|} \quad \text{for } \ \nu \in \mathbf{R}. \tag{6.32}$$

Equation (6.31) can be solved by Banach's contraction principle in the space

$$F = \{v \in C([0,\infty);\, E_1) : \ \ v(\tau) \to 0 \ \text{ as } \ \tau \to \infty\}.$$

We have

$$\begin{aligned}\|Lv(\tau)\|_{0*} &\le \int_0^\infty \|K(t-s)\|\,\|\mathbf{C}(s)\|\,\|v(s)\|_{1*}\,\mathrm{d}s\\ &\le M\Big(\int_{-\infty}^\infty \|K(\nu)\|\,d\nu\Big)\sup_{s\ge 0}\|v(s)\|_{1*},\end{aligned}$$

and hence by (6.32)

$$\|L\| \le M\int_{-\infty}^\infty \|K(\nu)\|\,\mathrm{d}\nu \le \frac{2M}{\delta} < 1.$$

Therefore, the solution of (6.31) is given by the converging series

$$v(\tau) = \sum_{j=0}^\infty L^j q(\tau).$$

Denote $q_0 = Q_k u_0 \in R(Q_k)$. The mapping T_k defined by $v(0) = T_k q_0$ is then bounded,

$$\|T_k\| \le \sum_{j=0}^\infty \Big(\frac{2M}{\delta}\Big)^j = \Big(1 - \frac{2M}{\delta}\Big)^{-1} < \infty. \tag{6.33}$$

Since $Q_k \circ T_k = I$ on $R(Q_k)$ and $T_k \circ Q_k = I$ on V_k, it follows that $Q_k : V_k \to R(Q_k)$ is an isomorphism; see [1, p. 435]. Hence, if $u_0 \in V_k$ for all $k \ge 1$, then $u_0 = T_k \circ Q_k u_0$ and

$$\|u_0\|_{0*} = \|T_k \circ Q_k u_0\|_{0*} \le \Big(1 - \frac{2M}{\delta}\Big)^{-1}\|Q_k u_0\|_{0*},$$

where

$$Q_k u_0 = \sum_{|\beta|>k} \langle u_0, \psi_\beta\rangle \psi^*_\beta \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, $u_0 = 0$. This completes the proof of Proposition 6.1. □

6.3. **Main result on blow-up formation of multiple zeros.** Thus, we arrive at the following classification of multiple zeros of solutions of parabolic PDEs (1.1).

Theorem 6.1. *Let, under given regularity assumptions* (6.2), (6.3)*, the rescaled solution* $u(\cdot,\tau) \in \tilde{H}^{2m}_{\rho^*}$ *of* (1.1) *create a multiple zero at* (0,0)*. Then there exists a finite* $l \geq 1$ *such that* (6.29) *holds, where* φ_l *is an eigenfunction* (6.12) *of* $\mathbf{B}^*$ *corresponding to the eigenvalue* $-\frac{l}{2m}$.

Next, we need to interpret the above asymptotic result by using the standard time-independent parabolic rescaling

$$u_\varepsilon(y,s) = \varepsilon^{-l} u(y\varepsilon, s\varepsilon^{2m}), \quad \text{with an arbitrary parameter } \varepsilon > 0, \tag{6.34}$$

where $s < 0$ is the new time variable. Then u_ε satisfies the perturbed equation (cf. (1.9))

$$u_s = \mathbf{B}_0 u + \mathbf{C}(\varepsilon) v, \tag{6.35}$$

with an asymptotically small perturbing operator

$$\mathbf{C}(\varepsilon) = \sum_{|\beta|=2m} \big[a_\beta(y\varepsilon, s\varepsilon^{2m}) - A_\beta\big] D_y^\beta + \sum_{|\beta|<2m} \varepsilon^{2m-|\beta|} a_\beta(y\varepsilon, s\varepsilon^{2m}) D_y^\beta. \tag{6.36}$$

By Theorem 6.1 we arrive at the following easy consequence.

Corollary 6.1. *With an* l *as in Theorem* 6.1, $\{u_\varepsilon\}_{\varepsilon>0}$ *is a compact subset, and uniformly on compact subsets from* $\mathbf{R}^N \times (-\infty, 0]$*, there holds*

$$u_\varepsilon(y,s) \to W(y,s) = (-s)^{\frac{l}{2m}} \varphi_l^*\big(y/(-s)^{\frac{1}{2m}}\big) \quad \text{as } \varepsilon \to 0^+. \tag{6.37}$$

Note that by (6.11), there exists a finite limit

$$W(y,0^-) = P_0(y) \equiv \frac{1}{\sqrt{l!}} \sum_{|\beta|=l} C_\beta y^\beta \not\equiv 0. \tag{6.38}$$

6.4. **Instantaneous collapse of multiple zeros.** We now consider the evolution of the above solutions for $t > 0$ describing collapse of multiple zeros. For the one-dimensional second-order parabolic equations, such an extension was also performed by Sturm [32]. He showed that, due to the established asymptotic behaviour as $t \to 0^+$ driven by the *adjoint polynomials* (not eigenfunctions of $\mathbf{B}$), most zero curves disappear at $t = 0$ in each of such collapses and new zero curves cannot appear at all. Recall that precisely this led Sturm to state his First Theorem, saying that the number of zeros of solutions does not increase with time; see [32, p. 431].

We briefly describe the collapse phenomenon of multiple zeros for the higher-order equations under consideration, where new zeros can appear for $t > 0$. We apply a time-evolution description of such a transition phenomenon from $\{t < 0\}$ to $\{t > 0\}$. Consider a general multiple-zero pattern (6.29). Bearing in mind the Sturm variable (1.5), we have that

$$u(x,t) = (-t)^{\frac{l}{2m}} \varphi_l^*\big(x/(-t)^{\frac{1}{2m}}\big) + \cdots, \tag{6.39}$$

where we omit higher-order terms. Since $\varphi_l^*(y)$ is a polynomial of order l, by (4.8)

$$\varphi_l^*(y) = \frac{1}{\sqrt{l!}} \sum_{|\beta|=l} C_\beta y^\beta + \cdots \equiv P_0(y) + \cdots \quad \text{as } \ y \to \infty, \tag{6.40}$$

where $P_0(y)$ denotes a non-trivial homogeneous polynomial of order l. Therefore, passing to the limit $t \to 0^-$ in (6.39) and using (6.40), we observe that, in the leading term, the time-dependent multipliers cancel each other. It follows from Corollary 6.1 that there exists a finite limit

$$u(x,0^-) = \frac{1}{\sqrt{l!}} \sum_{|\beta|=l} C_\beta x^\beta + \cdots \equiv P_0(x) + \cdots \quad \text{for small } \ |x|. \tag{6.41}$$

For convenience, we now perform a formal evolution analysis, to be justified later on. Using the forward continuation variables (1.19), we arrive at an exponentially perturbed equation of the form

$$u_\tau = (\mathbf{B} - \frac{N}{2m} I)u + \mathbf{C}(\tau)u, \tag{6.42}$$

with (6.41) as the initial data. The limit $t \to 0^+$ means $\tau \to -\infty$. As usual, this asymptotic problem with *a priori* prescribed initial data is easier than the above evolution problem.

Polynomial solutions Φ_l are not eigenfunctions of B. We then obtain that the asymptotic behaviour as $\tau \to -\infty$ for (6.42) is given by adjoint polynomials associated with the operator $\mathbf{B}$,

$$u(y,\tau) = \mathrm{e}^{\frac{l}{2m}\tau} \Phi_l(y) + \cdots. \tag{6.43}$$

Substituting into (6.42) and neglecting the perturbation, we have that Φ_l is a *polynomial solution* of the *eigenvalue equation*

$$\mathbf{B}\Phi_l = \mu_l \, \Phi_l, \quad \text{with} \quad \mu_l = \frac{N+l}{2m}. \tag{6.44}$$

We call such μ_l's *polynomial numbers*, for which (6.44) admits polynomial solutions. Obviously, $\{\mu_l\}$ are different from eigenvalues of $\mathbf{B}$ in L^2_ρ, and

Φ_l *are not* eigenfunctions of $\mathbf{B}$, since polynomials $\Phi_l \notin L^2_\rho$ with the exponentially growing weight (1.16). Similar to (6.11), we obtain the following representation of such polynomials:

$$\Phi_l(y) = P_0(y) + \sum_{j=1}^{[\frac{l}{2m}]} \frac{1}{j!} \mathbf{B}_0^j P_0(y). \tag{6.45}$$

Thus, in the original variables,

$$u(x,t) = t^{\frac{l}{2m}} \Phi_l\big(x/t^{\frac{1}{2m}}\big) + \cdots , \tag{6.46}$$

and hence taking into account the leading higher-order terms in polynomial (6.45), we see that $u(x,0^+) = P_0(x) + \cdots$, coinciding with (6.41).

The generating formulae of polynomials (6.11) (for $t < 0$) and (6.45) (for $t > 0$) describe all possible exchanges of zero surfaces at the focusing time $t = 0$ of multiple zeros. Combining both expansions (6.29) and (6.46), we have that if a multiple zero of $u(x,t)$ occurs at the origin $(0,0)$, then there exists a finite $l \geq 1$ such that, as $\varepsilon \to +0$,

$$\varepsilon^{-\frac{l}{2m}} u(y\varepsilon^{\frac{1}{2m}}, -\varepsilon) \to \varphi_l^*(y) \quad \text{and} \quad \varepsilon^{-\frac{l}{2m}} u(y\varepsilon^{\frac{1}{2m}}, \varepsilon) \to \Phi_l(y) \tag{6.47}$$

uniformly on compact subsets.

We now apply the rescaling argument based on the transformation (6.34) leading to the perturbed equations (6.35). Since this rescaling makes sense for both $s < 0$ and $s > 0$ and the equation and the asymptotically small perturbation operator in (6.36) are of the same structure, we arrive at the following result (see [6] for $m = 1$):

Corollary 6.2. *Under the assumptions of Corollary* 6.1*,* (6.37) *holds uniformly on compact subsets in* $\mathbf{R}^N \times [0,\infty)$.

We thus obtain that the function $W(y,s)$ in (6.37) is a polynomial solution of the linear homogeneous parabolic equation

$$W_s = \mathbf{B}_0 W \equiv \sum_{|\beta|=2m} A_\beta D_y^\beta W \quad \text{in } \mathbf{R}^N \times \mathbf{R}. \tag{6.48}$$

Hence,

$$\Phi_l(y) = (-1)^{\frac{l}{2m}} \varphi_l^*\big(y/(-1)^{\frac{1}{2m}}\big), \tag{6.49}$$

which is seen from (6.11) and (6.45).

Example 6.1. Let us discuss some types of multiple-zero formation of higher-order parabolic equations. For simplicity, we consider the 1D non-perturbed case and begin with the *bi-harmonic equation*

$$u_t = -u_{xxxx} \quad \text{in} \quad \mathbf{R} \times [-1,1]. \tag{6.50}$$

The simplest non-trivial polynomial explicit solution with zero at $(0,0)$ is

$$u(x,t) = \tfrac{1}{\sqrt{4!}}\,(x^4 - 24t).$$

Performing necessary scalings for $t<0$ and $t>0$ yields the following asymptotics corresponding to $l=4$, $\lambda_4 = -\frac{l}{4} = -1$:

$$u(x,t) = \begin{cases} (-t)\frac{1}{\sqrt{4!}}(y^4+24) \equiv \mathrm{e}^{-\tau}\psi_4^*(y),\ y=x/(-t)^{\frac14},\ \tau=-\ln(-t),\ t<0; \\ t\,\frac{1}{\sqrt{4!}}(y^4-24) \equiv \mathrm{e}^{\tau}\Phi_4(y),\ \ y=x/t^{\frac14},\ \ \tau=\ln t,\ \ t>0. \end{cases}$$

Here, as usual, $\psi_4^*(y)$ is the polynomial eigenfunction of the adjoint operator $\mathbf{B}^*$ in (1.10),

$$\mathbf{B}^* = -D_y^4 - \tfrac14\,yD_y.$$

$\Phi_4(y)$ is defined by (6.45). It is not an eigenfunction of $\mathbf{B}$, and is a polynomial solution of the corresponding eigenfunction equation (6.44) with $N=1$, $m=2$, and $l=4$, i.e.,

$$\mathbf{B}\Phi_4 \equiv -\Phi_4^{(4)} + \tfrac14\,y\Phi_4' + \tfrac14\,\Phi_4 = \tfrac54\,\Phi_4 \quad \big(\Phi_4 \notin L^2_\rho\big).$$

For the *tri-harmonic equation*

$$u_t = u_{xxxxxx} \quad \text{in} \quad \mathbf{R}\times[-1,1], \tag{6.51}$$

the corresponding explicit solution is

$$u(x,t) = \tfrac{1}{\sqrt{6!}}(x^6 + 720t).$$

This corresponds to $l=6$, with $\lambda_6 = -\frac{l}{6} = -1$, so that

$$u(x,t) = \begin{cases} (-t)\frac{1}{\sqrt{6!}}(y^6-720) \equiv \mathrm{e}^{-\tau}\psi_6^*(y),\ \ y=x/(-t)^{\frac16}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \tau=-\ln(-t),\ \ t<0; \\ t\frac{1}{\sqrt{6!}}(y^6+720) \equiv \mathrm{e}^{\tau}\Phi_6(y),\ \ y=x/t^{\frac16},\ \ \tau=\ln t,\ \ t>0. \end{cases}$$

Again, ψ_6^* is an eigenfunction of

$$\mathbf{B}^* = D_y^6 - \tfrac16\,yD_y, \tag{6.52}$$

and Φ_6 is the polynomial solution of (6.44) for $N=1$, $m=3$, and $l=6$,

$$\mathbf{B}\Phi_6 \equiv \Phi_6^{(6)} + \tfrac16\,y\Phi_6' + \tfrac16\,\Phi_6 = \tfrac76\,\Phi_6 \quad \big(\Phi_6 \notin L^2_\rho\big).$$

Notice an essential difference between these multiple-zero formations for the fourth- (6.50) and sixth-order (6.51) parabolic equations: for the former one two new zeros

$$x_\pm(t) = \pm\,(24\,t)^{\frac14} \quad \text{for} \quad t>0,$$

appear at $t = 0^+$ at the origin $x = 0$. In the latter equation, two zeros *disappear*

$$x_{\pm}(t) = \pm\,[720\,(-t)]^{\frac{1}{6}} \quad \text{for} \quad t < 0,$$

at $t = 0^-$. As Sturm's results say, the last feature is the only possible for the heat equation

$$u_t = u_{xx} \quad \text{in} \quad \mathbf{R} \times [-1, 1]. \tag{6.53}$$

This shows that certain weak traces of the Sturmian property of decreasing number of zeros for (6.53) can be observed for the $2m$-th order parabolic PDEs with odd $m = 3, 5, \cdots$ (this Sturm zero property is violated in other zero formation phenomena; see below). On the contrary, for even $m = 2, 4, ...$, such an "order-preserving" property of the parabolic flow is violated by this multiple-zero formation mechanism.

To show the obvious violation of Sturm's First Theorem for the flow (6.51), we consider the explicit solution with $l = 2$ and $\lambda_2 = -\frac{l}{6} = -\frac{1}{3}$,

$$u(x,t) = \tfrac{1}{\sqrt{2}}(-x^6 + x^2 - 720\,t),$$

so that

$$u(x,t) = \begin{cases} (-t)^{\frac{1}{3}}\,\frac{1}{\sqrt{2}}\big[y^2 - (y^6 - 720)(-t)^{\frac{2}{3}}\big] \equiv \mathrm{e}^{-\frac{1}{3}\,\tau}\,\psi_2^*(y) - \mathrm{e}^{-\tau}\sqrt{\frac{6!}{2}}\;\psi_6^*(y),\ t < 0; \\ t^{\frac{1}{3}}\,\frac{1}{\sqrt{2}}\big[y^2 - (y^6 + 720)t^{\frac{2}{3}}\big] \equiv \mathrm{e}^{\frac{1}{3}\,\tau}\,\Phi_2(y) - \mathrm{e}^{\tau}\,\Phi_6(y),\ t > 0. \end{cases} \tag{6.54}$$

Here, the normalized eigenfunctions of (6.52) with $\lambda_2 = -\frac{1}{3}$ is $\psi_2^*(y) = \frac{1}{\sqrt{2}}\,y^2$. Notice that two small zeros appeared at $t = 0^+$ having the behaviour

$$x_{\pm}(t) = \pm\sqrt{720}\,\sqrt{t}\;(1 + O(t^2)) \quad \text{for} \quad 0 < t \ll 1.$$

These zero curves are concentrated on smaller subsets $\sim \sqrt{t} \ll t^{1/6}$ for $t \approx 0^+$, than those given by the parabolic scaling $t^{1/2m}$. Nevertheless, the behaviour as $t \to 0$ in (6.54) is rather trivial and is governed by a "degenerate" eigenfunction $\psi_2^*(y) = \frac{1}{\sqrt{2}}\,y^2$ that does not have interesting internal structure. It is natural to qualify such a micro-structure as trivial.

One can construct explicit examples of polynomial solutions of the corresponding multi-dimensional PDEs

$$u_t = -\Delta^2 u \quad \text{and} \quad u_t = \Delta^3 u \quad \text{in} \quad \mathbf{R}^N \times [-1, 1],$$

where the adjoint basis $\{\psi_\beta^*\}$ for $\mathbf{B}^*$ and polynomial solutions $\{\Phi_l\}$ associated with $\mathbf{B}$ occur in multiple-zero evolution phenomena.

7. Unique continuation theorem

The first unique continuation theorem is a consequence of Theorem 6.1 establishing that a solution from the existence-uniqueness class of the parabolic equation (1.1) with sufficiently smooth coefficients cannot generate a multiple zero of infinite order unless $u = 0$.

Theorem 7.1. *Let, under given hypotheses on the coefficients, the solution* $u(\cdot, t) \in \tilde{H}^{2m}_{\rho^*}$ *of* (1.1) *satisfy*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^k} \int_{|x|<\varepsilon} |u(x,0)| \, \mathrm{d}x = 0 \quad \textit{for any } k > 0. \tag{7.1}$$

Then $u(x,t) \equiv 0$ *in* $\mathbf{R}^N \times (-1,1)$.

Proof. In view of (6.41), the integral condition (7.1) implies that the solution $u(x,t)$ has a zero of infinite order at $(0,0)$, and hence $u = 0$ by Proposition 6.1. □

Obviously, once we have achieved the optimal classification of multiple zeros (the micro-structure of the PDE), some backward uniqueness results are straightforward. Actually, one can characterize a variety of such optimal backward uniqueness approaches as follows:

if at $(0,0)$ a solution u violates (6.29) (or (6.41)) for any $l \in \mathbb{N}$, then $u = 0$. (7.2)

The results apply to systems of $2m$th-order *parabolic inequalities*

$$|u_t - \mathbf{B}_0 u| \le M \sum_{0 \le k < 2m} |D^k u| \quad \text{in } Q_1, \tag{7.3}$$

where $M > 0$ is a constant and $D^k u$ is the vector $\{D^\beta u, \ |\beta| = k\}$. These inequalities include the parabolic PDE (1.1) with constant coefficient $a_\beta = A_\beta$ for $|\beta| = 2m$ and arbitrary uniformly bounded coefficients $|a_\beta| \le M$ in the lower-order operators with $|\beta| < 2m$. We then arrive at a similar result.

Theorem 7.2. *Let* $u(\cdot,t) \in \tilde{H}^{2m}_{\rho^*}$ *be a solution of* (7.3)*, and* (7.1) *hold. Then* $u = 0$.

Proof. One can see that after Sturmian scaling (1.5), (1.6), the function $u(y,\tau)$ can be treated as a solution of the PDE (1.9) where the perturbation $\mathbf{C}(\tau)$ is uniformly exponentially small as $\tau \to \infty$, and the above conclusion applies. □

8. Dimension of nodal sets

Without loss of generality, we formulate the result on the Hausdorff dimension of nodal sets for the solutions of parabolic inequalities (7.3).

Theorem 8.1. *Let, under given hypotheses,* $u(\cdot,t) \in \tilde{H}^{2m}_{\rho^*}$*,* $u \not\equiv 0$*, be a sufficiently smooth solution of* (7.3)*. Then its nodal set* (1.22) *satisfies* (1.23)*.*

Estimates like (1.23) are well known for the second-order elliptic and parabolic equations with the proof based on a general idea of the dimensional reduction argument in geometric measure theory; see Simon [31, Section 2] and Lin [26].

Proof. We follow the lines of the analysis given in [6, Section 8, 9], which can be applied to solutions of higher-order inequalities or equations (or other functions exhibiting suitable asymptotic scaling properties at any point (x_0, t_0)) provided that two crucial results are available:

(i) The result of Corollary 6.2 makes it possible to introduce a locally asymptotically self-similar pair $(\mathcal{F}, \mathcal{L})$ as in [6, p. 627], where $\mathcal{F}$ is a collection of sufficiently smooth solutions u and $\mathcal{L}[u] = \{(x,0) \in \mathbf{R}^N \times \mathbf{R} : \; u(x,0) = 0\}$. The only difference is that according to (6.34) we define the scaling map $g(y,s;\lambda,\alpha)$ as follows:

$$(g(y,s;\lambda,\alpha)u)(x,t) = \alpha u(y+\lambda x, s+\lambda^{2m}t).$$

(ii) The polynomial structure of the limit function W in (6.37) makes it possible to apply Theorem 8.5 in [6] and to complete the proof. □

Therefore, geometric estimates on the parabolic dimension of various nodal sets obtained in [6] for $m = 1$ remain valid for higher-order differential parabolic equations.

9. Final conclusions: on backward blow-up analysis and polynomial zero structure for other PDEs

It is not remarkable that main results are extended to more general quasilinear *parabolic* equations. But it is rather unusual to observe that several typical features of the blow-up analysis via Sturmian backward continuation variable and polynomial eigenfunctions of various linear operators can be traced out for other types of PDEs. We will briefly discuss these applications, and finally summarize these results by introducing a wider class of integral equations with pseudo-differential operators that admit similar polynomial multiple zero micro-structure

9.1. **General $2m$th-order quasi-linear parabolic PDEs.** One can see that the present approach to multiple zero formations and the corresponding unique continuation theorems admits extensions to quasi-linear uniformly parabolic PDEs (or inequalities, cf. (7.3))

$$u_t = \sum_{|\beta| \le 2m} a_\beta(x,t,u) D^\beta u \quad \text{in} \;\; Q_1 \quad (m \ge 2), \tag{9.1}$$

with sufficiently smooth bounded coefficients $a_\beta(x,t,u)$ satisfying necessary hypotheses. Then $A_\beta = a_\beta(0,0,0)$ for $|\beta| = 2m$ and $\mathbf{B}_0$ is assumed to be uniformly elliptic.

Remark: on quasilinear degenerate parabolic equations. The structure of multiple zeros is still not fully justified even for simplest degenerate 1D parabolic PDEs such as the *porous medium equation* (PME) or the *p-Laplacian* equation

$$u_t = (|u|^n u)_{xx} \quad \text{and} \quad u_t = \big(|u_x|^n u_x\big)_x \quad \text{in} \quad \mathbf{R} \times [-1,1] \quad (n > 0). \tag{9.2}$$

The PME admits compactly supported solutions so zeros of infinite order are possible. We then mean *non-trivial* zeros at the origin $(0,0)$, where $u(x,t) \not\equiv 0$ in any of its neighbourhoods. Notice that equations (9.2) admit oscillatory solutions with *infinitely many* sign changes near finite interfaces [16, 4], so, formally, this means existence of non-trivial zeros of infinite order.

Some types of multiple zeros for (9.2) (but not all of them) are described by blow-up similarity solutions of the second kind and lead to complicated non-linear ODE problems generated *non-linear eigenfunctions* (unlike the linear spectral theory for $n = 0$). A complete classification of such multiple zeros is not available, so no unique continuation theorems of the pointwise sense can be formulated. We refer to a discussion in [14, pp. 30–33], where basic references and results can be found.

Little is known about complicated types of multiple zeros for fourth-order (or higher-order) degenerate PDEs of *porous medium* and *p-Laplacian* type

$$u_t = -\big(|u|^n u\big)_{xxxx} \quad \text{and} \quad u_t = -\big(|u_{xx}|^n u_{xx}\big)_{xx} \quad (n > 0),$$

to say nothing of the *thin film equation* (TFE) with the non-monotone and non-potential operator,

$$u_t = -\big(|u|^n u_{xxx}\big)_x \quad \text{in} \quad \mathbf{R} \times [-1,1] \quad (n \in (0,3)\,). \tag{9.3}$$

In the Cauchy problem, all these higher-order degenerate parabolic PDEs admit solutions that are infinitely oscillatory near interfaces with a difficult distributions of zeros; see [19, Chapter 3] and [10] (for the TFE (9.3)) for further references and basic results.

9.2. **Linear dispersion equations.** It turns out that, under necessary hypotheses, some of the results on multiple zeros can be extended to odd-order *linear dispersion* PDEs with sufficiently smooth coefficients

$$u_t = \sum_{|\beta| \le 2m+1} a_\beta(x,t) D^\beta u \quad \text{in} \quad Q_1 \quad (m \ge 1). \tag{9.4}$$

Semigroup approaches for the Cauchy problem for such PDEs have a long history, and the first systematic results go back to Kato and Kruzhkov–Faminskii in the 1970s; see Faminskii [11] for a full list of references and more recent results. A regularization approach for the KdV equation

$$u_t + uu_x = u_{xxx},$$

by adding to the right-hand side the parabolic term $-\varepsilon u_{xxxx}$ and passing to the limit $\varepsilon \to 0^+$, were used earlier by Temam in the 1960s; see the Lions classic book [27, Chapter 3].

Unlike parabolic equations, for PDEs (9.4), another version of spectral theory to reveal the structure of multiple zeros is needed. Notice that such $(2m+1)$th-order equations have a peculiar regularity theory and do have highly oscillatory fundamental solutions.

For instance, in the case $m = 1$, $N = 1$, with special constant coefficients α_β, we obtain the PDE (1.24), where the fundamental solution,

$$b(x,t) = t^{-\frac{1}{3}} F(y), \quad y = x/t^{\frac{1}{3}} \quad (\text{of the operator } D_t - D_x^3), \tag{9.5}$$

is given by the bounded Airy function $F = \mathrm{Ai}(y)$ satisfying

$$\mathbf{B}F \equiv F''' + \frac{1}{3}(Fy)' = 0 \quad \text{in} \quad \mathbf{R}, \quad \int F = 1. \tag{9.6}$$

Therefore, the fundamental kernel $F(y)$ has exponential decay as $y \to -\infty$ only, and as $y \to +\infty$ is oscillatory according to the asymptotics

$$F(y) \sim \begin{cases} |y|^{-\frac{1}{4}} \mathrm{e}^{-d|y|^{3/2}} & \text{as } y \to -\infty, \\ y^{-\frac{1}{4}} \cos\big(dy^{\frac{3}{2}} + A\big) & \text{as } y \to +\infty, \end{cases} \tag{9.7}$$

where $d = \frac{2}{9}\sqrt{3}$ and A is a constant.

This generates very oscillatory (as $y \to +\infty$) and unbounded (for any $l \ge 1$) eigenfunctions

$$\psi_l(y) = \frac{1}{\sqrt{l!}} D_y^l F(y), \text{ where } F = \mathrm{Ai}(y) \quad \Big(\psi_l^* = \frac{1}{\sqrt{l!}}(y^l + \dots)\Big), \tag{9.8}$$

of the linear operator $\mathbf{B}$ defined similarly to (1.15), e.g., for (1.24) and (9.6),

$$\mathbf{B} = D_y^3 + \tfrac{1}{3} yD_y + \tfrac{1}{3} I, \text{ with the point spectrum } \quad \sigma_p(\mathbf{B}) = \big\{\tfrac{1-l}{3},\, l \ge 0\big\}.$$

The space L^2_ρ has a non-symmetric continuous positive weight satisfying

$$\rho(y) = \begin{cases} \mathrm{e}^{-ay^{3/2}} & \text{for } \; y \ge 1, \\ \mathrm{e}^{a|y|^{3/2}} & \text{for } \; y \le -1. \end{cases} \tag{9.9}$$

Here $a \in (0, 2d)$ is a constant, so $\rho(y)$ is fast exponentially decaying as $y \to +\infty$ to compensate for the opposite behaviour of eigenfunctions (9.8), and is growing as $y \to -\infty$, where $\mathrm{Ai}(y)$ is similarly exponentially decaying as in (9.7).

The necessary adjoint operator that appears after blow-up rescaling of (1.24) with the Sturmian variable (cf. (1.5)) $y = x/(-t)^{\frac{1}{3}}$, takes the form (the analogy of (1.10))

$$\mathbf{B}^* = D_y^3 - \frac{1}{3}\, yD_y \quad \big(\text{with } \; \sigma_p(\mathbf{B}^*) = \{-\frac{l}{3},\, l \ge 0\}, \; \text{cf. (4.7)}\big). \tag{9.10}$$

One can see that $\mathbf{B}^*$ *is not adjoint* to $\mathbf{B}$ in the standard metric of L^2, since, obviously, then $\tilde{\mathbf{B}}^* = -\mathbf{B} + \frac{1}{3}\, I$, so $\mathbf{B} - \frac{1}{6}\, I$ is skew-symmetric.

$\mathbf{B}^*$ is adjoint to $\mathbf{B}$ in the sense of (4.2) in terms of the *indefinite metric*, with the product

$$\langle v, w \rangle_* = \int v(y)\overline{w(-y)}\, \mathrm{d}y \equiv \langle v, \overline{Jw} \rangle, \quad v, w \in L^2_{\rho^*}. \tag{9.11}$$

The spectrum of $\mathbf{B}^*$ is real, so we omit the complex conjugation and use the space over the field of real numbers. Here, the *canonical symmetry operator* $Jw(y) = w(-y)$ is bounded, self-adjoint, and unitary (it is the *Gramm operator* of this metric). The condition $Jw \in L^2_\rho$ determines the corresponding space $L^2_{\rho^*}$ with the symmetric exponentially decaying weight

$$\rho^*(y) = \rho(|y|) = \mathrm{e}^{-a|y|^{3/2}} \; \text{ for all } \; |y| \ge 1.$$

The set of even functions $E_+ = \{v(-y) \equiv v(y)\}$ is a *positive lineal* (a linear manifold) of the metric (9.11),

$$\langle v, v \rangle_* > 0 \quad \text{for} \quad v \in E_+, \; v \ne 0,$$

and odd functions $E_- = \{v(-y) \equiv -v(y)\}$ give the corresponding *negative lineal*. Therefore, $L^2_{\rho^*}$ with this metric is *decomposable*,

$$v = v_+ + v_- \equiv \frac{v(y) + v(-y)}{2} + \frac{v(y) - v(-y)}{2},$$

where $v_\pm \in E_\pm \implies L^2_{\rho^*} = E_+ \oplus E_-$, where, in addition, $E_+ \perp E_-$ in the metric (9.11). The corresponding positive *majorazing* metric is given by

$$|\langle v, v \rangle_*| \le [v, v]_* \equiv \langle v_+, v_+ \rangle_* - \langle v_-, v_- \rangle_*,$$

etc. This case of the decomposable space with indefinite metric with straightforward majorizing one is treated as rather trivial; see Azizov–Iokhvidov [2] for linear operators theory in spaces with indefinite metric. Metric (9.11) is widely used therein; see [2, pp. 13, 17, 23, 114]. Then the domain of B^* is defined as $H^3_{\rho^*}$, etc.

It is key that the bounded operator $\mathbf{B}^* : H^3_{\rho^*} \to L^2_{\rho^*}$ admits a complete set of *polynomial* eigenfunctions $\Phi = \{\psi_l^*(y)\}$, which are given in (9.8) and are constructed similarly to those obtained in (4.8). Quite analogously, these polynomials are responsible for formation of multiple zeros for the PDE (1.24). This study of spectral and other properties [12] leads to some difficult problems that do not appear in the parabolic case.

Example 9.1. For instance, take the simplest explicit solution of (1.24),

$$u(x,t) = \tfrac{1}{\sqrt{6}}\,(x^3 + 6\,t).$$

Rescaling it for $t < 0$ and $t > 0$ yields the following formation-collapse multiple-zero behaviour corresponding to $l = 3$, i.e., with $\lambda_l = -\frac{l}{3} = -1$:

$$u(x,t) = \begin{cases} (-t)\,\tfrac{1}{\sqrt{6}}\,(y^3 - 6) \equiv \mathrm{e}^{-\tau}\psi_3^*(y),\ y = x/(-t)^{\frac{1}{3}},\ \tau = -\ln(-t),\ t<0; \\ t\,\tfrac{1}{\sqrt{6}}\,(y^3 + 6) \equiv \mathrm{e}^{\tau}\Phi_3(y),\ y = x/t^{\frac{1}{3}},\ \tau = \ln t,\ t>0, \end{cases}$$

where $\psi_3^*(y)$ is the polynomial eigenfunction given in (9.8) of the adjoint operator (9.10), and $\Phi_3(y)$ is the polynomial defined similarly to (6.45), not an eigenfunction of $\mathbf{B}$.

Similarly, for the explicit polynomial solution of (1.24)

$$u(x,t) = \frac{1}{\sqrt{6!}}(x^6 + 120tx^3 + 360t^2),$$

we have $l = 6$, $\lambda_6 = -2$, and the formation-collapse expansions

$$u(x,t) = \begin{cases} (-t)^2\tfrac{1}{\sqrt{6!}}(y^6 - 120y^3 + 360) \equiv \mathrm{e}^{-2\tau}\psi_6^*(y),\ y = x/(-t)^{\frac{1}{3}},\ \tau = -\ln(-t); \\ t^2\tfrac{1}{\sqrt{6!}}(y^6 + 120y^3 + 360) \equiv \mathrm{e}^{2\tau}\Phi_6(y),\ y = x/t^{\frac{1}{3}},\ \tau = \ln t. \end{cases}$$

The approach to multiple zeros can be extended to *semi-linear* odd-order PDEs such as higher-order KdV-type equations

$$u_t = D_x^{2m+1}u + D_x^k(uD_x^n u), \quad \text{with any} \quad k+n < 2m+1.$$

Unlike the parabolic case (9.1), essentially quasi-linear PDEs are not allowed, since the corresponding non-linear semigroups are not continuous and solutions develop discontinuous shocks. For instance, such proper “entropy” shocks appear for the third- or fifth-order PDEs (see [15])

$$u_t = (uu_x)_{xx} \quad \text{and} \quad u_t + (uu_x)_{xxxx} = 0,$$

which exhibit stationary shocks and rarefaction waves that are similar to Euler's equation

$$u_t + uu_x = 0.$$

These singularity formation phenomena cannot be covered by linear spectral theory associated with operators of fundamental kernels such as (9.6).

More complicated spectral theory of multi-dimensional adjoint operators $\mathbf{B}^*$ and polynomial zero structures occur for N-dimensional counterparts of linear dispersion PDEs such as

$$u_t = (\Delta u)_{x_1} + \cdots \quad \text{or} \quad u_t = -(\Delta^2 u)_{x_1} + \cdots \quad \text{in} \quad \mathbf{R}^N \times [-1,1]. \tag{9.12}$$

9.3. **Hyperbolic wave equations.** As a typical example, consider the equation (1.25). Assuming that a multiple zero of $u(x,t)$ occurs at the point $(0,0)$, we perform the blow-up scaling near this point, now using the following rescaled variables (see details in [18]):

$$u(x,t) = w(y,\tau), \quad \text{with} \quad y = x/(-t) \quad \text{and} \quad \tau = -\ln(-t),$$

to get the rescaled PDE

$$w_{\tau\tau} + w_\tau + 2w_{y\tau}y = \mathbf{A}^* w \equiv (1-y^2)w_{yy} - 2w_y y. \tag{9.13}$$

Looking, as usual, for solutions in separate variables

$$w(y,\tau) = \mathrm{e}^{\lambda\tau}\psi^*(y), \tag{9.14}$$

yields the eigenvalue problem for a *quadratic pencil* of non-self-adjoint operators,

$$\mathbf{B}^*_\lambda \psi^* \equiv \{(\lambda^2+\lambda)I + 2\lambda y D_y - \mathbf{A}^*\}\psi^* = 0. \tag{9.15}$$

The higher-order operator $\mathbf{A}^*$ is singular (degenerate) at the *light cone* $\{|y| = 1\}$ (just two points in 1D), so the existence of eigenfunctions with smooth transitions through the cone is of principal importance. The behaviour as $y \to \infty$ (the second singular point) is not that crucial, and any L^2_ρ-space setting with $\rho(y) \sim \mathrm{e}^{-ay^2}$ (or $\mathrm{e}^{-a|y|}$), $a > 0$ small, would be enough. As usual, a key part is played by the following:

Proposition 9.1. *Eigenfunctions of the adjoint pencil* (9.15) *must be finite polynomials.*

Proof. As in Section 4.2, we have from (9.15) that the FT $\hat{\psi}^* = \mathcal{F}(\psi^*)$ satisfies a hyper-geometric ODE of the form

$$\xi^2(\hat{\psi}^*)'' - 2(\lambda+1)\xi(\hat{\psi}^*)' - (\xi^2 + \lambda^2 - \lambda)\hat{\psi}^* = 0 \quad \text{in} \quad \mathbf{R}. \tag{9.16}$$

Looking for a regular solution, one can see that the behaviour as $\xi \to \infty$ is governed by two terms containing the ξ^2-multiplier, i.e.,

$$\xi^2(\hat{\psi}^*)'' - \xi^2\hat{\psi}^* + \cdots = 0,$$

so that any non-trivial solutions has exponential growth $\sim \mathrm{e}^{\pm\xi}$ either as $\xi \to +\infty$ or as $\xi \to -\infty$. Therefore, the only admitted solution $\hat{\psi}^*$ of (9.16) must have support concentrated at the singular point $\xi = 0$, and hence ψ^* is a polynomial. □

In order to find the corresponding point spectrum of the pencil, looking for lth-order polynomial eigenfunctions

$$\psi_l^*(y) = y^l + \cdots$$

and substituting into (9.15) yields the following quadratic equation for eigenvalues:

$$\lambda_l^2 + (2l+1)\lambda_l + l^2 + l = 0.$$

Hence, there exist two sequences of eigenvalues

$$\lambda_l^+ = -l \quad \text{and} \quad \lambda_l^- = -l-1 \quad \text{for all} \quad l \ge 0. \tag{9.17}$$

Actually, the eigenfunctions $\psi_l^*(y)$ are lth-order polynomials given by finite Kummer's series corresponding to the operator in (9.15). Computations are given in [18, Section 4]. It is curious that

$$\mathbf{A}^* = \mathbf{A} = D_y((1-y^2)D_y) \ \text{ is symmetric in the topology of } L^2,$$

so that the formally adjoint (in L^2) pencil is

$$\mathbf{B}_\lambda = (\lambda^2 - \lambda)I - 2\lambda y D_y - \mathbf{A}^* \equiv \mathbf{B}_\lambda^* - 2\lambda I - 4\lambda y D_y. \tag{9.18}$$

For higher-order hyperbolic PDEs such as (1.26), the spectral problem becomes much more difficult, especially in $\mathbf{R}^N$. The operators in (9.15) are invariant under the symmetry $y \mapsto -y$, so we use a standard metric of L^2. Notice that basic methods of spectral theory of quadratic pencils can be associated with J-spaces having indefinite metrics (cf. the previous subsection); see fundamentals of pencil theory in Markus [29].

In view of completeness and closure of the eigenfunction set $\Phi = \{\psi_l^*\}$ in weighted L^2 spaces (see [25, p. 431]), we can use suitable eigenfunction expansions

$$w(y,\tau) = \sum_{(l)} a_l^\pm \mathrm{e}^{\lambda_l^\pm \tau}\psi_l^*(y) \quad (\lambda_l^\pm < 0),$$

in order to determine the structure of multiple zeros of solutions of (9.13). Here, for the second-order-in-time PDE (9.13), both sequences $\{\lambda_l^\pm\}$ of eigenvalues of (9.17) are used with two sequences of constants $\{a_l^\pm\}$ depending on initial data $w(0,y) = u_0(y)$ and $w_\tau(0,y) = u_1(y)$ prescribed at $t = -1$ ($\tau = 0$). Hence, the eigenfunctions of the pencil (9.15) give a countable set of different multiple zeros at $(0,0)$ for the wave equation (1.25) and hence determine its micro-structure,

$$u_l^{\pm}(x,t) = (-t)^{-\lambda_l^{\pm}} \tilde{\psi}_l^*(x/(-t)) \quad \big(\lambda_l^{\pm} < 0\big). \tag{9.19}$$

In particular, each multiple zero of lth order is formed as $t \to 0^-$ by zero curves focusing at $x = 0$ with the behavior $x_j(t) = y_j(-t)$, where $\tilde{\psi}_l^*(y_j) = 0$.

The polynomial eigenfunctions $\psi_l(y) \sim y^l + \cdots$, of the adjoint pencil (9.18) (cf. the analogy of (6.44), (6.45)) have the following eigenvalues:

$$\mathbf{B}_\lambda \psi_l = 0 \implies \lambda_l^2 - (2l+1)\lambda_l + l^2 + l = 0 \implies \lambda_l^+ = l+1 \text{ and } \lambda_l^- = l.$$

As usual, these polynomials ψ_l are used for the extensions of blow-up patterns (9.19) for $t > 0$. Here, the forward continuation rescaled variables are (cf. (9.13))

$$y = x/t, \;\; \tau = \ln t \quad \implies \quad w_{\tau\tau} - w_\tau - 2w_{y\tau}y = \mathbf{A}w. \tag{9.20}$$

Substituting $w(y,\tau) = \mathrm{e}^{\lambda\tau}\psi(y)$ yields the eigenvalue problem for the adjoint pencil (9.18).

There are still several open problems for perturbed wave equations in $\mathbf{R}^N$ and among related questions of unique continuation. For instance, in addition to spectral theory of operator pencils along the lines of Sections 3–5 (see [18]), another version of using the uniform eigenfunction expansions as in Section 6 is necessary.

Other applications of similar quadratic pencils occur [18, Section 4] in the study of blow-up in semi-linear wave equation

$$u_{tt} = \Delta u + |u|^{p-1}u \quad (p > 1), \tag{9.21}$$

for which polynomial eigenfunctions and spectra describe a countable subset of different blow-up patterns. These linear, together with *non-linear, eigenfunctions* [18, Section 2,3] that are unstable, are expected to be *evolutionarily complete*, i.e., describe all possible types of blow-up for (9.21).

Example 9.2. Take a solution of (1.25) describing a completely symmetric multiple zero at $(0,0)$ (here we do not care about normalizing multipliers of eigenfunctions)

$$u(x,t) = \tfrac{3}{2}\, tx^4 + 3t^3x^2 + \tfrac{3}{10}\, t^5.$$

Note that $u(x,0) \equiv 0$ but $u_t(x,0) = \frac{3}{2}\, x^4$ has a fourth-order zero at $x = 0$. The formation-collapse expansions of this multiple zero corresponds to $\lambda_4^- =$

$-l-1=-5$ in (9.17),

$$u(x,t)=\begin{cases}(-t)^5\big(\frac{3}{2}\,y^4+3y^2+\frac{3}{10}\big)\equiv \mathrm{e}^{-5\tau}\tilde\psi_4^*(y),\ y=x/(-t),\ \tau=-\ln(-t),\ t<0;\\ t^5\big(\frac{3}{2}\,y^4+3y^2+\frac{3}{10}\big)\equiv \mathrm{e}^{-5\tau}\tilde\Phi_4(y),\ y=x/t,\ \tau=\ln t,\ t>0.\end{cases}$$

For the solution

$$u(x,t)=x^3+tx^2+3t^2x+\tfrac{1}{3}\,t^3,$$

the multiple zero corresponds to $\lambda_3^+=-l=-3$ and is not symmetric,

$$u(x,t)=\begin{cases}(-t)^3\big(y^3-y^2+3y-\frac{1}{3}\big)\equiv \mathrm{e}^{-3\tau}\tilde\psi_3^*(y),\ y=x/(-t),\ \tau=-\ln(-t),\ t<0;\\ t^3\big(y^3+y^2+3y+\frac{1}{3}\big)\equiv \mathrm{e}^{3\tau}\tilde\Phi_3(y),\ y=x/t,\ \tau=\ln t,\ t>0.\end{cases}$$

For the fourth-order hyperbolic equation

$$u_{tt}=-u_{xxxx}\quad \text{in}\quad \mathbf{R}\times[-1,1],$$

consider the zero-formation mechanism described by the explicit solution

$$u(x,t)=x^4-12t^2.$$

Then a symmetric formation-collapse corresponds to $l=4$, with $\lambda_4=-4$,

$$u(x,t)=\begin{cases}(-t)^2(y^4-12)\equiv \mathrm{e}^{-4\tau}\tilde\psi_4^*(y),\ \ y=x/\sqrt{-t},\ \tau=-\ln(-t),\ t<0;\\ t^2(y^4-12)\equiv \mathrm{e}^{4\tau}\tilde\Phi_4(y),\ \ y=x/\sqrt{t},\ \tau=\ln t,\ t>0.\end{cases}$$

Here $\tilde\psi_4^*$ is an eigenfunction of a pencil, while $\tilde\Phi_4$ is a polynomial solution of the eigenfunction equation for $\mathbf{B}_\lambda$. An interesting situation is described by the solution

$$u(x,t)=-x^4+x^2+12t^2,$$

where a double zero occurs at $(0,0)$, but zero curves do not appear, neither at $t=0^-$, nor at $t=0^+$. Sturmian rescaling with $x=y\sqrt{-t}$ for $t<0$ gives the behaviour for $l=2$, $\lambda_2=-1$,

$$u(x,t)=(-t)y^2-(-t)^2(y^4-12)\equiv \mathrm{e}^{-\tau}\tilde\psi_2^*(y)-\mathrm{e}^{-2\tau}\tilde\psi_4^*(y).$$

9.4. **Unification class: integral evolution equations.** In fact, the origin of the above *polynomial* micro-structure of various PDEs under consideration is indeed connected with using Taylor's expansions of rescaled kernels (3.11), (3.12) for the semigroup (2.12) with the adjoint generator $\mathbf{B}^*$. Therefore, the polynomial structure of multiple zero formation via eigenfunctions of the adjoint operator has a more substantial range of application and occurs not only in various PDEs, but also in a wider unification class of *evolution integral equations* (this includes all suitable PDEs) with similar kernels.

Thus, consider (2.12), which used to be the semigroup representation of $\{e^{\mathbf{B}^*\tau}\}$,

$$w(y,\tau) = \mathcal{E}^*(\tau)u_0 \equiv (1-e^{-\tau})^{-\frac{N}{2m}} \int F\big((ye^{-\frac{1}{2m}\tau} - z)(1-e^{-\tau})^{-\frac{1}{2m}}\big)u_0(z)\,dz. \tag{9.22}$$

As we know, for any integer $m = 1, 2, \dots$ and the rescaled kernel $F(\cdot)$ taken from the fundamental solution (2.2), (9.22) is just the unique solution of the parabolic PDE (2.1).

Now, assume that $m > 0$ is an arbitrary number. Then, still, we can get by (9.22) solutions of other PDEs. For instance, taking, for $N = 1$, $m = \frac{3}{2}$, i.e., $\frac{1}{2m} = \frac{1}{3}$, and F being the Airy function as in (9.6), we obtain the solutions of the rescaled linear dispersion equation (1.24) with the adjoint operator (9.10),

$$w_\tau = \mathbf{B}^* w \equiv w_{yyy} - \tfrac{1}{3}\, w_y y.$$

As the next step, taking again arbitrary $m > 0$, we fix also an arbitrary analytic rescaled kernel $F(y)$ in (9.22), which admits a suitably converging Taylor's series in $\mathbf{R}^N$ and can have

(i) exponential decay as $y \to \infty$ as in the parabolic case (2.2), (2.3),

(ii) exponentially decaying or oscillatory tail as for the linear dispersion PDEs (9.7), or

(iii) both oscillatory tails for $N = 1$ or as $y_1 \to \pm\infty$ bearing in mind the multi-dimensional dispersion PDEs as in (9.12) (it seems such a behaviour of F is not possible for PDE operators), etc.

Thus, we then arrive at an integral evolution equation (9.22) that does not have any PDE counterpart, and, moreover, $\{e^{\mathbf{B}^*\tau}\}$ (for what $\mathbf{B}^*$ is, see below) cannot be interpreted as a semigroup, since in general such integral evolution is not invariant under translations. See, e.g., the *majorizing integral operator* for parabolic flows with the positive kernel $F > 0$ that, roughly speaking, is given on the right-hand side of (2.4) with a normalization condition, [17, Section 3].

Applying in (9.22) Taylor's expansion of the kernel $F(\cdot)$ as in (4.3), (4.4), we again obtain the point spectrum $\{\lambda_\beta\}$, (3.5), and eigenfunctions $\{\psi_\beta\}$ of the same form (3.6), which can be interpreted as spectral characteristics of a certain linear operator $\mathbf{B} = \mathbf{B}(\tau)$. In addition, we observe polynomial eigenfunctions $\{\psi_\beta^*\}$ of the adjoint operator $\mathbf{B}^*(\tau)$, which needs an extra interpretation and motivation, etc. See some details in [17, Section 2.3]. It is worth mentioning that, actually, assuming that $\mathbf{B}^*(\tau)$ is properly defined by its point spectrum and eigenfunction expansions, the integral evolution

(9.22) has its "pseudo-differential counterpart" (2.12), i.e.,

$$w_\tau = \mathbf{B}^*(\tau) w \quad \text{for} \quad \tau > 0, \quad w(0) = u_0 \ \big(\text{in } L^2_{\rho^*}\big). \tag{9.23}$$

Here $\mathbf{B}^*(\tau)$ is time-dependent in general and does not generate a semigroup.

Then, it is more or less clear that this multiple-zero structure remains unchanged for non-linear extensions of (9.23) by adding any sufficiently smooth non-linearity $f(u)$, e.g.,

$$w_\tau = \mathbf{B}(\tau) w \pm |w|^{p-1} w \quad \text{for} \quad \tau > 0, \quad w(0) = u_0 \quad (p > 1). \tag{9.24}$$

Using in (9.24) the eigenfunction expansion as in (4.6), we see that the non-linear term cannot affect the structure of multiple zeros.

The problem (9.24) admits its purely integral counterpart

$$w(\tau) = \mathcal{E}^*(\tau) u_0 \pm \int_0^t \mathcal{E}^*(\tau, s) * (|w|^{p-1} w)(s)\, \mathrm{d}s, \tag{9.25}$$

where $\mathcal{E}^*(\tau, s)$ is the "fundamental" operator of (9.23) satisfying $\mathcal{E}^*(s, s) = I$; see [13]. Local existence and uniqueness of solutions of (9.25) follows from typical contraction properties of such integral Hammerstein–Volterra operators in various metrics.

Hence, under the presence of well-posed spectral theory of such operators $\mathbf{B}^*(\tau)$ and $\mathbf{B}(\tau)$, completeness, closure, etc., of the eigenfunctions subsets, we can study and classify multiple zeros of solutions of linear or non-linear (integral) equations such as (9.22) (or equivalently (9.23) and (9.25) (or (9.24)), and develop further applications. Therefore, these linear and nonlinear equations, with differential or pseudo-differential pairs of operators $\{\mathbf{B}(\tau), \mathbf{B}^*(\tau)\}$ and differential or integral time-evolution, exhibit similar principles of formation of multiple zeros, and hence, under necessary hypotheses, such equations are expected to admit common typical concepts of unique continuation, nodal set properties (e.g., its Hausdorff dimension), etc.

A similar interpretation and extensions can be created for second- and higher-order evolution equations containing time-derivatives $w_\tau, w_{\tau\tau}, \ldots$ (see (9.13) and (9.20) as basic models for integral operator extensions and related spectral problems for operator pencils) in pseudo-differential equations such as (9.24).

Acknowledgment. The author would like to thank Yu.V. Egorov and S.I. Pohozaev for useful discussions.

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