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HELICOIDAL TRAJECTORIES OF A CHARGE IN A NONCONSTANT MAGNETIC FIELD

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1. Introduction

In this note we investigate the existence of helicoidal trajectories for a charged particle in a magnetic field. More precisely, denoting by $p = p(t)$ the position in $\mathbb{R}^3$ at the time t of the particle, we say that it moves along a helicoidal trajectory if there exists a versor n in $\mathbb{R}^3$ such that the component of $p(t)$ in the direction of n describes a uniform right motion, whereas the projection $p_\perp(t)$ of $p(t)$ on a plane orthogonal to n is periodic. In particular, if the closed curve supported by $p_\perp$ is simple; i.e., it has no self-intersections, the helicoidal trajectory $p(t)$ will be called simple.

From classical physics, in the presence of an external magnetic field B, the motion of a particle of mass m and charge e is driven by the Lorentz force, namely $p(t)$ is a solution of

$$m\ddot{p} = e\dot{p} \wedge B. \tag{1.1}$$

When B is a uniform, constant field, namely $B = b_0 n$ for some versor n and nonzero constant b_0, one can explicitly solve (1.1) and deduce that the particle admits helicoidal trajectories $p(t)$ which are coaxial with the magnetic field B. In particular the projection $p_\perp(t)$ of $p(t)$ on a plane orthogonal to B moves on a circle of radius r with constant angular speed ν. The values of ν and r are given respectively by

$$\nu = \frac{|eb_0|}{m}, \quad r = \frac{|\dot{p}_\perp(0)|}{\nu}. \tag{1.2}$$

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The situation can drastically change if one switches a perturbation on in the magnetic field, even considering the simplest situation

$$B(p) = B_\varepsilon(p) = (b_0 + \varepsilon b(p))n \tag{1.3}$$

where b is some scalar function defined on $\mathbb{R}^3$ and ε is a smallness parameter. Notice that the perturbation does not affect the direction of the magnetic field but only its modulus.

In this case again the component of $p(t)$ in the direction of n follows a uniform motion, but the projection $p_\perp(t)$ does not necessarily describe a closed (or also bounded) orbit, even for small nonzero $|\varepsilon|$. For instance the following nonexistence result holds:

Proposition 1.1. *Assume $B(p_1,p_2,p_3) = (b_0+\varepsilon\beta(p_1))e_3$ where $e_3 = (0,0,1)$ and $\beta\colon \mathbb{R} \to \mathbb{R}$ is of class C^1 and strictly monotone. Then for every $\varepsilon \neq 0$ there is no simple helicoidal trajectory.*

Our goal is to provide conditions on the perturbative term b ensuring the existence of helicoidal trajectories, at least for small $|\varepsilon|$. We will assume that b is constant in the direction of n, namely

$$\frac{\partial b}{\partial n}(p) = 0 \quad \text{for all } p \in \mathbb{R}^3. \tag{1.4}$$

Moreover, changing sign in n, b_0 and b in (1.3) if necessary, we can assume that $eb_0 < 0$.

Hereinafter we will denote by $p_\perp$ the projection of a vector $p \in \mathbb{R}^3$ on the plane $p \cdot n = 0$. Hence $p_\perp$ identifies with a vector in $\mathbb{R}^2 \approx \mathbb{C}$ and, according to (1.4), b depends just on the two components of $p_\perp$, so that we can write $b(p) = b(p_\perp)$, considering b as a mapping on $\mathbb{R}^2$. We will always assume b is at least of class C^1.

Noting that, by (1.3), $|\dot{p}_\perp(t)| = |\dot{p}_\perp(0)|$ for all $t \in \mathbb{R}$, the problem of helicoidal trajectories consists in studying the existence of solutions of

$$(P)_\varepsilon \qquad \begin{cases} m\ddot{p} = e(b_0 + \varepsilon b(p_\perp))\dot{p} \wedge n \\ |\dot{p}_\perp| = v, \quad p_\perp \text{ periodic} \end{cases}$$

where $v > 0$ is given.

In order to state our main results let us introduce the function $M\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$M(z) = \int_{D_r(z)} b(\zeta)\, d\zeta \quad \text{for every } z \in \mathbb{R}^2. \tag{1.5}$$

Here $D_r(z)$ denotes the two-dimensional disc centered at z and with radius r given by (1.2). The mapping M can be interpreted as the Poincaré-Melnikov function associated to the problem. In fact the existence of (branches of) helicoidal trajectories for small $|\varepsilon|$ turns out to be strongly related to the existence of critical points for M and to their stability properties.

Firstly let us state a necessary condition for the existence of a sequence of helicoidal trajectories converging in a suitable sense for $\varepsilon \to 0$.

Theorem 1.2. *Let $(\varepsilon_n) \subset \mathbb{R} \setminus \{0\}$ with $\varepsilon_n \to 0$, and for every $n \in \mathbb{N}$ let $p(t, \varepsilon_n)$ be a solution of problem $(P)_{\varepsilon_n}$ for some fixed $v > 0$. Let μ_n be the minimal period of $p_\perp(t, \varepsilon_n)$ and set*

$$z_n = \frac{1}{\mu_n} \int_0^{\mu_n} p_\perp(t, \varepsilon_n)\, dt.$$

If the sequence (μ_n) is bounded in $\mathbb{R}$ and far from 0, and $z_n \to \bar{z}$ then $\bar{z}$ is a critical point of M.

On the contrary, the presence of "stable" critical points of M constitutes a sufficient condition for the existence of helicoids. More precisely, we have the following:

Theorem 1.3. *If $\bar{z} \in \mathbb{R}^2$ is a nondegenerate critical point of M, then for every $v > 0$ and for small $|\varepsilon|$ (depending on v), problem $(P)_\varepsilon$ admits a solution $p(t, \varepsilon)$ drawing a simple helicoidal trajectory. Moreover, the mapping $\varepsilon \mapsto p_\perp(\cdot, \varepsilon)$ is of class C^1 (in the space of C^2 periodic functions) and, for $\varepsilon = 0$, $p_\perp(t, 0) = \bar{z} + re^{i\nu t}$, where ν and r are given by (1.2).*

In the presence of extremal points for M we can consider a weaker stability condition, as follows.

Theorem 1.4. *If there exists a nonempty, open, bounded set $A \subset \mathbb{R}^2$ such that $\max_{\partial A} M < \sup_A M$ (or $\min_{\partial A} M > \inf_A M$), then for every $v > 0$ and for small $|\varepsilon|$ (depending on v), problem $(P)_\varepsilon$ admits a solution $p(t, \varepsilon)$ drawing a simple helicoid. Moreover, denoting by $z_\varepsilon \in \mathbb{R}^2$ the average of $p_\perp(t, \varepsilon)$ over its minimal period, one has that $z_\varepsilon \in A$, $M(z_\varepsilon) \to \sup_A M$ (or $M(z_\varepsilon) \to \inf_A M$, respectively) and $p_\perp(t, \varepsilon) - z_\varepsilon \to re^{i\nu t}$ in the C^2 topology, as $\varepsilon \to 0$, where ν and r are given by* (1.2).

We can state further existence results by making explicit assumptions on b (rather than on M) when the ratio $|b_0|/v$ is sufficiently large. In particular we have that:

Theorem 1.5. *Assume that one of the following conditions is satisfied:*

(*i*) *b is of class C^2 and there exists a nondegenerate critical point $\bar{z} \in \mathbb{R}^2$ of b;*

(*ii*) *there exists a nonempty, open, bounded set* $A \subset \mathbb{R}^2$ *such that*

$$\max_{\partial A} b < \sup_A b \quad (\text{or } \min_{\partial A} b > \inf_A b).$$

Then for $\rho := |b_0|/v$ *large enough and for small* $|\varepsilon|$ *(depending on* ρ*), problem* $(P)_\varepsilon$ *admits a solution* $p(t, \varepsilon, \rho)$ *drawing a simple helicoidal trajectory. Moreover,*

$$\lim_{\rho\to+\infty} \lim_{\varepsilon\to 0} \|p_\perp(\cdot, \varepsilon, \rho) - z_{\varepsilon,\rho}\|_{C^2} \to 0,$$

where $z_{\varepsilon,\rho} \equiv \bar{z}$ *if* (*i*) *holds, or, in case* (*ii*)*,* $z_{\varepsilon,\rho}$ *is the average of* $p_\perp(\cdot, \varepsilon, \rho)$ *and satisfies:* $z_{\varepsilon,\rho} \in A$ *and*

$$\lim_{\rho\to+\infty} \lim_{\varepsilon\to 0} b(z_{\varepsilon,\rho}) = \sup_A b \quad \textit{(or} \quad \lim_{\rho\to+\infty} \lim_{\varepsilon\to 0} b(z_{\varepsilon,\rho}) = \inf_A b, \quad \textit{respectively).}$$

In addition, in case (*i*)*, for every large* ρ*, the map* $\varepsilon \mapsto p_\perp(\cdot, \varepsilon, \rho)$ *is of class* C^1 *in the space of* C^2 *periodic functions.*

As a consequence of Theorem 1.4 we can also prove the existence of helicoidal trajectories under some decay assumption on b, as follows.

Theorem 1.6. *If* $b \in L^1(\mathbb{R}^2) + L^2(\mathbb{R}^2)$ *then for every* $v > 0$ *and for small* $|\varepsilon|$ *(depending on* v*), problem* $(P)_\varepsilon$ *admits a solution* $p(t, \varepsilon)$ *corresponding to a simple helicoidal trajectory. Moreover, denoting by* $z_\varepsilon \in \mathbb{R}^2$ *the average of* $p_\perp(t, \varepsilon)$ *over its minimal period, one has that* (z_ε) *is bounded with respect to* ε*, and* $p_\perp(t, \varepsilon) - z_\varepsilon \to re^{i\nu t}$ *in the* C^2 *topology, as* $\varepsilon \to 0$*, where* ν *and* r *are given by (1.2).*

The main tool in the proof of Theorems 1.3 and 1.4 is the Lyapunov-Schmidt reduction method. In fact one can take advantage of the variational character of the problem (see Section 2) and follow a procedure introduced by Ambrosetti and Badiale [1]. This is developed in Section 3, devoted to the study of the "unperturbed" problem $(P)_0$, and in Section 4 where we make the finite-dimensional reduction of the "perturbed" problem $(P)_\varepsilon$. In Section 5 we give the proofs of the above results.

Finally in Section 6 we point out some geometrical problems, concerning closed curves in the plane with prescribed curvature, and (right) cylinders in $\mathbb{R}^3$ with prescribed mean curvature. These problems, in some cases, share the same analytical formulation of the problem of helicoids; hence in this geometrical frame we can state analogous versions of the previously stated theorems.

We conclude by observing that the method of the finite-dimensional reduction we follow has been widely and successfully used for a large class of variational-perturbative problems. We quote especially the recent monograph [2] and the references therein. We also mention the paper [3] which exhibits significant similarities with the problem discussed here.

2. Preliminaries

Let $B\colon \mathbb{R}^3 \to \mathbb{R}^3$ be as in (1.3), with $eb_0 < 0$, n a versor in $\mathbb{R}^3$ and b of class C^1 satisfying (1.4). Let us introduce a coordinate system such that $n = (0,0,1)$, let us denote by p_j the j-th component of p ($j = 1,2,3$) and let us set $p_\perp := (p_1, p_2)$. According to (1.4), the function b depends just on $p_\perp$ and the problem $(P)_\varepsilon$ can be equivalently written as follows:

$$\begin{cases} m\ddot{p}_\perp = -ie(b_0 + \varepsilon b(p_\perp))\dot{p}_\perp \\ |\dot{p}_\perp| = v, \quad p_\perp \text{ periodic,} \end{cases} \tag{2.1}$$

where $v > 0$ is given and, making the usual identification between $\mathbb{R}^2$ and $\mathbb{C}$, the product by the imaginary unit i acts as a counterclockwise rotation of $\pi/2$. Now set

$$r := -\frac{mv}{eb_0}, \quad \kappa(z) := \frac{1}{b_0} b(rz) \quad \text{for } z \in \mathbb{R}^2 \tag{2.2}$$

and observe that a function $p_\perp(t)$ solves (2.1) if and only if $\zeta(t) = r^{-1} p_\perp(\frac{rt}{v})$ solves

$$\begin{cases} \ddot{\zeta} = i(1 + \varepsilon\kappa(\zeta))\dot{\zeta} \\ |\dot{\zeta}| = 1, \quad \zeta \text{ periodic.} \end{cases} \tag{2.3}$$

Since the period of $\zeta(t)$ (or of $p_\perp(t)$) is *a priori* unknown, one makes a rescaling in order to include the unknown period in the equation. In particular one can consider the following problem:

$$\begin{cases} \ddot{u} = i\|\dot{u}\|_2(1 + \varepsilon\kappa(u))\dot{u} \qquad \text{in } [0,1] \\ u(0) - u(1) = 0 = \dot{u}(0) - \dot{u}(1) \\ u \text{ nonconstant,} \end{cases} \tag{2.4}$$

where

$$\|\dot{u}\|_2 := \Big(\int_0^1 |\dot{u}|^2\Big)^{1/2}.$$

The relationship between problems $(P)_\varepsilon$, (2.3) and (2.4) is expressed by the next lemma.

Lemma 2.1. *If $u \in C^2([0,1],\mathbb{R}^2)$ solves (2.4), then setting $\zeta(t) = u(t/\|\dot{u}\|_2)$ for $t \in [0, \|\dot{u}\|_2]$ and extending ζ by periodicity on $\mathbb{R}$, the mapping ζ belongs to $C^2(\mathbb{R},\mathbb{R}^2)$ and solves (2.3). Hence problem $(P)_\varepsilon$ admits a helicoidal solution $p(t)$ with $p_\perp(t) = r\zeta(vt/r)$.*

Conversely, if problem $(P)_\varepsilon$ admits a helicoidal solution $p(t)$ and $\mu > 0$ is the (minimal) period of $p_\perp(t)$, then the mapping $u(t) = r^{-1}p_\perp(\mu t)$ for $t \in [0,1]$ solves (2.4).

The proof of Lemma 2.1 is trivial.

According to Lemma 2.1 we are led to search for solutions of problem (2.4). This problem is variational in nature and its solutions can be found as critical points of a suitable energy functional associated to (2.4).

More precisely, let us introduce the following functional setting: let

$$H := \{u \in H^1([0,1],\mathbb{R}^2) : u(0) = u(1)\}$$

be the standard Sobolev space of 1-periodic mappings, endowed with the inner product

$$\langle u, v\rangle := [u]\cdot[v] + (\dot{u},\dot{v})_2,$$

where $[u] := \int_0^1 u$ is the average of u, and $(\cdot,\cdot)_2$ is the standard inner product in $L^2([0,1],\mathbb{R}^2)$.

It is well known that H is a Hilbert space with the above inner product and H is compactly embedded into the space of continuous, 1-periodic functions taking values in $\mathbb{R}^2$. We set

$$\|u\| := \sqrt{\langle u, u\rangle} \quad \text{for every } u \in H$$

and we point out that $\|\cdot\|$ is a norm in H equivalent to the standard norm of $H^1([0,1],\mathbb{R}^2)$, because of the Poincaré-Wirtinger inequality.

Given $\kappa \in C^0(\mathbb{R}^2)$, let $Q_\kappa\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field such that

$$\operatorname{div} Q_\kappa(z) = \kappa(z) \quad \text{for all } z \in \mathbb{R}^2. \tag{2.5}$$

A possible choice is

$$Q_\kappa(x,y) = \frac{1}{2}\Big(\int_0^x \kappa(s,y)\,ds, \int_0^y \kappa(x,s)\,ds\Big) \quad \text{for } z = (x,y) \in \mathbb{R}^2.$$

Then, for every $u \in H$ set

$$S_\kappa(u) := \int_0^1 iQ_\kappa(u)\cdot\dot{u}.$$

We point out that S_κ is well defined in H because if $u \in H$ then $Q_\kappa \circ u \in L^\infty$ and $\dot{u} \in L^2$.

Notice that in the case $\kappa(z) \equiv 1$ one can choose $Q_1(z) = \frac{1}{2}z$ and one obtains

$$S_1(u) = \frac{1}{2}\int_0^1 iu \cdot \dot{u}.$$

When $u \in H$ is regular and one-to-one, $S_1(u)$ measures (up to a sign) the area enclosed by the bounded component of $\mathbb{R}^2 \setminus u([0,1])$. More generally, the functional S_κ can be interpreted as the κ-weighted algebraic area of the inner domain bounded by $u([0,1])$. In particular, if

$$\omega(t) = e^{2\pi i t}, \tag{2.6}$$

using (2.5) and the divergence theorem, for every $z \in \mathbb{R}^2$ and $\rho > 0$ one has that

$$S_\kappa(\rho\omega + z) = \int_{D_\rho(z)} \kappa(q)\, dq \tag{2.7}$$

where $D_\rho(z)$ denotes the two-dimensional disc centered at z and with radius ρ. Finally, let $E_\kappa \colon H \to \mathbb{R}$ be the functional defined by

$$E_\kappa(u) := \|\dot{u}\|_2 - S_\kappa(u) \tag{2.8}$$

and let $\Omega = \{u \in H : u \text{ is nonconstant}\}$.

Lemma 2.2. *If $\kappa \in C^1(\mathbb{R}^2)$, then $E_\kappa \in C^2(\Omega, \mathbb{R})$ and for every $u \in \Omega$ the first and the second derivative of E_κ at u are given respectively by:*

$$E'_\kappa(u)h = \frac{1}{\|\dot{u}\|_2}\int_0^1 \dot{u}\cdot\dot{h} + \int_0^1 h \cdot i\kappa(u)\dot{u} \quad \text{for all } h \in H \tag{2.9}$$

$$\begin{aligned} E''_\kappa(u)[h,k] &= \frac{1}{\|\dot{u}\|_2}\int_0^1 \dot{h}\cdot\dot{k} - \frac{1}{\|\dot{u}\|_2^3}\Big(\int_0^1 \dot{u}\cdot\dot{h}\Big)\Big(\int_0^1 \dot{u}\cdot\dot{k}\Big) \\ &\quad + \int_0^1 k\cdot(i\kappa(u)\dot{h} + (\nabla\kappa(u)\cdot h)i\dot{u}) \quad \text{for all } h,k \in H. \end{aligned} \tag{2.10}$$

Moreover, a mapping $u\colon [0,1] \to \mathbb{R}^2$ is a classical solution of (2.4) if and only if u is a critical point for $E_{1+\epsilon\kappa}$ in Ω.

Remark 2.3. In case $\kappa \equiv 1$, after integration by parts one finds

$$E'_1(u)h = \frac{1}{\|\dot{u}\|_2}\int_0^1 \dot{u}\cdot\dot{h} - \int_0^1 \dot{h}\cdot iu \quad \text{for all } u \in \Omega \text{ and } h \in H. \tag{2.11}$$

Proof. It is well known that the functional $u \mapsto \|\dot u\|_2$ is of class C^∞ in Ω. Let us study the regularity of the functional S_κ. Fixing $u, h \in H$ and taking $\varepsilon \neq 0$ one has that

$$\frac{S_\kappa(u+\varepsilon h) - S_\kappa(u)}{\varepsilon} = -\int_0^1 \frac{Q_\kappa(u+\varepsilon h) - Q_\kappa(u)}{\varepsilon} \cdot i\dot u - \int_0^1 Q_\kappa(u+\varepsilon h)\cdot i\dot h.$$

Since $Q_\kappa \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and, in particular, Q_κ and ∇Q_κ are locally uniformly continuous, by standard arguments, using also the embedding of H into $C^0([0,1], \mathbb{R}^2)$, one can prove that

$$\lim_{\varepsilon\to 0} \frac{Q_\kappa(u+\varepsilon h) - Q_\kappa(u)}{\varepsilon} = \nabla Q_\kappa(u)h \quad \text{and}$$
$$\lim_{\varepsilon\to 0} Q_\kappa(u+\varepsilon h) = Q_\kappa(u) \quad \text{uniformly in } [0,1].$$

Therefore, by the Lebesgue dominated convergence theorem, we infer that

$$\lim_{\varepsilon\to 0} \frac{S_\kappa(u+\varepsilon h) - S_\kappa(u)}{\varepsilon} = -\int_0^1 \nabla Q_\kappa(u)[h, i\dot u] - \int_0^1 Q_\kappa(u)\cdot i\dot h$$
$$= -\int_0^1 \nabla Q_\kappa(u)[h, i\dot u] + \int_0^1 \nabla Q_\kappa(u)[\dot u, ih] \tag{2.12}$$
$$= -\int_0^1 \kappa(u) h\cdot i\dot u, \tag{2.13}$$

where (2.12) is obtained by integration by parts and (2.13) follows from (2.5) and from the algebraic formula

$$Av\cdot iw - Aw\cdot iv = (\mathrm{tr}\ A)v\cdot iw \quad \text{for every } v, w\in\mathbb{R}^2,$$

where A is any 2×2 matrix and tr A denotes its trace. Fixing $u \in H$, the mapping

$$h \mapsto -\int_0^1 \kappa(u)h\cdot i\dot u \tag{2.14}$$

is linear and continuous from H into $\mathbb{R}$. Hence S_κ is Gateaux-differentiable in H and for every $u \in H$ the Gateaux derivative of S_κ at u is the functional defined by (2.14). If (u_n) is a sequence in H converging to u, then (u_n) is bounded both in H and in L^∞ and, since κ is locally uniformly continuous, $\kappa\circ u_n \to \kappa\circ u$ uniformly in $[0,1]$. Then

$$\left|\int_0^1 (\kappa(u_n)h\cdot i\dot u_n - \kappa(u)h\cdot i\dot u)\right|$$
$$\le \int_0^1 |\kappa(u_n)-\kappa(u)|\ |h\cdot i\dot u_n| + \int_0^1 |\kappa(u)|\ |h\cdot i(\dot u_n - \dot u)|$$

$$\leq C\|\kappa\circ u_n-\kappa\circ u\|_\infty\|h\|+C\|\dot u_n-\dot u\|_2\|h\|.$$

Therefore, the Gateaux derivative is a continuous operator from H into H' and consequently S_κ is of class C^1 in H and

$$S'_\kappa(u)h=-\int_0^1\kappa(u)h\cdot i\dot u\quad\text{for every } u,h\in H. \tag{2.15}$$

Now let us prove that the mapping $S'_\kappa\colon H\to H'$ defined by (2.15) is of class C^1. Fixing $u,h\in H$ and taking $\varepsilon\in\mathbb{R}$, $\varepsilon\neq0$, we have that

$$\frac{S'_\kappa(u+\varepsilon h)-S'_\kappa(u)}{\varepsilon}=-\left(\frac{\kappa(u+\varepsilon h)-\kappa(u)}{\varepsilon}i\dot u,\cdot\right)_2-(i\kappa(u+\varepsilon h)\dot h,\cdot)_2.$$

Since $\kappa\in C^1(\mathbb{R}^2)$ and, in particular, $\nabla\kappa$ and κ are locally uniformly continuous, one obtains that

$$\lim_{\varepsilon\to0}\frac{\kappa(u+\varepsilon h)-\kappa(u)}{\varepsilon}=\nabla\kappa(u)\cdot h\quad\text{and}\quad\lim_{\varepsilon\to0}\kappa(u+\varepsilon h)=\kappa(u)$$

uniformly in $[0,1]$. As before, one infers that

$$\lim_{\varepsilon\to0}\frac{S'_\kappa(u+\varepsilon h)-S'_\kappa(u)}{\varepsilon}=-\Big((\nabla\kappa(u)\cdot h)i\dot u+\kappa(u)i\dot h,\cdot\Big)_2.$$

The operator

$$h\mapsto-\Big((\nabla\kappa(u)\cdot h)i\dot u+\kappa(u)i\dot h,\cdot\Big)_2=:L_uh \tag{2.16}$$

from H into H' is linear and continuous, because the mappings $\nabla\kappa\circ u$, $\kappa\circ u$ are bounded and because of the embedding of H into L^∞. Thus S'_κ is Gateaux-differentiable at any $u\in H$ and its Gateaux derivative at u is the operator $L_u\colon H\to H'$ defined in (2.16). Now let $(u_n)\subset H$ be such that $u_n\to u$ in H. We have to prove that $L_{u_n}\to L_u$ in the uniform topology of the space of linear, continuous operators from H into H'. Writing

$$\begin{aligned}L_{u_n}h-L_uh &= ((\nabla\kappa(u)-\nabla\kappa(u_n))\cdot h)i\dot u_n,\cdot)_2\\ &\quad+((\nabla\kappa(u)\cdot h)i(\dot u-\dot u_n),\cdot)_2+\Big((\kappa(u)-\kappa(u_n))i\dot h,\cdot\Big)_2,\end{aligned}$$

using again the embedding of H into L^∞, one infers that

$$\|L_{u_n}-L_u\|\leq C\|\nabla\kappa\circ u-\nabla\kappa\circ u_n\|_\infty\|\dot u_n\|_2+C\|\dot u-\dot u_n\|_2+\|\kappa\circ u-\kappa\circ u_n\|_\infty.$$

Since (u_n) is bounded in H, $u_n\to u$ uniformly, and κ and $\nabla\kappa$ are locally uniformly continuous mappings, we derive that $\|L_{u_n}-L_u\|\to0$. Finally let us prove the last part of the lemma. By (2.9) one can plainly check that any classical solution of (2.4) is a critical point of $E_{1+\epsilon\kappa}$. On the contrary, let

$u \in \Omega$ be such that $E'_{1+\epsilon\kappa}(u) = 0$. Setting $v = i(1+\varepsilon\kappa(u))\|\dot u\|_2\dot u$, one has that $v \in L^2$ and

$$\int_0^1 \dot u \cdot \dot h = -\int_0^1 v \cdot h \quad \text{for every } h \in H;$$

namely, $\dot u \in H$ and its weak derivative equals v. Hence $\dot u$ is an absolutely continuous function, with $\dot u(0) = \dot u(1)$ and

$$\dot u(t) = \dot u(0) + \int_0^t v \quad \text{for every } t \in [0,1].$$

Since v is continuous, $\dot u$ is of class C^1 and $\ddot u = v$; namely u is a classical 1-periodic solution of $\ddot u = i(1+\varepsilon\kappa(u))\|\dot u\|_2\dot u$. Furthermore u is nonconstant since $u \in \Omega$. Hence u solves (2.4) and the proof is complete. □

Throughout the rest of this paper, with a slight abuse of notation, for every functional $F \in C^2(\Omega, \mathbb{R})$ and for every $u \in \Omega$ we will identify the differential $F'(u) \in H'$ with its unique representative in H and similarly we will consider $F''(u)$ as a continuous linear operator in H, writing $F'(u)h = \langle F'(u), h\rangle$ and $F''(u)[h,k] = \langle F''(u)h, k\rangle$ for every $h, k \in H$.

3. The unperturbed problem

In this section we study the problem

$$\begin{cases} \ddot u = i\|\dot u\|_2\dot u & \text{in } [0,1] \\ u(0) - u(1) = 0 = \dot u(0) - \dot u(1) \\ u \text{ nonconstant} \end{cases} \tag{3.1}$$

and the properties of the linearized problem at any solution of (3.1).

For every $u \in \Omega$ let $F_0(u) := \|\dot u\|_2 - S_1(u)$. One has that $F_0 = E_1$ (compare with (2.8)) and thus, according to Lemma 2.2, $F_0 \in C^2(\Omega, \mathbb{R})$ and solutions to (3.1) correspond to critical points of F_0 in Ω.

In fact solutions of (3.1) parametrize unit circles anywhere placed in the plane. Indeed the following result holds true.

Lemma 3.1. *The solutions of* (3.1) *are given by* $u(t) = z + e^{2\pi i n(t+s)}$ *where* $z \in \mathbb{R}^2$, $s \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

Proof. A solution u of (3.1) satisfies the linear equation $\ddot u = i\ell\dot u$ with $\ell = \|\dot u\|_2 > 0$. Hence $u(t) = z + qe^{i\ell t}$ with $z, q \in \mathbb{C}$. Imposing the periodicity condition $u(0) = u(1)$, since u is nonconstant, we get $e^{i\ell} = 1$, which yields $\ell = 2\pi n$ with $n \in \mathbb{N}$. Finally, $\ell = \|\dot u\|_2$ implies $|q| = 1$, so that we can write $q = e^{2\pi i n s}$ with $s \in \mathbb{R}$. □

Let us introduce some notation. First of all, for every $s \in \mathbb{R}$, set $\tau_s\omega := \omega(\cdot + s)$, where ω is defined in (2.6). Now, set

$$Z := \{z + \tau_s\omega : z \in \mathbb{R}^2,\ s \in \mathbb{R}\}. \tag{3.2}$$

Z is a manifold in H diffeomorphically parametrized by $\mathbb{R}^2 \times \mathbb{S}^1$ (indeed $\tau_{s+n}\omega = \tau_s\omega$ if $n \in \mathbb{Z}$). Moreover, using Lemma 3.1, one has

$$Z = \{u \in \Omega : F_0'(u) = 0,\ F_0(u) = \pi\}. \tag{3.3}$$

Now we show that $F_0''(\omega)$ is a Fredholm operator of index zero. Notice that, by Lemma 2.2 and by definition of ω one has

$$\langle F_0''(\omega)h, k\rangle = \frac{(\dot{h}, \dot{k})_2}{2\pi} - \frac{(\dot{\omega}, \dot{h})_2(\dot{\omega}, \dot{k})_2}{(2\pi)^3} - (\dot{h}, ik)_2 \quad \text{for every } h, k \in H. \tag{3.4}$$

In particular

$$F_0''(\omega)\omega = -\frac{1}{2\pi}\omega. \tag{3.5}$$

Lemma 3.2. *One has that* ker $F_0''(\omega) = T_\omega Z = \mathbb{R}^2 \oplus \mathbb{R}i\omega$, *where* $T_\omega Z$ *denotes the tangent space of* Z *at* ω.

Proof. Considering the definition of Z, one has that

$$e_j = \partial_{z_j}(z + \tau_s\omega)\big|_{(z,s)=(0,0)} \quad (j = 1, 2) \quad \text{and} \quad i\omega = \partial_s(z + \tau_s\omega)\big|_{(z,s)=(0,0)}$$

namely, $T_\omega Z = \mathbb{R}^2 \oplus \mathbb{R}i\omega$. By explicit computation one can check that $\mathbb{R}^2 \oplus \mathbb{R}i\omega \subseteq$ ker $F_0''(\omega)$. It remains to prove the opposite inequality. Let $h \in$ ker $F_0''(\omega)$. By (3.5) and since $[\omega] = 0$ and $F_0''(\omega)$ is symmetric, one has that $(\dot{\omega}, \dot{h})_2 = 0$. Then, substituting into (3.4), one finds that

$$\frac{(\dot{h}, \dot{k})_2}{2\pi} + (i\dot{h}, k)_2 = 0 \quad \text{for every } k \in H,$$

namely h is a weak solution in H of $\ddot{h} = 2\pi i\dot{h}$, and hence $h(t) = z + qe^{2\pi it}$ with $z, q \in \mathbb{C}$. Since $(\dot{\omega}, \dot{h})_2 = 0$, it must be that $q \in i\mathbb{R}$. In conclusion $h \in \mathbb{R}^2 \oplus \mathbb{R}i\omega$. □

Let $(T_\omega Z)^\perp = \{u \in H : \langle u, h\rangle = 0 \ \text{ for every } h \in T_\omega Z\}$. Note that $\omega \in (T_\omega Z)^\perp$ (use the characterization of $T_\omega Z$ given by Lemma 3.2). Setting $H_\omega := \{u \in (T_\omega Z)^\perp : \langle u, \omega\rangle = 0\}$ one has that

$$(T_\omega Z)^\perp = H_\omega \oplus \mathbb{R}\omega. \tag{3.6}$$

Lemma 3.3. *One has that* im $F_0''(\omega) = (T_\omega Z)^\perp$ *and* $F_0''(\omega)\big|_{(T_\omega Z)^\perp}$ *is a bijection of* $(T_\omega Z)^\perp$ *onto itself.*

Proof. By Lemma 3.2 $F_0''(\omega)$ is one-to-one in $(T_\omega Z)^\perp$. $F_0''(\omega)$ being a self-adjoint operator, for every $u \in H$ we have $\langle F_0''(\omega)u, h\rangle = \langle h, F_0''(\omega)u\rangle = 0$ for all $h \in \ker F_0''(\omega)$ and then $\operatorname{im} F_0''(\omega) \subseteq (T_\omega Z)^\perp$. Now let us prove the opposite inclusion. Fix $v \in (T_\omega Z)^\perp$. According to (3.6) we can write $v = w + s\omega$ for some $w \in H_\omega$ and $s \in \mathbb{R}$ and we have to find $u \in H_\omega$ and $\sigma \in \mathbb{R}$ such that

$$F_0''(\omega)(u + \sigma\omega) = w + s\omega. \tag{3.7}$$

Multiplying (3.7) by ω and using (3.5), the fact that $\omega \perp H_\omega$ and $u, w \in H_\omega$, we infer that $\sigma = -2\pi s$. Hence, by (3.5), (3.7) reduces to solving

$$F_0''(\omega)u = w, \qquad u \in H_\omega. \tag{3.8}$$

More explicitly, by (3.4) and since $(\dot u, \dot\omega)_2 = 0$ and $[w] = 0$, (3.8) becomes

$$\begin{cases} \dfrac{1}{2\pi}(\dot u, \dot h)_2 - (iu, \dot h)_2 = (\dot w, \dot h)_2 & \text{for every } h \in H \\ u \in H_\omega. \end{cases} \tag{3.9}$$

Observe that, by Lemma 3.2 and by (3.6), one has $H_\omega = \{u \in H : [u] = 0,\ (\dot u, \dot\omega)_2 = (\dot u, i\dot\omega)_2 = 0\}$. We can write any $w \in H_\omega$ according to its Fourier expansion in H_ω with respect to the basis $\{\omega_n\}_{n\in\mathbb{Z}\setminus\{0,1\}}$, where $\omega_n(t) := e^{2\pi int}/(2\pi n)$, namely:

$$w = \sum_{n\in\mathbb{Z}\setminus\{0,1\}} c_n\omega_n.$$

Notice that the coefficients c_n can be complex numbers and the product $c_n\omega_n$ is meant as a product in $\mathbb{C}$. Setting

$$u := \sum_{n\in\mathbb{Z}\setminus\{0,1\}} \frac{2\pi n}{n-1} c_n\omega_n,$$

we can easily recognize that u satisfies (3.9). This concludes the proof. □

4. The finite-dimensional reduction

Fixing a mapping $\kappa \in C^1(\mathbb{R}^2)$ and a value $\varepsilon \in \mathbb{R}$, we are now interested in finding solutions to the problem (2.4) when $|\varepsilon|$ is small. According to Lemma 2.2, solutions to (2.4) can be sought as critical points of the functional $F_\varepsilon(u) := E_{1+\varepsilon\kappa}(u)$ which is defined and of class C^2 on Ω. Let us observe that

$$F_\varepsilon(u) = F_0(u) - \varepsilon S_\kappa(u) \quad \text{for } u \in \Omega.$$

Here F_0 plays the role of an unperturbed functional and S_κ can be viewed as a perturbation.

As proved in Section 3, the functional F_0 admits a three-dimensional, nondegenerate, critical manifold Z. Our goal is to construct, for every $R > 0$ and for $|\varepsilon|$ small, a "perturbed" three-dimensional manifold Z_ε^R which is close and diffeomorphic to $Z_0^R := \{\omega + z : z \in \mathbb{R}^2,\ |z| < R\}$ and which constitutes a so-called *natural constraint* for the perturbed functional F_ε, namely every stationary point for F_ε restricted to Z_ε^R is in fact a free critical point for F_ε. This yields the finite-dimensional reduction of the problem and it can be accomplished by means of a suitable adaptation, due to Ambrosetti and Badiale [1], of the Lyapunov-Schmidt reduction method.

As a first result, we state the existence of the perturbed manifold Z_ε^R.

Lemma 4.1. *For every $R > 0$ there exist $\varepsilon_R > 0$ and a unique C^1 mapping $(\varepsilon, z) \mapsto \eta_\varepsilon(z) \in H$, defined in $(-\varepsilon_R, \varepsilon_R) \times D_R$ (here $D_R = \{z \in \mathbb{R}^2 : |z| < R\}$), such that*

$$\eta_0(z) = 0 \tag{4.1}$$

$$\eta_\varepsilon(z) \in (T_\omega Z)^\perp \text{ and } \|\eta_\varepsilon(z)\| < \|\omega\| \tag{4.2}$$

$$\omega + z + \eta_\varepsilon(z) \in \Omega \text{ with derivative in } H \tag{4.3}$$

$$F'_\varepsilon(\omega + z + \eta_\varepsilon(z)) \in T_\omega Z \tag{4.4}$$

for every $\varepsilon \in (-\varepsilon_R, \varepsilon_R)$ and $z \in D_R$.

For every $R > 0$ and for $\varepsilon \in (-\varepsilon_R, \varepsilon_R)$ the perturbed manifold is defined using the previous lemma, as $Z_\varepsilon^R := \{\omega + z + \eta_\varepsilon(z) : |z| < R\}$.

Proof. Let us consider $(T_\omega Z)^\perp$ as a Hilbert space endowed with the inner product induced by H. Set $\mathcal{O} := \{\eta \in (T_\omega Z)^\perp : \|\eta\| < \|\omega\|\}$ and let $\mathcal{F} \colon \mathbb{R} \times \mathbb{R}^2 \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^2 \to H$ be defined by

$$\mathcal{F}(\varepsilon, z, \eta, \lambda, \alpha) := F'_\varepsilon(\omega + z + \eta) - \lambda i\omega - \alpha.$$

We want to apply the implicit function theorem to the equation

$$\mathcal{F}(\varepsilon, z, \eta, \lambda, \alpha) = 0$$

in order to obtain (η, λ, α) as a function of (ε, z). Note that $\mathcal{F}(0, z, 0, 0, 0) = F'_0(\omega + z) = 0$ for every $z \in \mathbb{R}^2$, thanks to (3.2) and (3.3). Moreover, by Lemma 2.2, $\mathcal{F}$ is of class C^1 in its domain and

$$\frac{\partial \mathcal{F}}{\partial(\eta, \lambda, \alpha)}(\varepsilon, z, \eta, \lambda, \alpha)[v, \mu, \beta] = F''_\varepsilon(\omega + z + \eta)v - \mu i\omega - \beta$$

for every $(v, \mu, \beta) \in (T_\omega Z)^\perp \times \mathbb{R} \times \mathbb{R}^2$. In particular,

$$\frac{\partial \mathcal{F}}{\partial(\eta, \lambda, \alpha)}(0, z, 0, 0, 0)[v, \mu, \beta] = F''_0(\omega)v - \mu i\omega - \beta,$$

since, by direct computation, $F_0''(\omega+z) = F_0''(\omega)$. We have to show that the continuous, linear operator

$$\mathcal{T} := \frac{\partial \mathcal{F}}{\partial(\eta,\lambda,\alpha)}(0,z,0,0,0)\colon (T_\omega Z)^\perp \times \mathbb{R} \times \mathbb{R}^2 \to H$$

is bijective.

Injectivity of $\mathcal{T}$: Let $(v,\mu,\beta) \in (T_\omega Z)^\perp \times \mathbb{R} \times \mathbb{R}^2$ be such that $\mathcal{T}(v,\mu,\beta) = 0$, namely $F_0''(\omega)v = \mu i\omega + \beta$. By Lemma 3.3, $F_0''(\omega)v \in (T_\omega Z)^\perp$, whereas, by Lemma 3.2, $\mu i\omega + \beta \in T_\omega Z$. Hence $\mu = 0$, $\beta = 0$, and, again by Lemma 3.3, $v = 0$.

Surjectivity of $\mathcal{T}$: Fixing $u \in H$ we have to find $(v,\mu,\beta) \in (T_\omega Z)^\perp \times \mathbb{R} \times \mathbb{R}^2$ such that $\mathcal{T}(v,\mu,\beta) = u$. By Lemma 3.2 we have that $H = \mathbb{R}^2 \oplus \mathbb{R}i\omega \oplus (T_\omega Z)^\perp$ and thus we can write $u = [u] + is\omega + h$ with $s \in \mathbb{R}$ and $h \in (T_\omega Z)^\perp$. By Lemma 3.3, there exists $v \in (T_\omega Z)^\perp$ such that $F_0''(\omega)v = h$. Hence, $\mathcal{T}(v,-s,-[u]) = F_0''(\omega)v + is\omega + [u] = u$.

Now we are in position to apply the implicit function theorem which ensures that for every $R > 0$, since $\overline{D}_R$ is compact, there exist $\varepsilon_R > 0$, a neighbourhood U_R of $\overline{D}_R$ and a C^1 map $\phi\colon (-\varepsilon_R,\varepsilon_R) \times U_R \to \mathcal{O} \times \mathbb{R} \times \mathbb{R}^2$ such that $\phi(0,z) = (0,0,0)$ and

$$\mathcal{F}(\varepsilon,z,\phi(\varepsilon,z)) = 0 \tag{4.5}$$

for every $z \in U_R$ and $\varepsilon \in (-\varepsilon_R,\varepsilon_R)$. Set

$$\phi(\varepsilon,z) =: (\eta_\varepsilon(z),\lambda_\varepsilon(z),\alpha_\varepsilon(z)) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^2 \quad \text{for } (\varepsilon,z) \in (-\varepsilon_R,\varepsilon_R) \times U_R. \tag{4.6}$$

Hence (4.1) and (4.2) are fulfilled. Moreover the fact that $\|\eta_\varepsilon(z)\| < \|\omega\|$ implies that $\omega + z + \eta_\varepsilon(z) \in \Omega$. In addition, writing explicitly (4.5) by means of (4.6), one obtains

$$F_\varepsilon'(\omega + z + \eta_\varepsilon(z)) = \alpha_\varepsilon(z) + \lambda_\varepsilon(z) i\omega. \tag{4.7}$$

Notice that (4.7) is equivalent to (4.4), by Lemma 3.2. Fixing $(\varepsilon,z) \in (-\varepsilon_R,\varepsilon_R) \times \overline{D}_R$ and setting $u = \omega + z + \eta_\varepsilon(z)$, it remains to prove that $\dot u \in H$. Since

$$F_\varepsilon'(u)h = \frac{(\dot u,\dot h)_2}{\|\dot u\|_2} + ((1+\varepsilon\kappa(u))i\dot u,h)_2 \quad \text{for every } h \in H,$$

from (4.7) it follows that

$$\begin{aligned}\frac{(\dot u,\dot h)_2}{\|\dot u\|_2} &= -((1+\varepsilon\kappa(u))i\dot u,h)_2 + [\alpha_\varepsilon(z)]\cdot[h] + \lambda_\varepsilon(z)(i\dot\omega,\dot h)_2 \\ &= -((1+\varepsilon\kappa(u))i\dot u,h)_2 + (\alpha_\varepsilon(z),h)_2 + 2\pi\lambda_\varepsilon(z)(\dot\omega,h)_2,\end{aligned} \tag{4.8}$$

where (4.8) is obtained by an integration by parts and because $i\dot{\omega} = -2\pi\omega$. Hence we get

$$(\dot{u}, \dot{h})_2 = -(v, h)_2 \quad \text{for every } h \in H,$$

where $v := \|\dot{u}\|_2 \left((1 + \varepsilon\kappa(u))i\dot{u} - \alpha_\varepsilon(z) - 2\pi\lambda_\varepsilon(z)\dot{\omega}\right)$. Since $v \in L^2$, we infer that $\dot{u} \in H$. This completes the proof. □

As a next step, we show that Z_ε^R is a natural constraint for F_ε. More precisely, define $f_\varepsilon \colon D_R \to \mathbb{R}$ by setting

$$f_\varepsilon(z) := F_\varepsilon(\omega + z + \eta_\varepsilon(z)) \quad \text{for } z \in D_R. \tag{4.9}$$

The regularity of F_ε and $\eta_\varepsilon(z)$ ensures that $f_\varepsilon \in C^1(D_R)$.

Lemma 4.2. *If $\bar{z} \in D_R$ is a critical point for f_ε, for some $\varepsilon \in (-\varepsilon_R, \varepsilon_R)$, then $F'_\varepsilon(\omega + \bar{z} + \eta_\varepsilon(\bar{z})) = 0$.*

Proof. By the definition (4.9) one has

$$\partial_{z_j} f_\varepsilon(z) = \langle F'_\varepsilon(\omega+z+\eta_\varepsilon(z)), e_j + \partial_{z_j}\eta_\varepsilon(z)\rangle = \langle F'_\varepsilon(\omega+z+\eta_\varepsilon(z)), e_j\rangle \quad (j = 1, 2) \tag{4.10}$$

because, by (4.2), also $\partial_{z_j}\eta_\varepsilon(z) \in (T_\omega Z)^\perp$ whereas $F'_\varepsilon(\omega + z + \eta_\varepsilon(z)) \in T_\omega Z$ (see (4.4)). Let $\bar{z} \in D_R$ be a critical point for f_ε, for some $\varepsilon \in (-\varepsilon_R, \varepsilon_R)$. Setting $u = \omega + \bar{z} + \eta_\varepsilon(\bar{z})$, by (4.10) one has $\langle F'_\varepsilon(u), e_j\rangle = 0$ for $j = 1, 2$. Therefore, by (4.4) and by Lemma 3.2, one has

$$F'_\varepsilon(u) = \lambda i\omega$$

for some $\lambda \in \mathbb{R}$. By (4.3) $\dot{u} \in H$ and we can compute

$$\langle F'_\varepsilon(u), \dot{u}\rangle = \langle \lambda i\omega, \dot{u}\rangle = (\lambda i\dot{\omega}, \ddot{u})_2 = -(\lambda i\ddot{\omega}, \dot{u})_2 = 2\pi\lambda(\dot{\omega}, \dot{u})_2 = 2\pi\lambda\langle\omega, u\rangle.$$

Using the explicit expression of $F'_\varepsilon(u)$ we also have

$$F'_\varepsilon(u)\dot{u} = \frac{(\dot{u}, \ddot{u})_2}{\|\dot{u}\|_2} + ((1 + \varepsilon\kappa(u))i\dot{u}, \dot{u})_2 = 0$$

by easy computations. Hence $\lambda\langle\omega, u\rangle = 0$. In fact

$$\langle\omega, u\rangle = \|\omega\|^2 + \langle\omega, \eta_\varepsilon(\bar{z})\rangle \geq \|\omega\|(\|\omega\| - \|\eta_\varepsilon(\bar{z})\|) > 0$$

by (4.2). In conclusion $\lambda = 0$ and $F'_\varepsilon(u) = 0$. □

As a last result of this section we provide the expansion of $f_\varepsilon(z)$ with respect to ε in a neighbourhood of $\varepsilon = 0$.

Lemma 4.3. *For every $R > 0$ one has that $f_\varepsilon(z) = F_0(\omega) - \varepsilon M_0(z) + o(\varepsilon)$ as $\varepsilon \to 0$, uniformly on compact sets of D_R, where for every $z \in \mathbb{R}^2$*

$$M_0(z) = \int_{D_1(z)} \kappa(q)\; dq. \tag{4.11}$$

Proof. Since the mapping $(\varepsilon, z) \mapsto f_\varepsilon(z)$ belongs to $C^1((-\varepsilon_R, \varepsilon_R) \times D_R)$ one has that $f_\varepsilon(z) = f_0(z) + \varepsilon\big[\partial_\varepsilon f_\varepsilon(z)\big]_{\varepsilon=0} + o(\varepsilon)$ as $\varepsilon \to 0$, uniformly on compact sets of D_R. From the definition of f_ε it follows that $f_0(z) = F_0(\omega+z) = F_0(\omega)$ and

$$\partial_\varepsilon f_\varepsilon(z) = F_0'(\omega+z+\eta_\varepsilon(z))\partial_\varepsilon\eta_\varepsilon(z) - S_\kappa(\omega+z+\eta_\varepsilon(z)) - \varepsilon S_\kappa'(\omega+z+\eta_\varepsilon(z))\partial_\varepsilon\eta_\varepsilon(z).$$

Noting that, by (4.2), $\partial_\varepsilon\eta_\varepsilon(z) \in (T_\omega Z)^\perp$ whereas $F_0'(\omega + z + \eta_\varepsilon(z)) \in T_\omega Z$, and, fixing $\varepsilon = 0$, one infers that

$$\big[\partial_\varepsilon f_\varepsilon(z)\big]_{\varepsilon=0} = -S_\kappa(\omega + z)$$

and the conclusion follows using (2.7). □

5. Proof of the main results

Proof of Proposition 1.1. By contradiction, assume that there exists a simple helicoidal trajectory for some $\varepsilon \neq 0$. Hence, there is a C^2 mapping $p_\perp \colon \mathbb{R} \to \mathbb{R}^2$ periodic with some positive period μ, injective on $[0, \mu)$ and solving $m\ddot{p}_\perp = -ie(b_0 + \varepsilon\beta(p_1))\dot{p}_\perp$ (here p_1 is the first component of $p_\perp$). Testing the equation with e_1 and integrating over $[0, \mu]$ one gets

$$0 = \varepsilon \int_0^\mu \beta(p_1)e_1 \cdot i\dot{p}_\perp. \tag{5.1}$$

Using the Gauss-Green theorem, since $\mathrm{div}_p(\beta(p_1)e_1) = \beta'(p_1)$, one can write

$$\Big|\int_0^\mu \beta(p_1)e_1 \cdot i\dot{p}_\perp\Big| = \Big|\int_D \beta'(x)\ dx\ dy\Big|, \tag{5.2}$$

where D is the bounded domain in $\mathbb{R}^2$ enclosed by range $p_\perp$. Since $\varepsilon \neq 0$ and β is strictly monotone, (5.1) and (5.2) yield a contradiction. □

Remark 5.1. By the definitions (1.5) and (4.11) of M and M_0 respectively, and by (2.2), one has

$$M_0(z) = \frac{1}{b_0 r^2} M(rz). \tag{5.3}$$

Moreover, by (2.7), it turns out that $M_0(z) = S_\kappa(\omega + z)$. Hence M_0 and consequently also M are of class C^2 (see Lemma 2.2). In particular,

$$\partial_{z_j} M_0(z) = S_\kappa'(\omega + z)e_j \quad \text{for all } z \in \mathbb{R}^2 \text{ and } j = 1, 2. \tag{5.4}$$

Proof of Theorem 1.2. Let ε_n, $p(t, \varepsilon_n)$, μ_n and z_n be given as in the statement of the theorem. Set

$$u_n(t) = \frac{1}{r} p_\perp(\mu_n t, \varepsilon_n).$$

One has that

$$u_n \in \Omega, \quad [u_n] = \frac{z_n}{r}, \quad |\dot{u}_n| = \frac{v\mu_n}{r}. \tag{5.5}$$

According to Lemma 2.1, u_n solves problem (2.4) with $\varepsilon = \varepsilon_n$. Hence $0 = F'_{\varepsilon_n}(u_n) = F'_0(u_n) - \varepsilon_n S'_\kappa(u_n)$. By (2.11) one has that $F'_0(u_n)e_j = 0$ and then, since $\varepsilon_n \neq 0$,

$$S'_\kappa(u_n)e_j = 0 \quad \text{for all } n \in \mathbb{N} \text{ and } j = 1, 2. \tag{5.6}$$

By (5.5) and by the assumptions on z_n and μ_n, the sequence (u_n) is bounded in H. After extracting subsequences, we may assume

$$\mu_n \to \mu, \quad u_n \to u \text{ weakly in } H, \tag{5.7}$$

for some $\mu \geq 0$ and $u \in H$. In particular, since the sequence (μ_n) is assumed to be bounded away from 0, one has $\mu > 0$. Set $\bar{\mu}_n = v\mu_n/r$ and $\bar{\mu} = v\mu/r$. Since

$$(\dot{u}_n, \dot{h})_2 + \bar{\mu}_n((1 + \varepsilon_n\kappa(u_n))h, i\dot{u}_n)_2 = 0 \quad \text{for all } h \in H, \tag{5.8}$$

passing to the limit, one obtains that u solves $\ddot{u} = \bar{\mu}i\dot{u}$. Moreover, using Rellich's theorem, one finds that $(u_n, i\dot{u}_n)_2 \to (u, i\dot{u})_2$, and then, taking $h = u_n$ in (5.8), one infers that $\|\dot{u}_n\|_2^2 \to \|\dot{u}\|_2^2$. Hence $u_n \to u$ strongly in H. As a consequence, u is a nonconstant 1-periodic solution of $\ddot{u} = i\|\dot{u}\|_2\dot{u}$. Therefore, using Lemma 3.1, $u(t) = [u] + e^{2\pi i\bar{n}(t+s)}$ for some $\bar{n} \in \mathbb{N}$ and $s \in \mathbb{R}$. In addition, one has that $[u] = \bar{z}/r$. Finally, passing to the limit in (5.6) and making easy computations, one obtains $0 = S'_\kappa(u)e_j = S'_\kappa(\omega + \bar{z}/r)e_j$ for $j = 1, 2$, namely $\nabla M_0(\bar{z}/r) = 0$, because of (5.4). Hence, by (5.3), $\nabla M(\bar{z}) = 0$. □

Proof of Theorem 1.3. Let $\bar{z} \in \mathbb{R}^2$ be a nondegenerate critical point of M. Then $z_0 := \bar{z}/r$ is a nondegenerate critical point of M_0. Fix $R > |z_0|$ and set $u_{\varepsilon,z} := \omega + z + \eta_\varepsilon(z)$ for any $\varepsilon \in (-\varepsilon_R, \varepsilon_R)$ and $z \in D_R$, where ε_R and $\eta_\varepsilon(z)$ are given by Lemma 4.1. Note that by (4.1) $u_{0,z} = \omega + z$. Define $G\colon (-\varepsilon_R, \varepsilon_R) \times D_R \to \mathbb{R}^2$ by setting:

$$G_j(\varepsilon, z) := S'_\kappa(u_{\varepsilon,z})e_j \quad (j = 1, 2).$$

In particular, by (5.4) one has

$$G(0, z) = \nabla M_0(z) \quad \text{for all } z \in D_R. \tag{5.9}$$

Moreover, by (2.11) and (4.10) one has that

$$-\varepsilon G_j(\varepsilon, z) = F'_\varepsilon(u_{\varepsilon,z})e_j = \partial_{z_j} f_\varepsilon(z) \quad \text{for all } (\varepsilon, z) \in (-\varepsilon_R, \varepsilon_R) \times D_R. \tag{5.10}$$

In addition, observe that G is of class C^1 on its domain, because of the C^1 regularity of S'_κ (see Lemma 2.2), and of the mapping $(\varepsilon, z) \mapsto u_{\varepsilon,z}$ (see

Lemma 4.1). Since z_0 is a nondegenerate critical point of M_0, by (5.9), $G(0,z_0)=0$ and $\partial_z G(0,z_0)=\nabla^2 M_0(z_0)$ is invertible. Therefore, by the implicit function theorem, there exist an open neighbourhood $I\subset(-\varepsilon_R,\varepsilon_R)$ of 0 and a C^1 mapping $\varepsilon\in I\mapsto z_\varepsilon\in D_R$ such that $G(\varepsilon,z_\varepsilon)=0$ for every $\varepsilon\in I$. Hence, by (5.10) we get $\nabla f_\varepsilon(z_\varepsilon)=0$ for every $\varepsilon\in I$. Then Lemmata 2.2 and 4.2 imply that $u_\varepsilon := u_{\varepsilon,z_\varepsilon}$ is a solution of (2.4) for every $\varepsilon\in I$. Since the mapping $\varepsilon\mapsto z_\varepsilon$ is of class C^1, also $\varepsilon\mapsto u_\varepsilon$ belongs to $C^1(I,H)$. Then a standard boot-strap argument provides the C^1 regularity from I into $C^2([0,1],\mathbb{R}^2)$. Finally, using Lemma 2.1, for $\varepsilon\in I$ problem $(P)_\varepsilon$ admits a helicoidal solution $p(t,\varepsilon)$ with $p_\perp(t,\varepsilon)=ru_\varepsilon(t/\mu_\varepsilon)$, being $\mu_\varepsilon=r\|\dot u_\varepsilon\|_2/v$. In particular one can check that $p_\perp(t,0)=\bar z+re^{i\nu t}$ where $\nu=|eb_0|/m$ and $r=mv/|eb_0|$. Notice that the convergence in the C^2 topology (but C^1 is enough) also ensures that the helicoidal trajectory corresponding to $p_\perp(t,\varepsilon)$ is simple, for small $|\varepsilon|$. □

Proof of Theorem 1.4. Let A be a nonempty open, bounded subset of $\mathbb{R}^2$ such that $\sup_{z\in A}M(z)>\max_{z\in\partial A}M(z)$. By (5.3) one has that $\sup_{z\in A_0}M_0(z)>\max_{z\in\partial A_0}M_0(z)$, where $A_0=\frac{1}{r}A$. Fix $R>0$ such that $\overline{A_0}\subset D_R$ and take ε_R and $\eta_\varepsilon(z)$ according to Lemma 4.1. Using Lemma 4.3, for $|\varepsilon|$ small, $\varepsilon\neq 0$, one obtains that $\sup_{z\in A_0}f_\varepsilon(z)>\max_{z\in\partial A_0}f_\varepsilon(z)$ or $\inf_{z\in A_0}f_\varepsilon(z)<\min_{z\in\partial A_0}f_\varepsilon(z)$ according to the sign of ε. In both cases there exists $\bar z_\varepsilon\in A_0$ which is an extremal point of f_ε in A_0. Hence by Lemmata 4.2 and 2.2 the mapping $u_\varepsilon=\omega+\bar z_\varepsilon+\eta_\varepsilon(\bar z_\varepsilon)$ is a solution of (2.4) for every ε in a neigbourhood of 0. As a consequence, by Lemma 2.1, for small $|\varepsilon|$ problem $(P)_\varepsilon$ admits a helicoidal solution $p(t,\varepsilon)$ with $p_\perp(t,\varepsilon)=ru_\varepsilon(t/\mu_\varepsilon)$, and $\mu_\varepsilon=r\|\dot u_\varepsilon\|_2/v$. Now, let z_ε be the average of $p_\perp(t,\varepsilon)$ over $[0,\mu_\varepsilon]$. One has that

$$z_\varepsilon=r[u_\varepsilon]=r\left([\omega]+\bar z_\varepsilon+[\eta_\varepsilon(\bar z_\varepsilon)]\right)=r\bar z_\varepsilon \tag{5.11}$$

(see (4.2)). In particular $z_\varepsilon\in A$. Moreover, by Lemma 4.3, one has $M_0(\bar z_\varepsilon)\to\sup_{A_0}M_0$ and then $M(z_\varepsilon)\to\sup_A M$ as $\varepsilon\to 0$. Now let us prove that

$$\|u_\varepsilon-\bar z_\varepsilon-\omega\|_{C^2}\to 0 \quad \text{as } \varepsilon\to 0. \tag{5.12}$$

Firstly, since the mapping $(\varepsilon,z)\mapsto\eta_\varepsilon(z)\in H$ is of class C^1 and $\bar z_\varepsilon$ runs through a bounded set, by (4.1) one has that $\|u_\varepsilon-\bar z_\varepsilon-\omega\|=\|\eta_\varepsilon(\bar z_\varepsilon)\|\to 0$ as $\varepsilon\to 0$. Then $\ddot u_\varepsilon=i\|\dot u_\varepsilon\|_2(1+\varepsilon\kappa(u_\varepsilon))\dot u_\varepsilon\to i\|\dot\omega\|_2\dot\omega=2\pi i\dot\omega=\ddot\omega$ strongly in L^2 as $\varepsilon\to 0$. Hence $u_\varepsilon-\bar z_\varepsilon\to\omega$ strongly in H^2 as $\varepsilon\to 0$. By standard arguments, we have convergence in C^2 and (5.12) is proved. Finally, recalling

(5.11), observe that

$$\rho_\varepsilon(t) := p_\perp(t,\varepsilon) - z_\varepsilon - re^{i\nu t} = ru_\varepsilon\left(t/\mu_\varepsilon\right) - r\bar{z}_\varepsilon - re^{i\|\dot{u}_\varepsilon\|_2 t/\mu_\varepsilon}$$

and then, by (5.12), $\rho_\varepsilon \to 0$ in the C^2 topology, as $\varepsilon \to 0$. This convergence also implies that the helicoidal trajectory corresponding to $p_\perp(t,\varepsilon)$ is simple, for small $|\varepsilon|$. □

Proof of Theorem 1.5. Let $\rho := |b_0|/v$ and $r := m/(|e|\rho)$. Moreover, set

$$\overline{M}(z,r) = \frac{1}{\pi}\int_{D_1} b(z+rq)\ dq \quad \text{for } z\in\mathbb{R}^2 \text{ and } r>0.$$

Observe that $\overline{M}(z,r) = M(z)/(\pi r^2)$, where $M(z)$, defined in (1.5), actually depends also on r. Notice that in the above definition of $\overline{M}(z,r)$ in fact we can take any $r\in\mathbb{R}$ and $b\in C^2(\mathbb{R}^2)$ implies $\overline{M}\in C^2(\mathbb{R}^2\times\mathbb{R})$. Moreover $\overline{M}(z,0) = b(z)$, $\partial_z\overline{M}(z,0) = \nabla b(z)$ and $\partial^2_{zz}\overline{M}(z,0) = \nabla^2 b(z)$ for all $z\in\mathbb{R}^2$. First, assume we are in the case (i), namely b admits a nondegenerate critical point $\bar{z}\in\mathbb{R}^2$. Then $\partial_z\bar{M}(\bar{z},0) = 0$ and $\partial^2_{zz}\overline{M}(\bar{z},0)$ is invertible. By the implicit function theorem there exists a mapping $r\mapsto z_r\in\mathbb{R}^2$ of class C^1 on $[0,\bar{r})$ such that $z_0 = \bar{z}$ and z_r is a nondegenerate critical point of $\overline{M}(\cdot,r)$, and consequently of M, for all $r\in(0,\bar{r})$. Hence we can apply Theorem 1.3, according to which, for every $r\in(0,\bar{r})$ there exist $\varepsilon_r>0$ and, for all $\varepsilon\in(-\varepsilon_r,\varepsilon_r)$ a helicoidal trajectory $p(t,\varepsilon,r)$ such that the mapping $\varepsilon\mapsto p_\perp(\cdot,\varepsilon,r)$ is of class C^1 from $(-\varepsilon_r,\varepsilon_r)$ into the space of C^2 periodic functions and moreover $p_\perp(t,0,r) = z_r + re^{i\nu t}$. Hence $\lim_{r\to0}\lim_{\varepsilon\to0}\|p_\perp(\cdot,\varepsilon,r) - \bar{z}\|_{C^2} = \lim_{r\to0}\|z_r - \bar{z} + re^{i\nu t}\|_{C^2} = 0$ which is the desired conclusion. Clearly one can pass from the parameter r into ρ according to the definition given at the beginning of the proof.

Now let us discuss the case (ii), assuming that there exists a nonempty open bounded set $A\subset\mathbb{R}^2$ such that $\max_{z\in\partial A} b(z) < \sup_{z\in A} b(z)$. Since $\overline{M}(\cdot,r)\to b$ as $r\to0$, uniformly on compact sets, for $r>0$ sufficiently small one has that $\max_{z\in\partial A}\overline{M}(z,r) < \sup_{z\in A}\overline{M}(z,r)$. Hence we can apply Theorem 1.4: for every $r>0$ small enough there exist $\varepsilon_r>0$ and, for all $\varepsilon\in(-\varepsilon_r,\varepsilon_r)$, a helicoidal trajectory $p(t,\varepsilon,r)$. Moreover, denoting by $z_{\varepsilon,r}$ the average of $p_\perp(\cdot,\varepsilon,r)$ over its minimal period, we know that $z_{\varepsilon,r}\in A$, $\lim_{\varepsilon\to0}(p_\perp(t,\varepsilon,r) - z_{\varepsilon,r}) = re^{i\nu t}$ in the C^2 topology, and $\lim_{\varepsilon\to0}\overline{M}(z_{\varepsilon,r},r) = \sup_{z\in A}\overline{M}(z,r)$. From the estimate

$$|b(z_{\varepsilon,r}) - \overline{M}(z_{\varepsilon,r},r)| \le \frac{1}{\pi r^2}\int_{D_r(z_{\varepsilon,r})}|b(z) - b(z_{\varepsilon,r})|\ dz \le r\sup_{z\in D_r(z_{\varepsilon,r})}|\nabla b(z)|$$

one infers that

$$\lim_{r\to 0}\lim_{\varepsilon\to 0}(b(z_{\varepsilon,r})-\overline{M}(z_{\varepsilon,r},r))=0.$$

Moreover, since $\overline{M}(\cdot,r)\to b$ as $r\to 0$, uniformly on compact sets, we have

$$\lim_{r\to 0}\sup_{z\in A}\overline{M}(z,r)=\sup_{z\in A}b(z).$$

In conclusion $\lim_{r\to 0}\lim_{\varepsilon\to 0}b(z_{\varepsilon,r})=\sup_{z\in A}b(z)$. □

Proof of Theorem 1.6. Since $b\in L^1(\mathbb{R}^2)+L^2(\mathbb{R}^2)$, also the function κ defined by (2.2) belongs to the same space. This easily implies that the mapping M_0 defined by (4.11) is such that $M_0(z)\to 0$ as $|z|\to+\infty$. If M_0 is not identically null, then the assumptions of Theorem 1.4 are satisfied with $A=D_R$ for some $R>0$. Hence the thesis follows. Now suppose that $M_0\equiv 0$. We shall show that $\kappa\equiv 0$ and then also in this case the conclusion trivially follows. Denoting by χ the characteristic function of the unit disc, for all $z\in\mathbb{R}^2$ we have that $\kappa*\chi=M_0$, where $*$ is the standard convolution operator. Hence $\kappa*\chi\equiv 0$. Since $\kappa\in L^1(\mathbb{R}^2)+L^2(\mathbb{R}^2)$ its Fourier transform $\hat{\kappa}$ belongs to $L^2_{loc}(\mathbb{R}^2)$ and $\widehat{\kappa*\chi}=\hat{\kappa}\,\hat{\chi}$. One knows that $\hat{\chi}(q)=J_1(2\pi|q|)/|q|$ for $q\in\mathbb{R}^2\setminus\{0\}$, where J_1 is the Bessel function of the first kind. In particular J_1 and consequently also $\hat{\chi}$ admit only a countable set of zeroes. Since $\hat{\kappa}\,\hat{\chi}=0$, we infer that $\hat{\kappa}=0$ almost everywhere on $\mathbb{R}^2$ and then $\kappa\equiv 0$. This concludes the proof. □

6. A related geometric problem

Let us consider a cylinder $\mathcal{C}$ in $\mathbb{R}^3$ with infinite length and circular section of radius 1. The mean curvature of $\mathcal{C}$ at any point $p\in\mathcal{C}$ is $1/2$. After introducing a Cartesian coordinate system in $\mathbb{R}^3$ such that the third axis coincides with the rotational axis of the cylinder $\mathcal{C}$, the mapping $\overline{U}\colon\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^3$ defined, in polar coordinates, by

$$\overline{U}(r,\theta)=(\cos\theta,\sin\theta,\log r)$$

provides a conformal parametrization of $\mathcal{C}$.

Given a regular mapping $\mathtt{H}\colon\mathbb{R}^3\to\mathbb{R}$, we call $\mathtt{H}$-*cylinder* a surface $\mathcal{M}$ in $\mathbb{R}^3$ having mean curvature $\mathtt{H}(p)$ at every point $p\in\mathcal{M}$ and admitting a conformal parametrization $U\colon\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^3$ of class C^2, diffeomorphic to $\overline{U}$.

Such a mapping U has to satisfy the equation of the prescribed mean curvature together with the conformality conditions. In polar coordinates they are respectively:

$$\Delta U=2\mathtt{H}(U)\frac{1}{r}U_r\wedge U_\theta \tag{6.1}$$

$$U_r \cdot U_\theta = 0 = r^2|U_r|^2 - |U_\theta|^2. \tag{6.2}$$

Now assume $\mathtt{H}$ of class C^1 and depending just on two variables; namely there exists a unit vector $n \in \mathbb{R}^3$ such that

$$\frac{\partial \mathtt{H}}{\partial n}(p) = 0 \quad \text{for every } p \in \mathbb{R}^3.$$

Without loss of generality, up to a rotation of the coordinate system, we can take $n = e_3$, so that $\mathtt{H}$ depends only on the two first Cartesian coordinates $p_1 = p \cdot e_1$ and $p_2 = p \cdot e_2$ (here (e_1, e_2, e_3) is the canonical basis in $\mathbb{R}^3$). In this situation we can look for $\mathtt{H}$-cylinders admitting conformal parametrizations of the form

$$U(r, \theta) = (v_1(\theta), v_2(\theta), \log r) \tag{6.3}$$

with $v_1, v_2 \colon \mathbb{R} \to \mathbb{R}$ periodic. Observe that $\mathtt{H}$-cylinders of this kind are invariant under translation with respect to the third axis and we will call them p_3-*invariant* $\mathtt{H}$-cylinders. Clearly there could exist also $\mathtt{H}$-cylinders which are not p_3-invariant; let us think, e.g., of the Delaunay surfaces (see [4] and also [5]).

We can see that for prescribed mean curvatures $\mathtt{H}$ depending only on the first two components p_1 and p_2, the problem of p_3-invariant $\mathtt{H}$-cylinders is equivalent to the problem of closed curves in the plane with prescribed curvature $\mathtt{K}(p_1, p_2) = 2\mathtt{H}(p_1, p_2)$.

Indeed, using complex notation and setting $v = (v_1, v_2) = v_1 + iv_2$, the equations (6.1) and (6.2) for U of the form (6.3) are equivalent, respectively, to

$$\ddot{v} = 2i\mathtt{H}(v)\dot{v} \tag{6.4}$$

$$|\dot{v}| = 1. \tag{6.5}$$

Notice that in the argument of $\mathtt{H}$ we can omit the entry corresponding to the third component, since we are assuming that $\partial_{p_3}\mathtt{H} \equiv 0$.

In fact, the system (6.4)–(6.5) describes analytically the problem of curves in the plane which are parametrized by arclength and with prescribed curvature $\mathtt{K} = 2\mathtt{H}$. Now we introduce the following definition.

Given a regular mapping $\mathtt{K} \colon \mathbb{R}^2 \to \mathbb{R}$ let us call $\mathtt{K}$-*loop* a closed curve Γ in $\mathbb{R}^2$ such that for every $z \in \Gamma$ the curvature of Γ at z equals $\mathtt{K}(z)$.

Hence, if $\mathcal{C}$ is a p_3-invariant $\mathtt{H}$-cylinder, then the intersection of $\mathcal{C}$ with the plane $p_3 = 0$ is a $2\mathtt{H}$-loop. Conversely, if Γ is a $\mathtt{K}$-loop and v is a parametrization of Γ by arclength, then the mapping U defined by (6.3) parametrizes a p_3-invariant $(\mathtt{K}/2)$-cylinder.

Let us focus on the perturbative case

$$\mathtt{H}_\varepsilon(p_1,p_2,p_3) = \tfrac{1}{2} + \varepsilon\kappa(p_1,p_2) \quad \text{for } (p_1,p_2,p_3)\in\mathbb{R}^3. \tag{6.6}$$

Under this assumption on $\mathtt{H}$, according to the previous discussion, the problem of p_3-invariant $\mathtt{H}_\varepsilon$-cylinders reduces to problem (2.3) and any result obtained in the previous sections and stated in terms of helicoids can be equivalently phrased in the geometrical frame considered here. For instance, the analogue of Theorem 1.3 is the following:

Theorem 6.1. *Let $\mathtt{H}_\varepsilon$ be as in* (6.6) *with $\kappa\in C^1(\mathbb{R}^2)$ and let M_0 be as in* (4.11)*. If $z\in\mathbb{R}^2$ is a nondegenerate critical point of M_0, then, for small $|\varepsilon|$, there exists a p_3-invariant $\mathtt{H}_\varepsilon$-cylinder $\mathcal{C}_\varepsilon$. In particular, $\Gamma_0 := \mathcal{C}_0\cap\{p_3=0\}$ is a unit circle centered at z. In addition, for small $|\varepsilon|$, $\Gamma_\varepsilon := \mathcal{C}_\varepsilon\cap\{p_3=0\}$ is a $2\mathtt{H}_\varepsilon$-loop admitting a uniform, 1-periodic parametrization u_ε such that the mapping $\varepsilon\mapsto u_\varepsilon$ is of class C^1 in the space of C^2 functions.*

Similarly, in correspondence of Theorem 1.4 we have the following:

Theorem 6.2. *Let $\mathtt{H}_\varepsilon$ and M_0 be as in Theorem* 6.1*. If there exists a nonempty, open, bounded set $A\subset\mathbb{R}^2$ such that $\max_{\partial A} M_0 < \sup_A M_0$ (or $\min_{\partial A} M_0 > \inf_A M_0$), then, for small $|\varepsilon|$, there exists a p_3-invariant $\mathtt{H}_\varepsilon$-cylinder $\mathcal{C}_\varepsilon$. Moreover, for every small $|\varepsilon|$ there exist a point $z_\varepsilon\in A$ and a uniform,* 1*-periodic parametrization u_ε of the $2\mathtt{H}_\varepsilon$-loop $\Gamma_\varepsilon=\mathcal{C}_\varepsilon\cap\{p_3=0\}$ satisfying: $M_0(z_\varepsilon)\to\sup_A M_0$ (or $M_0(z_\varepsilon)\to\inf_A M_0$, respectively) and $u_\varepsilon(t)-(z_\varepsilon+e^{2\pi i t})\to 0$ in C^2.*

Clearly, analogous results to Theorems 1.2, 1.5, 1.6 and Proposition 1.1 can be stated.

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